Minimum Rate Sampling and Reconstruction of Signals with Arbitrary Frequency Support

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***Abstract*—We examine the question of reconstruction of signals from periodic nonuniform samples. This involves discarding samples from a uniformly sampled signal in some periodic fashion. We give a characterization of the signals that can be reconstructed at exactly the minimum rate once a nonuniform sampling pattern has been fixed. We give an implicit characterization of the reconstruction system, and a design method by which the ideal reconstruction filters may be approximated. We demonstrate that for certain spectral supports the minimum rate can be approached or achieved using reconstruction schemes of much lower complexity than those arrived at by using spectral slicing, as in earlier work.**

Previous work on multiband signals have typically been those for which restrictive assumptions on the sizes and positions of the bands have been made, or where the minimum rate was approached asymptotically. We show that the class of multiband signals which can be reconstructed exactly is shown to be far larger than previously considered. When approaching the minimum rate, this freedom allows us, in certain cases to have a far less complex reconstruction system.

***Index Terms*— Multiband, nonuniform, reconstruction, sampling.**

I. Introduction

THE idea that a signal which has energy in only a limited range of frequencies may be represented by its samples is of fundamental importance in digital communications. Its formal statement is generally attributed to Shannon [1], E. T. Whittaker [2], J. M. Whittaker [3], and Kotel'nikov [4]. In the version given by Shannon it is stated that a real signal that has energy only in the range of frequencies $(-B, B)$ can be reconstructed without loss from uniform samples at rate $2B$. A variety of generalizations have been reported; the reader who is interested in the history of the subject might consult the comprehensive review by Jerri [5] or the articles and bibliography of [6].

An important class of generalizations are those where, instead of having a single train of samples at uniform rate $2B$, we have N uniform trains at rate $2B/N$. If these N trains were identical, obviously this would be no better than having one. If, however, prior to sampling, the signal is subjected to N different operations such as delay, differentiation, or is passed through N different linear systems then it is possible that the resulting N trains are independent (in a sense to be defined) and that the original signal can be reconstructed without loss. Thus instead of considering one high rate sample train, we consider several lower rate trains, the former obviously being a special case of the latter. Generalizations of this kind are important for two reasons.

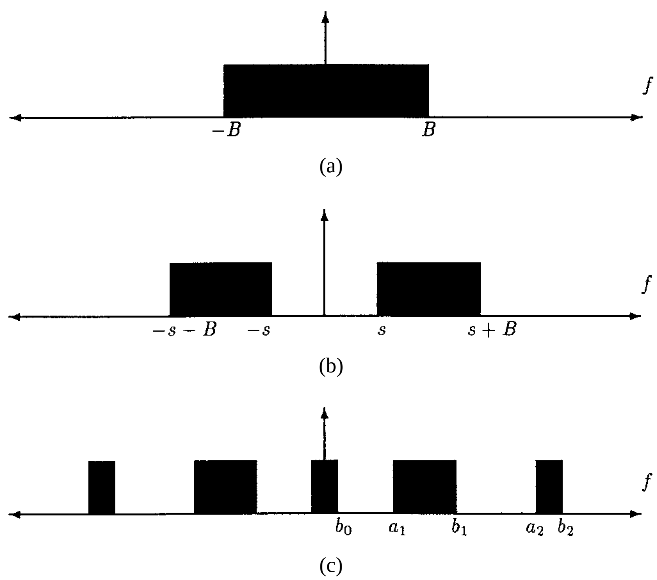

Fig. 1. Signals with various different frequency occupancies. The overall effective bandwidth is the same in all cases, but the required sampling differs. (a) Lowpass bandlimited case; uniform sampling at twice the highest frequency is sufficient. (b) Bandpass bandlimited case; uniform sampling is a function of the band position. (c) Multiband bandlimited case; uniform sampling is very redundant. Periodic nonuniform sampling allows minimum rate.

First, the classical sampling theorem [1]–[4] applies to lowpass bandlimited signals only, such as shown in Fig. 1(a). For signals that are bandpass, or multiband, such as shown in Fig. 1(b) and (c), respectively, sampling at $2f_{\max}$ entails considerable waste, since $2f_{\max} > B_{\text{eff}}$ (here B_{eff} is taken to represent the *total* support of the set of frequencies over which the spectrum of the signal is nonzero, and $f_{\max}$ is the highest frequency at which the signal has nonzero energy). Thus while the effective bandwidth for all of the signals shown in Fig. 1 is the same, the required rate to allow reconstruction from uniform sampling varies considerably. In such cases having N independent trains at rate $2B/N$ can allow reconstruction, while still requiring the same effective rate as in the lowpass case. A second reason for the importance of such sampling schemes is that the availability of several low-rate sampled

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trains rather than one high-rate train is often a consequence of the sampling mechanism. For example, practical considerations in the design of analog-to-digital converters [7] and antenna arrays [8] often result in just such a nonuniform sampling. Certain signals, such as video, are often sampled in a nonuniform way [9].

Since nonuniform sampling and multiband signals have been examined by researchers in areas as diverse as Mathematics, Signal Processing, Communications, and Circuit Design, and the work is spread over many decades in widely differing notations, it may be useful to summarize briefly and explain where our work fits in. N-channel sampling, as we shall term it, involves passing the signal through N distinct transformations before sampling. The case where those transformations are delays was treated in [10]–[15], and making them differentiation operators of various orders was examined in [16] and [17]. A generalization that encompasses both was introduced by Papoulis [18] and developed by Brown [19]. This shows that a function may be reconstructed from N sample trains derived by passing the function through N different linear filters. The reconstruction filters are found by solving an $N \times N$ matrix system, where the elements of the matrix are filters. An important exception is the work of Yen [13] which gives a closed form for the case of N channels of samples of a *lowpass* signal with various delays. This form does not extend easily to the case of multiband signals, and appears to be an isolated case of a closed form for an N-channel sampling system.

Recent treatments of nonuniform sampling are by Marvasti [20], Feichtinger and Gröchenig [21], and by Feng and Bresler [22]. Reconstruction from arbitrary sampling has been also considered [23]. Several interesting applications of nonuniform sampling are covered in [24]. A very interesting multidimensional construction by Cheung and Marks [25], [26] shows how nonuniform sampling may be used to exploit the spectral gaps that occur when sampling multidimensional signals. Their approach is to slice the spectrum into narrow bands, and handle separately those bands which contain signal energy and those which do not. A similar approach can be applied to multiband signals. One can show that asymptotically this achieves the minimum rate. However, while the sampling efficiency increases as the slices become narrower, so too does the complexity of the sampling and reconstruction scheme. The scheme that we present is not restricted to slicing the signal bands, and this means that, in some cases, much simpler sampling strategies can be found to achieve the same rate.

