Minimum-Time/Energy Output-Transitions in Linear Systems

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***Abstract*— This article addresses the optimal (minimum-time/energy) trajectory design for rapid output transitions, i.e. changing the output from one value to another, in linear systems. Furthermore, the output is required to be maintained constant (e.g. without vibration) outside the transition time interval. The output-transition problem is posed as a linear quadratic minimum-time (LQMT) optimal control problem, which avoids bang-bang controllers that result from solving the traditional time-optimal problem. Additionally, the LQMT approach allows the time-optimal requirement to be traded off with the energy requirement by selecting appropriated weighting factors. Current methods transform this LQMT output-transition problem into a state-transition problem by constraining the initial and final state of the output-transition interval. However, the choice of the initial and final states can be ad hoc and the resulting control law may not be optimal. In contrast, the proposed approach directly solves the LQMT output-transition problem by optimally choosing the initial and final states to minimize the output-transition cost. The novelty of the proposed approach is that inputs are not applied just during the output-transition time interval; rather, inputs are also applied before the beginning of and after the end of the output-transition time interval (these inputs are called pre- and post-actuation). The method is illustrated using a flexible structure model, and simulation results show substantial reduction in output-transition cost when compared with the cost of standard state-transition-based approaches, which do not use pre- and post-actuation.**

I. INTRODUCTION

This article proposes a feedforward trajectory design for rapid output transitions in linear systems. It is noted that the output-transition problem (i.e., changing the output of a system from one value to another) is a fundamental control problem, which appears in a wide range flexible structure applications. For example, such problems arise in (I) rapidly positioning the end-point of large-scale space manipulators (Refs. [1], [2], [3]); (II) positioning of read/write heads of disk-drive servo systems, which are relatively medium-scale flexible structures (Refs. [4], [5]); and (III) nano-scale positioning and manipulation using relatively small-scale piezo-actuators (Refs. [6], [7]). When performing fast maneuvers with flexible structures, it is critical to suppress residual vibrations that cause a loss of positioning precision. For example, in disk-drive applications, read and write operations cannot be performed (before and after the output transition) if the output position is not precisely maintained at the desired track. Such residual vibrations may take a prohibitively long time to settle down and are undesirable because the system cannot perform the next task until the vibrations reduce to an acceptable level. Therefore, it is important to achieve output transitions that are fast and do not cause residual vibrations. This article studies such vibration-free (rest-to-rest) output transitions, where the output is maintained at a constant value outside the output-transition time interval $[t_i, t_f]$, as shown in Figure 1. The main contribution of this article is the direct solution of the optimal (minimum-time/energy) output-transition (OOT) problem for linear systems. The method is illustrated on a flexible structure model and simulation results are presented.

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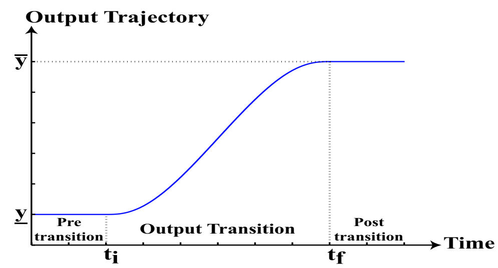

Fig. 1. Output-Transition Problem.

Existing techniques solve such optimal output-transition problem by first transforming it into a state-transition problem. In particular, output transitions without residual vibrations can be obtained by requiring that the flexible system maneuvers between equilibrium (rigid-body or rest) configurations. (*These equilibrium states, $\underline{x}$ and $\overline{x}$, are chosen to result in the initial and final output values, $\underline{y}$ and $\overline{y}$, respectively.*) Once the boundary states at the beginning and end of output transition ($x(t_i)$, and $x(t_f)$) are chosen to be the rest (equilibrium) states, i.e. $x(t_i) = \underline{x}$, and $x(t_f) = \overline{x}$, then a solution to the optimal output-transition problem can be found by solving the standard (minimum-time/energy) state-transition (SST) problem, Refs. [8], [9], [10]. However, the optimal SST solution found with this choice of the rest-to-rest boundary states $\{x(t_i) = \underline{x},\ x(t_f) = \overline{x}\}$ may not lead to the optimal output-transition (OOT) solution. On the other hand, arbitrary choices of the boundary states $\{x(t_i),\ x(t_f)\}$ are also not acceptable; they may not allow the output to be maintained at a constant value after the completion of the output-transition (i.e., without residual vibrations) for any choice of bounded inputs. Therefore, the existing state-to-state transition approaches cannot be used to directly solve the optimal output-transition problem.

In this article, the criterion used for choosing an optimal input to achieve the optimal output transition is obtained by adding a quadratic weight on the input to the time needed to achieve the output transition. This approach is referred to as a linear quadratic minimum-time (LQMT) problem (Refs. [8], [9], [10]). The LQMT approach avoids the bang-bang controller that results from the solution of traditional time-optimal problem, Refs. [1]-[4], [11]. The bang-bang controller tends to rapidly switch the control input between its maximum and minimum values [11]. Such rapid switching is undesirable since it tends to violate system constraints such as actuator bandwidth limitations and acceptable vibration levels required to avoid structural failure. In contrast, posing the OOT problem using the LQMT criterion allows one to balance the time-optimal requirement with the energy requirement without having to use a bang-bang controller. The solution to this optimal (LQMT) output-transition problem is presented in this article.

A novelty of the proposed output-transition approach, when compared to standard state-transition approaches, is that it uses both pre- and post actuation inputs to reduce the transition cost (Refs. [12] and [13]). In particular, the proposed approach uses an inversion-based control technique to find pre-actuation ($t < t_i$)

inputs that maintain output tracking ($y=\underline{y}$) before initiating the output transition, and similarly to find the post-actuation ($t>t_f$) inputs to maintain the output at the final value ($\overline{y}$) after completing the output transition. The inversion-based control approach, which finds the pre- and post-actuation inputs, is then integrated with the state-transition approach, e.g. in Ref. [8], during the output transition (between initial transition time t_i and final transition time t_f) to solve the optimal output-transition problem. It is noted that the proposed approach does not put a constraint on the boundary states $\{x(t_i), x(t_f)\}$; rather, we exploit the freedom in the choice of the boundary states to optimally reduce the output-transition cost. It is shown in this article that the optimal LQMT output-transition solution substantially reduces both transition time and input energy required to complete the output transition when compared to the LQMT state-transition (SST) approach.

