2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Minimum Variance Control over a Gaussian Communication Channel

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***Abstract*— We consider the problem of minimizing the response of a plant output to a stochastic disturbance using a control law that relies on the output of a noisy communication channel. We discuss a lower bound on the performance achievable at a specified terminal time using nonlinear time-varying communication and control strategies, and show that this bound may be achieved using strategies that are linear.**

I. Introduction

The standard minimum variance control problem consists of minimizing the variance of a plant output in response to a stochastic disturbance using a control law that depends on possibly noisy measurements of that output. A solution to this problem, in the case of a noise free measurement, is presented in [1], wherein transfer function methods are used to obtain the result. An alternate approach, that is applicable with noisy measurements, is to solve the "cheap control" Linear Quadratic Gaussian (LQG) problem, in which a state feedback gain is applied to an estimate of the plant state obtained from a Kalman filter.

In the present paper, we assume that the system output must be communicated to the controller over a Gaussian communication channel. In this scenario, it may be feasible to add precompensation (an encoder) before the channel. For example, we may transmit a filtered version of the system output, or a signal that depends on measurements of the plant states, if available. The only restriction is that the channel input must satisfy the power limit of the Gaussian channel. The flexibility available from channel precompensation does not come without a price: the certainty equivalence and separation properties present with LQG optimal control may no longer be present, thus complicating the design of communication and control strategies.

A special case of the minimum variance communication and control problem described above was treated in [6]. In that paper, it was assumed that the channel input is equal to a constant scalar multiple of the plant output, and that the control input is obtained by passing the channel output through a linear time invariant filter. In the present paper, we consider potential improvements using more general communication and control strategies.

A partial review of previous work on feedback performance over a communication channel follows. The authors of [8] derive a lower bound on a measure of disturbance attenuation that is stated in terms of channel capacity. The authors of [12] study performance limitations imposed by a vector Gaussian channel. The author of [4] relates the problem of feedback stabilization to that of communication over a channel with feedback. The authors of [9], [10] consider performance constraints imposed by noise free, data rate limited channels. The authors of [2] study the joint optimum design of communication and control strategies for feedback over noisy channels.

Because we wish to determine the performance limitations imposed by the channel, we assume that other known sources of limitations are not present: the plant is assumed to be minimum phase and relative degree one, a noiseless measurement of the plant output is available to the encoder, and there is no delay in the feedback path.

The remainder of this paper is outlined as follows. In Section II we note that the variance of the plant output is bounded below by that of the prediction estimation error, and present a control law that achieves this bound with equality. We also review the results of [6] discussed above. In Section III we present a lower bound on the variance of the output estimation error that holds for potentially nonlinear communication and control strategies. By the results of Section II, this lower bound also applies to the mean square plant output. We derive time varying communication and control strategies in Section IV that achieve this lower bound at a specified terminal time. The communication strategy proposed in Section IV assumes that the states of the plant are measurable. We show in Section V that, under appropriate conditions, these state measurements can be replaced by state estimates obtained from a filter that is driven only by the plant output. Communication and control strategies that are defined over an infinite horizon are briefly described in Section VI. The paper concludes in Section VII.

Notation

Denote a random sequence by $\{x_k\}$, and define the subsequence $x^k \triangleq \{x_\ell;\ \ell \leq k\}$. The z-transform of a discrete-time sequence is denoted by an upper case letter, e.g., $X(z) = \mathcal{Z}\{x_k\}$. The open and closed unit disks are denoted by $\mathbb{D}$ and $\bar{\mathbb{D}}$. A rational transfer function $G(z)$ is minimum phase if all its zeros lie in $\bar{\mathbb{D}}$, and is nonminimum phase (NMP) otherwise. The relative degree of $G(z)$ is defined as its excess of poles over zeros.