An outline of the paper is as follows. In the next section we present preliminaries, and discuss periodic nonuniform sampling schemes which are derived by deleting samples periodically from some uniform train. In Section III we give a characterization of the class of signals which can be exactly reconstructed at the minimum rate, once a sampling scheme has been chosen; this turns out to be a rediscovery of a result by Kahn and Liu [12]. It leads, however, in Section IV to an algorithm to design the filters of the reconstruction system, which is distinctly different from other design approaches. In Section V we examine the case where the spectral support of the signal is fixed and we wish to design a sampling and reconstruction scheme that will allow recovery at exactly the minimum average rate. We derive a condition in terms of the band-edge positions to allow exact reconstruction at the minimum average rate; this condition makes clear that a far wider class than was previously known can be sampled at the minimum rate. Further, the complexity of the reconstruction system can in some cases be much lower than is achievable by previous methods. We show that the minimum rate may be approached for any multiband signal, and sometimes with far lower complexity than previously thought. We examine the tradeoff between the oversampling factor and complexity in the design of sampling and reconstruction systems. A preliminary version of this work was presented in [27].

II. Periodic Nonuniform Sampling

The problem that we wish to solve is to find some strategy that will allow us to sample real-valued multiband signals $s(t)$ at the minimum rate. By multiband we mean that the set of frequencies $\mathcal{P}$ over which the power spectral density $S(f)$ is nonzero is a finite union of arbitrary nonoverlapping open intervals

$$\mathcal{P} = \bigcup_{i=0}^{L-1} \{(a_i, b_i) \cup (-b_i, -a_i)\}$$

where the a_i and b_i are any real numbers. For convenience, we will define $C_{\mathcal{P}}(f)$ as the *characteristic function* of the set $\mathcal{P}$

$$C_{\mathcal{P}}(f) \triangleq \begin{cases} 1, & S(f) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Fig. 1(c) shows an example of the characteristic function for a simple multiband signal. We also assume that $\mathcal{P}$ has some maximum frequency, $f_{\max} = b_{L-1}$. Following the notation of [28] we will refer to the set of all signals which have energy only at frequencies in $\mathcal{P}$ by $\mathcal{B}(\mathcal{P})$ and the effective bandwidth of this set as

$$B_{\text{eff}} = \mathcal{M}(\mathcal{P}) = 2 \cdot \sum_{i=0}^{L-1} (b_i - a_i)$$

where $\mathcal{M}(\cdot)$ denotes the Lebesgue measure of a set.

We know that reconstruction from uniform sampling at the rate $2f_{\max}$ or higher is always possible, but this entails waste of bandwidth since $2f_{\max} > B_{\text{eff}}$. The effective Bandwidth B_{eff} is generally regarded as the minimum possible (or Nyquist–Landau) rate [28], although special cases of reconstruction schemes below this rate have been reported [29]. Between these two extremes there may exist some rate at which uniform sampling becomes possible. An approach to finding out if this is the case is developed in [30], but the algorithm to find the lowest possible uniform rate is complicated, there is no guarantee that a rate lower than $2f_{\max}$ will be found, and only in very special cases is an average rate B_{eff} achievable. Of course it would be possible to explicitly bandpass filter for each of the bands, modulate to baseband, and uniformly sample at a rate appropriate for that band. Reconstruction would then involve L lowpass filtering and demodulation operations, and we would have to deal with L

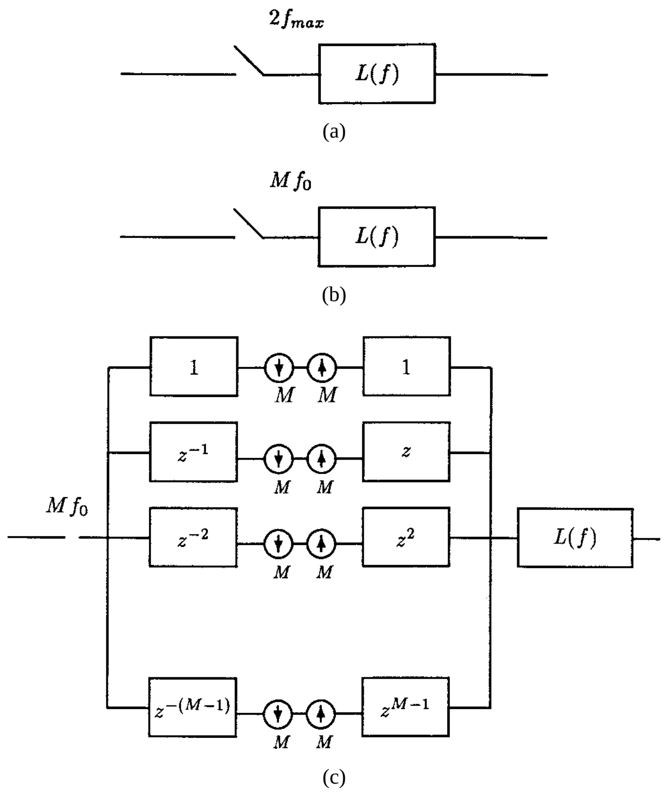

Fig. 2. Equivalent sampling and reconstruction structures for signals that have no energy at frequencies higher than $f_{\max}$, $L(f)$ is an ideal lowpass filter with cutoff $f_{\max}$. (a) Uniform sampling at the Nyquist rate. (b) Oversampling at a rate $Mf_0 > 2f_{\max}$. (c) Splitting the rate Mf_0 train into M trains at rate f_0.

sample trains at different rates. The complexity of such an approach clearly becomes forbidding as the number of bands L increases. Instead, we take the approach of reconstructing from periodic nonuniform sampling.

If we choose some integer M and frequency f_0 then sampling $s(t) \in \mathcal{B}(\mathcal{P})$ at Mf_0 is sufficient provided $Mf_0 \geq 2f_{\max}$. Thus the sampling alternatives shown in Fig. 2(a) and (b) are equivalent for all signals in $\mathcal{B}(\mathcal{P})$. Denote by $x(n)$ the discrete sequence which equals the rate-Mf_0 sampled signal at the sample points

$$x(n) = s(t)|_{t=n/(Mf_0)}. \tag{1}$$

We can obviously split a sample train at rate Mf_0 into M sample trains at rate f_0. Thus if we use the z-transform of the discrete-time signal we can write

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \sum_{i=0}^{M-1} \sum_{n=-\infty}^{\infty} x(nM-i)z^{-(nM-i)} \\ &= \sum_{i=0}^{M-1} z^i X_i(z^M). \end{aligned} \tag{2}$$

Clearly, (2) represents a decomposition into M sample trains at rate f_0. The M lower rate trains $X_i(z)$ are calculated as

$$X_i(z^M) = \sum_{n=-\infty}^{\infty} x(nM-i)z^{-nM}, \qquad i \in \{0, 1, \cdots, M-1\}.$$

We will refer to $X_i(z^M)$ as the ith M-phase component of $X(z)$. This involves delaying the signal by i samples, and then passing it through the combination of an M-fold downsampler followed by an M-fold upsampler. A downsampler is an operator that retains only every Mth sample from a sequence, an upsampler inserts $M-1$ zero samples between adjacent samples. The effect of the combination of a downsampler followed by an upsampler is that every Mth sample is retained, while all others are set to zero. Up and downsamplers are commonly used in the multirate and filter bank literatures, but the reader should not require any familiarity with this work for the rest of this paper. The essential point, the action of a downsampler followed by an upsampler, is illustrated in Fig. 3. There we see that the decomposition in (2) is carried out by the structure shown in Fig. 2(c), where the signal is first sampled at rate Mf_0, M trains are delayed by differing delays $k \in \{0, 1, \cdots, M-1\}$ and passed through the combination of a downsampler and upsampler. When added together the various components in Fig. 3 reproduce the original sampled signal. Thus the three sampling structures shown in Fig. 2 are equivalent for all signals in $\mathcal{B}(\mathcal{P})$.

