Minimum Moment Aberration
for Nonregular Designs and Supersaturated Designs

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Abstract: Nonregular designs are used widely in experiments due to their run size economy and flexibility. These designs include the Plackett-Burman designs and many other symmetrical and asymmetrical orthogonal arrays. Supersaturated designs have become increasingly popular in recent years because of the potential in saving run size and its technical novelty. In this paper, a novel combinatorial criterion, called minimum moment aberration, is proposed for assessing the goodness of nonregular designs and supersaturated designs. The new criterion, which is to sequentially minimize the power moments of the number of coincidence among runs, is a good surrogate with tremendous computational advantages for many statistically justified criteria, such as minimum G_2-aberration, generalized minimum aberration and $E(s^2)$. In addition, the minimum moment aberration is conceptually simple and convenient for theoretical development. The general theory developed here not only unifies several separate results, but also provides many novel results on nonregular designs and supersaturated designs.

Key words and phrases: Complementary design, fractional factorial design, generalized minimum aberration, orthogonal array, Pless power moment identity.

1 Introduction

Nonregular designs are used widely in experiments due to their run size economy and flexibility (Wu and Hamada 2000). These designs include the Plackett-Burman designs (with run size not a power of two) and many other symmetrical and asymmetrical *orthogonal arrays* (OA) as described in Dey and Mukerjee (1999), Hedayat, Sloane and Stufken (1999) and Wu and Hamada (2000). Nonregular designs are traditionally used for screening the main effects only. Hamada and Wu (1992) proposed an analysis strategy to demonstrate that some interaction effects in such designs can also be entertained and estimated. The success of their analysis strategy is due to the fact that nonregular designs have some (hidden) projection properties, which are further studied by Lin and Draper (1992), Wang and Wu (1995), Cheng (1995, 1998) and Box and Tyssedal (1996). Along this line of research, Deng and Tang (1999) and Tang and Deng (1999) proposed the concept of generalized resolution and aberration for assessing nonregular two-level designs. By studying ANOVA models and contrasts, Xu and Wu (2001) proposed a *generalized minimum aberration* (GMA) criterion for general nonregular designs. The GMA restrained on *regular* designs is the well-known *minimum aberration* (Fries and Hunter 1980). The literature on minimum aberration is rich and includes Franklin (1984), Chen and Wu (1991), Chen (1992, 1998), Chen, Sun and Wu (1993), Chen and Hedayat (1996), Tang and Wu (1996), Suen, Chen and Wu (1997), Cheng, Steinberg and Sun (1999), Cheng and Mukerjee (1998), and Fang and Mukerjee (2000).

Supersaturated designs have become increasingly popular in recent years because of the potential in saving run size and its technical novelty. Many authors have proposed methods and algorithms for constructing supersaturated designs. See, among others, Lin (1993, 1995), Wu (1993), Tang and Wu (1997), Nguyen (1996), Li and Wu (1997), Cheng (1997), Yamada and Lin (1997), Gupta and Chatterjee (1998), Deng, Lin and Wang (1999), Lu and Meng (2000), and Liu and Zhang (2000) for two-level supersaturated designs; Yamada and Lin (1999) and Yamada, Ikebe, Hashiguchi and Niki (1999) for three-level supersaturated designs and Fang, Lin and Ma (2000) for multi-level supersaturated designs. A popular criterion in the supersaturated design literature is the $E(s^2)$ criterion (Booth and Cox 1962), which is limited to the two-level case. The extensions to the multi-level case are not unique. One extension is an average χ^2 statistic (Yamada and Lin 1999), which measures the goodness of a three-level supersaturated design. Another extension is the GMA criterion (Xu and Wu 2001), which can assess the goodness of general supersaturated designs (including mixed-level cases). However, neither paper provides general optimality results, due to the complexity of the design problem itself and the lack of proper tools (though some general results are available for the two-level case).

At the initial stage of an experiment, it is often the case that a practitioner does not have enough confidence about which factorial effects will turn out to be significant. Therefore, it is vital for success to use a factorial design that is robust against the model uncertainty. GMA nonregular designs have the desirable model robustness as Xu and Wu (2001) showed that they tend to minimize the contamination of non-negligible two-factor and higher-order interactions on the estimation of the main effects.

It is, however, often a hard task to find an optimal nonregular design according to the GMA criterion. There are two major difficulties. The first difficulty arises from the complexity of nonregular designs. It is impossible, except for some special simple cases, to search over a complete list of all possible designs because either a complete list is not available in the literature or the list is too large. For instance, a list of all OAs of 18 runs, 6 factors and 3 levels is not available in the literature. A common practice is to search for the best design from some popular OAs. Nevertheless, it is still a difficult task because it is inevitable to enumerate and compare all possible subdesigns since nonregular designs may not have any structure. In addition, such designs may not have GMA because some OAs are maximal in the sense that they can not be embedded into any larger OAs (Mukerjee and Wu 1995; Beder 1998). Therefore, it is extremely difficult to show that a design has GMA. It calls for new technique to show that a design has GMA without searching over the complete design list. In particular, the same problems exist for supersaturated designs.

The second difficulty arises from the GMA criterion itself. Its definition involves the coding of all main effects and interactions, which is quite complicated in general. One consequence is that the GMA criterion is inefficient for computation. As a matter of fact, the complexity of the criterion is in an exponential order of the number of factors, which implies that it is infeasible to implement the GMA for many commonly used designs. Another consequence is that it is inconvenient to study the GMA criterion theoretically, which explains why few optimality results are known for the GMA criterion. Therefore, it is of both practical and theoretical interest to propose a new criterion that is statistically reasonable, cheap to compute, and convenient for theoretical development.

The purpose of this paper is to propose a new criterion that is conceptually simple and cheap to compute. The key innovation is to investigate the relationship between runs (i.e., rows), instead of studying the relationship between factors (i.e., columns). The new criterion, called minimum moment aberration, is to sequentially minimize the power moments of the number of coincidence among runs. Avoiding the complex coding of factorial effects, it has tremendous savings in computation over the GMA criterion. In addition, the minimum moment aberration is statistically sound because it is equivalent to the GMA for symmetrical designs and weakly equivalent to the GMA for asymmetrical designs.

The minimum moment aberration criterion is also convenient to study theoretically. The conceptual simplicity of the criterion allows us to investigate some hard problems in depth. Sufficient conditions are given to show when a design has minimum moment aberration; therefore, it becomes possible to assess the GMA property without searching over a complete list of designs. Furthermore, based on the new criterion, a unified theory is developed for nonregular designs and supersaturated designs, which includes several separated results in the literature as special cases.

The paper is organized as follows. Preliminary notation and results are given in Section 2. The minimum moment aberration criterion is introduced in Section 3 and a unified theory is developed for nonregular designs and supersaturated designs in Section 4. Applications and extensions of the new concept and theory are given in Section 5 and Section 6, respectively. Concluding remarks are given in Section 7. For the simplicity of the presentation, all proofs are given in the appendix.

2 Preliminary Notation and Results

For a set S, let $|S|$ be its cardinality. For an integer $k > 0$, let $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$. For convenience, let $\binom{x}{0} = 1$ and $\binom{x}{k} = 0$ if $k < 0$. For integers $k, j \geq 0$, let $S(k,j)$ be a Stirling number of the second kind, i.e., the number of ways of partitioning a set of k elements into j nonempty sets. Clearly $S(k,k) = 1$, $S(k,k-1) = \binom{k}{2}$ and $S(k,j) = 0$ if $j > k$. It is well known that $S(k,j) = (1/j!)\sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i}i^k$ for $k \geq j \geq 0$. For convenience, let $0^0 = 1$.