II. Preliminaries

We consider the linear plant

$$x_{k+1} = Ax_k + Bu_k + Ed_k,\ x_k \in \mathbb{R}^n,\ u_k, d_k \in \mathbb{R}, \quad (1)$$

$$y_k = Cx_k, \quad y_k \in \mathbb{R}, \quad (2)$$

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where d_k is a zero mean Gaussian white noise sequence of variance σ_d^2. Denote the transfer functions from the control and disturbance to the output by $G_u(z) = C\Phi(z)B$ and $G_d(z) = C\Phi(z)E$, respectively, where $\Phi(z) \triangleq (zI - A)^{-1}$. Measurements of the plant output must be processed and communicated to the controller over a Gaussian communication channel

$$r_k = s_k + n_k, \tag{3}$$

where n_k is a zero mean Gaussian white noise sequence of variance σ_n^2, and the channel input satisfies the power limit

$$\mathcal{E}\{s_k^2\} \leq P, \ \forall k. \tag{4}$$

We discuss several control and communication strategies in the paper. The most general of these are time-varying and nonlinear of the form

$$s_k = f_k(y^k), \tag{5}$$

$$u_k = g_k(r^k). \tag{6}$$

In later sections we shall also discuss various *linear* control and communication strategies that, depending on context, may be static, dynamical, or time-varying.

Define the conditional expectation of the plant state x_{k+1} given the history of the channel output r^{k-1} or r^k by

$$\hat{x}_{k|k-1} = \mathcal{E}\{x_k | r^{k-1}\}, \tag{7}$$

$$\hat{x}_{k|k} = \mathcal{E}\{x_k | r^k\}. \tag{8}$$

It is well known that the conditional expectations (7)-(8) are optimal with respect to minimizing the mean square estimation error [7, p. 97]. Denote the associated state estimation errors by

$$\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}, \qquad \tilde{x}_{k|k} = x_k - \hat{x}_{k|k}. \tag{9}$$

Similarly, denote the corresponding output estimates and estimation errors by $\hat{y}_{k|k-1}$, $\hat{y}_{k|k}$, $\tilde{y}_{k|k-1}$, and $\tilde{y}_{k|k-1}$. We shall denote the covariance of $\tilde{x}_{k|k-1}$ by

$$\Sigma_{k|k-1} = \mathcal{E}\{\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^T\}. \tag{10}$$

In particular, the covariance of the initial state estimate is given by $\Sigma_{0|-1}$.

We are interested in the problem of selecting communication and control strategies to minimize the mean square value of the system output, $\mathcal{E}\{y_k^2\}$. Our first result provides a lower bound, and follows immediately from properties of the conditional expectation [7, p. 97].

Lemma II.1 *Consider the plant* (1)-(2)*, channel* (3)*, and the communication and control strategies* (5)-(6)*. Then*

$$\mathcal{E}\{y_k^2\} \geq \mathcal{E}\{\tilde{y}_{k|k-1}^2\}. \tag{11}$$

■

Our next result is an immediate consequence of plant linearity and the assumption that d_k is white noise.

Lemma II.2 *Consider the plant* (1)-(2)*, the channel* (3)*, and the communication and control strategies* (5)-(6)*. Then*

$$\mathcal{E}\{\tilde{y}_{k+1|k}^2\} = \mathcal{E}\{(CA\tilde{x}_{k|k})^2\} + (CE)^2\sigma_d^2. \tag{12}$$

■

Since the channel output sequence r^k may be influenced by the choice of communication strategy, one may think of choosing such a strategy to minimize $\mathcal{E}\{\tilde{y}_{k+1|k}^2\}$. The significance of Lemma II.2 lies in the fact that the problem of minimizing $\mathcal{E}\{\tilde{y}_{k+1|k}^2\}$ is equivalent to that of minimizing $\mathcal{E}\{(CA\tilde{x}_{k|k})^2\}$. We shall return to this observation in Section IV.

Our final preliminary result states conditions under which the lower bound of Lemma II.1 is achieved with equality.

Lemma II.3 *Consider system* (1)-(2)*, the channel* (3)*, and the communication strategy* (5)*. Assume that $CB \neq 0$. Then, under the control law*

$$u_k = -(CB)^{-1}CA\hat{x}_{k|k}, \tag{13}$$

the output and its conditional estimate satisfy

$$y_{k+1} = \tilde{y}_{k+1|k}, \qquad \hat{y}_{k+1|k} = 0. \tag{14}$$

■

As we shall see in the sequel, a communication strategy chosen to minimize the variance of the output estimation error may require that a specific control law be applied. Such a control law may be different than (13), with the result that communication and control strategies chosen to minimize estimation error may differ from those that minimize disturbance response.

