Minimum LM Unit Root Test with One Structural Break

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Abstract

In this paper, we propose a minimum LM unit root test that endogenously determines a structural break in intercept and trend. Critical values are provided, and size and power properties are compared to the endogenous one-break unit root test of Zivot and Andrews (1992). Nunes, Newbold, and Kuan (1997) and Lee and Strazicich (2001) previously demonstrated that the Zivot and Andrews test exhibits size distortions in the presence of a break under the null. In contrast, the one-break minimum LM unit root test exhibits no size distortions in the presence of a break under the null. As such, rejection of the null unambiguously implies a trend stationary process.

JEL classification: C12, C15, and C22

Key words: Lagrange Multiplier, Unit Root Test, Structural Break, and Endogenous Break

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1. INTRODUCTION

The importance of allowing for structural breaks in unit root tests is now well documented in the literature. Whereas Perron (1989) assumed that the break point was known *a priori*, or exogenously given, subsequent literature has allowed for the break point to be determined from the data. Zivot and Andrews (1992, ZA hereafter) suggested adopting a minimum statistic that determines the break point where the unit root *t*-test statistic is minimized (i.e., the most negative). Perron (1997) and Vogelsang and Perron (1998) suggest selecting the break by examining the significance of the dummy variables in the testing regression that capture the structural break. We refer to these and other similar tests as *endogenous break unit root tests*.

One important issue regarding these augmented Dickey-Fuller (ADF) type endogenous break unit root tests is that they omit the possibility of a unit root with break. If a break exists under the unit root null, two undesirable results can follow. First, these endogenous break unit root tests will exhibit size distortions such that the unit root null hypothesis is rejected too often. When utilizing such tests, researchers may incorrectly conclude that a time series is stationary with break when in fact the series is nonstationary with break. As such, "spurious rejections" might occur and more so as the magnitude of the break increases. This problem has been previously noted in Nunes, Newbold, and Kuan (1997), Vogelsang and Perron (1998), and Lee and Strazicich (2001). It is important to note that this nuisance parameter problem is restricted to the endogenous break tests and does not occur with the exogenous break unit root test. The asymptotic distribution of Perron's (1989) exogenous test does not depend on the magnitude of the break, even when a break occurs under the null. Thus, there is no size distortion in the exogenous break test, even when the magnitude of a break is very large. A second consequence of utilizing these ADF type endogenous break unit root tests is that the break point is incorrectly estimated. Lee and Strazicich (2001) note that these tests tend to identify the break point at one period prior to the true break point (i.e., at T_{B-1} rather than T_B), where the bias in estimating the persistence parameter is maximized and spurious rejections are the greatest. This problem occurs under both the null and alternative hypotheses. [1]

In this paper, we propose an alternative one-break unit root test that does not lead to the above problems. We utilize the theoretical findings presented in Lee and Strazicich (2003), who propose an endogenous two-break Lagrange Multiplier (LM) unit root test that is unaffected by structural breaks under the null. Similar to the two-break LM test, the one-break test proposed here is invariant to the magnitude of a structural break under the null and alternative hypotheses. Thus, spurious rejections will not occur in either case. Finally, by combining the two-break unit root test of Lee and Strazicich (2003) with the one-break test developed here, researchers can more accurately determine the correct number of breaks.

The remainder of the paper is organized as follows. In Section 2, we discuss properties of the minimum LM unit root test in the presence of a structural break. Section 3 discusses asymptotic properties of the LM unit root test and derives the

[1] The problem of size distortions with the endogenous break unit root test is not restricted to behavior under the null. In the presence of a break, the null distribution shifts to the left and causes more rejections than the true power of the test. Accordingly, the null hypothesis is rejected too often under the alternative unless size-corrected critical values are employed. Thus, these endogenous break unit root tests may appear more powerful, but this is often simply a reflection of the size distortions under the null. This point has not been clearly emphasized in the literature.

invariance results. Section 4 provides simulation results on the performance of the minimum LM test and comparisons to the ZA test. The paper summarizes and concludes in Section 5.

