Minimum Satisfying Assignments for SMT

Işıl Dillig, Tom Dillig
College of William & Mary

Ken McMillan
Microsoft Research

Alex Aiken
Stanford U.

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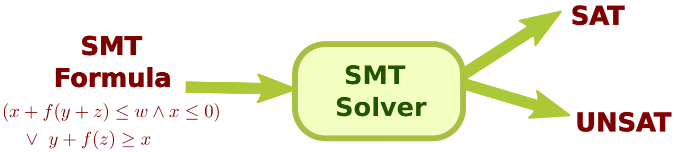

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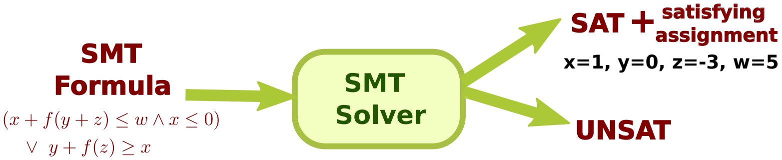

- An assignment σ for formula ϕ is a mapping from free variables of ϕ to values

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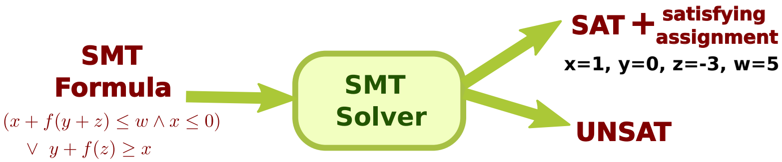

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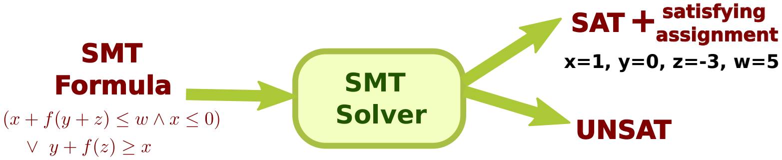

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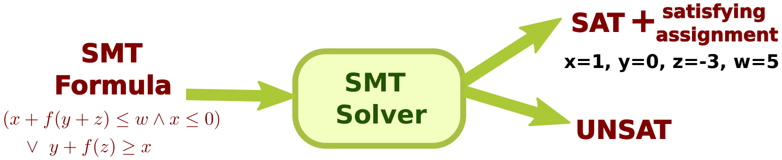

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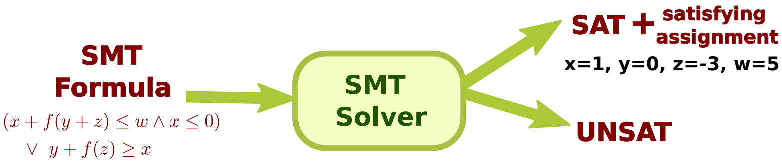

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- For formula $x < 0 \vee x + y \geq 0$, $x = -1$ is a partial satisfying assignment

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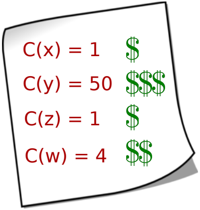

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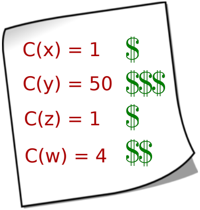

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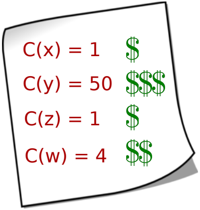

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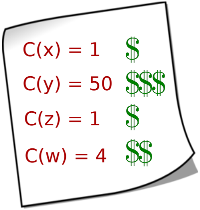
- If variables have equal cost, an MSA is partial sat assignment with fewest variables



Example and Applications

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MSAs have many applications in verification:

✓ Finding small counterexamples in BMC

✓ Classifying and diagnosing error reports

✓ Abductive inference

✓ Minimizing # of predicates in pred abstraction

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✓ **First algorithm for computing min sat assignments for SMT formulas**