This paper is organized as follows. In Section II, the output-transition problem is formulated. The problem is solved in Section III. The application to flexible structure model and the simulation results are presented in Sections IV. Our conclusions are in Section V.

II. Problem Formulation

We consider a *square* (same number of inputs as outputs), linear, time-invariant dynamical system in the state-space form, described as

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t) \\ y(t) & = & Cx(t) \end{array} \right. , \quad \forall\, t \in (-\infty,\infty) \qquad (1)$$

where the system state is $x(t) \in \Re^n$, the input is $u(t) \in \Re^p$, and the output is $y(t) \in \Re^p$. It is assumed that the system (Eq. 1) is *controllable*. Then, the output-transition problem is defined as following.

Definition 1: **The output-transition problem** is to find bounded input-state trajectories $[u(\cdot), x_{ref}(\cdot)]$ that satisfy the system equations in (Eq. 1) and the following two conditions:

1. The output-transition condition: The output is transferred from an initial value $\underline{y}$ to a final value $\overline{y}$ within the output-transition time interval $[t_i, t_f]$, and is maintained constant at the desired value before and after the output transition, i.e.,

$$\begin{array}{rcccll} y_{ref}(t) & = & \underline{y} & = & C\underline{x} & \text{for } \forall\, t \le t_i \\ y_{ref}(t) & = & \overline{y} & = & C\overline{x} & \text{for } \forall\, t \ge t_f \end{array} \qquad (2)$$

where $y_{ref}(t) := Cx_{ref}(t)$, t_i and t_f denote the times when the output transition starts and completes, respectively. Furthermore, the states $\underline{x}$ and $\overline{x}$ denote the initial and final equilibrium states (i.e., $A\underline{x} = A\overline{x} = 0$), which are chosen to result in the desired initial and final output values ($\underline{y}$ and $\overline{y}$).

2. The delimiting-state condition: The state approaches the equilibrium configuration as time goes to (plus or minus) infinity, i.e.,

$$x_{ref}(t) \to \underline{x} \text{ as } t \to -\infty; \text{ and } x_{ref}(t) \to \overline{x} \text{ as } t \to \infty. \qquad (3)$$

For controllable systems, there exists at least one input that achieves the desired output transition, e.g. by setting the state $x(t) = \underline{x}$ during the pre-transition time ($t \le t_i$) and the state $x(t) = \overline{x}$ during the post-transition time ($t \ge t_f$) the output-transition problem becomes a state-transition problem which can be solved by existing techniques (see, for example, Ref. [14]). In this article, we want to choose the input that minimizes the time/energy performance. The optimal output-transition (OOT) problem is stated as following.

Definition 2: **The optimal (linear-quadratic/minimum-time) output-transition problem (OOT)** is to find bounded input-state trajectories $[u(\cdot), x_{ref}(\cdot)]$ and a final transition time (t_f) that solve the output-transition problem (see Definition 1), and minimizes the following time/input-energy cost functional,

$$\begin{aligned} J\{t_f, u(t;t_f)\} &= \rho(t_f - t_i) + \int_{-\infty}^{\infty} u(t)^T R u(t)\; dt \quad (4) \\ &:= \rho T_{tran} + J_{LQ}\{t_f, u(t;t_f)\} \quad (5) \end{aligned}$$

where the positive constant ρ and the symmetric positive-definite matrix R represent weighting factors between the elapsed output-transition time and the input energy, respectively. The time interval $T_{tran} := t_f - t_i$ is the elapsed output-transition time, and the cost component J_{LQ} denotes the contribution of the input-energy to the total cost. Throughout the rest of the article, the initial transition time (t_i) is assumed to be constant.

III. The LQMT Output-transition (OOT) Solution

We approach the LQMT output-transition problem in two steps. First, we consider the OOT problem with a fixed final transition time, and find the optimal input that minimizes the quadratic input-energy term J_{LQ}, defined in (Eq. 5). Second, the *optimal* input-energy cost (which is treated as a parameterized function of final transition time) is substituted into the total cost equation (Eq. 4). Then, we can formulate the OOT problem as a parameterized optimization problem in the final transition time, i.e.

$$\min_{t_f,u} J\{t_f, u(t;t_f)\} := \min_{t_f} \left\{ \rho(t_f - t_i) + \min_{u} \left[J_{LQ}\{t_f, u(t;t_f)\} \right] \right\}. \qquad (6)$$

A. Fixed final transition time

In this subsection, we consider the output-transition solution that minimizes the quadratic input-energy cost J_{LQ}, see (Eq. 6) for a given fixed final transition time $t_f > t_i$. In the following, we assume that the system can be represented in the output-tracking form or normal form (see Ref. [15]).

Assumption 1: Throughout the rest of the article, we assume the following. There exist

I.) a state transformation Φ, defined by

$$\begin{aligned} x(t) = {} & \Phi \left[\xi(t)^T \;\; \eta_s(t)^T \;\; \eta_u(t)^T \;\; \eta_c(t)^T \right]^T \\ := {} & \left[\Phi_\xi | \Phi_{\eta_s} | \Phi_{\eta_u} | \Phi_{\eta_c} \right] \left[\xi(t)^T | \eta_s(t)^T | \eta_u(t)^T | \eta_c(t)^T \right]^T \quad (7) \end{aligned}$$

where the component $\xi(t) := \left[y_1(t)^T, \ldots, y_1^{(\rho_1 - 1)}(t)^T, \ldots y_p(t)^T, \ldots, y_p^{(\rho_p - 1)}(t)^T \right]$ represents the output and its time-derivatives up to order $\rho - 1$ (the parameters $\rho_1, \rho_2, \ldots, \rho_p$ denote the corresponding relative degrees to each output), and the components $\eta_s(t), \eta_u(t), \eta_c(t)$ represent the stable, unstable, and center subspaces of the internal dynamics (see Ref. [15]), respectively, and