To illustrate the preceding remarks, we now review the results of [6], in which it is assumed that the channel input is a scalar multiple of the plant output, $s_k = \lambda y_k$, and the control input is the response of a linear time invariant filter to the channel output, denoted in the transform domain by $U(z) = -K(z)R(z)$, where $K(z)$ is the transfer function of the filter. The scalar λ and the filter $K(z)$ solve the infinite horizon minimum variance problem

$$J^* \triangleq \min_{K,\lambda} \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{E}\{y_k^2\}. \tag{15}$$

Since the system is linear and time invariant, stability implies that all signals will have stationary distributions in the limit, and thus $J^* \triangleq \min_{K,\lambda} \mathcal{E}\{y_k^2\}$. For a fixed value of λ, this problem may be solved using standard LQG methods. Under the assumptions that $G_u(z)$ is minimum phase and relative degree one, the optimal control input is stabilizing and is given by (13). The state estimate $\hat{x}_{k|k}$ satisfies

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + L_p(r_k - \lambda C\hat{x}_{k|k-1}), \tag{16}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_f(r_k - \lambda C\hat{x}_{k|k-1}),$$

where

$$L_p = AL_f, \ L_f = \lambda\Sigma(\lambda)C^T/(\lambda^2 C\Sigma(\lambda)C^T + \sigma_n^2),$$

and $\Sigma(\lambda)$ is the unique positive semidefinite solution to the algebraic Riccati equation

$$\Sigma(\lambda) = A\Sigma(\lambda)A^T - \frac{\lambda^2 A\Sigma(\lambda)C^T C\Sigma(\lambda)A^T}{\lambda^2 C\Sigma(\lambda)C^T + \sigma_n^2} + \sigma_d^2 EE^T.$$

Since the optimal control input is given by (13), Lemma II.3 implies that, $\forall k$,

$$\mathcal{E}\{y_k^2\} = \mathcal{E}\{\tilde{y}_{k|k-1}^2\} = C\Sigma(\lambda)C^T,$$
$$\mathcal{E}\{s_k^2\} = \mathcal{E}\{\tilde{s}_{k|k-1}^2\} = \lambda^2 C\Sigma(\lambda)C^T.$$

To choose a value of λ, we note from [6] that (a) $\mathcal{E}\{\tilde{s}_{k|k-1}^2\}$ is monotonically increasing with λ, becoming unbounded as $\lambda \to \infty$, and (b) $\mathcal{E}\{\tilde{y}_{k|k-1}^2\}$ is monotonically decreasing with λ. Hence $\mathcal{E}\{y_k^2\}$ can be set equal to P/λ^{*2}, where λ^* is the value of the λ for which the channel input satisfies the power constraint (4) with equality.

Example II.4 Consider the system (1)-(2) with

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Assume that the disturbance has variance $\sigma_d^2 = 1$. The optimal value of $\mathcal{E}\{y_k^2\}$ is plotted as a function of the channel SNR in Figure 1. Note that $\mathcal{E}\{y_k^2\}$ becomes unbounded as the SNR approaches the limit required for stabilization [3],

$$P/\sigma_n^2 > \prod_{i=1}^{m} |\phi_i|^2 - 1, \tag{17}$$

where $\phi_i, i = 1, \dots, m$ denote the eigenvalues of A that satisfy $|\phi_i| > 1$. As the channel SNR becomes large, $\mathcal{E}\{y_k^2\}$ approaches the lower limit $(CE)^2\sigma_d^2$ [6]. ■

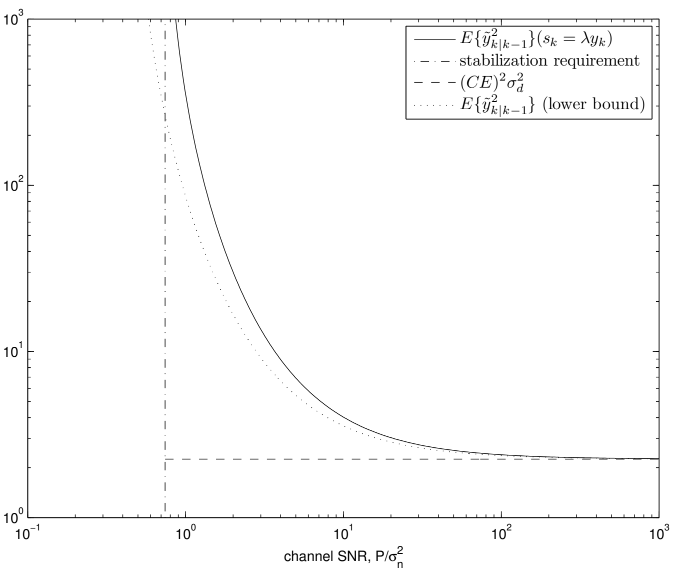

Fig. 1. Estimation error $\mathcal{E}\{\tilde{y}_{k|k-1}^2\}$ for $s_k = \lambda y_k$ and the asymptotic lower bound (20) vs. channel SNR.