✓ **Our algorithm applicable to any theory for which full first-order logic including quantifiers is decidable**

Maximum Universal Subset

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X **is an MUS of** ϕ $\Leftrightarrow$ **MSA is a sat assignment of** $\forall X.\phi$

Our approach first computes an MUS X and extracts an MSA from a sat assignment of $\forall X.\phi$

Algorithm to Compute MUS

- Recursive branch-and-bound style algorithm with input:

find_mus(ϕ, C, V, L) {

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- We do this by comparing cost of universal subsets with and without x

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Set $Y =$ find_mus($\forall x.\phi$, C, V' $L - C(x)$);

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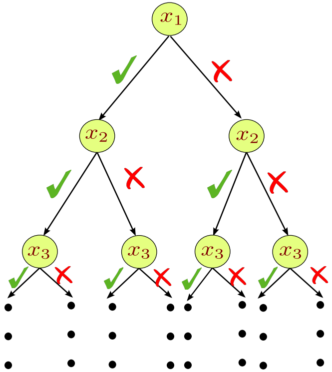

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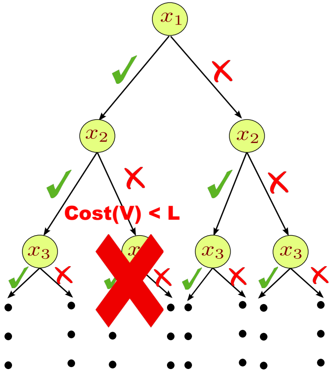

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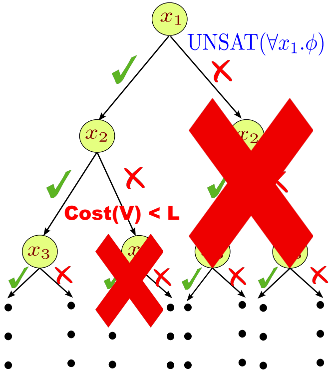

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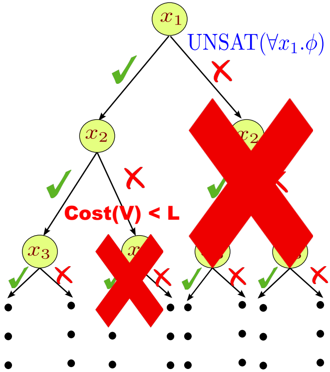

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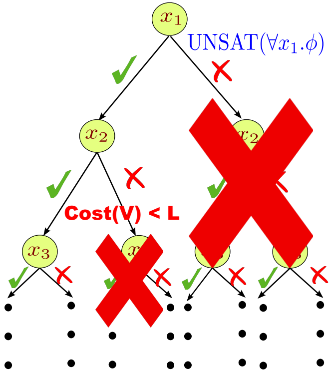
- To make algorithm practical, must consider more optimizations



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   - Turns out better to consider variables likely to be in MSA first

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How can we "quickly" find good enough approximate MSA?

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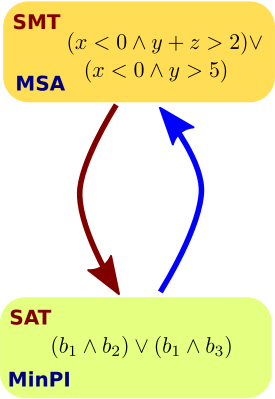

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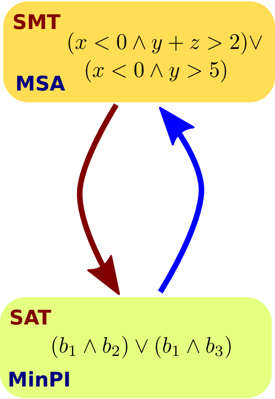

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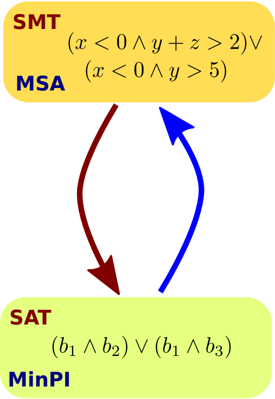

