MINIMUM VARIANCE ESTIMATES

OF

SIGNAL DERIVATIVES

A PROBLEM IN INSTRUMENT LANDING SYSTEMS

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ABSTRACT

As in the case for the rate of descent of an aircraft, frequently the derivative of a state cannot be observed with sufficient accuracy. The state itself can however; but, as with the radar altimeter, the signal is too noisy for differentiation.

The approach presented in this paper, uses the form of the Kalman filter as an observer for the derivatives of the observed signals or states. The gain matrix for the filter is derived to minimize the variance of the estimate of the derivative of the state, instead of the state itself. The derived gain matrix and the covariance of the (derivative) estimation error are related to those of the Kalman filter through the system's dynamics.

A quantitative evaluation of this method's improvement over taking the derivative of the otimum filter of the state and/or augmenting the state vector, is included. The improvement over the former is shown to increase linearly with the covariance of the state's estimation error (i.e., the advantage of this approach is greater, the worse the signal estimation). For the unusual case of infinite observation time, this error and the improvement disappear as would be expected from Zakai's relation for Wiener filters. Over augmenting the state vector, the improvement uses the lower order, observable states more directly through the systems dynamics.

Simulation of the aircraft's dynamics, used real-time with the digital filter, corroborated the theoretical advantages presented.

INTRODUCTION

As in the case for the rate of descent of an aircraft, frequently the derivatives of a state cannot be observed with sufficient accuracy. The state itself can however, but as with the radar altimeter, the signal is too noisy for differentiation.

Three approaches have been considered to the estimation of state derivatives (observed state, unobserved derivatives). They are:

1. a. Taking the "optimum derivative" of the minimum variance estimate of the state.

   b. Taking the minimum variance estimate of the "optimum derivative" of the state.

2. Using Phase-Locked Loops; they offer noise advantages over the optimum low pass filter for certain noise spectra. This is discussed below.

3. Using the Kalman filter as an observer for the derivative of the observed state.

A straightforward comparison of the error covariances for approaches (1a) and (1b) show them to be equal.[1] Their covariances are used for comparison with approach (3). Regarding the Phase-Locked Loop approach, it is superior to a low-pass filter only for cases of one over f^n type noise.[2,3] Although at this writing, data (with noise) for actual landings (made at Ames Research Center, Moffett Field, California) have not been reduced, it is felt certain that the noise is near white in the frequency range of the system. Thus, approach (2) was rejected for (3).

THE KALMAN OBSERVER ESTIMATOR

The minimum variance, gain matrix is developed using the Kalman filter as an observer for the derivative of the state. This observer is different from those of Kalman[4] and Luenberger[5] in that the measurement-noise-free case is not required.

The observable $\overline{y}$ is related to the states by

$$\overline{y}(t) = C(t)\,\overline{x}(t) + \overline{q}(t),$$

where $\overline{q}(t)$ is observation noise,

and the system equation (the dynamics) is

$$\dot{\overline{x}}(t) = A(t)\,\overline{x}(t) + B(t)\,\overline{u}(t)$$

where $\overline{u}(t)$ is the control.

All of the variables may be time varying; however, the time auguments will be left off in subsequent equations.

The Kalman, state estimation equation for such a system is[6]

$$\dot{\hat{x}} = A\,\hat{x} + K[\overline{y} - C\,\hat{x}] \qquad \text{(I)}$$

On replacing K by H + $\varepsilon\eta$ the gain matrix H can be determined[1] which minimizes the covariance of the error, defined by

$$\overline{e} = \dot{\overline{x}} - \hat{\dot{\overline{x}}}$$

Note in the Kalman equation H, which replaces K, is derived, so as to minimize the covariance of the unobserved derivative error rather than that of the state (completely observable in the proposed system). Thus the estimate of the derivative is defined

$$\hat{\dot{\overline{x}}} = A\overline{x} + H(\overline{y} - C\overline{x}) \qquad \text{(II)}$$

or

$$\overline{e} = \dot{\overline{x}} - A\overline{x} - H(\overline{y} - C\overline{x})$$

$$= \overline{w} - H\overline{v}$$

where $\bar{w}$ and $\bar{v}$ are defined by comparing the last two equations.

The expression for H has been determined three ways. The first was by using Calculus of Variation,[1] the second, by taking the derivative of the covariance of $\bar{e}$ with respect to the matrix H[1]. The third and simplest is presented below. It has the added advantage of giving the expression for the minimum error covariance (minimum trace).