Note the implicit dependence of the communication strategy (i.e., choice of λ) upon the control law, which sets $y_k = \tilde{y}_{k|k-1}$, and thus allows λ to be selected based on the solution to an estimation problem. The estimator gains and the estimation error do not themselves depend on the control law. Indeed, using Lemma II.3 implies that the estimation error satisfies the difference equation

$$\tilde{x}_{k+1|k} = (Ax_k + Ed_k) - (A\hat{x}_{k|k-1} + L_p n_k + L_p \lambda \tilde{y}_{k|k-1}),$$

which is diagrammed in Figure 2. The effect of the control law may thus be viewed as one of inverting the plant and introducing a feedback path around the communication channel.

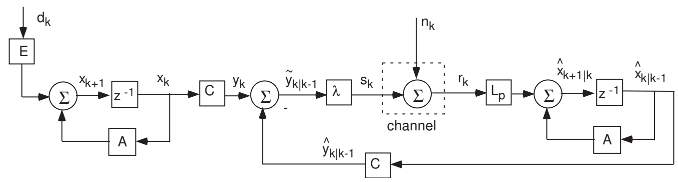

Fig. 2. Optimal control reduces the feedback system to a communication channel with feedback.

III. A Lower Bound on the Disturbance Response

We now present a lower bound on the disturbance response that is valid for the potentially nonlinear communication and control strategies (5)-(6). A proof is found in [5].

Proposition III.1 *Consider the plant* (1)-(2)*, channel* (3)*, and the communication and control scheme* (5)-(6)*. Then the variance of the output estimation error satisfies the lower bound*

$$\mathcal{E}\{\tilde{y}_{k+1|k}^2\} \geq CA^{k+1}\Sigma_{0|-1}A^{(k+1)T}C^T \left(\frac{\sigma_n^2}{P + \sigma_n^2}\right)^{k+1} + \sigma_d^2 \sum_{j=1}^{k+1} (CA^{k+1-j}E)^2 \left(\frac{\sigma_n^2}{P + \sigma_n^2}\right)^{k+1-j}. \tag{18}$$

■

It follows from Lemma II.1 that the right hand side of (18) is also a lower bound for the mean square value of the plant output.

The lower bound (18) holds for any finite value of k, and for any channel signal to noise ratio P/σ_n^2. Our next result, whose proof is straightforward, reveals the asymptotic behavior of (18) for large values of k.

Corollary III.2 *Assume in Proposition III.1 that*

$$P/\sigma_n^2 > \rho^2(A) - 1, \tag{19}$$

where $\rho(A)$ denotes the spectral radius of A. Then, in the limit as $k \to \infty$, the right hand side of (18) *remains bounded, and the estimation error variance satisfies*

$$\lim_{k \to \infty} \mathcal{E}\{\tilde{y}_{k+1|k}^2\} \geq \sigma_d^2 \sum_{\ell=0}^{\infty} (CA^{\ell}E)^2 \left(\frac{\sigma_n^2}{P + \sigma_n^2}\right)^{\ell}. \tag{20}$$

■

A sufficient condition for (19) to be satisfied is that the channel SNR satisfies the lower bound (17) required for stabilization. The fact that the lower bound (20) remains finite for SNRs that are incompatible with closed loop stability is not a contradiction, since the bound need not be tight. We shall, in fact, show that the bound (20) can be achieved at any given *finite* value of k.

Example III.3 Consider again the system in Example II.4. The estimation error computed in that example for the channel input $s_k = \lambda y_k$ is plotted, together with the asymptotic lower bound (20), in Figure 1. The difference between these curves describes the potential to achieve lower estimation error by using communication and control strategies more general than those in Example II.4. ■

IV. ACHIEVABILITY OF LOWER BOUND AT A FIXED TIME

We now exhibit a communication strategy that achieves the lower bound (18) on estimation error for a fixed finite value of k. We then present a control strategy, whose details depend upon those of the communication strategy, that results in $\mathcal{E}\{y_k^2\}$ achieving the same lower bound.