2. TESTING PROCEDURES

Consider the following data generating process (DGP) based on the unobserved components model:

$$y_t = \delta' Z_t + X_t , \quad X_t = \beta X_{t-1} + \varepsilon_t , \qquad (1)$$

where Z_t contains exogenous variables. The unit root null hypothesis is described by $\beta = 1$. If $Z_t = [1, t]'$, then the DGP is the same as that shown in the no break LM unit root test of Schmidt and Phillips (1992, hereafter SP). We consider two models of structural change. "Model A" is known as the "crash" model, and allows for a one-time change in intercept under the alternative hypothesis. Model A can be described by $Z_t = [1, t, D_t]'$, where $D_t = 1$ for $t \geq T_B+1$, and zero otherwise, T_B is the time period of the structural break, and $\delta' = (\delta_1, \delta_2, \delta_3)$.[2] 'Model C" allows for a shift in intercept and change in trend slope under the alternative hypothesis and can be described by $Z_t = [1, t, D_t, DT_t]'$, where $DT_t = t - T_B$ for $t \geq T_B + 1$, and zero otherwise.

According to the LM (score) principle, unit root test statistics are obtained from the following regression:

$$\Delta y_t = \delta' \Delta Z_t + \phi \tilde{S}_{t-1} + u_t , \qquad (2)$$

[2] When $Z_t = [1, t, DT_t^*]'$, the model becomes the "changing growth" Model B, where $DT_t^* = t$ for $t \geq T_B+1$ and zero otherwise. Model B will not be examined here as most economic time series can be adequately described by Model A or C (see, for example, Table VII in Perron, 1989).

where $\tilde{S}_t = y_t - \tilde{\psi}_x - Z_t\tilde{\delta}$, $t=2,..,T$; $\tilde{\delta}$ are the coefficients in the regression of Δy_t on ΔZ_t; and $\tilde{\psi}_x$ is the restricted MLE of ψ_x $(\equiv \psi + X_0)$ given by $y_1 - Z_1\tilde{\delta}$. Note that the testing regression (2) involves ΔZ_t instead of Z_t. Therefore, ΔZ_t is described by $[1, B_t]'$ in Model A and $[1, B_t, D_t]'$ in Model C, where $B_t = \Delta D_t$ and $D_t = \Delta DT_t$. Thus, B_t and D_t correspond to a change in intercept and trend under the alternative, and to a one period jump and (permanent) change in drift under the null hypothesis, respectively. The unit root null hypothesis is described by $\phi = 0$ and the LM *t*-test statistic is given by:

$$\tilde{\tau} = t\text{-statistic testing the null hypothesis } \phi = 0 \,. \qquad (3)$$

To correct for autocorrelated errors, we include augmented terms $\Delta\tilde{S}_{t-j}$, $j = 1,..., k$ in (2) as in the standard ADF test.[3] Ng and Perron (1995) suggest utilizing a general to specific procedure to determine the optimal number of *k* augmented terms. The location of the break (T_B) is determined by searching all possible break points for the minimum (i.e., the most negative) unit root test *t*-test statistic as follows:

$$Inf\ \tilde{\tau}(\tilde{\lambda}) = \underset{\lambda}{Inf}\ \tilde{\tau}(\lambda)\ , \qquad (4)$$

where $\lambda = T_B/T$.

3. ASYMPTOTICS AND INVARIANCE PROPERTY

[3] We determine *k* by following the "general to specific" procedure suggested by Perron (1989). We begin with a maximum number of lagged first-differenced terms k = 8 and examine the last term to see if it is significantly different from zero at the 10% level (critical value in an asymptotic normal distribution is 1.645). If insignificant, the maximum lagged term is dropped and the model re-estimated with k = 7 terms. The procedure is repeated until either the maximum term is found or k = 0, at which point the procedure stops. This technique has been shown to perform well as compared to other data-dependent procedures to select the number of augmented terms in unit root tests (Ng and Perron, 1995).