Using the expressions above and the apostrophe to indicate a transpose, the covariance of the error is

$$\mathrm{Cov}(\bar{e}) = E(\bar{e}\,\bar{e}') = E(\bar{w} - H\bar{v})(\bar{w} - H\bar{v})'$$

$$= \Phi_w - H\Phi_{vw} - \Phi_{wv}H' + H\Phi_v H'.$$

where E(w) is the ensemble average operation. Since real stationary ergodic processes are considered, the covariance $\Phi_w = E(w\,w')$, for w zero mean.

Analogous to "completing the square" for H, a matrix indentity,[7] on noting $\Phi_{xy} = \Phi'_{yx}$, gives

$$\mathrm{Cov}(\bar{e}) = \Phi_w + (H - \Phi_{wv}\Phi_v^{-1})\Phi_v(H - \Phi_{wv}\Phi_v^{-1})' - \Phi_{wv}\Phi_v^{-1}\Phi_{vw} \quad (1)$$

Since all terms are nonnegative-definite matrices, the minimum covariance is

$$\mathrm{Cov}(\bar{e})_{min} = \Phi_w - \Phi_{wv}\Phi_v^{-1}\,\Phi_{vw}$$

for the optimum gain matrix,

$$H = \Phi_{wv}\Phi_v^{-1}$$

In the appendix, the following expressions are developed:

$$\Phi_w = ARA' \qquad \Phi_{wv} = ARC'$$

$$\Phi_{vw} = CRA' \qquad \Phi_v = CRC' + Q$$

defining $\tilde{\tilde{x}} = (\bar{x} - \hat{\bar{x}})$, the state estimation error for the Kalman filter, and $R = E(\tilde{\tilde{x}}\ \tilde{\tilde{x}}')$, $Q = E(\bar{q}\ \bar{q}')$, the covariances of the state estimation error and observation noise, respectively.

Their use gives,

$$Cov(\bar{e})_{min} = ARA' - ARC'(CRC' + Q)^{-1}CRA' \quad (2)$$

and

$$H = ARC'(CRC' + Q)^{-1} \quad (3)$$

Since the derivatives of the system's states are related to its states by A, the coefficient matrix of the system's dynamics, the minimum error covariance for the estimate of the derivatives could be no less than the minimum error covariance of the states, pre and post multiplied by A.[8] Equation (2) is recognized to be just that minimum value (i.e., the error covariance for the optimum (Kalman) estimate of the state so multiplied by A). As 20-20 hindsight would suggest Equation (3) is just AK where K is the optimum filter for state estimation.[9]

COMPARISON OF ERROR COVARIANCES

The Kalman-observer estimator (KOE), developed for minimum variance estimate of state derivatives, will now be compared with the results of (1), taking the derivative of the optimum estimate of the states (DFC for derivative-filter combination) and (2), augmenting the state vector to include its derivatives (ASF).

Since the Kalman filter is linear, the derivative of the output must equal the input to its intergrator, or just Equation (I). It is noted that it has the same form as Equation (II) for the KOE with K replacing H. Thus the error covariance is the same as Equation (1) with H replaced by K. Substituting for K (the terms after A in Equation (3)) and the Φ's.*

$$\mathrm{Cov}(\bar{e})_{DFC} = ARA' - ARC'(CRC' + Q)^{-1}CRA'$$
$$+ (RC' - ARC')[(RC' - ARC')(CRC' + Q)^{-1}]'$$

Comparison of this equation with (2) shows it greater than (2) by the terms on the second line. After manipulation using the identities for transposes and inverses of products.

$$\Delta_{KOE/DFC} = (I - A)RC'(CRC' + Q)^{-T}CR(I - A)'$$

If measurements were near perfect ($Q \simeq 0$)

$$\Delta_{KOE/DFC} = (I - A)R(I - A).$$

This expression shows the advantage of the KOE increases with greater estimation error covariance, R.

SIMULATION

The filter, operating on a typical aircraft during flight, was modeled using CSMP/360 supplemented with several user subprograms. Four aircraft states were considered, that of the pitch angle, the altitude, and the first derivative of each. Both DFC and the KOE filters were concurrently employed, and error analysis performed using the resulting data. The simulation may be grouped into two major divisions, that of system dynamics and that of the operation and updating of each filter.

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*As seen in the appendix, the Φ's are independent of the form of H, thus their expressions are invariant on changing H to K.

Before "flying" the system, the discrete state transition matrix was calculated for use in the updating of the filters. A time increment of 0.05 seconds was used. (However, the time increment parameter could be varied at will.) Also at this point the Kalman filter and the derivative filter were developed for the initial time state.