Suppose now it were possible to reconstruct $x(n)$ exactly using only N of the M components in (2) (i.e., reconstruct $X(z)$ from $X_i(z)$ where $i \in \mathcal{A}$ and $\mathcal{A}$ is a set that contains only N of the indices $\{0, 1, \cdots, M-1\}$). Since we can in turn reconstruct $s(t)$ from $x(n)$ this would mean that we could recover $s(t)$ from an average of Nf_0 samples per unit time. If in addition we could have

$$Nf_0 = B_{\text{eff}} \tag{3}$$

then the average number of samples required to reconstruct the analog signal $s(t)$ would be the same as the effective bandwidth.

III. Reconstruction of a Discrete-Time Signal from N Samples out of M

In the last section we gave a brief decription of our attack on the problem. We now demonstrate under what conditions a sampled signal may be reconstructed from N samples out of M. This material was presented in an earlier form by Foster and Herley [31] and is a development of an idea in [32].

Modifications of the structure in Fig. 2(c) will be key to our derivation. Consider two branches i and k of Fig. 2(c); we have redrawn them in Fig. 4(a), with $X(z)$ as input. Here we illustrate the effect of the combination of downsampler and upsampler. Observe that the outputs of the two channels are $z^i X_i(z^M)$ and $z^k X_k(z^M)$, which can be interpreted as two of the terms in the sum (2). Now consider the modification shown in Fig. 4(b), where we have changed the "filter" on the left-hand side of branch i from

$$z^{-i} \quad \text{to} \quad z^{-i} - z^{-k}\phi(z^M)$$

and the "filter" on the right-hand side of branch k from

$$z^k \quad \text{to} \quad z^k + z^i\phi(z^M).$$

We first point out that the factor $\phi(z^M)$ can be placed on either side of the downsampler/upsampler combination.

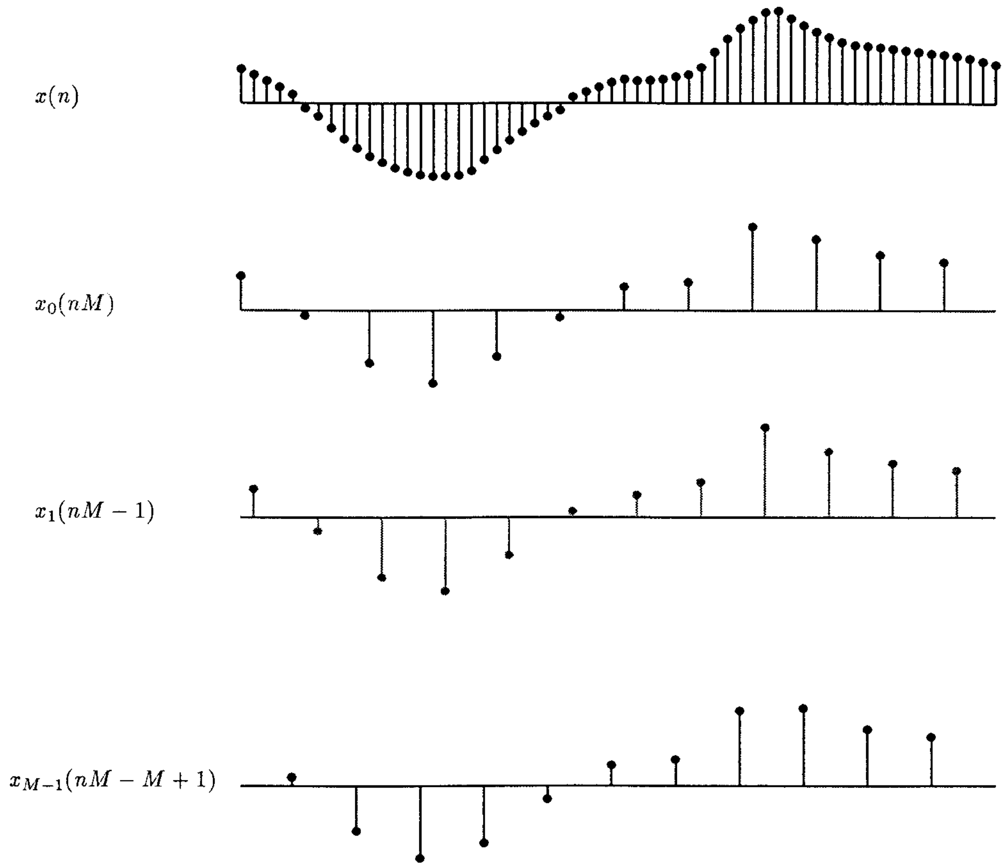

Fig. 3. Breaking a train at rate Mf_0 into M trains at rate f_0. The top line represents the original signal sampled at Mf_0. Below it we show the components $X_i(z^M)$ in the sum (2) for the case of $M = 5$. The ith train can be found by delaying the original by i samples, using the combination of an $M = 5$ downsampler followed by an upsampler, and advancing the signal by i samples. Such a decomposition is carried out in Fig. 2(c).

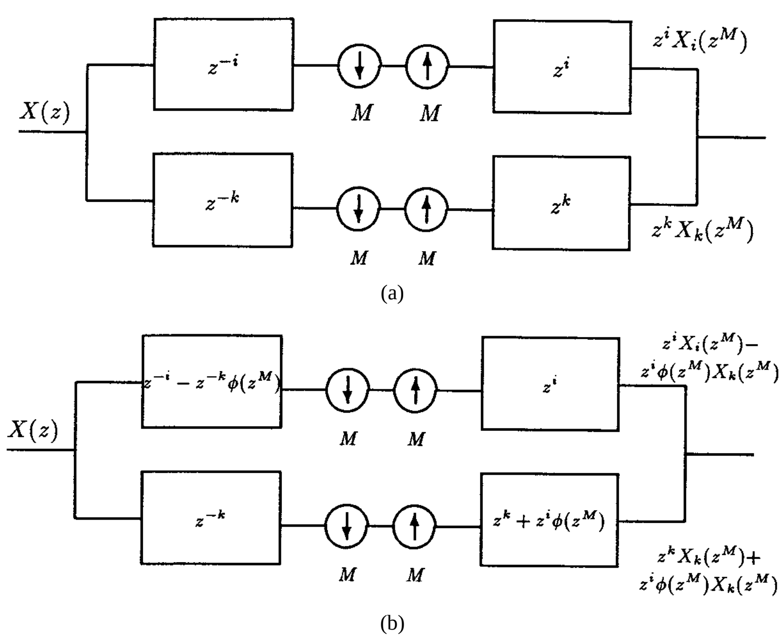

Fig. 4. Modification of two channels of the structure shown in Fig. 2(c). Observe that after the modification the summed output is unchanged. (a) Two channels of the structure. (b) Modification by adding to the right and subtracting from the left so that the overall output is unchanged.