To examine the asymptotic distribution of the minimum LM test, we define *V(r)* as a standard Brownian bridge over the interval [0, 1], and $\underline{V}(r)$ as the demeaned Brownian bridge (see SP, equation (23)). The asymptotic distribution of the minimum LM unit root test in Model A is described as follows.[4]

Theorem 1. *Assume that (i) the data are generated according to (1) with* $Z_t = (1, t, D_t)'$*, (ii) the innovations* ε_t *satisfy the regularity conditions of Phillips and Perron (1998, p. 336), and (iii)* $T_B/T \to \lambda$ *as* $T \to \infty$*. Then, under the null hypothesis that* $\beta = 1$*,*

$$Inf\ \tilde{\tau}(\tilde{\lambda}) \to \underset{\lambda}{Inf}\ [-\frac{1}{2}\int_0^1 \underline{V}(r)^2]^{-1/2} \quad . \tag{5}$$

(See Appendix for proof.) [5]

An important implication of Theorem 1 is that the asymptotic distribution in (5) does not depend on the size (δ) or location ($\lambda = T_B/T$) of the break under the null. The result that the asymptotic null distribution does not depend on the size or location of the break follows from the same invariance property found in the exogenous one-break LM unit root test of Amsler and Lee (1995). Thus, an important advantage of the endogenous break LM unit root test is revealed. In the presence of a break under the null it is not necessary to simulate new critical values in empirical applications, as critical values for the minimum LM unit root test are invariant to the magnitude and location of the break.

The invariance property is an important feature of the endogenous break LM unit root test that makes the test free of spurious rejections. As shown by Amsler and Lee,

[4] Throughout the paper, the symbol "$\to$" denotes weak convergence of the associated probability measure.

allowing for one known or "exogenous" structural break will not affect the asymptotic null distribution of the SP type LM unit root test statistic. Regardless of the presence or absence of a structural break, the null distribution of the minimum LM unit root test statistic remains compact and well defined, implying that it will be unaffected by the magnitude of the break or correctly estimating its location.

The invariance results of the LM unit root test do not apply to the ZA test. In particular, the asymptotic null distribution of the ZA test depends on the location of the break (λ) through the projection residual $W(\lambda, r)$ of a Brownian motion projected onto the subspace generated by $[1, r, d(\lambda,r)]$, where $d(\lambda,r) = 1$ if $r > \lambda$ and 0 otherwise. The dependence of the asymptotic distribution of the sequential minimum ADF-type test on the nuisance parameter λ is cumbersome in applied work, and provides a rationale for ZA to assume no structural break under the null. That is, ZA omit B_t under the null by assuming that $d = 0$ in (6a), and thus omit $\alpha_2 = 0$ from their test regression as follows:

Null $$y_t = \mu_0 + dB_t + y_{t-1} + v_t \qquad (6a)$$

Testing Regression $$y_t = \alpha_0 + \alpha_1 t + \alpha_2 B_t + \alpha_3 D_t + \beta y_{t-1} + \sum_{j=1}^{k} c_j \Delta y_{t-j} + e_t \, . \qquad (6b)$$

As previously suggested, the ZA test depends upon the magnitude of the break term under the null and exhibits size distortions and spurious rejections as the break increases in absolute value. Thus, while the ZA test is valid if $d = 0$ in (6a), the test may give misleading inference results when $d \neq 0$. Lee and Strazicich (2001) investigate this issue further, and find that regardless of whether B_t is included or excluded in the ZA test

[5] See Lee and Strazicich (2003) for a proof of Model C in the context of two breaks in level and trend. The proof is similar for the one break minimum LM test and is omitted here to conserve space.

regression, the size distortions and spurious rejections remain. The problem is that the ZA test (and other similar ADF-type endogenous break tests) tends to select the break point incorrectly at T_{B-1}, where bias in estimating β, the coefficient that tests for a unit root, and spurious rejections are the greatest.[6]

While the accuracy of estimating the break point with the minimum LM unit root test does not matter under the null, it does matter when the alternative is true. Namely, as Perron (1989) initially showed, failure to allow for an existing structural break leads to a bias in unit root tests that makes it more difficult to reject a false null hypothesis. If the magnitude of the break is large, the minimum LM test estimates the break point fairly well. When the magnitude of the break is small, the break point cannot be accurately estimated, but the LM test does not suffer a significant loss of power in this case, as this is similar to having no break.[7]