Motivated by the structure of the minimum variance controller at the close of Section II, we consider the estimation scheme depicted in Figure 3, with channel input

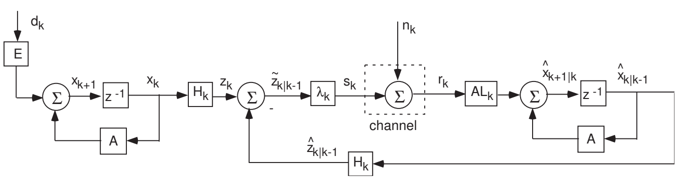

Fig. 3. Estimation over a channel with feedback.

$$s_k = \lambda_k H_k \tilde{x}_{k|k-1}. \qquad (21)$$

The sequence of estimator gains L_k minimizes the mean square state estimation error for given sequences H_k and λ_k, and is given by

$$L_k = \lambda_k \Sigma_{k|k-1} H_k^T / (\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2), \qquad (22)$$

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{\lambda_k^2 A\Sigma_{k|k-1}H_k^T H_k \Sigma_{k|k-1} A^T}{\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2} + \sigma_d^2 EE^T, \qquad (23)$$

with initial condition $\Sigma_{0|-1}$. The output estimation error $\tilde{y}_{k+1|k}$ satisfies

$$C\Sigma_{k+1|k}C^T = CA\Sigma_{k|k-1}A^TC^T - \frac{\lambda_k^2 CA\Sigma_{k|k-1}H_k^T H_k \Sigma_{k|k-1} A^T C^T}{\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2} + \sigma_d^2 (CE)^2. \qquad (24)$$

We shall choose the sequences λ_k and H_k, for $k = 0, \ldots, N$, so that the lower bound (18) on estimation error is achieved at time $k = N$.

We consider first the problem of minimizing the variance of the output estimation error at a fixed time step, given the variance of the state estimation error one time step earlier.

Lemma IV.1 *For a given time $k = N$ and covariance matrix $\Sigma_{N|N-1}$, the choices of λ_N and H_N that minimize $C\Sigma_{N+1|N}C^T$, the output estimation error at the subsequent time step, subject to the power constraint* (4)*, are given by*

$$H_N = CA,$$

and, assuming that $H_N \Sigma_{N|N-1} H_N^T \neq 0$,

$$\lambda_N^2 H_N \Sigma_{N|N-1} H_N^T = P. \qquad (25)$$

Furthermore, with these choices of H_N and λ_N,

$$\mathcal{E}\{\tilde{y}_{N+1|N}^2\} = CA\Sigma_{N|N-1}A^TC^T \left(\frac{\sigma_n^2}{P+\sigma_n^2}\right) + \sigma_d^2 (CE)^2. \qquad (26)$$

Proof: Any positive semidefinite matrix $X \in \mathbb{R}^{n\times n}$, whose rank is equal to m, has a matrix square root $Y \in \mathbb{R}^{n\times m}$ that has rank m and satisfies $X = YY^T$. Denote such a square root for $\Sigma_{N|N-1}$ by Y_N. It follows that

$$CA\Sigma_{N|N-1}H^T = \|CAY_N\| \|H_N Y_N\| \cos\phi_N, \qquad (27)$$

where $\|\cdot\|$ denotes the Euclidean vector norm, and

$$\cos\phi_N \triangleq |CAY_N Y_N^T H_N^T| / (\|CAY_N\| \|H_N Y_N\|).$$

Substituting (27) into (24) and rearranging yields

$$C\Sigma_{N+1|N}C^T = \sigma_d^2(CE)^2 + CA\Sigma_{N|N-1}A^TC^T \left(\frac{\lambda_N^2 H_N \Sigma_{N|N-1} H_N^T \sin^2\phi_N + \sigma_n^2}{\lambda_N^2 H_N \Sigma_{N|N-1} H_N^T + \sigma_n^2}\right).$$

It is straightforward to show that the coefficient of $CA\Sigma_{N|N-1}A^TC^T$ is a monotonically decreasing function of λ_N^2. Hence, for any value of H_N, λ_N should be chosen so that (25) is satisfied. Doing so yields

$$C\Sigma_{N+1|N}C^T = \sigma_d^2(CE)^2 + CA\Sigma_{N|N-1}A^TC^T \left(\frac{P\sin^2\phi_N + \sigma_n^2}{P+\sigma_n^2}\right). \qquad (28)$$

Since we assume that $\Sigma_{N|N-1}$ is given, it follows that H_N should be chosen as a scalar multiple of CA, so that $\phi_N = 0$. ■

Our next result builds on Lemma IV.1 to exhibit choices of H_k and λ_k such that the variance of the output estimation error achieves the lower bound from Proposition III.1.