For a real number x, let $\lfloor x \rfloor$ be the largest integer that does not exceed x. For integers $m, n \geq 0$, let

$$h(m,n) = \lfloor m/n \rfloor^2 n + (2\lfloor m/n \rfloor + 1)(m - \lfloor m/n \rfloor n).$$

Clearly $h(m,n) = m^2/n$ if m is a multiple of n. The following minimization problem, related to $h(m,n)$, is elementary and quite useful in the theoretical development for the minimum moment aberration.

Lemma 1. *Suppose that $x_1, \ldots, x_n$ are nonnegative integers and that $\sum x_i = m$. Then $\sum x_i^2 \geq h(m,n)$ with equality if and only if all x_i equals $\lfloor m/n \rfloor$ or $\lfloor m/n \rfloor + 1$.*

An *asymmetrical* (or mixed-level) design of N runs, n factors and with levels $s_1, \ldots, s_n$ is denoted by $(N, s_1 \cdots s_n)$. An $(N, s_1 \cdots s_n)$-design is an $N \times n$ matrix $[r_{ij}]_{N\times n}$ with r_{ij} from a set of s_j symbols, say, $\{0, 1, \ldots, s_j - 1\}$. For example, an $(N, s_1^{n_1} s_2^{n_2})$-design has n_1 factors of s_1 levels and n_2 factors of s_2 levels. In particular, an (N, s^n)-design is *symmetrical.* Two designs are *isomorphic* if one can be obtained from the other through permutations of rows, columns and symbols in each column.

An asymmetrical (or mixed-level) *orthogonal array* (OA) of N runs, n factors, strength t and with levels $s_1, \ldots, s_n$, denoted by $OA(N, s_1 \cdots s_n, t)$ or $OA(t)$, is an $(N, s_1 \cdots s_n)$-design in which all possible level combinations for any t factors appear equally often. A balanced design is an $OA(1)$. For an $OA(N, s_1 \cdots s_n, 2)$, the Rao bound says that $N - 1 \geq \sum_{i=1}^n (s_i - 1)$. An $(N, s_1 \cdots s_n)$-design is saturated if $N - 1 = \sum_{i=1}^n (s_i - 1)$ and supersaturated if $N - 1 < \sum_{i=1}^n (s_i - 1)$. A supersaturated design does not have enough degrees of freedom to estimate all the main effects. In the literature, nonregular designs are often referred to $OA(2)$'s that are not completely specified by some defining relations among factors.

The definition of $OA(t)$ requires that all level combinations for any t factors appear equally often. This condition is often too strong to satisfy. The following concept of weak strength t is more useful in many cases.

A design is called an OA of weak strength t, denoted by $OA(t^-)$, if all level combinations for any t columns appear as equally often as possible, that is, the difference of occurrence of level combinations does not exceed one. It is easy to show that an $OA(t^-)$ is always an $OA(t)$ if the latter exists. It is important to note that an $OA(t^-)$ is not necessary an $OA((t-1)^-)$.

Now we briefly describe the GMA criterion proposed by Xu and Wu (2001). For an $(N, s_1 \cdots s_n)$-design D, consider the following ANOVA model

$$Y = X_0\beta_0 + X_1\beta_1 + \cdots + X_n\beta_n + \varepsilon,$$

where Y is the vector of N observations, β_j is the vector of all j-factor interactions, X_j is the matrix of contrast coefficients for β_j and ε is the vector of independent random errors. For $j = 0, \ldots, n$, if $X_j = [x_{ik}]$, let

$$A_j(D) = N^{-2} \sum_k \left| \sum_{i=1}^N x_{ik} \right|^2. \tag{1}$$

The $A_j(D)$ defined in (1) are invariant with respect to the choice of orthonormal contrasts. The vector $(A_1(D), \ldots, A_n(D))$ is called the *generalized wordlength pattern*. Xu and Wu showed that the generalized wordlength pattern has the following important property.

LEMMA 2. *D is an $OA(t)$ if and only if $A_j(D) = 0$ for $1 \le j \le t$.*

DEFINITION 1. For two $(N, s_1 \cdots s_n)$-designs D_1 and D_2, D_1 is said to have less aberration than D_2 if there exists an r, $1 \le r \le n$, such that $A_r(D_1) < A_r(D_2)$ and $A_j(D_1) = A_j(D_2)$ for $j = 1, \ldots, r-1$. D_1 is said to have generalized minimum aberration if there is no other design with less aberration than D_1.

Xu and Wu showed that the GMA reduces to the *minimum aberration* (Fries and Hunter 1980) for regular designs and the *minimum G_2-aberration* (Tang and Deng 1999) for two-level nonregular designs.

Finally, we turn to optimality criteria for supersaturated designs. For an $(N, 2^n)$-design D, the popular $E(s^2)$ criterion (Booth and Cox 1962) can be defined as

$$E(s^2) = N^2 A_2(D)/[n(n-1)/2].$$

For an (N, s^n)-design $D = [r_{ij}]_{N \times n}$, let $n_{kl}(a,b) = |\{i : r_{ik} = a, r_{il} = b\}|$ and

$$\chi^2_{kl} = \sum_{a=0}^{s-1} \sum_{b=0}^{s-1} [n_{kl}(a,b) - N/s^2]^2/(N/s^2).$$

The average χ^2 statistic (Yamada and Lin 1999) is

$$\operatorname{ave}\chi^2 = \sum_{1 \le k < l \le n} \chi^2_{kl}/[n(n-1)/2].$$

Yamada and Lin showed that $E(s^2) = N \operatorname{ave}\chi^2$ for a balanced $(N, 2^n)$-design. As mentioned in the introduction, the GMA criterion can serve as an optimality criterion for supersaturated designs. It will be shown in Section 5 that both $E(s^2)$ and $\operatorname{ave}\chi^2$ are special cases of the GMA.

3 Minimum Moment Aberration

For simplicity of the presentation, only symmetrical designs are considered in this and the next two sections. Extensions to asymmetrical designs are given in Section 6.

For an (N, s^n)-design $D = [r_{ij}]_{N\times n}$ and a positive integer t, define the tth power moment to be

$$K_t(D) = [N(N-1)/2]^{-1} \sum_{1\le i<j\le N} [\delta_{ij}(D)]^t \,,$$

where

$$\delta_{ij}(D) = \sum_{k=1}^{n} \delta(r_{ik}, r_{jk}) \tag{2}$$

is the number of coincidence between the ith and jth rows and $\delta(x, y)$ is the Kronecker delta function, which equals 1 if $x = y$ and 0 otherwise. It is important to note that $n - \delta_{ij}(D)$ is known as the *Hamming distance* between the ith and jth rows in algebraic coding theory.

The minimum moment aberration criterion is to sequentially minimize the power moments. A formal definition is given below.

Definition 2. For two (N, s^n)-designs D_1 and D_2, D_1 is said to have less moment aberration than D_2 if there exists a t, $1 \le t \le n$, such that $K_t(D_1) < K_t(D_2)$ and $K_i(D_1) = K_i(D_2)$ for $i = 1, \ldots, t-1$. D_1 is said to have *minimum moment aberration* if there is no other design with less moment aberration than D_1.