3. SIMULATION RESULTS

This section provides critical values and simulation results for the one-break minimum LM unit root test. To perform our simulations, we generate pseudo-iid $N(0,1)$ random numbers using the Gauss (version 3.2.12) RNDNS procedure, where the DGP

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[6] Harvey, Leybourne, and Newbold (2001) suggest adapting the ADF-type endogenous break test by moving the break point forward one period to: $\hat{T}_B{*} = 1 + \hat{T}_B$, where $\hat{T}_B{*}$ is the revised break point and $\hat{T}_B$ is the estimated break. Given that the break point tends to be determined correctly at the true break when using the LM test, there is no need to move the break point as in Harvey *et al*. (2001).

[7] Asymptotic properties for the one-break minimum LM unit root test are similar for Model C, except that the test statistic is no longer invariant to the location of a break under the null. However, simulation results show that even though the minimum LM test for Model C is not invariant to the location of a break under the null, it is nearly so. See Lee and Strazicich (2003) for discussion of the asymptotic properties of Model C in the context of two breaks in level and trend.

has the form described in equation (1). Initial values of y_0 and ε_0 are assumed to be zero and σ_ε^2 is assumed to equal 1. All simulations are performed using 5,000 replications with T = 100 and a break at T_B = 50. Critical values for Models A and C are provided in Table 1. Since critical values for Model C depend (somewhat) on the location of the break, we provide critical values for a variety of break locations. Critical values at additional break points can be interpolated. Properties of size and power, and accuracy of estimating the break, are examined in simulations displayed in Table 2 for Model A.[8] Simulations are first performed for the case where the unit root null hypothesis is true (β = 1), and then where the alternative is true (β = 0.8). The size (frequency of rejections under the null) and power (frequency of rejections under the alternative) properties of the test are evaluated at the 5% significance level in each case. While the primary goal in this section is to examine the performance of the minimum LM test, simulation results for the ZA test are provided for comparison.[9]

To begin our investigation, we examine the one-break minimum LM unit root test for properties of size. Using a 5% critical value, Table 2 (a) examines the rate of rejecting the null hypothesis (β = 1) given that the DGP is a unit root with break. For example, with no break under the null (δ_3 = 0), column 3 indicates a 5.7% rejection rate, which is close to the nominal size of 5%. In the presence of a unit root with break (i.e., $\delta_3 \neq 0$), the LM unit root test statistic is relatively stable with correct size across all break

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[8] Size and power properties are examined only for Model A to conserve space. For detailed simulations of Model C in a two-break framework please see Lee and Strazicich (2003).

[9] See Lee and Strazicich (2001) for more detailed simulation results that examine properties of the ZA test. The ZA test simulation results repeated here are provided for convenience of comparison. Copies of computer code to run the minimum LM unit root test for Model A and Model C are available on the web site http://www.cba.ua.edu/~jlee/gauss/.

magnitudes. Desirable size properties can also be observed when examining the 5% empirical critical values in column 4. Results show that the empirical critical values are mostly invariant to the magnitude of a break under the null. Overall, we see that the one-break minimum LM test is free of size distortions and spurious rejections in the presence of a unit root with break. This is true even for a relatively large break of δ_3 = 10. In contrast, the minimum ZA unit root test exhibits significant size distortions and spurious rejections in the presence of a break under the null, which increase as the magnitude of the break increases.

To further examine performance of the one-break LM unit root test, Table 2 (b) displays simulations under the alternative hypothesis when β = 0.8. For the case of no break, δ_3 = 0, we see that the power to reject the null when the alternative is true is relatively high at 71%. As the magnitude of the break increases, the power of the test decreases (equal to 45% when δ_3 = 10), but remains relatively strong.