II.) an input law that yields the exact output tracking with the following general form,

$$u_{inv}(t) := U_s \eta_s(t) + U_u \eta_u(t) + U_c \eta_c(t) + U_\xi \mathbb{Y}_d(t) \qquad (8)$$

where $\mathbb{Y}_d(t) := \left[\xi_d(t)^T,\; y_{1_d}^{(\rho_1)}(t)^T, \ldots,\; y_{p_d}^{(\rho_p)}(t)^T \right]^T$ with the subscript d denoting the desired (or known) values,

III) such that the original system (Eq. 1) can be transformed into the following *output-tracking form* (by using the above state transformation Φ and input law u_{inv}),

$$\begin{aligned} \dot{\xi}(t) &= \dot{\xi}_d(t) \\ \begin{bmatrix} \dot{\eta}_s(t) \\ \dot{\eta}_u(t) \\ \dot{\eta}_c(t) \end{bmatrix} &= \begin{bmatrix} A_s & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_c \end{bmatrix} \begin{bmatrix} \eta_s(t) \\ \eta_u(t) \\ \eta_c(t) \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \\ B_c \end{bmatrix} \mathbb{Y}_d(t). \quad (9) \end{aligned}$$

Remark 1: Assumption 1 is satisfied if the system has a well-defined vector relative degree, $\rho := [\rho_1, \rho_2, \ldots, \rho_p]$, see Ref. [15].

The minimum-energy output-transition cost is computed in two steps. In the first step, we quantify the control inputs, required before the initiation of the output-transition (pre-actuation) and after the completion of the output-transition (post-actuation), that allow the output to be maintained at constant value before and after the output-transition interval (i.e, satisfying the conditions for output-transition problem in Definition 1). Associated with these pre- and post-actuation inputs, we identify the acceptable set of the boundary states, i.e. the state x_i at the initial transition time (t_i) and the state x_f at the final transition time (t_f) that can be chosen while satisfying the conditions for the output-transition problem. We note that if the acceptable boundary states $\{x_i, x_f\}$ are specified, the optimal input during the output-transition interval ($t_i \leq t \leq t_f$) can be found by using the standard LQ optimal control technique (see, for example, Chapter 3 in Ref. [14]).

In the second step, we integrate the pre- and post-actuation inputs with the optimal state-transition during the output-transition time interval $[t_i, t_f]$ to find the optimal boundary states that yield the minimum cost. The detailed of the minimum-energy output-transition solution can be found in Ref. [12], and the results are briefly stated in the next Theorem.

Theorem 1: Let Assumption 1 be satisfied. Then for any fixed final transition time $t_f > t_i$, we obtain the following results.

I) The pre-actuation input that satisfies the output-tracking conditions (Eqs. 2 and 3 in Definition 1) is uniquely specified in terms of the unstable internal state component η_{u_i} at the initial transition time, and is given by

$$u_{pre}(t) := U_u e^{A_u(t-t_i)}(\eta_{u_i} - \underline{\eta}_u), \qquad \text{for} \quad t < t_i \tag{10}$$

where $[\underline{\xi}^T, \underline{\eta}_s^T, \underline{\eta}_u^T, \underline{\eta}_c^T]^T = \Phi^{-1}\underline{x}$. Similarly, the post-actuation input that satisfies the output-tracking conditions (Eqs. 2 and 3 in Definition 1) is uniquely specified in terms of the stable internal-state component η_{s_f} at the final transition time, and is given by

$$u_{post}(t) := U_s e^{A_s(t-t_f)}(\eta_{s_f} - \overline{\eta}_s), \qquad \text{for} \quad t > t_f \tag{11}$$

where $[\overline{\xi}^T, \overline{\eta}_s^T, \overline{\eta}_u^T, \ \overline{\eta}_c^T]^T = \Phi^{-1}\overline{x}^T$. Furthermore, the costs associated with these pre- and post-actuation inputs are equal to

$$\begin{array}{rcl} J_{pre} & = & (\eta_{u_i} - \underline{\eta}_u)^T W_{pre}(\eta_{u_i} - \underline{\eta}_u) \quad \text{, and} \\ J_{post} & = & (\eta_{s_f} - \overline{\eta}_s)^T W_{post}(\eta_{s_f} - \overline{\eta}_s) \end{array} \tag{12}$$

where $W_{pre} = \int_0^\infty e^{-A_u^T\tau} U_u^T R U_u e^{-A_u\tau} d\tau$, and $W_{post} = \int_0^\infty e^{A_s^T\tau} U_s^T R U_s e^{A_s\tau} d\tau$.

II) The only components of the boundary states $\{x_i, x_f\}$ that can be varied while satisfying the output-transition conditions (Eqs. 2 and 3) are the unstable component η_{u_i} (of internal state at the initial transition time) and the stable component η_{s_f} (of internal state at the final transition time). Therefore, the acceptable boundary states $\{x_i, x_f\}$ must be chosen as

$$x_i = \Phi\left[\underline{\xi}^T | \underline{\eta}_s^T | \eta_{u_i}^T | \underline{\eta}_c^T\right]^T \text{ and } \ x_f = \Phi\left[\overline{\xi}^T | \eta_{s_f}^T | \overline{\eta}_u^T | \overline{\eta}_c^T\right]^T \tag{13}$$

Furthermore, we define the *boundary condition* Ψ which is the components of the state, at the initiation and completion of the output transition, that can be freely varied while satisfying the conditions for the output-transition problem, i.e.

$$\Psi := \left[\eta_{s_f}^T \quad \eta_{u_i}^T\right]^T. \tag{14}$$

III) Given a pair of acceptable boundary states $\{x_i, x_f\}$, the minimum-energy control input that transfers the system from the initial state x_i to the final state x_f within a transition time $T_{tran} = t_f - t_i$ is given by

$$u_{tran}(t) := R^{-1}B^T e^{A^T(t_f-t)} G^{-1} d_x, \ \text{for} \quad t_i \leq t \leq t_f \tag{15}$$

where G is the *invertible* controllability grammian, defined by

$$G := \int_{t_i}^{t_f} e^{A(t_f-\tau)} B R^{-1} B^T e^{A^T(t_f-\tau)} d\tau, \tag{16}$$

and d_x denotes the transition-state difference, given by

$$\begin{array}{rcl} d_x & := & x_f - e^{A(t_f-t_i)}x_i \qquad (17) \\ & := & H_1\hat{f} + H_2\Psi \qquad (18) \end{array}$$

where $H_1 := [\Phi_\xi \ \Phi_{\eta_u} \ \Phi_{\eta_c} \ \ -\Gamma_\xi \ \ -\Gamma_{\eta_s} \ \ -\Gamma_{\eta_c}]$,
$H_2 := [\Phi_{\eta_s} \ \ -\Gamma_{\eta_u}]$,
$[\Gamma_\xi \ \Gamma_{\eta_s} \ \Gamma_{\eta_u} \ \Gamma_{\eta_c}] := e^{A(t_f-t_i)}[\Phi_\xi \ \Phi_{\eta_s} \ \Phi_{\eta_u} \ \Phi_{\eta_c}]$,
and $\hat{f} := \left[\overline{\xi}^T \quad \overline{\eta}_u^T \quad \overline{\eta}_c^T \quad \underline{\xi}^T \quad \underline{\eta}_s^T \quad \underline{\eta}_c^T\right]^T$.