During "flight," the dynamics of the system were modelled and disturbed by various forms of Gaussian noise. The pitch angle and the altitude of the aircraft were observed, and to these observations proper Gaussian noise was added. The noise produces observation errors commensurate with the real instrument.* The corrupted system was filtered with both filters. Every 0.05 seconds of the flight both filters and their respective covariances were updated.

Pertinent simulation information is included in the following:

1. The state transition matrix was produced using the first seven terms of an exponential power series expansion.

2. Gaussian noise was produced using a pseudorandom, congruential noise generator, from which noise of uniform density was obtained. This noise was then sampled twelve times and averaged to produce normal noise. (Central Limit Theorem)

3. The integral equations involved in the simulation, both in the system dynamics and the filter calculations, were solved using a fourth order Runge-Kutta integration technique. A variable step version was employed.

4. As stated earlier Equation (3) for H is just AK. On using it, $\Delta_{KOE/DFC}$ becomes $\Delta_{KOE/DFC} = (RC' - ARC')(K - H)'$

The computer flowcharts are appended (pages A-4, A-5, and A-6).

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*The radar altimeter and pitch gyro, used on the test flight at Ames Research Center have variances of 25 and 5×10^{-6}, respectively.

CONCLUSIONS

For continuous systems (such as aircraft in flight, possibly observed at discrete times) the diagonal terms of A, corresponding to the augmented states (as to include their derivatives) are zero.[9] Thus the KOE has the special advantage of using the system's dynamics associated with the lower order states.

For discrete cases, the diagonal terms of A are unity for unobserved states,[9] showing that when there is no knowledge of the system's dynamics (as in the stock market for most people), the rate of change can be estimated only by pass errors as in ASF.

The above analysis has been based on the covariance of the estimation and measurement errors. Since they are not affected by the input control, its consideration has been obviated.

In the section on "Comparision of Error Covariances," the advantage of the KOE over the DFC, is strongly dependent on the A matrix. Equation II, the derivative estimate by definition, is recognized as Kalman's optimal estimate of the state[9] multiplied by A. The dependence is on the form of A rather than its accuracy, for inaccuracies can be corrected by observation. The form is controlled by the type of the dynamic process. In the simulation above, the A matrix has all zeros in the fourth column. This condition shows that the associated state (altitude) is not in a Markov process of any order,[10] as the model is structured. For this reason, the covariance of the rate of change of altitude, is not affected by the observered data from any state; thus, the advantage of the KOE is minimal. The simulation showed it is less than one percent. This could have been due to such as round-off errors; however, the advantage was always positive for the entire run (20 sec.).

The A matrix shows that the pitch angle, while not in a simple (first order) Markov process, is in one of multiple order.[10] The results of the simulation give improvements of the KOE as expected. The advantage initially exceeds one hundred percent but decreases rapidly (e.g., fourteen percent in .25 sec.) as expected by its dependences on R, the covariance of the state estimation error. Recall, both filters cause R to decrease. It is felt that the KOE's faster response from zero time, is of paramount importances in systems such as aircraft.

In continued work, an aircraft model will be structured so that the altitude will be in a multiple order Markov process. This will be approached in using the altitude and its first three derivatives as a four state vector. Concurrent with this work optimum mixing of information from other sources as rate sensitive gyros will be pursued.

As noted above, both filters cause R to decrease. In steady-state as on occassions of no perturbations, R could be small compared to Q, the covariance of the measurement error. On neglecting R relative to Q, the advantage of KOE would be

$$\Delta_{KOE/DFC} = (I - A)\ RC'Q^{-T}CR(I - A)',$$

where Q to the minus T is used to indicate the inverse of Q transpose. This expression shows how the improvement increases with smaller Q (i.e., better measuring instruments).

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Appendix

Reduction of Covariance Matrices

Recall: System equations

$$\dot{\bar{x}} = A\,\bar{x}(t) + B\,\bar{u}(t)$$

$$\bar{y} = C\,\bar{x}(t) + \bar{q}(t)$$

Hencefore the t will be dropped.