To see that this is so, recall that the downsampler/upsampler combination retains only every Mth sample of a signal, so that if

$$X(z) = \sum_{i=0}^{M-1} z^i X_i(z^M)$$

is input to the combination, then $X_0(z^M)$ is the output. Placing a filter $\phi(z^M)$ before the downsampler/upsampler combination causes

$$X(z) \cdot \phi(z^M) = \sum_{i=0}^{M-1} z^i X_i(z^M) \cdot \phi(z^M)$$

to be input to the combination and $X_0(z^M) \cdot \phi(z^M)$ to be output. Since the same result, $X_0(z^M) \cdot \phi(z^M)$, is obtained by filtering $X_0(z^M)$ by $\phi(z^M)$, we can move this filter to either side of the downsampler/upsampler combination.

Returning to the output of Fig. 4(b), we observe that the summed output is

$$\begin{aligned} z^i[X_i(z^M) - \phi(z^M)X_k(z^M)] + z^k X_k(z^M) \\ + z^i \phi(z^M) X_k(z^M) \\ = z^i X_i(z^M) + z^k X_k(z^M). \end{aligned}$$

Thus the outputs of Fig. 4(a) and (b) are the same, *independently of the choice of* $\phi(z)$.

Repeated application of this procedure allows us to change the form of the filters used in Fig. 2(c). We already denoted by $\mathcal{A}$ the sized-N subset of the indices $\{0, 1, 2, \cdots, M-1\}$ that we are going to use for reconstruction. For each branch $i \notin \mathcal{A}$ change the filter on the left-hand side from

$$z^{-i} \quad \text{to} \quad z^{-i} - \sum_{k \in \mathcal{A}} a_{ik}(z^M) z^{-k}.$$

Thus we have *subtracted* N terms from that filter, where N is the number of indices in the set $\mathcal{A}$. We can compensate for the term $a_{ik}(z^M)z^{-k}$ by *adding* $a_{ik}(z^M)z^i$ to the filter on the right-hand side of the kth channel. Thus we change the kth filter on the right

$$z^k \quad \text{to} \quad z^k + \sum_{i \notin \mathcal{A}} a_{ik}(z^M) z^i, k \in \mathcal{A}.$$

Thus in subtracting N terms from one channel $i \notin \mathcal{A}$ on the left-hand side we compensate by adding one term each to N channels $k \in \mathcal{A}$ on the right-hand side. After we have carried out these changes for all of the channels $i \notin \mathcal{A}$ we end up with the following filters on the left:

$$H_i(z) = \begin{cases} z^{-i}, & i \in \mathcal{A} \\ z^{-i} - \sum_{k \in \mathcal{A}} a_{ik}(z^M) z^{-k}, & i \notin \mathcal{A} \end{cases} \tag{4}$$

and the compensating filters on the right.

$$G_i(z) = \begin{cases} z^i + \sum_{k \notin \mathcal{A}} a_{ki}(z^M) z^k, & i \in \mathcal{A} \\ z^i, & i \notin \mathcal{A}. \end{cases} \tag{5}$$

The important point is that this choice of filters, (4) and (5), produces an output that is equivalent to that of Fig. 2(c), irrespective of the choices of the $a_{mn}(z)$ and of the set $\mathcal{A}$.

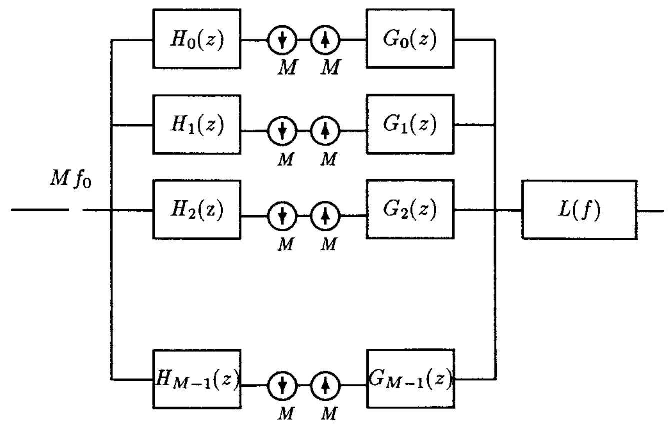

Fig. 5. Choice of filters as in (4) and (5) makes this structure equivalent to those shown in Fig. 2.

Thus the system in Fig. 5 with filters as in (4) and (5) is still equivalent to all of the systems in Fig. 2 for any signal in $\mathcal{B}(\mathcal{P})$. Readers familiar with multirate filter banks will recognize this structure as an M-channel perfect reconstruction filter bank, although the filters are very constrained. The technique used for modifying the filters was used in the context of filter banks in [33].

Observe from (4) that the analysis filters $H_i(z), i \in \mathcal{A}$ are simply delay elements, and thus the input to the ith synthesis filter $G_i(z), i \in \mathcal{A}$ is the ith component $X_i(z^M)$. The goal is to reconstruct $X(z)$ from these components alone. The key observation in [31] is that if the channels $i \in \mathcal{A}$ suffice for reconstruction

$$X(z) = \sum_{i \in \mathcal{A}} X_i(z^M) G_i(z) \tag{6}$$

then the other channels, $i \notin \mathcal{A}$ contribute nothing to the solution. For these channels to contribute nothing it is clearly sufficient that

$$H_i(e^{j\omega}) X(e^{j\omega}) = 0, \qquad i \notin \mathcal{A}, \qquad \forall \omega. \tag{7}$$

This means that $H_i(e^{j\omega}) = 0$ when $X(e^{j\omega}) \neq 0$. This leads directly to [31], [27] the following theorem.

Theorem 3.1: If no more than N elements in the set

$$R(\omega) = \{X(e^{j(\omega + 2\pi p/M)}): p = 0, 1, \cdots, M-1\}$$

are nonzero for any $\omega \in [0, 2\pi/M)$, then $X(e^{j\omega})$ can be exactly reconstructed from some set $\mathcal{A}$ of N of its M-phase components

$$X(z) = \sum_{i \in \mathcal{A}} X_i(z^M) G_i(z) \tag{8}$$

where the $G_i(z)$ are determined by (5) and the $a_{ik}(z)$ are fixed by the requirement (7).