The minimum moment aberration has a geometrical interpretation. The number of coincidence is a similarity measure. The power moments measure the overall similarity among all possible pairs of rows (runs). Minimizing the first power moment means minimizing the average similarity (or maximizing the average dissimilarity or distance) among runs. Given the first power moment, minimizing the second power moment means minimizing the variance of the dissimilarity (or distance) among runs. Sequentially minimizing higher-order power moments makes all runs be as dissimilar as possible.

It is important to note that the power moments measure not only the row similarity directly, but also the column nonorthogonality implicitly. Indeed, the first power moment measures the overall balance within each column. The second power moment measures the overall nonorthogonality between all pairs of columns. In general, the tth power moment measures the overall nonorthogonality among all possible t columns. It will be proven in the next section that sequentially minimizing the power moments is equivalent to sequentially minimizing the generalized wordlength patterns. Therefore, the minimum moment aberration is indeed equivalent to the GMA though they are quite different in definition. As a consequence, the former can be used as a surrogate for the latter, which is statistically well justified.

The minimum moment aberration has tremendous computational advantages over the GMA. The complexity of computing A_j according to the definition (1) is $O(\binom{n}{j}(s-1)^j N)$ because $X_j = [x_{ik}]$ is an $N \times \binom{n}{j}(s-1)^j$ matrix; hence, the complexity of computing the generalized wordlength pattern is $O(Ns^n)$. The exponential order implies that it is prohibitive to implement the GMA in practice. In contrast, the complexity of computing K_j is $O(N^2 n)$ for any j. Thus, the complexity of computing the first n power moments is $O(N^2 n^2)$, which is much less than $O(Ns^n)$, the complexity of computing the generalized wordlength pattern, if n is large.

There are also substantial savings in computation when the minimum moment aberration is used to assess the goodness of a supersaturated design. A practical exercise for supersaturated designs is to compute and compare A_2 or K_2, which includes $E(s^2)$ and ave χ^2 as special cases. The complexity of A_2 (and ave χ^2) is $O(n^2(s-1)^2 N)$, which is greater than the complexity of K_2, $O(N^2 n)$, for a supersaturated design. The difference is enormous when the number of factors, n, is much larger than the number of runs, N, which is common for supersaturated designs. This observation implies that many algorithms will speed up significantly if we replace $E(s^2)$ with K_2 as the objective function.

REMARK 1. A related but different concept is *optimal moments* proposed by Franklin (1984). The moments in his definition are functions of wordlengths of defining contrasts among factors while our moments are functions of the number of coincidence among runs. In addition, the minimum moment aberration defined here is equivalent to the GMA (see the next section) while the optimal moments is not.

4 Theory of Minimum Moment Aberration

Our first theorem shows that the power moments are linear combinations of the generalized wordlength patterns. The proof of this theorem involves the generalized *Pless power moment identities*, a deep and fundamental result in algebraic coding theory.

THEOREM 1. *For an (N, s^n)-design D and $t = 1, 2, \ldots,$*

$$K_t(D) = \alpha_t A_t(D) + \alpha_{t-1} A_{t-1}(D) + \ldots + \alpha_1 A_1(D) + \alpha_0 - c_0, \tag{3}$$

where

$$\alpha_i = \alpha_i(t; N, n, s) = [N/(N-1)] \sum_{k=0}^{t} (-1)^{k+i} \binom{t}{k} n^{t-k} \left[\sum_{j=0}^{k} j! S(k,j) s^{-j} (s-1)^{j-i} \binom{n-i}{j-i} \right],$$

$c_0 = n^t/(N-1)$ *and* $S(k,j)$ *are Stirling numbers of the second kind. In particular,* $\alpha_t = t!N/[(N-1)s^t]$, $\alpha_{t-1} = t![n+(t-1)(s-2)/2]N/[(N-1)s^t]$.

Because the leading coefficient α_t in (3) is positive, it is clear that sequentially minimizing $K_t(D)$ for $t = 1, 2, \ldots$ is equivalent to sequentially minimizing $A_t(D)$ for $t = 1, 2, \ldots$. Therefore, we have the following important result.

THEOREM 2. *For symmetrical designs the minimum moment aberration is equivalent to the GMA. In particular, a symmetrical design has GMA if and only if it has minimum moment aberration.*

Another important consequence of Theorem 1 is that results of the power moments can be obtained through that of the generalized wordlength pattern, and vice versa. For example, Theorem 1 and Lemma 2 together lead to the following result regarding the power moments.

COROLLARY 1. *For an* $OA(N, s^n, e)$*-design* D *and* $t = 1, 2, \ldots, e$, $K_t(D) = \alpha_0(t; N, n, s) - n^t/(N-1)$ *is a constant depending only on* t, n, N *and* s.

The identities in Theorem 1 involving Stirling numbers of the second kind are complicated in general. The first three identities of (3) are of most interest in practice and therefore are given below explicitly.

$$
\begin{aligned}
K_1(D) &= \{[A_1(D) + n]N - ns\}/[(N-1)s], \\
K_2(D) &= \{[2A_2(D) + (2n+s-2)A_1(D) + n(n+s-1)]N - (ns)^2\}/[(N-1)s^2], \\
K_3(D) &= \{[6A_3(D) + 6(n+s-2)A_2(D) + (3n^2 + 6ns + s^2 - 9n - 6s + 6)A_1(D) \\
&\quad + n(n^2 + 3ns + s^2 - 3n - 3s + 2)]N - (ns)^3\}/[(N-1)s^3].
\end{aligned}
$$

With these identities and the fact that $A_j(D) \geq 0$, we can establish a series of lower bounds for $K_t(D)$. For example, we have the following lower bounds:

COROLLARY 2. *(i)* $K_1(D) \geq [n(N-s)]/[(N-1)s]$ *with equality if and only if* D *is an* $OA(1)$.
(ii) $K_2(D) \geq [Nn(n+s-1) - (ns)^2]/[(N-1)s^2]$ *with equality if and only if* D *is an* $OA(2)$.
(iii) $K_3(D) \geq [Nn(n^2 + 3ns + s^2 - 3n - 3s + 2) - (ns)^3]/[(N-1)s^3]$ *with equality if and only if* D *is an* $OA(3)$.

These lower bounds in Corollary 2 are valuable; nevertheless, they provide no more information than Lemma 2. In the following, we shall develop new lower bounds for $K_t(D)$, which are more useful than those in Corollary 2. Note that the lower bound of $K_t(D)$ in Corollary 2 is tight if and only if an $OA(t)$ exists. Recall that all level combinations of any t columns of an $OA(t)$ appear equally often. When the equal occurrence cannot be met, it is reasonable to expect that a design of which all level combinations of any t columns appear as equally often as possible should have a minimum $K_t(D)$ value. Formally, we have the following results.

Theorem 3. *$K_t(D)$ is minimized if D is an $OA(i^-)$ for $i = 1, \ldots, t$.*

Corollary 3. *(i) $K_1(D) \geq [nh(N,s) - Nn]/[N(N-1)]$.*
(ii) $K_2(D) \geq [n(n-1)h(N,s^2) + nh(N,s) - Nn^2]/[N(N-1)]$.
(iii) $K_3(D) \geq [n(n-1)(n-2)h(N,s^3) + 3n(n-1)h(N,s^2) + nh(N,s) - Nn^3]/[N(N-1)]$.

Corollary 4. *An $OA(t)$ has minimum moment aberration if its projection onto any $t+1$ columns has no repeated run.*

It is clear that Corollary 3 improves Corollary 2. In addition, Theorem 3 and Corollary 4 provide a sufficient condition when $K_t(D)$ is minimized and when a design has minimum moment aberration. Recall that it is often infeasible to search over a complete list of all possible nonregular designs because such a list is either unknown or extremely large. Therefore, the sufficient condition is highly valuable because it avoids searching over a complete list. Examples will be given in the next section.