Furthermore, the cost during the output transition when using this optimal state-transition control input is then equal to

$$J_{tran} = d_x^T G^{-1} d_x. \tag{19}$$

IV) The optimal input that minimizes the input-energy cost J_{LQ} (defined in Eq. 5) over all acceptable sets of boundary states is given by

$$u^*(t;t_f) = \begin{cases} U_u e^{A_u(t-t_i)}[\eta_u^* - \underline{\eta}_u] & \text{if} \quad t < t_i \\ U_s e^{A_s(t-t_f)}[\eta_s^* - \overline{\eta}_s] & \text{if} \quad t > t_f \\ R^{-1}B^T e^{A^T(t_f-t)} G(t_f)^{-1}[x_f^* - e^{A(t_f-t_i)}x_i^*] \\ \quad \text{if} \quad t_i \leq t \leq t_f \end{cases} \tag{20}$$

where the optimal boundary condition and the optimal boundary states are given by

$$\Psi^* := \begin{bmatrix} \eta_s^* \\ \eta_u^* \end{bmatrix} := \begin{cases} \Lambda^{-1}b & \text{, if } \Lambda \text{ is invertible} \\ \Lambda^\dagger b & \text{, otherwise} \end{cases},$$

$$x_i^* = \Phi\left[\underline{\xi}^T \ \underline{\eta_s}^T \ \eta_u^{*T} \ \underline{\eta_c}^T\right]^T, \ x_f^* = \Phi\left[\overline{\xi}^T \ \eta_s^{*T} \ \overline{\eta_u}^T \ \overline{\eta_c}^T\right]^T,$$

respectively. Furthermore, the matrix Λ is defined by

$$\Lambda := \begin{bmatrix} W_{post} & 0 \\ 0 & W_{pre} \end{bmatrix} + H_2^T G^{-1} H_2, \tag{21}$$

and $\Lambda^\dagger$ denotes the pseudo (generalized) inverse of Λ (Ref. [16]). This optimal input $u^*(t;t_f)$ is referred to as the solution to the minimum input-energy output-transition problem.

V) The minimum input-energy cost for a given final transition time t_f is equal to

$$J_{LQ}^*\{t_f, u^*(t;t_f)\} := \Psi^{*T}\Lambda\Psi^* - 2\Psi^{*T}b + c. \tag{22}$$

$$\text{where} \quad b := \begin{bmatrix} W_{post}\overline{\eta}_s \\ W_{pre}\underline{\eta}_u \end{bmatrix} - H_2^T G^{-1} H_1 \hat{f}, \ \text{and}$$
$$c := \overline{\eta}_s^T W_{post}\overline{\eta}_s + \underline{\eta}_u^T W_{pre}\underline{\eta}_u + \hat{f}^T H_1^T G^{-1} H_1 \hat{f}. \tag{23}$$

Proof: Parts **I)** and **II)** follow directly from Lemmas 3 and 5 in Ref. [12]. Part **III)** is derived in Chapter 3 in Ref. [14]. For parts **IV)** and **V)**, see Theorem 1 in Ref. [12]. ■

B. Free final transition time

The total LQMT cost $J\{t_f, u(t;t_f)\}$ in (Eq. 4) is a functional of the final transition time t_f and the control input $u(t;t_f)$ where the notation t_f in the input term indicates the parameterized dependency on the final transition time. If the final transition time t_f is specified, then one can obtain the optimal quadratic cost J_{LQ}^* and the optimal control input $u^*(t;t_f)$ by using Theorem 1. Thus, by substituting this optimal quadratic term (Eq. 22) into the total LQMT cost function (Eq 4), the LQMT output-transition problem can be formulated as a parameterized optimization problem in the final transition time (as shown in Eq. 6), i.e.

$$J\{t_f, u^*(t;t_f)\} = J(t_f) = \rho(t_f - t_i) + J^*_{LQ}\{t_f, u^*(t;t_f)\}. \quad (24)$$

As shown in previous Section, the optimal quadratic cost J^*_{LQ} directly depends on the boundary state Ψ, which, in turn, is an implicit function of the final transition time t_f. The challenge is to show the existence of an optimal final transition time t_f^*. If such a solution exists, then one can be computed by minimizing the total LQMT cost in (Eq. 24) over the parameter $t_f > t_i$. We begin by investigating some properties of the optimal quadratic cost $J^*_{LQ}\{t_f, u^*(t;t_f)\}$ as a parameterized function of the final transition time t_f when $t_f > t_i$.

Proposition 1: The optimal quadratic cost $J^*_{LQ}\{t_f, u^*(t;t_f)\}$, as defined in (Eq. 22), is a continuous function of the final transition time t_f for all $t_f > t_i$.