Proof: We need to solve (7) for $i \notin \mathcal{A}$ which implies

$$e^{-j\omega i} = \sum_{k \in \mathcal{A}} a_{ik}(e^{j\omega M}) e^{-j\omega k}, \qquad \forall \omega \text{ s.t. } X(e^{j\omega}) \neq 0. \tag{9}$$

Finding the a_{ik} such that this holds solves (7) and hence (8) is true. But the $a_{ik}(e^{j\omega M})$ are $2\pi/M$ periodic and cannot be specified independently at all frequencies.

Thus for a particular $\omega \in [0, 2\pi/M)$ and

$$\omega_p = \omega + \frac{2\pi p}{M}, \qquad p = 0, 1, \cdots M-1$$

(9) becomes

$$e^{-j\omega_p i} = \sum_{k \in \mathcal{A}} a_{ik}(e^{j\omega M})e^{-j\omega_p k}, \qquad p = 0, 1, 2 \cdots, M-1.$$

Since $R(\omega)$ contains no more than N nonzero elements we need to satisfy no more than N linear equations corresponding to the frequencies where $X(e^{j\omega_p}) \neq 0$. We wish to solve for the unknowns $a_{ik}(e^{j\omega M})$ at each frequency $\omega \in [0, 2\pi/M)$. Thus for each $i \notin \mathcal{A}$ we must solve the N equations

$$e^{-j\omega_p i} = \sum_{k \in \mathcal{A}} a_{ik}(e^{j\omega M})e^{-j\omega_p k}, \qquad p \in \mathcal{C}$$

where $\mathcal{C}$ is the subset of the N indices from the set $\{0, 1, \cdots, M-1\}$ for which $X(e^{j\omega_p}) \neq 0$. In other words, $p \in \mathcal{C} \Rightarrow X(e^{j\omega_p}) \neq 0$. We have freedom in choosing $\mathcal{A}$; for example, if we choose $\mathcal{A} = \{0, 1, 2, \cdots, N-1\}$ then this is a Vandermonde system which is always invertible. □

Note that the choice $\mathcal{A} = \{0, 1, 2, \cdots, N-1\}$ guarantees that the system of equations can always be solved, any other choice of $\mathcal{A}$ that guarantees invertibility is also acceptable. Also, note that while it is not central to our concerns, a simple linear independence argument gives that (7) is necessary as well as sufficient. Theorem 3.1 says that when $x(n)$ is subsampled by M, to reconstruct from N components, there should be no more than $N-1$ overlaps in the spectrum of the subsampled signal. This is equivalent a discrete-time version of the result by Kahn and Liu [12]. One of the strengths of the result lies in the characterization of the reconstruction. For example, given a sampling strategy (i.e., the set $\mathcal{A}$), it is difficult to characterize the space of signals that can be reconstructed. Observe that in Fig. 5 the lowpass filter $L(f)$ acts as an inverse for the MF_0 sampler (at least for all signals bandlimited to $2f_{\max}$), and the system of filters $H_i(z)$ comprise the inverse of the system of filters $G_i(z)$ (since the back-to-back system is an identity). This back-to-back system consists of M channels; denote by $\boldsymbol{R}$ the channels $i \in \mathcal{A}$ and by $\boldsymbol{S}$ the channels $i \notin \mathcal{A}$ so that $\boldsymbol{R} + \boldsymbol{S} = \boldsymbol{I}$. Of course we are not interested in reconstructing the signals bandlimited to $2f_{\max}$, but the smaller set $\mathcal{B}(\mathcal{P})$, and we will not use the whole back-to-back system $\boldsymbol{R} + \boldsymbol{S}$ but only the system $\boldsymbol{R}$. The actual sampling and reconstruction system used will then consist of the rate Mf_0 sampler, the N-channel system $\boldsymbol{R}$, and the lowpass filter $L(f)$. We found it difficult to characterize the span of the reconstruction system, i.e., the rangespace of $\boldsymbol{R}$. Instead, we characterized the nullspace of its complement $\boldsymbol{S}$ using (7), which turned out to be a far simpler task. While the channels $i \notin \mathcal{A}$ play no role in our sampling reconstruction system, characterizing the nullspace of that subset of the channels was the key ingredient in characterizing the range of the reconstruction system. The set of signals that can be reconstructed using (6) is the same as the set that satisfies (7).

Note: we need not restrict ourselves to linear filters. The modifications made to Fig. 2(c) hinged on the fact that a filter that affects only every Mth sample could be moved to either side of the downsampler/upsampler combination. This is true even if the filters action on the samples is nonlinear. For example, suppose that the operator $\psi_M[x(n)]$ acts on every Mth sample of $x(n)$ in a nonlinear fashion, but leaves all other samples alone. Placing such an operator at the input to a downsampler/upsampler combination will give as input

$$\psi_M\left[\sum_{i=0}^{M-1} x_i(nM - i)\right]$$

and as output $\psi_M[x_0(nM)]$. Placing such an operator at the output of the downsampler/upsampler combination with $x(n)$ as input will produce $\psi_M[x_0(nM)]$ as output. Just as in the linear case it can be seen that an operator that acts on only every Mth sample can be moved to either side of the downsampler/upsampler combination. Just as in the linear case, nonlinear operators $a_{mn}[\cdot]$ can be constructed that yield identity systems much as ws done in (4) and (5). Design of nonlinear filter banks has been examined by de Queiroz, Florencio, and Schafer in [34], although not from a sampling point of view.

IV. Filter Design Using POCS

Observe that the filters $H_i(z), i \notin \mathcal{A}$ satisfy a spectral constraint, given by (7), and also a structural constraint, given by (4). We point out that the set of filters that satisfies each of these constraints is a subspace of the set of all filters. To find a filter that satisfies both, i.e., a point of the intersection, we can use the algorithm of Projection On Convex Sets (POCS) [35], provided we can find a way of carrying out the orthogonal projection onto each of these spaces. The algorithm is guaranteed to converge provided that the intersection is not empty, which in turn requires that we satisfy the conditions of Theorem 3.1. We can find the orthogonal projection onto the space of filters that satisfy (7), by zeroing out the frequencies of $H_i(e^{j\omega})$ where $X(e^{j\omega}) \neq 0$ (ideal multiband filtering). We can carry out the orthogonal projection onto the space of filters that satisfy (4) by zeroing out the $k \notin \mathcal{A}$ M-phase components of the filter (i.e., imposing the form implied by (4)), and setting the ith component to $\delta(n)$, the discrete delta. Thus POCS provides a method of satisfying both constraints and designing the filters $H_i(z), i \notin \mathcal{A}$. The procedure is (for each $i \notin \mathcal{A}$) as follows.