We next examine the accuracy of estimating the break point. Lee and Strazicich (2001) examined the frequency of estimating the structural break point correctly using the ADF-type minimum tests of ZA (1992) and Perron (1997). They found that these tests tend to determine the break point incorrectly where bias and spurious rejections are the greatest. Frequency of estimating the break point at different locations is shown in columns 5-11 in Table 2 (a) and (b). Setting the break at T_B = 50 in the DGP, the frequency of estimating the break at $T_B - 5$ to $T_B - 2$, $T_B - 1$, T_B, $T_B + 1$, $T_B + 2$ to $T_B + 5$, $T_B \pm 10$, and $T_B \pm 30$ is reported for different magnitudes of the break term, δ_3. As the magnitude of a break under the null increases, the frequency of estimating the break point correctly at T_B increases (becoming 48% at δ_3 = 10). The ability to determine the break point improves with the magnitude of the break. Under the alternative, the story is similar, only more pronounced. As the size of the break increases, the minimum LM test estimates the break point accurately with increasing frequency (90% at δ_3 = 10).

4. CONCLUSION

This paper proposes a minimum LM unit root test that endogenously determines one structural break in level and trend. Properties of the test were discussed and critical values presented. In contrast to similar ADF-type endogenous break tests, the one-break minimum LM unit root test tends to estimate the break point correctly and is free of size distortions and spurious rejections in the presence of a unit root with break. In addition, the test is invariant to the magnitude of a break under the null and is mostly invariant to its location. Finally, by combining the one-break LM unit root test presented here with the two-break LM test in Lee and Strazicich (2003), researchers can more accurately determine the correct number of structural breaks in their unit root test.

REFERENCES

Amsler, C. and J. Lee. (1995). "An LM Test for a Unit-Root in the Presence of a Structural Change," *Econometric Theory*, 11, 359-368.

Harvey, D., Leybourne, S. and P. Newbold, 2001, "Innovational Outlier Unit Root Tests with An Endogenously Determined Breaks in Level." *Oxford Bulletin of Economics and Statistics* 63, 559-575.

Lee, Junsoo and Mark C. Strazicich, 2001, "Break Point Estimation and Spurious Rejections with Endogenous Unit Root Tests." *Oxford Bulletin of Economics and Statistics* 63, 535-558.

Lee, Junsoo and Mark C. Strazicich, 2003, "Minimum LM Unit Root Test with Two Structural Breaks." *Review of Economics and Statistics* 85(4), 1082-1089.

Ng, S. and P. Perron, 1995, "Unit Root Tests in ARMA Models with Data-Dependent Methods for the Selection of the Truncation Lag," *Journal of the American Statistical Association* 90, 269-281.

Nunes, L., P. Newbold, and C. Kuan, 1997, "Testing for Unit Roots with Breaks: Evidence on the Great Crash and the Unit Root Hypothesis Reconsidered." *Oxford Bulletin of Economics and Statistics*, 59, 435-448.

Perron, P., 1989, "The Great Crash, the Oil Price Shock, and the Unit Root Hypothesis." *Econometrica* 57, 1361-1401.

Perron, P., 1997, "Further Evidence on Breaking Trend Functions in Macroeconomic Variables." *Journal of Econometrics* 80, 355-385.

Phillips, P.C.B. and P. Perron. (1988). "Testing for a Unit Root in Time Series Regression," *Biometrika*, 75, 335-346.

Schmidt, P. and P.C.B. Phillips, 1992, "LM Tests for a Unit Root in the Presence of Deterministic Trends." *Oxford Bulletin of Economics and Statistics* 54, 257-287.

Vogelsang, T. and P. Perron, 1998, "Additional Tests for a Unit Root Allowing for a Break in the Trend Function at an Unknown Time." *International Economic Review* 39(4), 1073-1100.

Zivot, E. and D. W. K. Andrews, 1992, "Further Evidence on the Great Crash, the Oil-Price Shock and the Unit Root Hypothesis." *Journal of Business and Economic Statistics* 10, 251-270.