Derivative estimate

$$\hat{\dot{\bar{x}}} = \dot{\hat{\bar{x}}} = A\,\hat{\bar{x}} + H(\bar{y} - C\hat{\bar{x}})$$

Error vectors,

$$\bar{w} = \dot{\bar{x}} - A\hat{\bar{x}} \qquad \tilde{x} = \bar{x} - \hat{\bar{x}}$$

$$\bar{v} = \bar{y} - C\hat{\bar{x}} \qquad R = E(\tilde{\bar{x}}\tilde{\bar{x}}')$$

On using the system equations and the error vectors,

$$\dot{\hat{\bar{x}}} = A\hat{\bar{x}} + H\bar{v}$$

or

$$A\hat{\bar{x}} = \dot{\hat{\bar{x}}} - H\bar{v}$$

and

$$\bar{w} = \dot{\bar{x}} - \dot{\hat{\bar{x}}} + H\bar{v}$$

$$= \dot{\tilde{x}} + H\bar{v}$$

Again by the system's (observation equation),

$$\bar{v} = C\bar{x} + \bar{q} - C\hat{\bar{x}}$$

$$= C\tilde{\bar{x}} + \bar{q}$$

$$\phi_{\bar{v}} = E(C\tilde{\bar{x}} + \bar{q})(C\tilde{\bar{x}} + \bar{q})'$$

$$= CRC' + Q \qquad \text{(A-1)}$$

where from above $R = \phi_{\tilde{\bar{x}}} = E(\tilde{\bar{x}}\tilde{\bar{x}}')$ and $E(\bar{q}\bar{q}') = Q$, and the noise is considered uncorrelated with either the state or its estimate.

$$\phi_W = E(\dot{\tilde{\bar{x}}} + H\bar{v})(\dot{\tilde{\bar{x}}} + H\bar{v})'$$

$$= \phi_{\dot{\tilde{\bar{x}}}} + H\phi_{\bar{v}\dot{\tilde{\bar{x}}}} + \phi_{\dot{\tilde{\bar{x}}}\bar{v}}H' + H\phi_{\bar{v}}H \qquad \text{(A-2)}$$

where on using the system equation*

$$\dot{\bar{x}} = A\bar{x}$$

and the derivative estimate

$$\dot{\hat{\bar{x}}} = A\hat{\bar{x}} + H(\bar{y} - C\hat{\bar{x}})$$

$$= A\hat{\bar{x}} + H(C\tilde{\bar{x}} + \bar{q})$$

$$\dot{\tilde{\bar{x}}} = \dot{\bar{x}} - \dot{\hat{\bar{x}}} = A\bar{x} - A\hat{\bar{x}} - H(C\tilde{\bar{x}} + \bar{q})$$

$$= (A - HC)\tilde{\bar{x}} - H\bar{q}$$

Thus

$$\phi_{\dot{\tilde{\bar{x}}}} = (A - HC)R(A - HC)' + HQH'$$

*The control $B\bar{u}$ is neglected for it does not affect $\tilde{\bar{x}}$.

Now

$$\phi_{\bar{v}\dot{\tilde{x}}} = E(C\tilde{\tilde{x}} + \bar{q})(A\bar{x} - A\hat{\bar{x}} - H\bar{v})'$$

$$= E(C\tilde{\tilde{x}} + \bar{q})(\tilde{\tilde{x}}' A' - \bar{v}'H')$$

$$= CRA' - \phi_v H'$$

$$= CRA' - CRC'H' - QH' \qquad \text{(A-3)}$$

Thus recognizing $\phi_{\dot{\tilde{x}}\bar{v}}$ as the transpose of $\phi_{\bar{v}\dot{\tilde{x}}}$, substituting from above and on using (A-1), (A-2) becomes

$$\phi_w = (A-HC) R (A-HC)' + HQH' + HCRA'$$

$$- HCRC'H' - HQH' + ARC'H' - HCRC'H'$$

$$- HQH' + HCRC'H + HQH'$$

After multiplying out and cancelling,

$$\phi_w = ARA' \qquad \text{(A-4)}$$

Lastly,

$$\phi_{wv} = E(\dot{\tilde{\tilde{x}}} + H\bar{v})\bar{v}'$$

$$= \phi_{\dot{\tilde{x}}\bar{v}} + H\phi_{\bar{v}}$$

On substituting from (A-1) and using the transpose of (A-3)

$$\phi_{wv} = ARC' - HCRC' - HQ + HCRC' + HQ$$

$$= ARC' \qquad \text{(A-5)}$$

FLOWCHART DATA

1. State variables

$x' = (\dot{e}, e, \dot{h}, h)$; where e : pitch angle
h : altitude

2. A matrix

$$A = \begin{bmatrix} -0.6 & -0.76 & 0.003 & 0.0 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.025 & -0.4 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