1) Set $H_i(e^{j\omega}) = 0 \; \forall \; \omega$ where $X(e^{j\omega}) \neq 0$.
2) Set $h_i(k + nM) = 0$, $k \notin \mathcal{A}, \forall n, h_i(i + nM) = \delta_n$.
3) GOTO 1) unless convergence.

We can start the algorithm with any guess for $H_i(z)$. Once these $H_i(z)$ have been designed, as before, we substitute the values of the $a_{mn}(z)$ into (5) to get the filters $G_i(z), i \in \mathcal{A}$. Of course, just as in the case of bandlimited reconstruction, the filters are ideal and nonrealizable. In particular, no realizable filter can ever satisfy condition 1). In practice, of course, just as in the bandlimited reconstruction case, we will have to approximate the ideal response with a finite impulse response (FIR) or realizable infinite impulse response (IIR) filter. Empirically we observe that when we constrain the filter to have a finite impulse response the POCS design approach converges very well in most cases, even with a random initial guess for $H_i(e^{j\omega})$. In this case, POCS will converge to the filter that satisfies condition 2), and most closely approximates condition 1). It should be noted that there are no simple bounds on the rate of convergence to be expected; we have observed that cases where the sampling structure is bunched tend to converge more slowly than those which have well-spaced samples. If the reconstruction is very ill-conditioned (e.g., when the M-phase components are bunched together) it is important to choose a good initial guess, since the convergence in this case can be slow and the computational noise incurred at each iteration can have a cumulative detrimental affect on the quality of the approximation to condition 1). A good initial guess can be found by solving (9) for $a_{ik}(e^{j\omega M})$ at some set of frequencies, and using these to assist our first guess. This condition can also be included in the POCS iteration to speed convergence.

The computational effort required to design the filters is small, since (using either the approach here or alternative approaches such as in [22]) the filters are designed once. As we pointed out, exactly satisfying the constraints is not possible with realizable filters; the reader should not expect that good approximation of the constraints will be achieved with low-order filters.

V. Minimum Rate Sampling of Multiband Signals

We have shown how the N-channel reconstruction system is characterized, and have demonstrated a simple procedure for its design. We are now ready to use it to tackle the problem of minimum rate sampling of multiband signals. Recall that $x(n)$ was derived from the rate-Mf_0 sampled signal, and that $C_{\mathcal{P}}(f)$ is the characteristic function of the of set $\mathcal{P}$ over which the spectrum is nonzero; we denote its inverse Fourier transform as $c_{\mathcal{P}}(t)$. In addition, we will use the rate f_0 sampled version of $c_{\mathcal{P}}(t)$. This is related to the spectrum of the continuous time signal by

$$c_{\mathcal{P}}(n/f_0) = \int_{-\infty}^{\infty} C_{\mathcal{P}}(f) e^{j2\pi n f/f_0}\, df.$$

The requirement of Theorem 3.1, that no more than N elements of the set $R(\omega)$ be nonzero at any frequency in $[0, 2\pi/M)$ is equivalent to requiring that when $c_{\mathcal{P}}(t)$ is sampled at rate f_0 no frequency of the spectrum of the subsampled signal have nonzero contribution from more than N alias copies

$$\max_{f\in[-f_0/2, f_0/2)} C_{\mathcal{P}}(f/f_0) \le N. \tag{10}$$

When this condition is satisfied we can sample $s(t)$ at an average rate Nf_0. The effective bandwidth is, of course,

$$B_{\text{eff}} = \int_{-f_{\max}}^{f_{\max}} C_{\mathcal{P}}(f)\, df = \int_{-f_0/2}^{f_0/2} C_{\mathcal{P}}(f/f_0)\, df. \tag{11}$$

Subject to (10), clearly we can satisfy (3) if and only if

$$C_{\mathcal{P}}(f/f_0) = N, \qquad \forall f. \tag{12}$$

This requires that exactly N elements of the set $R(\omega)$ are nonzero at every frequency in $[0, 2\pi/M)$. At no frequency are more than N elements nonzero, and at no frequency are fewer elements nonzero. This is the natural generalization of the uniform sampling case, where to have minimum rate sampling without aliasing, at no frequency can we have more than one aliased copy of the soectrum, and at no frequency can we have less than one.

We have seen that when $Mf_0 \ge 2f_{\max}$ we can recover $s(t)$ from the sequence $x(n)$. Further, when (12) is satisfied we can recover $x(n)$ from only N of its M-phase components. Since (3) holds we have minimum rate sampling. If we can find an f_0 that satisfies (12), we can then choose any M that satisifes $Mf_0 \ge 2f_{\max}$. It is not yet obvious whether finding such an f_0 is possible in general. The following theorem helps.

Theorem 5.1: To satisfy

$$C_{\mathcal{P}}(f/f_0) = N, \qquad \forall f \tag{13}$$

it is necessary and sufficient that

$$\sum_{i=0}^{L-1} [e(g+b_i) - e(g+a_i) - e(g-b_i) + e(g-a_i)] = 0, \quad \forall g \tag{14}$$

where

$$e(f) \triangleq \sum_{n\neq 0} \frac{(-1)^{n+1} f_0}{2\pi j n}\, e^{j2\pi f n/f_0}. \tag{15}$$

Proof: Observe that in the time domain (12) gives

$$c_{\mathcal{P}}(n/f_0) = 0, \qquad \forall n \neq 0. \tag{16}$$

Since a function that is zero everywhere has all-zero Fourier coefficients, (16) is equivalent to

$$\sum_{n\neq 0} (-1)^{n+1} c_{\mathcal{P}}(n/f_0)\, e^{j2\pi n g/f_0} = 0, \qquad \forall g. \tag{17}$$

Evaluating the integral for $c_{\mathcal{P}}(n/f_0)$

$$\begin{aligned} c_{\mathcal{P}}(n/f_0) &= \int_{-\infty}^{\infty} C_{\mathcal{P}}(f) e^{j2\pi f n/f_0}\, df \\ &= \sum_{i=0}^{L-1} \left[\int_{a_i}^{b_i} e^{j2\pi f n/f_0}\, df + \int_{-b_i}^{-a_i} e^{j2\pi f n/f_0}\, df \right] \\ &= \sum_{i=0}^{L-1} \frac{f_0}{j2\pi n} \Big[e^{(j2\pi n b_i/f_0)} - e^{(j2\pi n a_i/f_0)} \\ &\qquad + e^{(-j2\pi n a_i/f_0)} - e^{(-j2\pi n b_i/f_0)} \Big]. \end{aligned}$$

Substituting into (17) we get (14). □

The reason that this theorem is helpful is that $e(f)$ is the Fourier series expansion of the periodic piecewise-linear function shown in Fig. 6. Observe from the figure, or by substituting into (15), that for any α and β whose difference is an integer times f_0, we have $e(\alpha) = e(\beta)$. This allows us to see more clearly how solutions to (14) may be found. For example, suppose that we had $b_i - a_i = k_i f_0$ for some integer $k_i, \forall i$. This would give $e(g+b_i) = e(g+a_i), \forall g$, and then (17), and hence (14), would be satisfied. Thus if we could find