Proof: First we expand the optimal quadratic cost (Eq. 22) in terms of individual components as

$$\begin{aligned} J^*_{LQ}\{t_f, u^*(t;t_f)\} &= \Psi^*(t_f)^T \Lambda(t_f)\Psi^*(t_f) - 2\Psi^*(t_f)^T b(t_f) + c(t_f) \\ &= J^*_{pre} + J^*_{tran} + J^*_{post} = \left[\eta^*_u(t_f) - \underline{\eta}_u\right]^T W_{pre}\left[\eta^*_u(t_f) - \underline{\eta}_u\right] \\ &+ \left[H_1(t_f)\hat{f} + H_2(t_f)\Psi^*(t_f)\right]^T G(t_f)^{-1}\left[H_1(t_f)\hat{f} + H_2(t_f)\Psi^*(t_f)\right] \\ &+ \left[\eta^*_s(t_f) - \overline{\eta}_s\right]^T W_{post}\left[\eta^*_s(t_f) - \overline{\eta}_s\right] \end{aligned}$$

where the dependency on the final transition time is explicitly shown by the notation (t_f) after the variables. Note that the matrices $H_1(t_f)$ and $H_2(t_f)$ are continuous in term of t_f since they are composed of terms from the constant transformation matrix Φ and from the matrix exponential $e^{A(t_f - t_i)}$ which is continuous in t_f (see Eq. 18).

The controllability grammian $G(t_f)$, as defined in (Eq. 16), is continuous in t_f since the integrand is a continuous function. Note that if a matrix function is continuous in its variable and nonsingular, then the inverse of the matrix is also continuous in its variable (Ref. [17]). Since the controllability of the system implies that the grammian $G(t_f)$ is nonsingular, the inverse grammian function $G^{-1}(t_f)$ is also continuous in t_f. The matrices $\Lambda(t_f)$, $b(t_f)$, and $c(t_f)$ (as given in Eqs. 21 and 23) are also continuous functions of t_f since they are products of the matrices $H_1(t_f)$, $H_2(t_f)$, and $G^{-1}(t_f)$. The optimal boundary condition Ψ^* (as given in Theorem 1) is a product of two continuous matrices. Therefore the cost function $J^*_{LQ}\{t_f, u^*(t;t_f)\}$ is continuous in the final transition time t_f since each components that depend on t_f is continuous in t_f. ∎

In the following, let the operator $\|\cdot\|$ denotes the Euclidean norm of a vector and the l_2-operator norm (spectral norm) of a matrix, see Ref. [17].

Proposition 2: Suppose the output $\underline{y} \neq \overline{y}$, and the boundary states $\{x_i, x_f\}$ are chosen so that satisfy the conditions output-transition problem (in Definition 1). Then there exist some time t_f^α greater than the initial transition time t_i and a constant K_α greater than 0 such that magnitude of the transition-state difference $\|d_x(t_f)\| > K_\alpha$ whenever the final transition time is in between $t_i < t_f < t_f^\alpha$.

Proof: Since the boundary condition Ψ represents the only internal-state components that can be varied while satisfying the conditions for the output-transition problem (see Theorem 1), the boundary states must be chosen among the acceptable set described in (Eq. 13). So the transition-state difference d_x with the acceptable boundary states $\{x_i, x_f\}$ can be written in terms of the output-tracking coordinates (ξ, η) as

$$\begin{aligned} d_x &= x_f - e^{A(t_f - t_i)}x_i \\ &= \Phi\left\{\left[\overline{\xi}^T, \eta^T_{s_f}, \overline{\eta}^T_u, \overline{\eta}^T_c\right]^T - \Phi^{-1}e^{A(t_f - t_i)}\Phi\left[\underline{\xi}, \underline{\eta}^T_s, \eta^T_{u_i}, \underline{\eta}^T_c\right]^T\right\} \\ &:= \Phi\, d_{\xi,\eta}. \end{aligned}$$

where the transformation Φ is given in (Eq. 7). Without loss of generality, assume that the system coordinates are shifted so that the initial equilibrium state is at the origin, i.e. let $\underline{x} = 0$, thus the transition-state difference becomes

$$d_x = \Phi\left\{\left[\overline{\xi}^T, \eta^T_{s_f}, \overline{\eta}^T_u, \overline{\eta}^T_c\right]^T - \Phi^{-1}e^{A(t_f - t_i)}\Phi\left[0, 0, \eta^T_{u_i}, 0\right]^T\right\}.$$

Next, partition the transition matrix $Phi^{-1}e^{A(t_f - t_i)}\Phi$ (corresponding to the output-tracking coordinate) as

$$\Phi^{-1}e^{A(t_f - t_i)}\Phi := \begin{bmatrix} \Theta_{\xi\xi}(t_f) & \Theta_{\xi\eta_s}(t_f) & \Theta_{\xi\eta_u}(t_f) & \Theta_{\xi\eta_c}(t_f) \\ \Theta_{\eta_s\xi}(t_f) & \Theta_{\eta_s\eta_s}(t_f) & \Theta_{\eta_s\eta_u}(t_f) & \Theta_{\eta_s\eta_c}(t_f) \\ \Theta_{\eta_u\xi}(t_f) & \Theta_{\eta_u\eta_s}(t_f) & \Theta_{\eta_u\eta_u}(t_f) & \Theta_{\eta_u\eta_c}(t_f) \\ \Theta_{\eta_c\xi}(t_f) & \Theta_{\eta_c\eta_s}(t_f) & \Theta_{\eta_c\eta_u}(t_f) & \Theta_{\eta_c\eta_c}(t_f) \end{bmatrix}.$$

Note that the matrix functions $\Theta_{\xi\eta_u}(t_f)$ and $\Theta_{\eta_u\eta_u}(t_f)$ are continuous functions of the final transition time t_f with the initial conditions $\lim_{t_f \to t_i}\Theta_{\xi\eta_u}(t_f) = 0$ and $\lim_{t_f \to t_i}\Theta_{\eta_u\eta_u}(t_f) = I$, since $\lim_{t_f \to t_i}\Phi^{-1}e^{A(t_f - t_i)}\Phi = I$. Then define a new a variable,

$$\psi(t_f;\eta_{u_i}) = \begin{bmatrix} \overline{\xi} - \Theta_{\xi\eta_u}(t_f)\eta_{u_i} \\ \overline{\eta_u} - \Theta_{\eta_u\eta_u}(t_f)\eta_{u_i} \end{bmatrix}.$$

Let $\sigma_{\Phi,min}$ denotes the smallest singular value of the transformation matrix Φ (note that $\sigma_{\Phi,min} > 0$ since the matrix Φ is nonsingular). Since $\|d_x\| \geq \sigma_{\Phi,min}\|d_{\xi,\eta}\| \geq \sigma_{\Phi,min}\|\psi(t_f,\eta_{u_i})\|$, it suffices to show that if there exists some time instant $t_f^\alpha > t_i$ and a constant $K_\alpha > 0$ such that $\|\psi(t_f,\eta_{u_i})\| > K_\alpha/\sigma_{\Phi,min}$ whenever $t_i < t_f < t_f^\alpha$, then $\|d_x\| > K_\alpha$.