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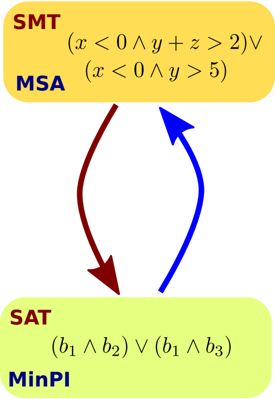

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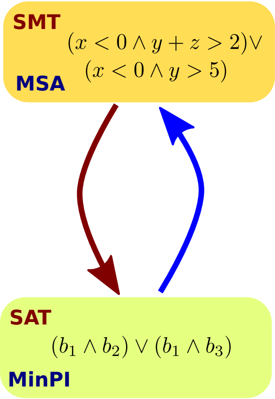

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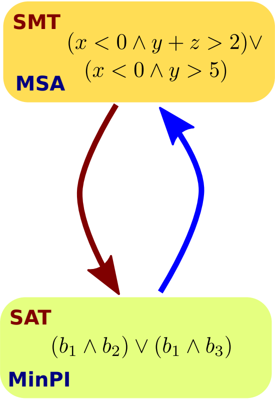

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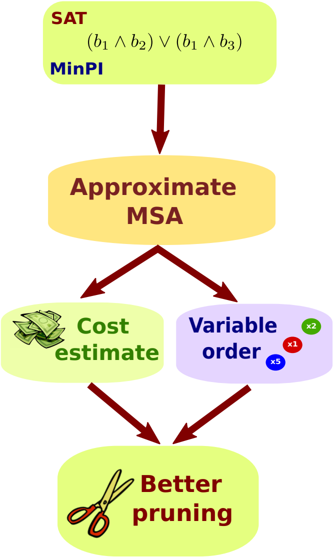

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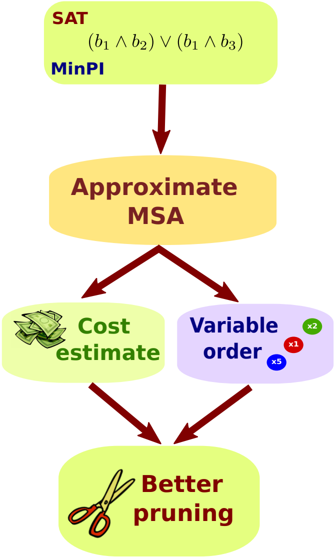

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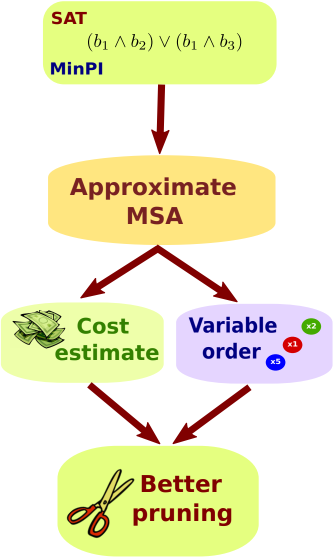

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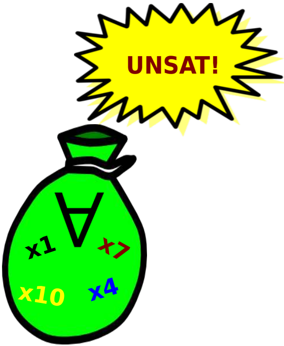
How can we "quickly" find implicates with small non-universal subsets?