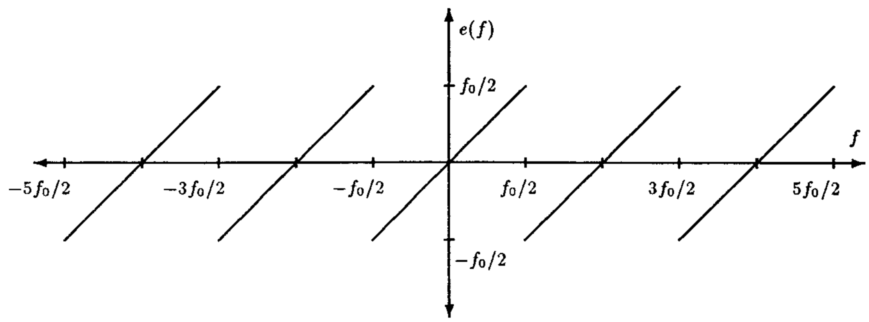

Fig. 6. Periodic piecewise-linear function $e(f)$.

f_0 such that each pair $b_i - a_i$ had f_0 as an integer factor we would have a solution. This is sufficient, but not necessary. In fact, we could pair b_i with a_j or $-b_j$ for any j, and still satisfy (14) provided that each pair has f_0 as an integer factor.

Compare with the description in [36] of the signals known to be reconstructable from samples at an average of the Nyquist–Landau rate: "This means that the bands of $\mathcal{P}$ have lengths that are all integer multiples of some basic length, and the same is true of the gaps between them." This is the class of multiband signals considered in [36]–[39]. In the notation that we have been using it is equivalent to requiring $b_i - a_i = k_i f_0$ for $i = 0, 1, \cdots, L-1$ and and $a_{i-1} - b_i = l_i f_0$ for $i = 0, 1, \cdots, L-2$, and $2a_0 = l_0 f_0$ for some integers k_i and l_i. We have just seen that something far less restrictive is in fact required. To reconstruct at the Nyquist–Landau rate requires that (14) hold. This does not imply that the bands, and the gaps between them, be integer multiples of some f_0, but merely that in some pairing each pair have such a factor.

Exactly satisfying this condition may be difficult in practice, since the a_i and b_i are real numbers, and pairing them such that any other real number is precisely an integer factor of all pairs is not possible in general. However, by examining a slightly larger set $\mathcal{Q} \supset \mathcal{P}$ we can find a solution easily. Thus we manipulate the positions of the bands slightly: given signals in $\mathcal{B}(\mathcal{P})$ we find a strategy to sample at minimum rate signals in the larger set $\mathcal{B}(\mathcal{Q})$. The oversampling implied can be made arbitrarily small, a conclusion that had already been reached in [25], [22], and [40].

Theorem 5.2: For signals bandlimited to $\mathcal{P}$, where $\mathcal{P}$ is a finite union of open intervals, and given any $\epsilon > 0$ there exists a set $\mathcal{Q} \supset \mathcal{P}$, with

$$\int_{-\infty}^{\infty} |C_{\mathcal{Q}}(f) - C_{\mathcal{P}}(f)|\, df < \epsilon \tag{18}$$

such that signals bandlimited to $\mathcal{Q}$ can be recovered from periodic nonuniform samples at the minimum rate.

Proof: Begin by pairing the band edges in some way; for example, we can pair each b_i with a_i. For any f_0, and each pair we can find $b_i' \geq b_i$ and $a_i' \leq a_i$ such that

$$b_i' - a_i' = k_i \cdot f_0 \geq b_i - a_i.$$

The worst case bound on the excess bandwidth introduced for each band is f_0 since

$$b_i' - a_i' < b_i - a_i + f_0.$$

Define the set

$$\mathcal{Q} \triangleq \bigcup_i \{(a_i', b_i') \cup (-b_i', -a_i')\}.$$

Observe that $e(b_i') = e(a_i') \forall i$, and hence (12) is satisfied for $\mathcal{Q}$. Thus signals bandlimited to the set $\mathcal{Q}$ can be sampled at the minimum rate.

Clearly, we have

$$\int_{-\infty}^{\infty} |C_{\mathcal{Q}}(f) - C_{\mathcal{P}}(f)|\, df = \left| 2 \cdot \sum_{i=0}^{L-1} b_i' - a_i' - (b_i - a_i) \right| \leq 2L \cdot f_0.$$

Hence if we choose $f_0 < \epsilon/(2L)$ (18) holds. □

Since the effective sampling rate is $\sum_{i=0}^{L-1} k_i \cdot f_0$, the sampling efficiency will be

$$1 \geq \frac{B_{\text{eff}}}{N f_0} = \frac{2 \cdot \sum_{i=0}^{L-1} (b_i - a_i)}{2 \cdot \sum_{i=0}^{L-1} (b_i' - a_i')} \geq \frac{\sum_{i=0}^{L-1} (b_i - a_i)}{\sum_{i=0}^{L-1} (b_i - a_i) + \epsilon}$$

which can be made arbitrarily close to one.

Notes:

1) The choice $f_0 < \epsilon/(2L)$ was taken to simplify the demonstration of the result. In practice, this might be an undesirable solution, since it will require M to be large. It is possible to use the freedom in pairing the a_i and b_i to choose a more reasonable solution. We explore this in the next section.
2) To recover $s(t)$ then we find a set $\mathcal{Q}$, N, M, and f_0 for the given ϵ. We reconstruct $x(n)$ from N of its M-phase components, and recover $s(t)$ from $x(n)$.
3) The literature on multiband sampling has typically fallen into two types: the work where a particular structure in the relations between the size of the bands and the gaps was assumed [36], [38], [39], and those where a slicing approach was taken to asymptotically approach the minimum rate [25], [22], [40]. For the first of these, we have shown in Theorem 5.1 a necessary and sufficient condition on the band edges to allow minimum rate reconstruction, and this condition is far more general than the previous literature had assumed. For the second, we will demonstrate in Section VI,