By continuity of the matrix function $\Theta_{\xi\eta_u}(t_f)$ and $\lim_{t_f \to t_i}\Theta_{\xi\eta_u}(t_f) = 0$, there exists some time t_1 greater than the initial transition time t_i and a constant ϵ_1 with $0 < \epsilon_1 < \|\overline{\xi}\|/\{2(\|\overline{\xi}\| + \|\overline{\eta_u}\|)\}$ such that $\|\Theta_{\xi\eta_u}(t_f)\| < \epsilon_1$ whenever $t_i < t_f < t_1$. Next, consider 2 possible cases of the internal state component η_{u_i}.

Case (I): Suppose $\|\eta_{u_i}\| \leq \|\overline{\xi}\| + \|\overline{\eta_u}\|$.

Note that $\|\psi(t_f,\eta_{u_i})\| \geq \|\overline{\xi} - \Theta_{\xi\eta_u}(t_f)\eta_{u_i}\| \geq \left|\, \|\overline{\xi}\| - \|\Theta_{\xi\eta_u}(t_f)\eta_{u_i}\| \,\right|$. Since $\|\Theta_{\xi\eta_u}(t_f)\eta_{u_i}\| \leq \|\Theta_{\xi\eta_u}(t_f)\| \cdot \|\eta_{u_i}\| < \|\overline{\xi}\|/2$, therefore $\|\psi(t_f,\eta_{u_i})\| > \|\overline{\xi}\|/2$.

Case (II): Suppose $\|\eta_{u_i}\| > \|\overline{\xi}\| + \|\overline{\eta_u}\|$.

Since the matrix function $\Theta_{\eta_u\eta_u}(t_f)$ is continuous function of the final transition time t_f and $\lim_{t_f \to t_i}\Theta_{\eta_u\eta_u}(t_f) = I$, there exists some time $t_I > t_i$ such that the matrix $\Theta_{\eta_u\eta_u}(t_f)$ is invertible whenever the final transition time $t_i < t_f < t_I$. Let $\sigma_{\Theta,min}(t_f)$ denotes the smallest singular value of the matrix $\Theta_{\eta_u\eta_u}(t_f)$. Since $\lim_{t_f \to t_i}\sigma_{\Theta,min}(t_f) = 1$, by continuity of the singular value function there exist some time t_2 with $t_i < t_2 < t_I$ and a constant ϵ_2 with $0 < \epsilon_2 < \epsilon_1$ such that $|\ \sigma_{\Theta,min}(t_f) - 1\ | < \epsilon_2$ whenever $t_i < t_f < t_2$. Then

$$\begin{aligned} \|\Theta_{\eta_u\eta_u}(t_f)\eta_{u_i}\| &\geq \sigma_{\Theta,min}(t_f) \cdot \|\eta_{u_i}\| \\ &> (1 - \epsilon_2) \cdot (\|\overline{\xi}\| + \|\overline{\eta_u}\|) \\ &> (1 - \epsilon_1) \cdot (\|\overline{\xi}\| + \|\overline{\eta_u}\|) \\ &= (\|\overline{\xi}\| + \|\overline{\eta_u}\|) - \epsilon_1 \cdot (\|\overline{\xi}\| + \|\overline{\eta_u}\|) \\ &> (\|\overline{\xi}\| + \|\overline{\eta_u}\|) - \|\overline{\xi}\|/2 = \|\overline{\xi}\|/2 + \|\overline{\eta_u}\| \end{aligned}$$

Note that $\|\psi(t_f,\eta_{u_i})\| \geq \|\overline{\eta_u} - \Theta_{\eta_u\eta_u}(t_f)\eta_{u_i}\| \geq |\,\|\overline{\eta_u}\| - \|\Theta_{\eta_u\eta_u}(t_f)\eta_{u_i}\|\,|$, therefore $\|\psi(t_f,\eta_{u_i})\| > \|\overline{\xi}\|/2$.

Set $t_f^\alpha = \min\{t_1, t_2\}$ and $K_\alpha = \sigma_{\Phi,min}\|\overline{\xi}\|/2$. Then the state difference, $\|d_x\| > K_\alpha$ whenever the final transition time $t_i < t_f < t_f^\alpha$, which completes the proof. ∎

Remark 2: The smallest singular value $\sigma_{\Theta,min}$ is a specialized case of the matrix lower bound $\|\Theta_{\eta_u\eta_u}(t_f)\|_L$, see Ref. [18]. The matrix lower bound is also a continuous function of the final transition time on the interval (t_i, t_I) by Corollary 4.3 in Ref. [18].

Proposition 3: Suppose the output $\underline{y} \neq \overline{y}$, and the boundary states $\{x_i, x_f\}$ are chosen to satisfy the conditions for output-transition problem (in Definition 1). Then there exist some time $t_f^\epsilon > t_i$ and a constant $K_\epsilon > 0$ such that the optimal quadratic cost $J_{LQ}^*\{t_f, u^*(t;t_f)\} \geq K_\epsilon/(t_f - t_i)$ when the final transition time t_f is in between the interval $(t_i, t_f^\epsilon]$.