The definition of power moments allows us to obtain another series of lower bounds of $K_t(D)$ easily. It is well known that for a random variable X, $(E|X|^r)^{1/r}$ is nondecreasing in $r > 0$. This fact indicates the following inequality:

$$K_t(D)^{1/t} \geq K_r(D)^{1/r} \text{ for } t \geq r \geq 1. \tag{4}$$

Combining Corollary 2(i), we obtain the following lower bounds.

Theorem 4. *For an (N, s^n)-design D,*

$$K_t(D) \geq [n(N-s)/(s(N-1))]^t \textit{ for } t \geq 2.$$

The equality holds if and only if D is an $OA(1)$ and the number of coincidence between any pair of distinct rows is a constant.

An important class of designs which satisfy the conditions in Theorem 4 are saturated $OA(2)$'s. It is easy to verify that the lower bound of $K_2(D)$ in Theorem 4 is tight for an $OA(N, s^n, 2)$ if $N - 1 = n(s-1)$. As a consequence, we obtain the following important property regarding saturated $OA(2)$'s (Mukerjee and Wu 1995).

Corollary 5. *The number of coincidence between any distinct pair of rows of a saturated $OA(2)$ is a constant.*

A direct outcome of Corollary 5 and Theorem 4 is that any saturated $OA(2)$ has minimum moment aberration. In addition, removing one column from (or adding a balanced column to) a saturated $OA(2)$ results a minimum moment aberration design. In general, we have the following result.

Theorem 5. *If D is an $OA(1^-)$ and the difference among all $\delta_{ij}(D)$, $i < j$, does not exceed one, then D has minimum moment aberration.*

Along the direction of Theorem 4, we can establish many other lower bounds of $K_t(D)$. For example, by Corollary 2(ii), $K_2(D)$ is a known constant for an $OA(2)$. Then the inequality (4) provides a new lower bound of $K_t(D)$ for $t \geq 3$. The procedure is straightforward and the details are omitted.

5 Applications

In this section, we present some applications of the concept and theory of minimum moment aberration on the GMA criterion, complementary designs and supersaturated designs.

5.1 Generalized Minimum Aberration

The minimum moment aberration theory developed in the previous section provides a way of assessing the GMA property without searching over all possible nonregular designs because the minimum moment aberration is equivalent to the GMA.

Example 1. Consider an $OA(18, 3^7, 2)$ given in Table 1. Xu and Wu (2001) showed that any design not containing the first column has GMA among all subdesigns from Table 1. However, they failed to show that it has GMA among all possible nonregular designs (including other designs that are not part of Table 1). In contrast, using the new technique, we can show that such a design has GMA. Specifically, it is easy to verify that any design not containing the first column is an $OA(2)$ and its projection onto any 3 columns has no repeated run. Thus, it has minimum moment aberration by Corollary 4 and hence has GMA.

The minimum moment aberration theory also provides new lower bounds for the generalized wordlength patterns via key identities in Theorem 1. For example, the following lower bounds of $A_t(D)$ are obtained through Corollary 3, Theorem 1 and Lemma 2.

Corollary 6. *(i)* $A_1(D) \geq n[h(N, s)s/N^2 - 1]$.
(ii) $A_2(D) \geq \binom{n}{2}[h(N, s^2)s^2/N^2 - 1]$ *for an* $OA(1)$.
(iii) $A_3(D) \geq \binom{n}{3}[h(N, s^3)s^3/N^2 - 1]$ *for an* $OA(2)$.

The following lower bounds are obtained through the inequality (4), Corollary 2, Theorem 1 and Lemma 2.

Corollary 7. *(i)* $A_2(D) \geq [n(s-1)(ns - n - N + 1)]/[2(N-1)]$ *for an* $OA(1)$.
(ii) $A_3(D) \geq \{[Nn(n+s-1) - (ns)^2]^{3/2}(N-1)^{-1/2} + (ns)^3 - Nn(n^2+3ns+s^2-3n-3s+2)\}/(6N)$ *for an* $OA(2)$.

The lower bounds in Corollary 6 are tight if an OA exists. They are useful for assessing the nonorthogonality of a design. On the other hand, the lower bounds in Corollary 7 are more useful for assessing nearly saturated or supersaturated designs. Note that these lower bounds are not available in Xu and Wu (2001).

Example 2. Consider three-level designs of 18 runs (i.e., $N = 18, s = 3$). The lower bounds of A_3 in Corollary 6 are $0.5, 2, 5, 10$ for $n = 3, 4, 5, 6$, respectively. These bounds are tight and achieved by the GMA designs from Table 1. However, for $n = 7$, the lower bound of A_3 in Corollary 6 is 17.5 and not tight. It is less than the lower bound of A_3 in Corollary 7, which is 18.2. The latter bound may be used for assessing the efficiency of an $(18, 3^7)$-design. For instance, the A_3 efficiency of the $OA(18, 3^7, 2)$ given in Table 1 is $18.2/22 = 82.8\%$ with respect to the lower bound in Corollary 7.

5.2 Complementary Designs

Many authors have studied the characterization of GMA designs in terms of their complementary designs. Here we revisit this technique with the minimum moment aberration.

Suppose H is an (N, s^p)-design. Call $(D, \overline{D})$ a pair of *complementary designs* from H if they are a *column* partition of H. The characterization problem is to express the generalized wordlength pattern of D in terms of that of its complementary design $\overline{D}$. This is a hard problem and has been tackled by Tang and Wu (1996) and Tang and Deng (1999) via a combinatorial approach and by Suen, Chen and Wu (1997) and Xu and Wu (2001) via an algebraic coding approach. Here we revisit this problem with the minimum moment aberration approach. It turns out to be surprisingly trivial and straightforward.

If H is a saturated $OA(2)$, by Corollary 5, for $i < j$,

$$\delta_{ij}(D) + \delta_{ij}(\overline{D}) = \gamma,$$

where γ is a constant independent of D and $\overline{D}$. Then by definition,

$$K_t(D) = \sum_{i=0}^{t} \binom{t}{i} (-1)^i \gamma^{t-i} K_i(\overline{D}). \tag{5}$$

By applying these identities and Theorem 1 recursively, we can express the generalized wordlength pattern of D in terms of that of its complementary design $\overline{D}$:

$$A_t(D) = (-1)^t A_t(\overline{D}) + (-1)^t [1 + (s-2)(t-1)] A_{t-1}(\overline{D}) + \text{lower order terms} \tag{6}$$

for $t = 1, 2, \ldots$. We reach the same general relations derived by Tang and Wu (1996), Suen, Chen and Wu (1997), Tang and Deng (1999) and Xu and Wu (2001).

The complementary design technique is very powerful for regular designs but is not so powerful for nonregular designs. For example, by (6), Tang and Wu (1996) and Suen, Chen and Wu (1997) showed that any design obtained by removing one column from a saturated regular design has minimum aberration. However, we cannot conclude from (6) that such a design has GMA among all possible nonregular designs. In contrast, it is easy to show that such a design has minimum moment aberration by Theorem 5 and hence has GMA.

5.3 Supersaturated Designs

Here we use the concept of minimum moment aberration to study supersaturated designs. As done in the literature, we shall consider only balanced designs, which minimize the first power moment $K_1(D)$. In the spirit of minimum moment aberration, a good optimality criterion for supersaturated designs is the minimization of $K_2(D)$.