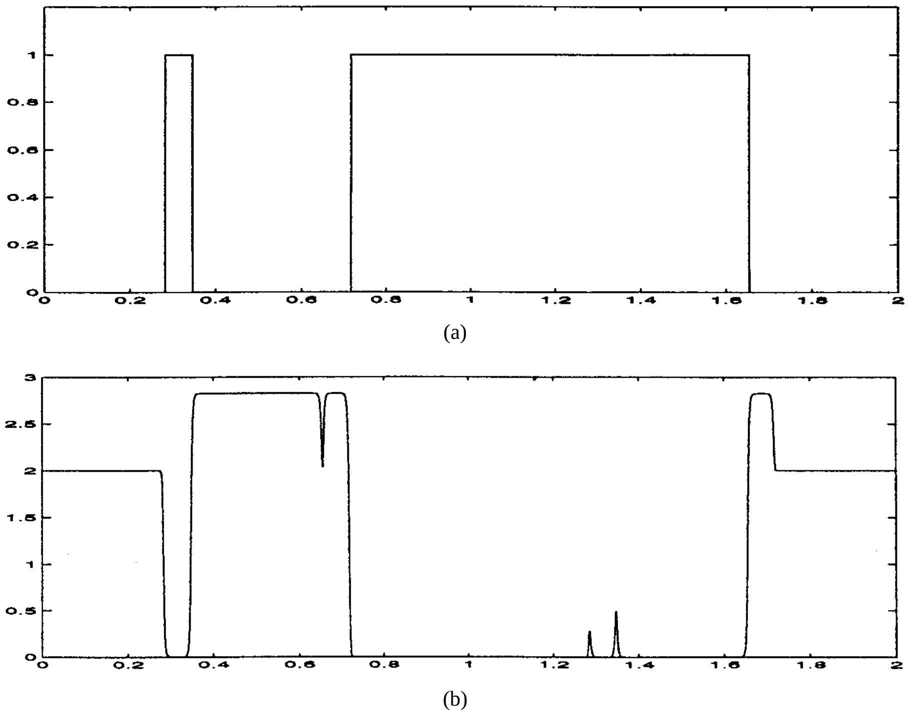

Fig. 7. (a) Characteristic function of the signal, the signal has energy where $C(f)$ is one and has no energy elsewhere. (b) Magnitude response of the filter $H_3(z)$ for the nulling system. Note that this filter has response close to zero at frequencies where the signal is nonzero, thus approximately satisfying (7).

that far less complex reconstruction systems can often be designed even in the case where we approach the minimum rate asymptotically.

VI. Examples and Applications

We have seen that reconstruction of multiband signals with arbitrary spectral support is possible at an average rate arbitrarily close to the Nyquist–Landau rate. Similar results have previously been reported in [25], [22], and [40]. We wish to make clear the distinction between our approach and earlier works. In the proof of Theorem 5.2 we merely paired the band edges in the most obvious fashion (each a_i paired with the corresponding b_i), and tried to force $b_i - a_i \approx k_i \cdot f_0$ for some integer k_i. This solution actually resembles the approach of [25] for multidimensional signals, and [22], [40] for multiband signals, in that we slice the spectrum into narrow bands and derive a sampling based on the number of bands which contain signal and which do not. As we decrease the size of the slices, the spectral efficiency improves, but the complexity of the sampling and reconstruction system becomes greater. By complexity we mean the number of filters that will be involved in the reconstruction system (i.e., N in the notation we have been using). We now demonstrate that this approach is far from optimal, and that a careful examination of the condition (14) shows that our solutions are less constrained than those found by slicing the spectrum, and that the additional freedon can be exploited to find less complex reconstruction systems in certain cases.

Consider a simple two-band signal, i.e., $L = 2$, with $a_0 = \sqrt{2}/5$, $b_0 = \sqrt{3}/5$, $a_1 = 1 - \sqrt{2}/5$, and $b_1 = 2 - \sqrt{3}/5$. The effective bandwidth is 2 Hz, and the characteristic function (on the positive frequency axis only) is shown in Fig. 7(a). Observe that the minimum uniform rate to allow reconstruction would be $2 \cdot (2 - \sqrt{3}/5)$, giving an efficiency of $1/(2 - \sqrt{3}/5) \approx 0.6047$.

Next try the approach of slicing the spectrum [25], [22], [40]. Clearly, the efficiency will be dominated by the smallest of the signal bands. Choosing f_0 small enough we can find a solution with $N = 32$ and $M = 63$. The efficiency $B_{\text{eff}}/N f_0 \approx 0.9846$ and can be improved only at the cost of increased complexity. For example, if we make f_0 smaller the efficiency improves but M and N become larger. Alternatively, we can exploit the fact that to satisfy (14) we can pair band edges in any fashion we wish. For example, pairing b_0 and $-b_1$ and a_0 and $-a_1$ we can find a solution with $f_0 = 1$ Hz. We indeed then have $b_0 + b_1 = 2f_0$ and $a_0 + a_1 = f_0$. Thus (14) is solved with $N = 2$ and $M = 4$. The minimum rate is actually achieved in this case, so the efficiency is one. This serves to illustrate the important distinction between our approach and those derived from the slicing of bands used in [25], [22], and [40]. It also shows that the minimum average rate is attainable for classes of multiband signals that do not adhere to the band structure studied in [36], [38], and [39]. This example shows that greater spectral efficiency, and much simpler sampling structures ($N/M = 2/4$ instead of $32/63$) are possible by carefully exploiting the properties of (14). Further, the filters are more easily designed in the case where N and M are small. Choosing $\mathcal{A} = \{0, 1\}$ we design the filters $H_3(z)$ and $H_4(z)$, using the procedure of Section IV, and hence we design the reconstruction filters $G_0(z)$ and $G_1(z)$. In Fig. 7(b) the magnitude response of the filter $H_3(z)$ is shown. As required by (7) the filter is almost zero everywhere when the signal is nonzero.

The question of how to pair the band edges in general to solve (14), and how to choose f_0, N, and M for best performance is an open one. Recall that to solve (14) we must pair each b_i with one of the a_j or one of the $-b_j$, and each pair must be an integer times f_0. Asymptotically, we can acheieve any desired efficiency by making f_0 small. In practice, however, we probably wish to minimize the spectral waste, but we might also want to constrain N and M to be moderate as well. The larger N and M the more complex the reconstruction and the more difficult the filters are to design in general. Examining all possible band pairings becomes prohibitive for a large number of bands. We have found the following scheme to work well, but make no claims of optimality.

- Pair a_i with $-a_{L-i-1}$ and b_i with $-b_{L-i-1}$ for $i = 0, 1, \cdots, L/2 - 1$.
- Let $d_i = a_i + a_{L-i-1}$ and $d_{L/2+i} = b_i + b_{L-i-1}$ $i = 0, 1, \cdots, L/2 - 1$.
- Order and reindex the d_i so that: $d_0 \le d_1 \le \cdots \le d_{L-1}$.
- Try $f_0 = d_{L-1}/p$ for $p = 1, 2, 3, \cdots$ until

$$\sum_i \lceil d_i/f_0 \rceil - d_i < \epsilon.$$

VII. Conclusion

We have examined in some detail the problem of periodic nonuniform sampling. The strength of our approach is that it allows very simple characterization of the range of the reconstruction system and design of the filters. We have given a characterization of the set of signals that can be reconstructed from particular sampling structures, and have shown that this is wider than had been considered in the existing literature. We have shown for multiband signals that minimum rate can be approached, or even acheived, with reconstruction systems which are far less complex than attainable by slicing.

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