Proof: (This following proof is adapted from the LQMT state-transition solution, see Theorem 2.1 in Ref. [8].) Note that the optimal quadratic cost $J_{LQ}^*\{t_f, u^*(t;t_f)\}$ can be partitioned into the pre-transition cost J_{pre}, the post-transition cost J_{post} (Eq. 12), and the cost during output transition J_{tran}, see (Eq. 19). Since the pre- and post-transition costs are always greater than or equal to zero, we obtain the inequality

$$\begin{aligned} J_{LQ}^*\{t_f, u^*(t;t_f)\} &\geq J_{tran}\{u^*(t;t_f)\} \\ &= \left[x_f^* - e^{A(t_f-t_i)}x_i^*\right]^T G(t_f)^{-1}\left[x_f^* - e^{A(t_f-t_i)}x_i^*\right] \\ &= d_x^*(t_f)^T G(t_f)^{-1} d_x^*(t_f), \end{aligned}$$

where the optimal boundary state $\{x_i^*, x_f^*\}$ are defined in Theorem 1. Next consider the operator norm of the controllability grammian $G(t_f)$,

$$\begin{aligned} \|G(t_f)\|_{op} &\leq \int_{t_i}^{t_f} \|e^{A(t_f-\tau)}BR^{-1}B^T e^{A^T(t_f-\tau)}\|d\tau \\ &= \int_0^{t_f-t_i} \|e^{A\upsilon}BR^{-1}B^T e^{A^T\upsilon}\|d\upsilon \\ &\leq \int_0^{t_f-t_i} \max_{\upsilon\in[0,t_f-t_i]} \|e^{A\upsilon}\|^2\|BR^{-1}B^T\|d\upsilon. \end{aligned}$$

Note that the function norm $\|e^{A\upsilon}\|$ is continuous in υ and $\|e^{A\upsilon}\| = 1$ when $\upsilon = 0$. So there exist constants $K_\beta > 1$ and $\upsilon_\beta > 0$ such that $\|e^{A\upsilon}\| < K_\beta$ whenever $0 < \upsilon < \upsilon_\beta$. Let the time $t_f^\beta = t_i + \upsilon_\beta$, we obtain that, for all $t_f \in (t_i, t_f^\beta]$,

$$\|G(t_f)\|_{op} \leq K_\beta\|BR^{-1}B^T\|(t_f - t_i).$$

Note that the grammian $G(t_f)$ is symmetric, so the largest eigenvalue of $G(t_f)$ equals to the operator norm of $G(t_f)$ (Ref. [16]), i.e., for all $t_f \in (t_i, t_f^\beta]$,

$$\lambda_{max}[G(t_f)] = \|G(t_f)\|_{op} \leq K_\beta\|BR^{-1}B^T\|(t_f - t_i).$$

Consider the Rayleigh-Ritz inequality,

$$\begin{aligned} d_x(t_f)^T G(t_f)^{-1} d_x(t_f) &\geq \lambda_{min}[G(t_f)^{-1}]\;\|d_x(t_f)\|^2 \\ &= \lambda_{max}[G(t_f)]^{-1}\|d_x(t_f)\|^2 \end{aligned}$$

since $G(t_f)$ is square and symmetric. From Proposition 2, there exist $K_\alpha > 0$ and $t_f^\alpha > t_i$ such that for all $t_f \in (t_i, t_f^\alpha)$, $\|d_x(t_f)\| > K_\alpha$. Set $t_f^\epsilon = min(t_f^\alpha, t_f^\beta)$ and $K_\epsilon = K_\alpha/\left\{K_\beta\|BR^{-1}B^T\|\right\}$, then

$$J_{LQ}^*\{t_f, u^*(t;t_f)\} \geq K_\epsilon/(t_f - t_i), \text{ for all } t_f \in (0, t_f^\epsilon].$$

∎

Proposition 4: There exists a time $t_f^a > t_i$ such that the total cost $J\{t_f, u^*(t;t_f)\} \geq J\{t_f^a, u^*(t, t_f^a)\}$ for any t_f in the time interval $(t_i, t_f^a]$.

Proof: For any final transition time t_f in the time interval $(t_i, t_f^\epsilon]$, Proposition 3 states that the total cost

$$\begin{aligned} J\{t_f, u^*(t;t_f)\} &= \rho(t_f - t_i) + J_{LQ}^*\{t_f, u^*(t;t_f)\} \\ &\geq \rho(t_f - t_i) + K_\epsilon/(t_f - t_i), \end{aligned}$$

for some constant $K_\epsilon > 0$. Set $t_f^a := min(t_f^\epsilon, t_i + (K_\epsilon/\rho)^{1/2})$, then the total cost $J\{t_f, u^*(t;t_f)\} \geq J\{t_f^a, u^*(t, t_f^a)\}$ for any t_f in the time interval $(t_i, t_f^a]$ since the lower bound $\rho(t_f - t_i) + K_\epsilon/(t_f - t_i)$ is monotonically decreasing. ∎

Theorem 2: The solution to the LQMT output-transition problem (Definition 2) exists, and the optimal final transition time (t_f^*) is either equal to the initial transition time $(t_f^* = t_i)$ or lies inside the interval $[t_f^a,\; t_i + J\{t_f^a, u^*(t, t_f^a)\}/\rho]$ for some time $t_f^a > t_i$ defined by Proposition 2.

Proof: If the initial and final output are the same, i.e. $\underline{y} = \overline{y}$, the solution is trivial with the optimal final transition time $t_f^* = t_i$, i.e. $J\{t_f, u^*(t;t_f)\} = 0$. If the output $\underline{y} \neq \overline{y}$, then the optimal final transition time $t_f^* \geq t_f^a$, since from Proposition 4 the total cost $J\{t_f, u^*(t;t_f)\} \geq J\{t_f^a, u^*(t, t_f^a)\}$ for all $t_f \in (t_i, t_f^a]$. Let a time instant $t_f^b := t_i + J\{t_f^a, u^*(t, t_f^a)\}/\rho$. Then the optimal $t_f^* \leq t_f^b$ since the total cost $J\{t_f, u^*(t;t_f)\} \geq \rho(t_f - t_i) \geq J\{t_f^b, u^*(t, t_f^b)\}$ for all $t_f \geq t_f^b$. Therefore, the optimal final transition time t_f^* exists and lies inside the closed interval $[t_f^a, t_f^b]$ since the cost functional $J\{t_f, u^*(t;t_f)\}$ is continuous function on a compact interval, which always have a minimum (see Theorem 3.17.21 in Ref. [17]). ∎

IV. Example: Two-mass/flexible rod system

In this example, we consider two masses linked by a flexible rod, shown in Figure 2, as an illustrative example. The input $u(t)$ is the force applied to the mass m_2 on the left-side of the rod, and the output is the displacement of the mass m_1 on the right-side of the rod. The goal is to change the output (the position of the mass m_1) by one unit length with the minimum elapsed time and minimum input-energy effort.