Proposition IV.2 *Consider the communication channel with feedback depicted in Figure 3. Choose the channel input to satisfy* (21)*, where*

$$H_k = CA^{N+1-k}, \qquad k = 0, \ldots, N. \qquad (29)$$

Assume that $H_k \Sigma_{k|k-1} H_k^T \neq 0$, and choose λ_k such that

$$\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T = P, \qquad k = 0, \ldots, N. \qquad (30)$$

Then the variance of the estimation error at time $k = N+1$ satisfies the lower bound (18) *with equality:*

$$\mathcal{E}\{\tilde{y}^2_{N+1|N}\} = CA^{N+1}\Sigma_{0|-1}A^{(N+1)T}C^T \left(\frac{\sigma_n^2}{P+\sigma_n^2}\right)^{N+1} + \sigma_d^2 \sum_{j=1}^{N+1} (CA^{N+1-j}E)^2 \left(\frac{\sigma_n^2}{P+\sigma_n^2}\right)^{N+1-j}. \quad (31)$$

Proof: We have shown in Lemma IV.1 that the choices of H_N and λ_N given by (29) and (30) minimize the estimation error for a given value of $\Sigma_{N|N-1}$, whose variance is then given by (26). The problem of choosing H_{N-1} and λ_{N-1} to minimize $C\Sigma_{N+1|N}C^T$ thus reduces to that of choosing these variables to minimize $CA\Sigma_{N|N-1}A^TC^T$ for a given value of $\Sigma_{N-1|N-2}$. Computations similar to those in the proof of Lemma IV.1 thus show that the values of H_{N-1} and λ_{N-1} given by (29) and (30) are optimal. Repeating this process yields (31). ■

Corollary IV.3 *Consider the plant* (1)-(2) *and the channel* (3). *Assume that the plant states are measurable, and that $\hat{x}_{0|-1} = 0$. Choose the channel input to satisfy*

$$s_k = \lambda_k H_k x_k, \quad (32)$$

where

$$H_k = CA^{N+1-k}, \qquad k = 0, \ldots, N, \quad (33)$$

and, if $H_k\Sigma_{k|k-1}H_k^T \neq 0$,

$$\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T = P, \qquad k = 0, \ldots, N. \quad (34)$$

Assume that $H_{k+1}B \neq 0,\ k = 0, \ldots, N$, and set

$$u_k = -(H_{k+1}B)^{-1}H_{k+1}A\hat{x}_{k|k}, \qquad k = 0, \ldots, N, \quad (35)$$

with $\hat{x}_{k|k}$ given by

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + AL_k(r_k - \lambda_k H_k \hat{x}_{k|k-1}),$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(r_k - \lambda_k H_k \hat{x}_{k|k-1}),$$

where $L_k = \lambda_k \Sigma_{k|k-1} H_k^T/(P+\sigma_n^2)$, and $\Sigma_{k|k-1}$ is the solution to the Riccati difference equation

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{A\Sigma_{k|k-1}H_k^T H_k \Sigma_{k|k-1}A^T}{H_k\Sigma_{k|k-1}H_k^T}\frac{P}{P+\sigma_n^2} + \sigma_d^2 EE^T.$$

Then at time $k = N+1$, the mean square value of the plant output satisfies $\mathcal{E}\{y^2_{N+1}\} = \mathcal{E}\{\tilde{y}^2_{N+1|N}\}$, where $\mathcal{E}\{\tilde{y}^2_{N+1|N}\}$ is given by (31).

Proof: We first need to show that the channel input sequence (32) for the feedback control system is identical to the channel input sequence (21) for the communication system in Figure 3 for $k = 0, \ldots, N$. Doing so will imply that the estimation error at time $k = N+1$ satisfies (31). We then need to show that $y_{N+1} = \tilde{y}_{N+1}$, which implies that the mean square value of the system output is also equal to (31).

