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MINIMUM OUTPUT ENTROPY OF A GAUSSIAN BOSONIC CHANNEL

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The entropy at the output of a Bosonic channel is analyzed when coherent fields are randomly added to the signal. Coherent-state inputs are conjectured to provide the minimum output entropy. Supporting physical and mathematical evidence is provided.

Keywords: quantum channels; entropy; additivity; capacity; bosonic channels.

A memoryless quantum channel[1] can be described by a completely positive, trace preserving, linear super-operator $\mathcal{M}$ on the Hilbert space $\mathcal{H}$, that maps an input state ρ to an output state $\rho' \equiv \mathcal{M}(\rho)$. In this paper we analyze the minimal von Neumann entropy at the output of the channel[2], i.e.,

$$\mathbb{S} \equiv \min_{\rho \in \mathcal{H}} \{S(\rho')\} \equiv \min_{\rho\in\mathcal{H}} \{-\mathrm{Tr}\,[\rho' \ln \rho']\} \ , \tag{1}$$

where the minimization is performed over all possible input density matrices ρ. The quantity $\mathbb{S}$ defined in Eq. (1) provides a "measure" of the minimum amount of noise introduced by the channel $\mathcal{M}$ during the communication. It is also connected with the channel capacity[3] for the transmission of classical information. Here we focus on Bosonic channels[4] where the message is encoded onto electromagnetic field modes and where the map describes the noise encountered during the transmission. In particular, we consider a single mode channel with a Gaussian map[a] that randomly

[a]These are maps that transform states with Gaussian characteristic functions into outputs which also have Gaussian characteristic functions.

displaces the input signal according to a Gaussian probability distribution[5,6,7], i.e.

$$\rho' \equiv \mathcal{N}_n(\rho) = \int d^2\mu \; P_n(\mu) \; D(\mu)\rho D^\dagger(\mu) \tag{2}$$

where

$$P_n(\mu) = \frac{e^{-|\mu|^2/n}}{\pi n}, \tag{3}$$

and $D(\mu) \equiv \exp(\mu a^\dagger - \mu^* a)$ is the displacement operator associated with the input mode. This channel can be seen as a simplified version of a map in which the signal photons interact linearly with a thermal environment[6,7]. In particular, $\mathcal{N}_n$ transforms the vacuum input $|0\rangle$ into the thermal output

$$\rho_0' \equiv \frac{1}{n+1} \sum_{m=0}^{\infty} \left(\frac{n}{n+1}\right)^m |m\rangle\langle m| \equiv \int d^2\mu \; P_n(\mu) \; |\mu\rangle\langle\mu| \;, \tag{4}$$

where $|m\rangle$ and $|\mu\rangle$ are Fock states and coherent states respectively. This output has von Neumann entropy equal to

$$S(\rho_0') = g(n) \equiv (1+n)\ln(1+n) - n \ln n \;. \tag{5}$$

Analogously one can show that the output associated with a coherent input $|\alpha\rangle$ is obtained by displacing ρ_0' of Eq. (4), i.e.

$$\rho_\alpha' = D(\alpha) \; \rho_0' \; D^\dagger(\alpha). \tag{6}$$

Thus, the invariance of the von Neumann entropy under unitary transformation guarantees that all coherent-state inputs have the same output entropy.

1. A conjecture

From the concavity of the von Neumann entropy we know that the minimum in Eq. (1) can be achieved with pure input states $\rho = |\psi\rangle\langle\psi|$. Recently, it has been conjectured[6] that the minimum output entropy of a wide class of Gaussian Bosonic channels is achieved by coherent-state inputs. In the case of $\mathcal{N}_n$ this is equivalent to having

$$\mathbb{S} = g(n) \;. \tag{7}$$

The physical intuition behind this conjecture lies in the fact that the input state is contaminated by noise from a reservoir [characterized by the probability distribution of Eq. (2)] whose quantum phase is completely random. It is hence reasonable to expect that no coherence can be extracted from the reservoir to reduce the output entropy below the level when no photons are sent through the channel. Theoretical and numerical evidence[6] suggests that an even stronger version of this conjecture should apply, namely that the output states produced by coherent inputs *majorize*[8] all the other output states. Here we will focus only on the (weaker) version (7) of the conjecture.

Even though the relation (7) has not been proven yet, ample supporting evidence that supports its validity has been obtained[6,7,9]. In the following we discuss some of the main results.

1.1. *Local minimum*

Coherent states produce local minima in the output von Neumann entropy[6]. In fact, an output state σ_0' is a local minimum of the output entropy if the following directional derivative is non-negative, i.e.

$$\frac{\partial}{\partial t} S(\sigma_0'(1-t)+\sigma' t)\bigg|_{t=0^+} = \mathrm{Tr}[(\sigma_0'-\sigma')\ln\sigma_0'] \geqslant 0, \tag{8}$$

for any output σ'. In the case of vacuum input $\sigma_0 = |0\rangle\langle 0|$, the output state (4) can be written as $\sigma_0' \propto \exp[-\zeta a^\dagger a]$ with $\zeta > 0$. Since σ_0' is the output state with minimum average photon number, we have $\mathrm{Tr}[(\sigma_0' - \sigma')a^\dagger a] \leqslant 0$ and Eq. (8) is satisfied for vacuum input (an analogous derivation applies in the case of generic coherent-state inputs). If one could show that the inequality (8) is true only for coherent-state inputs, then the conjecture would be proved.