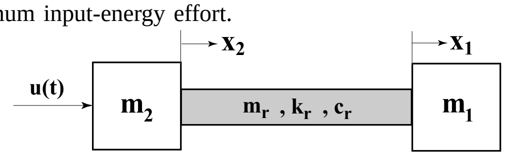

Fig. 2. Two-mass/flexible rod system

a) System description: The dynamics of the system, derived by using a simplified finite element model (FEM) with one element for the flexible rod, can be represented by

$$\left\{[M^l] + [M^r]\right\}\ddot{\mathbb{X}}(t) + [C^r]\dot{\mathbb{X}}(t) + [K^r]\mathbb{X}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{25}$$

$[M^l] := \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$, $[M^r] := \frac{\rho_r A_r l_r}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $[K^r] := \frac{A_r E_r}{l_r}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $[C^r] := \alpha_r[K^r]$ where $\mathbb{X}(t) := [x_1(t) \;\; x_2(t)]^T$ is the position vector of the two lumped masses attached at both ends of the rod, $[M^l]$ is the diagonal mass-matrix term associated with the lumped masses, $[M^r]$ is the mass-matrix term representing the distributed mass of the rod, $[K^r]$ is the stiffness matrix, $[C^r]$ is the structural damping matrix, and the parameters α_r, ρ_r, A_r, E_r, and l_r represent damping factor, density, cross-sectional area, elasticity modulus, and length of the flexible rod, respectively. In the simulations, the system parameters were chosen to be $m_1 = m_2 = 10kg$, $m_r = \frac{\rho_r A_r l_r}{6} = 1kg$, $k_r = \frac{A_r E_r}{l_r} = 1.4N/m$, and $\alpha_r = 2.8sec$, i.e. corresponding to the natural frequency $\omega_n = 0.5rad/sec$, and the damping coefficient $\zeta = 0.707$ ($\approx 1/\sqrt{2}$) of the flexible mode. The state of the system is defined as $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ where x_3 and x_4 represent the velocities of masses m_1 and m_2, respectively. During simulations, the equilibrium states for the output transitions were chosen to be $\underline{x} = [0 \ 0 \ 0 \ 0]^T$ and $\overline{x} = [1 \ 1 \ 0 \ 0]^T$. In this example, the output transition begins at the fixed initial transition time $t_i = 10sec$. The terminal-time weighted factor is chosen as $\rho = 3000$, and the input-energy weighted matrix is chosen as $R = 1$.

b) The LQMT output-transition (OOT) solution: The optimal final transition time (t_f^*) is found as the least elapsed time required to complete the output transition. As shown in Figure 3(a), the optimal LQMT output-transition solution occurs when the output-transition time ($t_f - t_i$) is equal to $2.85sec$, i.e. the optimal final transition time is $t_f^* = 12.85sec$, and the optimal output-transition cost is $J_{OOT}^* = 10565.22$. The comparisons between the optimal quadratic cost J_{LQ}^* (Eq. 22), the final time penalty $\rho(t_f - t_i)$, and the total LQMT output-transition cost J_{OOT} (Eq. 4) when the output-transition time ($t_f - t_i$) is varied are also shown in Figure 3(a). The optimal input and output trajectories are presented in Figure 4(a). It is noted that, for the OOT approach, the pre- and post-actuation inputs were applied outside the output-transition interval to maintain the output at constant value ($\underline{y}$ or $\overline{y}$).

c) Comparison to the LQMT solution for state transition (SST): The conventional LQMT approach (e.g. in Ref. [8]) for the state transition, between the initial equilibrium state $\underline{x}$ and the final equilibrium state $\overline{x}$, results in the optimal final transition time $t_f^* = 16.3sec$, and the corresponding optimal LQMT state-transition cost $J_{SST}^* = 21637.96$. The costs as functions of the output-transition time are shown in Figure 3(b), and the optimal SST input and output trajectories when using the state-transition approach are presented in Figure 4(b).

It is noted that the LQMT output-transition (OOT) approach substantially reduces the elapsed time required to complete the output-transition maneuver by 54.7% (i.e., from $6.3sec$ to $2.85sec$) and reduce the total cost by 51%, when compared to the LQMT state-transition approach (SST).

V. Conclusion

The minimum-time/input-energy output-transition problem was posed and solved in this article. The approach was applied to a two-mass/flexible-rod system model, and simulation results were presented. It was shown that the proposed approach of using pre- and post-actuation inputs can substantially reduce the overall time/energy cost of the output transition when compared to current approaches, such as the state-transition-based approach, that does not use pre- and post-actuation.

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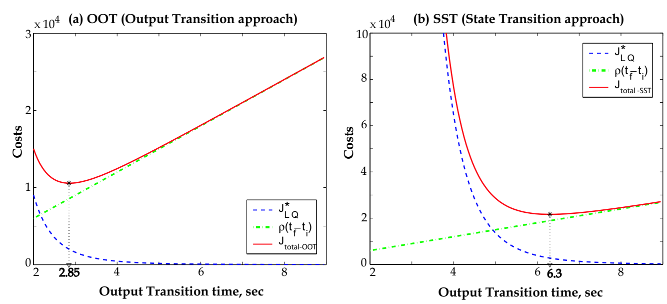

Fig. 3. The optimal quadratic cost (J_{LQ}^*), the terminal-time penalty ($\rho(t_f - t_i)$), and the total LQMT cost (J_{total}) when the output-transition time ($t_f - t_i$) is varied: *(a)* using the LQMT output-transition (OOT) approach as presented in Section 3, and *(b)* using the LQMT state-transition approach (SST) from Ref. [8].

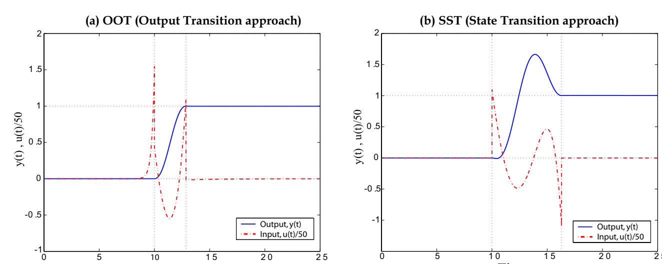

Fig. 4. The optimal input and output trajectories: *(a)* using the LQMT output-transition (OOT) approach as presented in Section 3, and *(b)* using the LQMT state-transition approach (SST) from Ref. [8].