To proceed, we note the assumption $\hat{x}_{0|-1} = 0$ implies that $s_0 = \lambda_0 H_0 x_0 = \lambda_0 H_0 \tilde{x}_0$, and thus (32) and (21) are equal for $k = 0$. At subsequent times, use of the control law (35) implies that $H_k x_k = H_k \tilde{x}_{k|k-1},\ k = 1, \ldots, N+1$, and thus (32) and (21) are equal for $k = 1, \ldots, N$, completing the first step. At time $k = N+1$, $H_{N+1}x_{N+1} = Cx_{N+1} = C\tilde{x}_{N+1|N}$, thus completing the second step. ■

Example IV.4 Consider again the plant model from Example II.4, and let $P = 10$, $\sigma_n^2 = 5$. Suppose we wish to achieve the lower bound (18) on the variance of the estimation error and the mean square value of the plant output at time $k = 20$; to do so, we apply Proposition IV.2 and Corollary IV.3 with $N = 19$. Time histories of the lower bound (18) together with the estimation error that results from applying the communication strategy of Proposition IV.2 are plotted in Figure 4 in the case that $\Sigma_{0|-1} = 0$. Note that the estimation error is equal to the lower bound only at $k = 20$. Corollary IV.3 implies that $\mathcal{E}\{y^2_{20}\} = \mathcal{E}\{\tilde{y}^2_{20|19}\}$. At prior times the mean square plant output is guaranteed to be no smaller than the variance of the estimation error. The poor transient behavior of the estimation error and plant output is feasible given that the communication and control strategies were chosen only to optimize at the final time. ■

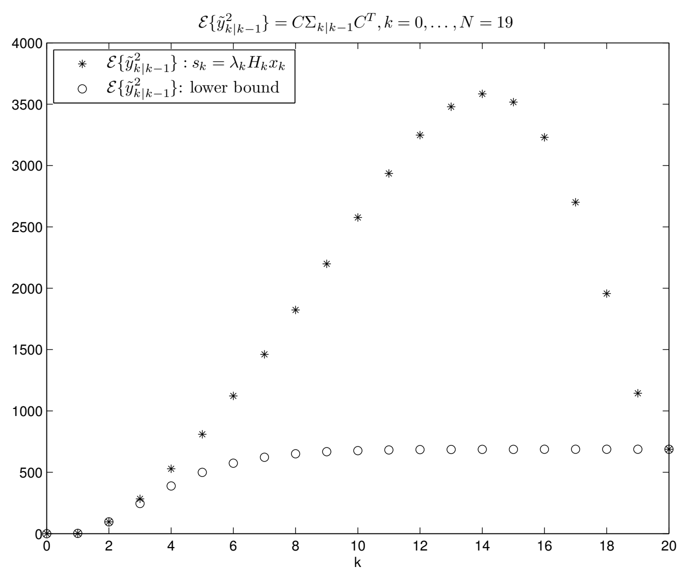

Fig. 4. Estimation error $\mathcal{E}\{\tilde{y}^2_{k|k-1}\} = C\Sigma_{k|k-1}C^T$ for channel input $s_k = \lambda_k H_k \tilde{x}_{k|k-1}$, $H_k = CA^{N+1-k}$, $N = 19$, together with the theoretical lower bound (18).

V. Channel Input a Filtered Version of the Plant Output

We have assumed heretofore that an arbitrary linear combination of the plant states can be measured and transmitted over the channel at each time step. We now show that, under appropriate hypotheses, the results obtained with measurable states also hold when only a noise-free measurement of the system output is available, provided that this output can be processed by a linear filter before transmission over the channel. As we shall see, the filter has the form of a state estimator that *does not* require knowledge of the control input to the plant.

To distinguish state estimates based on the plant output from those obtained by processing the channel output, we denote them by $\hat{x}^0_{k|k-1} = \mathcal{E}\{x_k|u^{k-1}, y^{k-1}\}$ and $\hat{x}^0_{k|k} = \mathcal{E}\{x_k|u^{k-1}, y^k\}$.