Finding Non-Universal Subsets

- For complete theories, such as Presburger arithmetic, if $\neg\psi$ sat, then $\forall \mathrm{free}(\psi).\psi$ unsat



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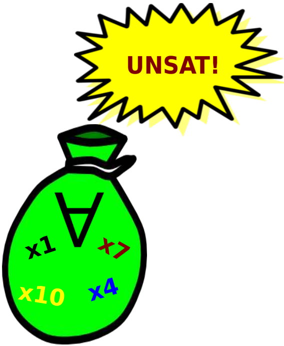
- For complete theories, such as Presburger arithmetic, if $\neg\psi$ sat, then $\forall \mathrm{free}(\psi).\psi$ unsat
- Thus, if ψ is an implicate of ϕ whose negation is sat, free(ψ) is a non-universal set



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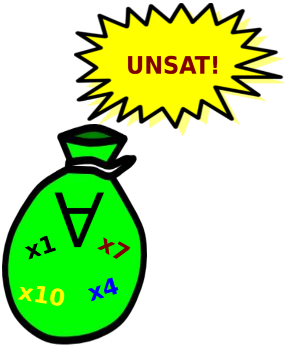

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- Thus, if ψ is an implicate of ϕ whose negation is sat, free(ψ) is a non-universal set
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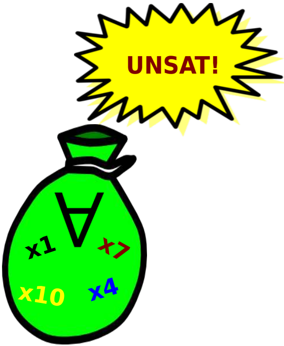
- When all variables in ψ are $\forall$-quantified, backtrack without checking satisfiability



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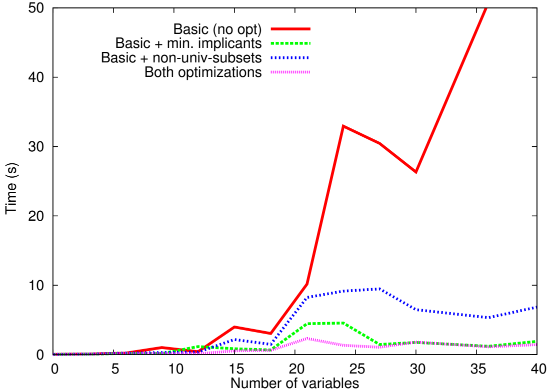
- Evaluated algorithm on 400 Presburger arithmetic formulas
- Formulas taken from static analysis tool that uses MSAs for performing abduction, in turn used for diagnosing error reports
- Formulas contain up to 40 variables and several hundred boolean connectives

Experimental Results



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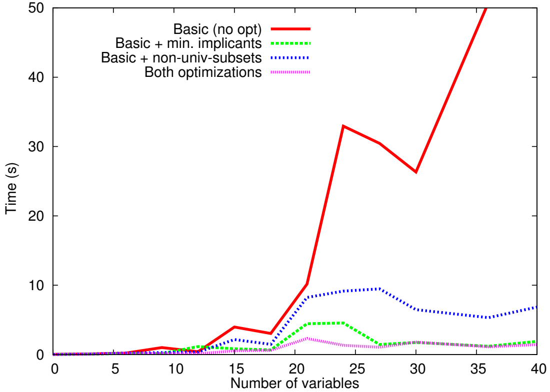

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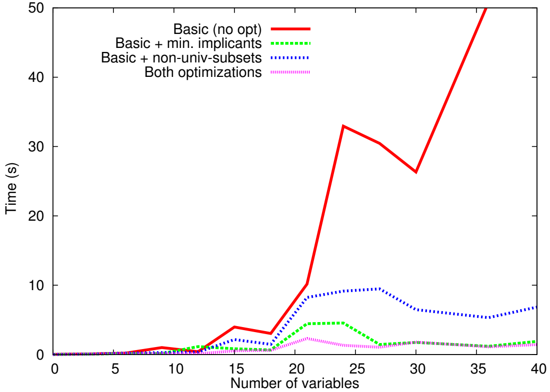