Table 1. Critical Values of the One-Break Minimum LM_τ Test

Model A

1%	5%	10%
-4.239	-3.566	-3.211

Model C

λ	1%	5%	10%
.1	-5.11	-4.50	-4.21
.2	-5.07	-4.47	-4.20
.3	-5.15	-4.45	-4.18
.4	–5.05	-4.50	-4.18
.5	-5.11	-4.51	-4.17

Note: All critical values were derived in samples of size T = 100. Critical values in Model C (intercept and trend break) depend (somewhat) on the location of the break ($\lambda = T_B/T$) and are symmetric around λ and (1-λ). Model C critical values at additional break locations can be interpolated.

Table 2. Rejection Rates and Frequency of Estimated Break Points

Test	δ_3	5% Rej.	Emp. Crit.	Frequency of Estimated Break Points in the Range						
				T_B-5 ~ T_B -2	T_B-1	$\mathbf{T_B}$	T_B+1	T_B+2 ~ T_B+5	T_B± 10	T_B± 30
(a) Under the Null (β = 1)										
LM	0	.057	-3.62	.048	.015	.013	.010	.054	.259	.721
	4	.046	-3.53	.020	.006	.325	.005	.019	.446	.809
	6	.050	-3.56	.023	.013	.401	.009	.022	.519	.832
	8	.049	-3.56	.035	.019	.448	.018	.035	.598	.861
	10	.039	-3.48	.051	.031	.480	.029	.046	.682	.877
ZA	0	.060	-4.89	.048	.009	.013	.012	.049	.248	.726
	4	.081	-5.04	.099	.191	.003	.003	.019	.414	.801
	6	.169	-5.66	.108	.367	.003	.001	.009	.552	.859
	8	.325	-6.75	.085	.584	.002	.000	.002	.719	.918
	10	.506	-7.87	.064	.758	.005	.000	.000	.850	.963
(b) Under the Alternative (β = .8)										
LM	0	.710	-5.20	.057	.012	.014	.015	.062	.305	.745
	4	.581	-4.91	.040	.026	.553	.027	.048	.746	.907
	6	.537	-4.70	.041	.028	.737	.028	.047	.908	.962
	8	.492	-4.64	.041	.018	.834	.017	.036	.967	.982
	10	.454	-4.63	.026	.014	.898	.013	.024	.985	.991
ZA	0	.389	-5.77	.054	.015	.013	.015	.050	.276	.735
	4	.472	-6.15	.197	.493	.001	.001	.013	.778	.916
	6	.472	-6.15	.197	.493	.001	.001	.013	.778	.916
	8	.921	-8.48	.042	.949	.000	.000	.000	.996	.998
	10	.987	-9.94	.011	.988	.000	.000	.000	1.0	1.0

Note: All simulations were performed in samples of size T = 100.

APPENDIX

Proof of Theorem 1

We employ the functional limit theory used in Phillips and Perron (1988) and utilize the results of Zivot and Andrews (1992) on continuity of the composite functional. First, we consider the following regression:

$$\Delta y_t = \delta(\lambda)'\Delta Z_t(\lambda) + \phi(\lambda)\,\tilde{S}_{t-1}(\lambda) + e_t\,, \qquad t = 2,..,T\,, \qquad \text{(A.1)}$$

where $\tilde{S}_t(\lambda) = \sum_{j=2}^{t}\varepsilon_j - (\tilde{\delta}(\lambda)' - \delta(\lambda)')(Z_t(\lambda) - Z_1(\lambda))$, and the vector $Z_t(\lambda)$ includes deterministic terms such that $Z_t(\lambda) = [1, t, D_t]'$. Let $S_t = \sum_{j=2}^{t}\varepsilon_j$ and $[rT]$ be the integer part of rT, for $r \in [0,1]$. Following a procedure similar to ZA, we let $P_{\Delta Z}(\lambda) = \Delta z_T(\lambda)[\Delta z_T(\lambda)'\Delta z_T(\lambda)]^{-1}\Delta z_T(\lambda)$, and $M_{\Delta Z}(\lambda) = I - P_{\Delta Z}(\lambda)$, where $\Delta z_T(\lambda) = (\Delta z_{1,T}(\lambda),..,\Delta z_{T,T}(\lambda))'$. Pre-multiplying (A.1) by $M_{\Delta Z}(\lambda)$, we obtain:

$$M_{\Delta Z}(\lambda)\Delta Y = \phi(\lambda)\,M_{\Delta Z}(\lambda)\tilde{S}_1(\lambda) + M_{\Delta Z}(\lambda)\,e\,, \qquad \text{(A.2)}$$

where $\Delta Y = (\Delta y_2,.., \Delta y_T)'$, $\tilde{S}_1(\lambda) = (\tilde{S}_1(\lambda),..,\tilde{S}_{T-1}(\lambda))'$ and $e = (e_{2,..,}e_T)'$. Then, the $Inf\ \tilde{\tau}(\tilde{\lambda})$ statistic can be written as:

$$Inf\ \tilde{\tau}(\tilde{\lambda}) = \underset{\lambda}{Inf}\ [T^{-2}\tilde{S}_1(\lambda)'M_{\Delta Z}(\lambda)\,\tilde{S}_1(\lambda)]^{-1/2}[T^{-1}\tilde{S}_1(\lambda)'M_{\Delta Z}(\lambda)\,e] / s_T(\lambda)\,, \qquad \text{(A.3)}$$

where $s_T(\lambda)$ is the corresponding standard error of the regression. We then obtain:

$$T^{-2}\tilde{S}_1(\lambda)'M_{\Delta Z}(\lambda)\,\tilde{S}_1(\lambda) = \sigma^2\int_0^1[S_T(r) - P_{\Delta Z}(\lambda)\,S_T(r)]^2\,dr\,, \qquad \text{(A.4)}$$

$$T^{-1}\tilde{S}_1(\lambda)'M_{\Delta Z}(\lambda)\,e = \sigma^2\int_0^1 S_T(r)dS_T(r) - \sigma^2\int_0^1 P_{\Delta Z}(\lambda)\,S_T(r)dS_T(r)\,. \qquad \text{(A.5)}$$

The effect of applying $M_{\Delta Z}(\lambda)$ or $P_{\Delta Z}(\lambda)$ to the above expressions is twofold; one is to demean the process, and the other is to de-trend the structural dummy effect. We can establish the result that the effect of de-trending the structural break in the minimum LM unit root test vanishes asymptotically. To see this, we note that:

$$\int_0^1 \Delta z_T(\lambda,s)\Delta z_T(\lambda,s)'ds = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \qquad \text{(A.6)}$$

This is so because the term B_t in ΔZ_t is asymptotically negligible, whereas $\Delta Z_t = [1, B_t]'$. Then, it is clear that:

$$\int_0^1 [S_T(r) - P_{\Delta Z}(\lambda)\, S_T(r)]^2\, dr = \sigma^2 \int_0^1 \underline{V}_T(r)^2\, dr \ , \qquad \text{(A.7)}$$

where $\underline{V}_T(r)$ is the demeaned Brownian bridge, $\underline{V}_T(r) = V_T(r) - \int_0^1 V_T(r)dr$. Therefore, using the results in Schmidt and Phillips (p. 286), and (A.7), we can establish the limiting distribution of the minimum LM test. In particular, it does not depend on λ. The remaining procedure of the proof is to show continuity of a composite function. We simply follow ZA (1991) and Perron (1997), and express the $Inf\ \tilde{\tau}(\tilde{\lambda})$ t-statistic as:

$$Inf\ \tilde{\tau}(\tilde{\lambda}) = \mathrm{g}[S_T(r), \underline{V}_T(r), \int_0^1 S_T(r)dS_T(r), \int_0^1 P_{\Delta Z}(\lambda)\, S_T(r)dS_T(r), s^2] + o_p(1) \ , \qquad \text{(A.8)}$$

where $g = h^*[h[H_1(\bullet), H_2(\bullet), s_T(\lambda)]]$, with $h^*(m) = Inf\ m(\bullet)$ for any real function $m(\bullet)$, and $h[m_1, m_2, m_3] = m_1^{-1/2} m_2/m_3$. The functionals H_1 and H_2 are defined by (A.4) and (A.5). Continuity of h^* and h is proved in ZA.