It can be shown (in the appendix) that for a balanced (N, s^n)-design D

$$\text{ave}\,\chi^2 = [(N-1)s^2K_2(D) - Nn(n+s-1) + (ns)^2]/[n(n-1)]. \tag{7}$$

Then by Theorem 1 and Lemma 2,

$$\text{ave}\,\chi^2 = NA_2(D)/[n(n-1)/2].$$

Since $E(s^2)$ and $\text{ave}\,\chi^2$ optimality are special cases of the minimum moment aberration and GMA, we obtain many results for free. For example, Corollary 7 implies the following lower bounds:

$$E(s^2) \geq N^2(n-N+1)/[(n-1)(N-1)],$$

$$\text{ave}\,\chi^2 \geq [N(s-1)(ns-n-N+1)]/[(n-1)(N-1)].$$

Nguyen (1996) and Tang and Wu (1997) derived independently the lower bound of $E(s^2)$. Yamada and Lin (1999) reported a special case of the lower bound of $\text{ave}\,\chi^2$ for three-level supersaturated designs.

The theory of minimum moment aberration also provides many optimality results for supersaturated designs. For example, Theorem 5 and Corollary 5 together imply the following result.

COROLLARY 8. *If $D_1, \ldots, D_m$ are m saturated $OA(2)$'s, their column juxtaposition $D = (D_1, \ldots, D_m)$ has minimum moment aberration. In addition, removing one column from or adding one column to D results a minimum moment aberration design.*

The special case of Corollary 8 for two-level supersaturated designs and $E(s^2)$ optimality is first obtained by Tang and Wu (1997) (for the first statement) and Cheng (1997). Furthermore, the $E(s^2)$ optimality of Lin's (1993) half-Hadamard designs, proved by Nguyen (1996) and Cheng (1997), also follows from Theorem 5 and Corollary 5.

As another application, we propose a novel construction method which is an extension of Lin's (1993) half-Hadamard construction method. The new method is illustrated with a saturated $OA(27, 3^{13}, 2)$. Taking any three-level column as the branching column, we obtain three one-third fractions according to the level of the branching column. Each one-third fraction is an $OA(9, 3^{12}, 1)$ and any two-third fraction is an $OA(18, 3^{12}, 1)$ after removing the branching column. Following Theorem 5 and Corollary 5, it is easy to show that all these designs have minimum moment aberration and thus have GMA.

6 Extensions

In this section we extend the concept and theory of minimum moment aberration to the asymmetrical case.

Consider an $(N, s_1 \cdots s_n)$-design $D = [r_{ij}]_{N\times n}$. In order to handle mixed levels, we introduce weights and modify the definition of $\delta_{ij}(D)$ in (2). For the kth column, assign weight $w_k > 0$. Let

$$\delta_{ij}(D) = \sum_{k=1}^{n} w_k \delta(r_{ik}, r_{jk}) \tag{8}$$

be the weighted coincidence number between the ith and jth rows. With this modification, the definitions of power moments and minimum moment aberration remain the same. Then most results developed earlier can be extended easily to the asymmetrical case. In particular, Theorems 3 and 5 remains unchanged, and Theorem 4 becomes

Theorem 6. *For an $(N, s_1 \cdots s_n)$-design D,*

$$K_t(D) \geq \left[\sum w_k (N/s_k - 1)/(N-1)\right]^t \text{ for } t \geq 2.$$

The equality holds if and only if D is $OA(1)$ and $\delta_{ij}(D)$ defined in (8) is a constant for all $i < j$.

On the other hand, the results regarding to the GMA need more attention. Recall that for symmetrical designs, the power moments are linear combinations of the generalized wordlength patterns; thus, minimum moment aberration is equivalent to GMA. For asymmetrical designs, the relationship between the power moments and the generalized wordlength patterns is more complicated and minimum moment aberration is not equivalent to GMA in general. Nevertheless, minimum moment aberration is still a good surrogate for GMA because these two criteria are weakly equivalent, which is expressed in the following theorem.

Theorem 7. *For an asymmetrical $(N, s_1 \cdots s_n)$-design D, if $w_k = \lambda s_k$ for all k, then*

$$K_t(D) = \lambda^t [N(N-1)^{-1} t! A_t(D) + \gamma_t] \text{ for } t = 1, 2, \ldots, e+1,$$

where e is the strength of D and γ_t are constants depending on t, n, N and the levels $s_1, \ldots, s_n$.

For convenience, the choice of $w_k = \lambda s_k$ is called *natural weights*. Natural weights provide a reasonable connection between minimum moment aberration and GMA. An important property regarding natural weights is the following result (Mukerjee and Wu 1995).

Lemma 3. *Suppose D is an saturated $OA(N, s_1 \cdots s_n, 2)$. Then $\delta_{ij}(D)$ defined in (8) is a constant for all $i < j$ if $w_k = \lambda s_k$ for all k.*

Now consider complementary designs. Suppose that D and $\overline{D}$ are a pair of complementary designs of a saturated (asymmetrical) $OA(2)$. Then the relationship between $K_t(D)$ and $K_t(\overline{D})$ in (5) still holds with natural weights. In contrast, the relationship between $A_t(D)$ and $A_t(\overline{D})$ in (6) no longer holds. Nevertheless, the following weak result can be obtained through Theorem 7 and (5):

$$A_3(D) = -A_3(\overline{D}) + constant.$$

Finally, as an application, consider constructing minimum moment aberration designs from a commonly used $OA(36, 3^{12}2^{11}, 2)$, which is given in Table 2 and Wu and Hamada (2000, Table 7C.7). It can study up to 12 three-level factors and 11 two-level factors simultaneously. Natural weights are considered here. To find a minimum moment aberration design of n_3 three-level factors and n_2 two-level factors, it is necessary to enumerate all $\binom{12}{n_3}\binom{11}{n_2}$ subdesigns. To reduce the burden of computation, the criterion is relaxed to compare only K_3, K_4 and K_5, which should meet the practical need. Indeed it makes no difference if the first eight moments are used. The complementary design technique is used to further reduce the computation if $n_3 + n_2 > 11$. In particular, no computation is needed if $n_3 + n_2 = 21$ or 22 because the complementary designs have only one or two columns and thus are indistinguishable under minimum moment aberration. Table 3 lists minimum moment aberration designs from Table 2 with n_3 three-level factors and n_2 two-level factors for $n_3 \leq 12$ and $n_2 \leq 11$. No design is given in Table 3 if all possible subdesigns are indistinguishable under minimum moment aberration.

7 Concluding Remarks

The concept of minimum moment aberration is proposed for assessing the goodness of nonregular designs and supersaturated designs. It is to sequentially minimize the power moments of the number of coincidence among runs. The minimum moment aberration is conceptually simple, cheap to compute, and convenient for theoretical development.

The statistical justification developed by Xu and Wu (2001) for the GMA also works for the minimum moment aberration because the two criteria are equivalent for symmetrical designs and weakly equivalent for asymmetrical designs.

Without explicitly handling with the coding of treatment contrasts, the power moments are easy to program. The minimum moment aberration provides a unified and efficient treatment for regular and nonregular designs, nonsaturated and supersaturated designs, orthogonal and nonorthogonal designs, symmetrical and asymmetrical designs. As an application, Xu (2000) developed an efficient algorithm for constructing a variety of mixed-level orthogonal and nearly orthogonal arrays.