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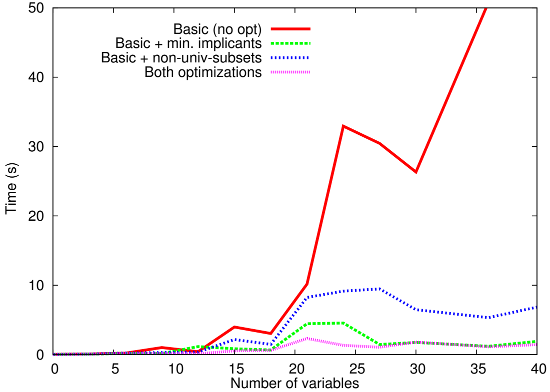

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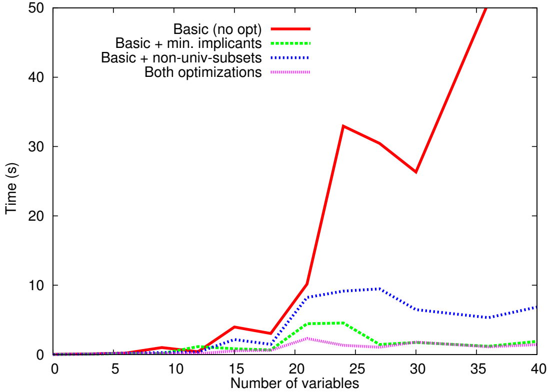
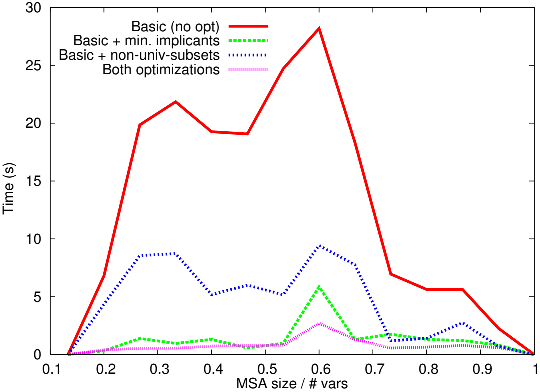


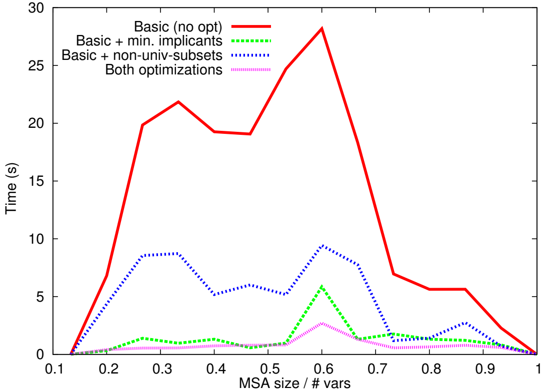
- Basic algorithm very sensitive to # vars
- Optimizations have dramatic impact on performance
- Optimized version grows slowly in # of variables

Even with both optimizations, computing MSAs 25 times more expensive than checking satisfiability

Experimental Results, cont.

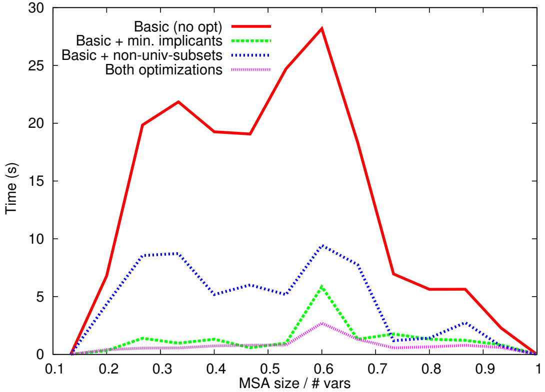


Experimental Results, cont.



- Problem easier if # vars in MSA very small or very large

Experimental Results, cont.



- Problem easier if # vars in MSA very small or very large
- Problem hardest for formulas when ratio of vars in MSA to free vars is ≈ 0.6

Summary

- First algorithm for finding MSAs of SMT formulas
- Recursive branch-and-bound style algorithm with two crucial optimizations
- MSAs can be computed in reasonable time for a set of benchmakrs obtained from static analysis
- But finding MSAs much more expensive than finding full sat assignment
- We believe significant improvements are still possible