Proposition V.1 *Assume that $E = B$, and that $G_u(z)$ is minimum phase and has relative degree one. Then the state estimate $\hat{x}^0_{k|k}$ may be obtained from the recursion*

$$\hat{x}^0_{k|k} = A\hat{x}^0_{k-1|k-1} + L_f(y_k - CA\hat{x}^0_{k-1|k-1}),$$

where $L_f = B(CB)^{-1}$. Furthermore, if $F_x(z)$ denotes the transfer function from the plant output y_k to the state estimate $\hat{x}^0_{k|k}$, then $F_x(z)G(z) = (zI - A)^{-1}B$.

Proof: The form of the optimal gain L_f follows from [11]. The state estimates $\hat{x}^0_{k|k}$ and $\hat{x}^0_{k|k-1}$ satisfy

$$\hat{x}^0_{k+1|k} = A\hat{x}^0_{k|k-1} + Bu_k + AL_f(y_k - C\hat{x}^0_{k|k-1})$$
$$\hat{x}^0_{k|k} = \hat{x}^0_{k|k-1} + L_f(y_k - C\hat{x}^0_{k|k-1}).$$

Combining these equations yields

$$\hat{x}^0_{k+1|k+1} = A\hat{x}^0_{k|k} + Bu_k + L_f(y_{k+1} - CA\hat{x}^0_{k|k} - CBu_k),$$

and the result follows by using the expression for L_f. ■

It follows from Proposition V.1 that *the response of the state estimate $\hat{x}^0_{k|k}$ to disturbance and control inputs is identical to the response of the system state to these signals*. Hence if the estimator is initialized with the plant initial condition, $\hat{x}_{0|-1} = x_0$, then $\hat{x}^0_{k|k} = x_k$, $\forall k$. If the initial plant state is unknown, then $\hat{x}^0_{k|k} \to x_k$ as $k \to \infty$.

VI. INFINITE HORIZON PROBLEMS

The communication and control strategies presented in Section IV are optimal at a single specified time, but are only defined over a finite time interval, and may exhibit poor transient performance during that interval. Consider instead a communication and control strategy

$$s_k = \lambda_k H x_k \quad (36)$$
$$u_k = -(HB)^{-1}HA\hat{x}_{k|k}. \quad (37)$$

If the plant $H\Phi(z)B$ is minimum phase, then this control law is stabilizing, and the channel input is given by $s_k = \lambda_k H\hat{x}_{k|k-1}$. Suppose that λ_k is adjusted so that $\lambda_k^2 H\Sigma_{k|k-1}H^T = P$. Then $\Sigma_{k|k-1}$ is the solution to the "SNR constrained" Riccati difference equation

$$\Sigma_{k+1|k} = A\Sigma_{k|k-1}A^T - \frac{A\Sigma_{k|k-1}H^T H\Sigma_{k|k-1}A^T}{H\Sigma_{k|k-1}H^T}\frac{P}{P+\sigma_n^2} + \sigma_d^2 EE^T, \quad (38)$$

with initial condition $\Sigma_{0|-1}$. The "SNR constrained" algebraic Riccati equation corresponding to (38) is

$$\Sigma = A\Sigma A^T - \frac{A\Sigma H^T H\Sigma A^T}{H\Sigma H^T}\frac{P}{P+\sigma_n^2} + \sigma_d^2 EE^T, \quad (39)$$

The following result, whose proof will be presented elsewhere, allows study of the asymptotic behavior of systems governed by the communication and control laws (36)-(37).

Proposition VI.1 *(a) Assume that (A, E) is stabilizable, (A, H) is detectable, and the ratio P/σ_n^2 satisfies the lower bound* (17). *Then the "SNR constrained" algebraic Riccati equation* (39) *has a unique positive semidefinite solution $\hat{\Sigma}$, and $A - \hat{L}H$ has stable eigenvalues, where $\hat{L} = (A\hat{\Sigma}H^T/H\hat{\Sigma}H^T)(P/(P+\sigma_n^2))$.*

(b) In the limit as $k \to \infty$, the solution to the difference equation (38) *converges to the unique positive semidefinite solution to the algebraic Riccati equation* (39). ■

VII. CONCLUSIONS AND FUTURE DIRECTIONS

We have derived communication and control strategies that are optimal with respect to minimizing the mean square plant output and the variance of the output estimation error at a single specified time. In contrast to the standard LQG problem, the tasks of control and estimation cannot be separated. We have also presented communication and control strategies that are valid over an infinite horizon. These will be suboptimal for any specific time, but may yield better transient properties.

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