Finally, data from an experiment using nonregular designs or supersaturated designs can be analyzed by stepwise selection or Bayesian variable selection procedure. Examples and details are available in Hamada and Wu (1992), Lin (1993, 1995), Chipman, Hamada, and Wu (1997), and Wu and Hamada (2000, Chapter 8).

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Appendix: Proofs

Some concepts and results in algebraic coding theory are necessary to prove Theorem 1. The readers are referred to MacWilliams and Sloane (1977), Pless (1989) and van Lint (1999) for details.

For an (N, s^n)-design D, let $d_{ij}(D) = n - \delta_{ij}(D)$ and

$$B_k(D) = N^{-1} \left|\{(i,j) : d_{ij}(D) = k,\ i,j = 1,\ldots,N\}\right| \text{ for } k = 0,\ldots,n.$$

In coding theory, $d_{ij}(D)$ is called the *Hamming distance* and the vector $(B_0(D), B_1(D), \ldots, B_n(D))$ is the *distance distribution*. It is clear that for $k = 0, 1, \ldots,$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}[d_{ij}(D)]^k = N\sum_{i=0}^{n} i^k B_i(D). \tag{A.1}$$

Xu and Wu (2001) showed that the distance distributions are linear combinations of the generalized wordlength patterns, that is, for $j = 0, \ldots, n$,

$$B_j(D) \quad = \quad N\, s^{-n}\sum_{i=0}^{n} A_i(D)P_j(i; n, s), \tag{A.2}$$

where $P_j(x;n,s) = \sum_{i=0}^{j}(-1)^i(s-1)^{j-i}\binom{x}{i}\binom{n-x}{j-i}$ are the *Krawtchouk polynomials*.

The following identities, extensions of the *Pless power moment identities* (Pless 1963), relate the moments of the distance distribution and the generalized wordlength pattern.

LEMMA 4. *For an (N, s^n)-design D and integers $k \geq 0$,*

$$\sum_{i=0}^{n} i^k B_i(D) = N\sum_{i=0}^{n}(-1)^i A_i(D)\theta_i(k;n,s),$$

where $\theta_i(k;n,s) = \sum_{j=0}^{k} j!S(k,j)s^{-j}(s-1)^{j-i}\binom{n-i}{j-i}$ and $S(k,j)$ is a Stirling number of the second kind.

Proof of Lemma 4. Let $f(z) = (1-z)^x[1+(s-1)z]^{n-x}$ and D_z be the differentiation operator with respect to z. It is known that, for an integer x, $0 \leq x \leq n$, $f(z) = \sum_{j=0}^{n} P_j(x;n,s)z^j$. Thus, $\sum_{j=0}^{n} j^k P_j(x;n,s) = (zD_z)^k f(z)|_{z=1}$. It is also known that $(zD_z)^k = \sum_{j=0}^{k} S(k,j)z^j(D_z)^j$. Noting that

$$f(z) = (1-z)^x[s+(s-1)(z-1)]^{n-x} = (-1)^x \sum_{i=0}^{n-x}\binom{n-x}{i}s^{n-x-i}(s-1)^i(z-1)^{x+i},$$

we have $(D_z)^j f(z)|_{z=1} = (-1)^x j!\binom{n-x}{j-x}s^{n-j}(s-1)^{j-x}$ and

$$\sum_{j=0}^{n} j^k P_j(x;n,s) = \sum_{j=0}^{k} S(k,j)z^j(D_z)^j f(z)|_{z=1} = (-1)^x \sum_{j=0}^{k} j!S(k,j)\binom{n-x}{j-x}s^{n-j}(s-1)^{j-x}.$$

Finally, by (A.2), we obtain

$$\sum_{j=0}^{n} j^k B_j(D) = \sum_{j=0}^{n} j^k N s^{-n}\sum_{i=0}^{n} P_j(i;n,s)A_i(D) = N\sum_{i=0}^{n} A_i(D)\left[\sum_{j=0}^{n} s^{-n} j^k P_j(i;n,s)\right]$$
$$= N\sum_{i=0}^{n} A_i(D)\left[(-1)^i\sum_{j=0}^{k} j!S(k,j)s^{-j}(s-1)^{j-i}\binom{n-i}{j-i}\right].$$

□

Proof of Theorem 1. By (A.1) and Lemma 4, for $t = 1, 2, \ldots$,

$$\begin{aligned} K_t(D) &= [N(N-1)]^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}[\delta_{ij}(D)]^t - (N-1)^{-1}n^t \\ &= [N(N-1)]^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}[n-d_{ij}(D)]^t - c_0 \end{aligned}$$

$$
\begin{aligned}
&= [N(N-1)]^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=0}^{t}(-1)^k\binom{t}{k}n^{t-k}[d_{ij}(D)]^k - c_0\\
&= (N-1)^{-1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}n^{t-k}\sum_{i=0}^{n} i^k B_i(D) - c_0\\
&= (N-1)^{-1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}n^{t-k}\left(N\sum_{i=0}^{n}(-1)^i A_i(D)\theta_i(k;n,s)\right) - c_0\\
&= N(N-1)^{-1}\sum_{i=0}^{n} A_i(D)\left(\sum_{k=0}^{t}(-1)^{k+i}\binom{t}{k}n^{t-k}\theta_i(k;n,s)\right) - c_0\\
&= \sum_{i=0}^{n}\alpha_i(t;N,n,s)A_i(D) - c_0.
\end{aligned}
$$

It is easy to verify from definition that $\alpha_t(t;N,n,s) = t!N/[(N-1)s^t]$, $\alpha_{t-1}(t;N,n,s) = t![n+(t-1)(s-2)/2]N/[(N-1)s^t]$ and $\alpha_i(t;N,n,s) = 0$ if $i > t$. □

Proof of Theorem 3. We state a proof for $t = 2$ only. The general case is essentially the same with more complicated notation.

For an (N, s^n)-design $D = [r_{ij}]_{N\times n}$, let $n_{kl}(a,b) = |\{i : r_{ik} = a, r_{il} = b\}|$. Then

$$
\begin{aligned}
N(N-1)K_2(D) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\sum_{k=1}^{n}\delta(r_{ik}, r_{jk})\right]^2 - Nn^2\\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\sum_{k=1}^{n}\sum_{l=1}^{n}\delta(r_{ik}, r_{jk})\delta(r_{il}, r_{jl})\right] - Nn^2\\
&= \sum_{k=1}^{n}\sum_{l=1}^{n}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\delta(r_{ik}, r_{jk})\delta(r_{il}, r_{jl})\right] - Nn^2\\
&= \sum_{k=1}^{n}\sum_{l=1}^{n}\left[\sum_{a=0}^{s-1}\sum_{b=0}^{s-1}n_{kl}(a,b)^2\right] - Nn^2\\
&= \sum_{k=1}^{n}\left[\sum_{a=0}^{s-1}n_{kk}(a,a)^2\right] + \sum_{1\le k\ne l\le n}\left[\sum_{a=0}^{s-1}\sum_{b=0}^{s-1}n_{kl}(a,b)^2\right] - Nn^2.
\end{aligned}
$$

Then, by Lemma 1, the first term is minimized if D is an $OA(1^-)$ and the second term is minimized if D is an $OA(2^-)$. □

Proof of Theorem 5. First, by Theorem 3, $K_1(D)$ is minimized for $OA(1^-)$. Second, by definition and Lemma 1, $K_2(D)$ is minimized. Finally, by Lemma 1 again, all other $K_t(D)$'s are determined uniquely given $K_1(D)$ and $K_2(D)$. □