1.2. *Lower bounds*

The output entropy for coherent inputs gives an upper bound for the minimum output entropy of a channel. Lower bounds for $\mathbb{S}$ have been derived in Ref. 6. Here we only analyze a few of them, i.e.,

$$\mathbb{S} \geqslant \begin{cases} g(n-1) \\ \\ \ln(2n+1)\ . \end{cases} \tag{9}$$

The first one is a tight bound for $n \gg 1$. It is obtained by considering the Husimi expansion[10,11] of the input state ρ,

$$\rho = \int d^2\alpha\ Q(\alpha)\ \sigma(\alpha)\ , \tag{10}$$

where $Q(\alpha) = \langle\alpha|\rho|\alpha\rangle/\pi$ and $\sigma(\alpha)$ is a convolution of displacement operators of the field. It implies

$$\rho' = \int d^2\alpha\ Q(\alpha)\ \sigma'(\alpha)\ , \tag{11}$$

where $\sigma'(\alpha)$ is the evolution of $\sigma(\alpha)$ through the channel. The first inequality of (9) then follows from the convexity of the von Neumann entropy. The second inequality of (9) gives a tight bound for $n \sim 0$. It is derived by observing that for any state ρ, the von Neumann entropy satisfies

$$S(\rho) \geqslant -\ln(\mathrm{Tr}[\rho^2]), \tag{12}$$

and by using the bound on $\mathrm{Tr}[\rho'^2]$ described below.

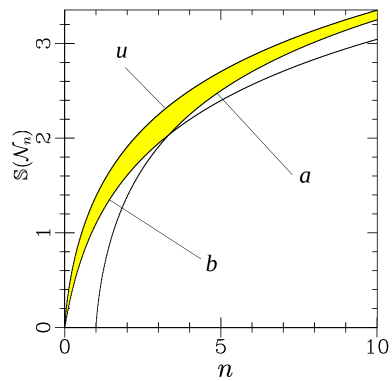

Fig. 1. Bounds on the minimum output entropy of the channel $\mathcal{N}_n$ as a function of the noise parameter n. The curve u is the upper bound obtained feeding the channel with a coherent input; a and b are the first and second lower bounds of Eq. (9) respectively. The minimal output entropy $\mathbb{S}$ must reside between the upper and lower bounds, i.e. in the shaded area.

1.3. *Rényi entropy*

Generalizing an argument by Caves[12] we proved[7] that the output Rényi entropies S_k of integer order k greater than or equal to 2 are minimized by coherent-state inputs, i.e.,

$$S_k(\rho') \equiv \frac{\ln(\mathrm{Tr}[(\rho')^k])}{1-k} \geqslant \frac{\ln[(n+1)^k - n^k]}{k-1} \quad k = 2, 3, \ldots \tag{13}$$

To derive this inequality we express $\mathrm{Tr}[(\rho')^k]$ as an expectation value of a Gaussian operator A acting on an extended Hilbert space of k modes, i.e.,

$$\mathrm{Tr}[(\rho')^k] \equiv \mathrm{Tr}[\underbrace{\rho \otimes \rho \otimes \cdots \otimes \rho}_{k \text{ times}} A]. \tag{14}$$

The maximum eigenvalue of A is an upper bound for $\mathrm{Tr}[(\rho')^k]$ (i.e., a lower bound for $S_k(\rho')$) which is achieved by coherent-state inputs. Notice that the Rényi entropy tends to the von Neumann entropy in the limit of $k \to 1$. If we could generalize Eq. (13) to any $k \in]1, 2[$ then the conjecture would follow from the continuity of Rényi entropy.

For the Gaussian Bosonic channel, the above technique also proved successful[9] in analyzing the additivity properties[13] of the minimum integer-order Rényi entropies over successive channel uses.

1.4. *Wehrl entropy*

The Wehrl entropy of a state ρ is the Shannon entropy of its Husimi function, i.e.

$$W(\rho) \equiv -\int \frac{d^2\alpha}{\pi} \, \langle\alpha|\rho|\alpha\rangle \, \ln\langle\alpha|\rho|\alpha\rangle. \tag{15}$$

This quantity measures the localization of a state in phase space and is minimized by coherent states[10,11], as Lieb proved. Using Young's inequality and Hausdorff-Young's inequality one can show[7] that the Wehrl entropy at the output of $\mathcal{N}_n$ achieves its minimum over coherent inputs, i.e.,

$$W(\rho') \geqslant 1 + \ln(1+n). \tag{16}$$

In the same way, one can prove[7] that the Rényi-Wehrl entropies[14] W_k with order k greater than 1 are minimized by coherent inputs, namely[b]

$$W_k(\rho') \equiv \frac{\ln[\int \frac{d^2\alpha}{\pi} \langle\alpha|\rho'|\alpha\rangle^k]}{1-k} \geqslant \frac{\ln[k(n+1)^{k-1}]}{k-1}. \tag{17}$$

Notice that in the limit of $k \to 1$, Eq. (17) gives Eq. (16).

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[b]N.B.: here it is not necessary to restrict k to integer values, as this property is valid for any $k > 1$.