3. C matrix

$$C = \begin{bmatrix} 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 1. \end{bmatrix}$$

4. R matrix (at t = 0.0 sec.)

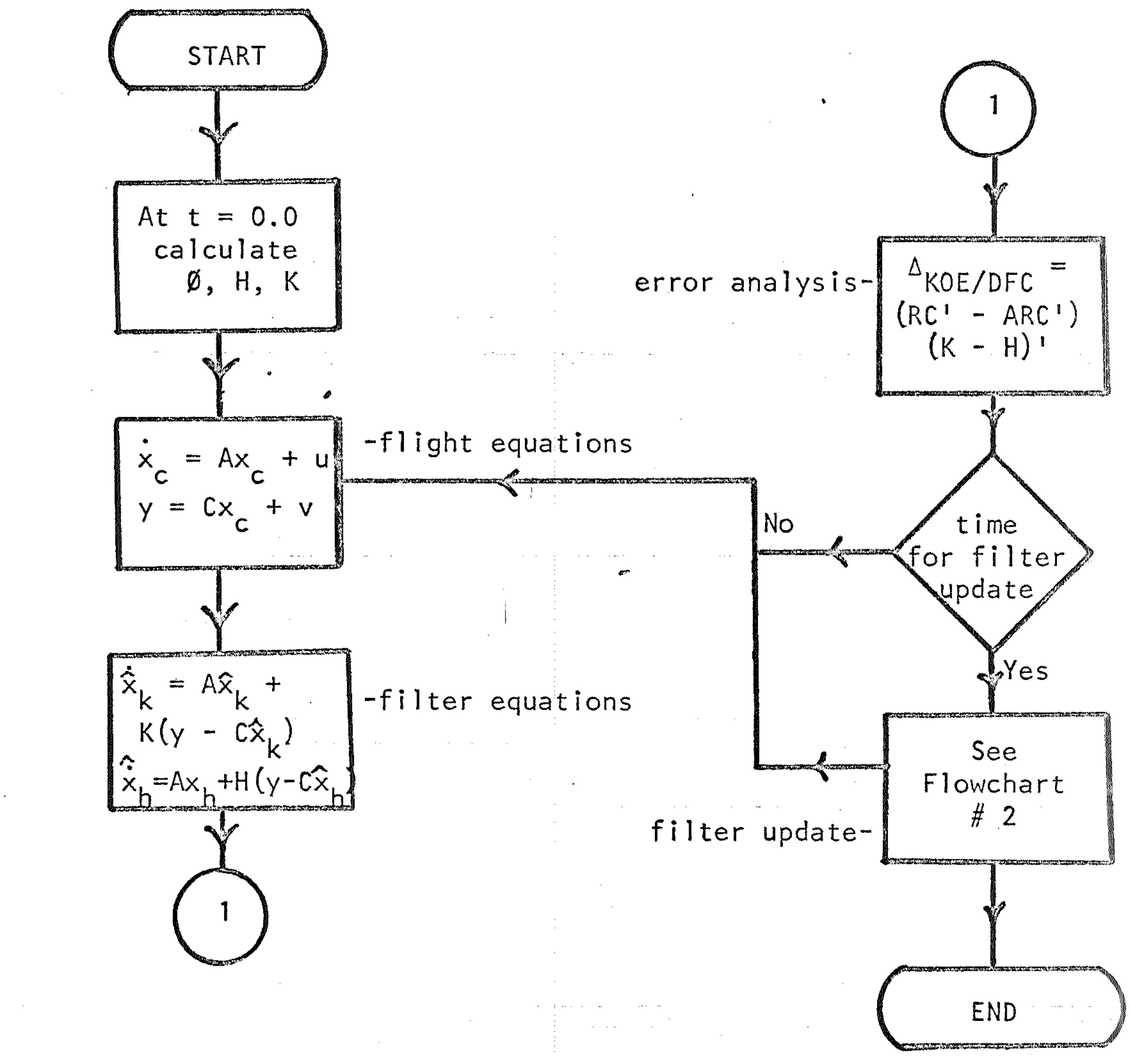
$$R = \begin{bmatrix} 1.E-6 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.E-4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.25 & 0.0 \\ 0.0 & 0.0 & 0.0 & 25.0 \end{bmatrix}$$

5. Q matrix

$$Q = \begin{bmatrix} 5.E-6 & 0.0 \\ 0.0 & 25.0 \end{bmatrix}$$

FLOWCHART # 1

System Simulation



FLOWCHART # 2

Filter Update