Proof of Equation (7). It is easy to verify that for a balanced (N, s^n)-design D

$$\chi^2_{kl} = s^2/N \sum_{a=0}^{s-1}\sum_{b=0}^{s-1} n_{kl}(a,b)^2 - N.$$

Then, following the proof of Theorem 3,

$$\begin{aligned} N(N-1)K_2(D) &= \sum_{k=1}^{n}[N^2/s] + \sum_{1\le k\ne l\le n}[(N/s^2)(\chi^2_{kl}+N)] - Nn^2 \\ &= nN^2/s + (N/s^2)n(n-1)(\text{ave}\,\chi^2 + N) - Nn^2, \end{aligned}$$

and equation (7) follows. □

Proof of Theorem 6. Following the proof of Theorem 3, $K_1(D) \ge \sum w_k(N/s_k - 1)/(N-1)$ with equality if and only if D is an $OA(1)$. Then the theorem follows from inequality (4). □

Proof of Theorem 7. The proof is similar to that of Theorem 1 with the generalized Pless power moment identities for asymmetrical designs. The details are omitted. □

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Table 1: $OA(18, 3^7, 2)$

Run	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1
3	0	2	2	2	2	2	2
4	1	0	0	1	1	2	2
5	1	1	1	2	2	0	0
6	1	2	2	0	0	1	1
7	2	0	1	0	2	1	2
8	2	1	2	1	0	2	0
9	2	2	0	2	1	0	1
10	0	0	2	2	1	1	0
11	0	1	0	0	2	2	1
12	0	2	1	1	0	0	2
13	1	0	1	2	0	2	1
14	1	1	2	0	1	0	2
15	1	2	0	1	2	1	0
16	2	0	2	1	2	0	1
17	2	1	0	2	0	1	2
18	2	2	1	0	1	2	0

Table 2: $OA(36, 3^{12}2^{11}, 2)$

Run	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0	0	0	1	1	0	0	1	0	2	2	0	1	0	1	0	0	0	1	1	1	0	1
2	0	0	0	0	2	0	2	0	2	0	0	1	1	1	0	1	0	0	0	1	1	1	0
3	0	0	1	0	0	2	1	2	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1
4	0	0	2	2	0	1	0	0	1	1	0	0	1	0	1	1	0	1	0	0	0	1	1
5	0	1	2	2	0	0	1	1	2	0	2	2	1	1	0	1	1	0	1	0	0	0	1
6	0	1	2	1	2	1	2	2	2	2	1	0	1	1	1	0	1	1	0	1	0	0	0
7	0	1	0	0	2	2	0	2	1	1	2	2	0	1	1	1	0	1	1	0	1	0	0
8	0	1	1	2	1	2	2	0	0	2	0	2	0	0	1	1	1	0	1	1	0	1	0
9	0	2	1	2	1	0	0	2	2	1	1	1	0	0	0	1	1	1	0	1	1	0	1
10	0	2	1	0	0	1	2	1	1	2	2	1	1	0	0	0	1	1	1	0	1	1	0
11	0	2	2	1	2	2	1	1	0	1	0	1	0	1	0	0	0	1	1	1	0	1	1
12	0	2	0	1	1	1	1	0	1	0	1	2	0	0	0	0	0	0	0	0	0	0	0
13	1	1	1	2	2	1	1	2	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1
14	1	1	1	1	0	1	0	1	0	1	1	2	1	1	0	1	0	0	0	1	1	1	0
15	1	1	2	1	1	0	2	0	1	1	2	1	0	1	1	0	1	0	0	0	1	1	1
16	1	1	0	0	1	2	1	1	2	2	1	1	1	0	1	1	0	1	0	0	0	1	1
17	1	2	0	0	1	1	2	2	0	1	0	0	1	1	0	1	1	0	1	0	0	0	1
18	1	2	0	2	0	2	0	0	0	0	2	1	1	1	1	0	1	1	0	1	0	0	0
19	1	2	1	1	0	0	1	0	2	2	0	0	0	1	1	1	0	1	1	0	1	0	0
20	1	2	2	0	2	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0	1	0
21	1	0	2	0	2	1	1	0	0	2	2	2	0	0	0	1	1	1	0	1	1	0	1
22	1	0	2	1	1	2	0	2	2	0	0	2	1	0	0	0	1	1	1	0	1	1	0
23	1	0	0	2	0	0	2	2	1	2	1	2	0	1	0	0	0	1	1	1	0	1	1
24	1	0	1	2	2	2	2	1	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0
25	2	2	2	0	0	2	2	0	2	1	1	2	1	0	1	0	0	0	1	1	1	0	1
26	2	2	2	2	1	2	1	2	1	2	2	0	1	1	0	1	0	0	0	1	1	1	0
27	2	2	0	2	2	1	0	1	2	2	0	2	0	1	1	0	1	0	0	0	1	1	1
28	2	2	1	1	2	0	2	2	0	0	2	2	1	0	1	1	0	1	0	0	0	1	1
29	2	0	1	1	2	2	0	0	1	2	1	1	1	1	0	1	1	0	1	0	0	0	1
30	2	0	1	0	1	0	1	1	1	1	0	2	1	1	1	0	1	1	0	1	0	0	0
31	2	0	2	2	1	1	2	1	0	0	1	1	0	1	1	1	0	1	1	0	1	0	0
32	2	0	0	1	0	1	1	2	2	1	2	1	0	0	1	1	1	0	1	1	0	1	0
33	2	1	0	1	0	2	2	1	1	0	0	0	0	0	0	1	1	1	0	1	1	0	1
34	2	1	0	2	2	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	0
35	2	1	1	0	1	1	0	0	2	0	2	0	0	1	0	0	0	1	1	1	0	1	1
36	2	1	2	0	0	0	0	2	0	2	0	1	0	0	0	0	0	0	0	0	0	0	0

Table 3: Minimum Moment Aberration Designs from Table 2

$n_3.n_2$	Three-Level Columns	Two-Level Columns
0.5		13 14 15 16 17
0.6		13 14 15 16 17 22
1.5	1	13 14 15 16 17
1.6	1	13 14 15 16 17 22
2.1	1 2	20
2.2	1 3	15 16
2.3	2 3	13 15 23
2.4	1 3	15 16 19 20
2.5	1 3	15 16 19 20 22
2.6	1 3	13 15 16 19 20 22
2.7	1 3	13 14 15 16 19 20 22
2.8	1 3	13 14 15 16 18 19 20 22
2.9	1 3	13 14 15 16 18 19 20 22 23
2.10	9 11	13 14 15 16 17 18 19 20 22 23
2.11	8 11	13 14 15 16 17 18 19 20 21 22 23
3.0	1 2 3	
3.1	1 2 8	20
3.2	4 9 10	16 21
3.3	2 3 4	15 21 23
3.4	2 3 4	13 15 21 23
3.5	2 3 4	13 15 21 22 23
3.6	1 3 4	13 15 18 19 20 23
3.7	1 3 4	13 15 18 19 20 22 23
3.8	2 4 7	14 15 16 18 20 21 22 23
3.9	7 8 10	13 14 15 17 19 20 21 22 23
3.10	6 8 11	13 14 15 16 17 18 19 20 21 22
3.11	1 6 8	13 14 15 16 17 18 19 20 21 22 23
4.0	1 2 3 7	
4.1	1 5 9 10	21
4.2	1 5 9 10	16 21
4.3	1 5 9 10	16 21 23
4.4	1 5 9 10	16 21 22 23
4.5	2 8 11 12	15 19 20 21 23
4.6	2 8 11 12	13 15 19 20 21 23
4.7	2 8 11 12	13 15 17 19 20 21 23
4.8	2 8 11 12	13 15 17 19 20 21 22 23
4.9	2 8 11 12	13 15 17 18 19 20 21 22 23
4.10	7 8 10 11	13 14 15 17 18 19 20 21 22 23
4.11	5 7 10 12	13 14 15 16 17 18 19 20 21 22 23
5.0	1 2 3 7 8	
5.1	1 2 6 7 11	21
5.2	1 2 6 7 11	18 21
5.3	1 5 8 9 10	21 22 23
5.4	1 5 8 9 10	16 21 22 23
5.5	1 5 8 9 10	13 16 21 22 23
5.6	1 5 6 7 11	13 15 16 17 18 21
5.7	1 5 9 10 12	13 16 17 19 21 22 23
5.8	1 7 9 11 12	13 14 15 17 18 19 21 23
5.9	2 3 5 10 12	13 14 15 16 17 18 19 21 22
5.10	5 7 8 10 11	13 14 15 17 18 19 20 21 22 23
5.11	5 7 8 10 11	13 14 15 16 17 18 19 20 21 22 23
6.0	1 2 3 7 8 9	
6.1	1 2 5 6 7 11	21
6.2	1 5 8 9 10 12	21 22
6.3	1 2 5 6 7 11	16 18 21
6.4	1 2 5 6 7 11	15 16 18 21
6.5	1 2 5 6 7 11	15 16 18 19 21
6.6	1 5 8 9 10 12	16 17 19 21 22 23
6.7	1 5 8 9 10 12	13 16 17 19 21 22 23
6.8	1 5 8 9 10 12	13 16 17 18 19 21 22 23
6.9	1 5 8 9 10 12	13 16 17 18 19 20 21 22 23
6.10	1 2 5 6 7 11	13 15 16 17 18 19 20 21 22 23
6.11	2 3 5 7 10 12	13 14 15 16 17 18 19 20 21 22 23

Table 3: Minimum Moment Aberration Designs from Table 2 (Continued)

$n_3.n_2$	Three-Level Columns	Two-Level Columns
7.0	1 2 3 4 6 8 10	
7.1	1 2 3 5 9 10 11	16
7.2	1 3 5 6 7 8 12	16 17
7.3	1 3 5 6 7 8 12	16 17 20
7.4	1 3 5 6 7 8 12	16 17 19 20
7.5	1 3 5 6 7 8 12	16 17 19 20 22
7.6	1 3 5 6 7 8 12	16 17 18 19 20 22
7.7	1 2 3 5 6 7 11	13 15 16 18 19 21 23
7.8	1 2 3 5 6 7 11	13 15 16 17 18 19 21 23
7.9	1 3 5 6 7 8 12	13 15 16 17 18 19 20 22 23
7.10	2 4 5 6 7 9 12	13 14 15 16 17 18 20 21 22 23
7.11	2 3 5 7 10 11 12	13 14 15 16 17 18 19 20 21 22 23
8.0	1 2 3 4 5 9 10 11	
8.1	1 3 4 5 6 7 8 12	17
8.2	2 6 7 8 9 10 11 12	15 19
8.3	1 3 4 5 6 7 8 12	16 17 19
8.4	1 3 4 5 6 7 8 12	16 17 19 22
8.5	1 3 4 5 6 7 8 12	16 17 19 20 22
8.6	1 3 4 5 6 7 8 12	16 17 18 19 20 22
8.7	1 3 4 5 6 7 8 12	15 16 17 18 19 20 22
8.8	1 3 4 5 6 7 8 12	15 16 17 18 19 20 22 23
8.9	1 3 4 5 6 7 8 12	13 15 16 17 18 19 20 22 23
8.10	1 3 4 5 6 7 8 12	13 14 15 16 17 18 19 20 22 23
8.11	1 3 4 5 6 7 8 12	13 14 15 16 17 18 19 20 21 22 23
9.0	1 2 3 4 6 7 8 9 10	
9.1	1 2 4 5 6 7 9 11 12	18
9.2	1 3 4 6 7 9 10 11 12	14 20
9.3	1 3 4 6 7 9 10 11 12	14 17 20
9.4	1 3 4 6 7 9 10 11 12	14 17 20 21
9.5	1 3 4 6 7 9 10 11 12	14 17 20 21 23
9.6	1 3 4 6 7 9 10 11 12	14 17 18 20 21 23
9.7	1 3 5 6 8 9 10 11 12	14 16 18 20 21 22 23
9.8	1 3 5 6 8 9 10 11 12	14 15 16 18 20 21 22 23
9.9	1 3 5 6 8 9 10 11 12	13 14 15 16 18 20 21 22 23
9.10	1 2 3 5 6 7 10 11 12	13 14 15 16 18 19 20 21 22 23
9.11	1 4 5 7 8 9 10 11 12	13 14 15 16 17 18 19 20 21 22 23
10.1	1 2 3 4 5 6 7 8 11 12	21
10.2	2 3 5 6 7 8 9 10 11 12	13 18
10.3	1 2 3 4 5 6 8 9 10 11	13 16 18
10.4	2 3 5 6 7 8 9 10 11 12	13 15 18 19
10.5	1 3 4 5 6 7 8 9 11 12	13 14 16 17 18
10.6	1 3 4 5 6 7 8 9 11 12	13 14 16 17 18 22
10.7	1 3 4 5 6 7 8 9 11 12	13 14 16 17 18 20 22
10.8	2 4 5 6 7 8 9 10 11 12	13 15 16 18 19 20 22 23
10.9	2 4 5 6 7 8 9 10 11 12	13 14 15 16 18 19 20 22 23
10.10	1 2 3 4 5 6 7 8 11 12	13 14 16 17 18 19 20 21 22 23
11.1	1 2 3 5 6 7 8 9 10 11 12	16
11.2	1 2 3 5 6 7 8 9 10 11 12	16 22
11.3	1 2 3 5 6 7 8 9 10 11 12	16 20 22
11.4	1 2 3 5 6 7 8 9 10 11 12	13 16 20 22
11.5	1 2 3 5 6 7 8 9 10 11 12	13 16 20 22 23
11.6	1 2 3 5 6 7 8 9 10 11 12	13 16 19 20 22 23
11.7	1 2 3 5 6 7 8 9 10 11 12	13 14 16 18 20 22 23
11.8	1 2 3 5 6 7 8 9 10 11 12	13 14 15 16 18 20 22 23
11.9	1 2 3 5 6 7 8 9 10 11 12	13 14 15 16 18 19 20 22 23
12.1	1 2 3 4 5 6 7 8 9 10 11 12	22
12.2	1 2 3 4 5 6 7 8 9 10 11 12	18 23
12.3	1 2 3 4 5 6 7 8 9 10 11 12	18 22 23
12.4	1 2 3 4 5 6 7 8 9 10 11 12	18 20 22 23
12.5	1 2 3 4 5 6 7 8 9 10 11 12	13 18 20 22 23
12.6	1 2 3 4 5 6 7 8 9 10 11 12	13 18 20 21 22 23
12.7	1 2 3 4 5 6 7 8 9 10 11 12	13 14 16 18 20 22 23
12.8	1 2 3 4 5 6 7 8 9 10 11 12	13 14 16 18 20 21 22 23