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Minkowski Decomposition of Associahedra and Related Combinatorics

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Abstract Realisations of associahedra with linear non-isomorphic normal fans can be obtained by alteration of the right-hand sides of the facet-defining inequalities from a classical permutahedron. These polytopes can be expressed as Minkowski sums and differences of dilated faces of a standard simplex as described by Ardila et al. (Discret Comput Geom, 43:841–854, 2010). The coefficients y_I of such a Minkowski decomposition can be computed by Möbius inversion if tight right-hand sides z_I are known not just for the facet-defining inequalities of the associahedron but also for all inequalities of the permutahedron that are redundant for the associahedron. We show for certain families of these associahedra:

(1) How to compute the tight value z_I for any inequality that is redundant for an associahedron but facet-defining for the classical permutahedron. More precisely, each value z_I is described in terms of tight values z_J of facet-defining inequalities of the corresponding associahedron determined by combinatorial properties of I.

(2) The computation of the values y_I of Ardila, Benedetti & Doker can be significantly simplified and depends on at most four values $z_{a(I)}$, $z_{b(I)}$, $z_{c(I)}$ and $z_{d(I)}$.

(3) The four indices $a(I)$, $b(I)$, $c(I)$ and $d(I)$ are determined by the geometry of the normal fan of the associahedron and are described combinatorially.

(4) A combinatorial interpretation of the values y_I using a labeled n-gon. This result is inspired from similar interpretations for vertex coordinates originally described by Loday and well-known interpretations for the z_I-values of facet-defining inequalities.

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1 Introduction

Postnikov defined in [18] generalised permutahedra as a subfamily of all convex polytopes that have the following H-description:

$$P_n(\{z_I\}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \sum_{i\in[n]} x_i = z_{[n]} \text{ and } \sum_{i\in I} x_i \geq z_I \text{ for } \varnothing \subset I \subset [n] \right\},$$

where $[n]$ denotes the set $\{1, 2, \dots, n\}$. The classical $(n-1)$-dimensional permutahedron, as described for example by Ziegler, [29], corresponds to $z_I = \frac{|I|(|I|+1)}{2}$ for $\varnothing \subset I \subseteq [n]$ (we distinguish between $\subset$ and $\subseteq$!). Obviously, some of the above inequalities may be redundant for $P_n(\{z_I\})$ and, unless the value z_I is tight, sufficiently small increases and decreases of z_I for a redundant inequality do not change the combinatorial type of $P_n(\{z_I\})$. Although the encoding by all values z_I is not efficient, Proposition 1.2 below gives a good reason to specify tight values z_I for all $I \subseteq [n]$. The subfamily of generalised permutahedra is now characterised by the additional requirement that $P_n(\{z_I\})$ is an element of the deformation cone of the classical permutahedron. Equivalently, this means that the normal fan of the generalised permutahedron is a coarsening of the normal fan of the classical permutahedron or that no facet-defining hyperplane of the permutahedron is moved *past any vertices*, compare Postnikov et al. [19]. This fine distinction and additional condition is easily overlooked but essential. For example, Proposition 1.2 does not hold for arbitrary polytopes $P_n(\{z_I\})$, we illustrate this by a simple example in Sect. 5. Fundamental examples of generalised permutahedra are dilations of the standard simplex $\Delta_n = \operatorname{conv}\{e_1, e_2, \dots, e_n\}$ where e_i denotes the ith standard basis vector of $\mathbb{R}^n$.

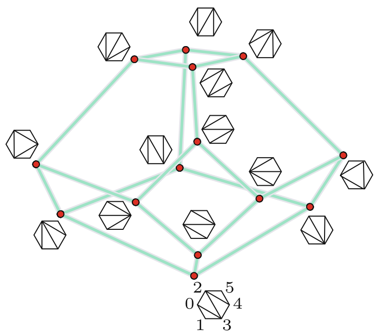
For any two polytopes P and Q, the Minkowski sum $P + Q$ is defined as

$$\{p + q \mid p \in P,\ q \in Q\}.$$

In contrast, we define the Minkowski difference $P - Q$ of P and Q only if there is a polytope R such that $P = Q + R$. For more details on Minkowski differences we refer to [23]. We are interested in decompositions of generalised permutahedra into Minkowski sums and differences of dilated faces $\lambda_I \Delta_I$ of the $(n-1)$-dimensional standard simplex Δ_n, where the faces Δ_I of Δ_n are given by $\operatorname{conv}\{e_i\}_{i\in I}$ for $I \subseteq [n]$. If a polytope P is the Minkowski sum and difference of dilated faces of Δ_n, we say that P has a Minkowski decomposition into faces of the standard simplex. The following two results are known key observations.

Lemma 1.1 ([1, Lemma 2.1]). $P_n(\{z_I\}) + P_n(\{z'_I\}) = P_n(\{z_I + z'_I\})$.

If we consider the function $I \longmapsto z_I$ that assigns every subset of $[n]$ the corresponding tight value z_I of $P_n(\{z_I\})$, then the Möbius inverse of this function assigns to I the coefficient y_I of a Minkowski decomposition of $P_n(\{z_I\})$ into faces of the standard simplex:



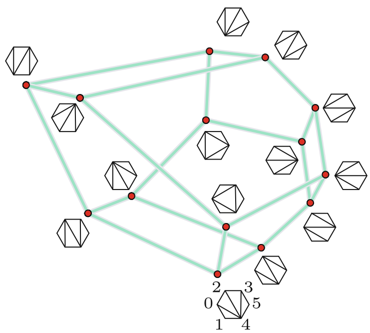


Fig. 1 Two 3-dimensional associahedra $\mathsf{As}_3^c = P_4(\{\tilde{z}_I^c\})$ with vertex coordinates computed for differently chosen Coxeter elements according to [11]. The different Coxeter elements are encoded by different labelings of hexagons as indicated. The images shown are isometric copies of 3-polytopes contained in the affine hyperplane $x_1 + x_2 + x_3 + x_4 = 10$ of $\mathbb{R}^4$

Proposition 1.2 ([1, Proposition 2.3]) *Every generalised permutahedron $P_n(\{z_I\})$ can be written uniquely as a Minkowski sum and difference of faces of Δ_n:*

$$P_n(\{z_I\}) = \sum_{I\subseteq[n]} y_I \Delta_I,$$

where $y_I = \sum_{J\subseteq I}(-1)^{|I\setminus J|} z_J$ for each $I \subseteq [n]$.

In particular, we also have $z_I = \sum_{J\subseteq I} y_J$. A basic example is the classical permutahdron: it is known to be a zonotope and it is the Minkowski sum of the edges and vertices of Δ_n. The reader is invited to check that the corresponding z_I-values obtained by this formula yield precisely the right-hand sides mentioned earlier.

We will study Minkowski decompositions of generalised permutahedra that have the same normal fan as As_{n-1}^c. Two 3-dimensional examples of As_3^c (with distinct normal fans) are shown in Fig. 1, we describe their construction in detail in Sect. 2. The normal fans of these polytopes are determined by a Coxeter element c of the symmetric group, but we will avoid the explicit use of Coxeter elements and use a partition $\mathsf{D}_c \sqcup \mathsf{U}_c$ of $[n]$ induced by c instead. The main result is that the relation between z_I- and y_I-coordinates of Proposition 1.2 simplifies significantly: each y_I can be computed from at most four values z_J which depend on I and the normal fan of the polytope (or, equivalently, the Coxeter element c or the corresponding partition of $[n]$). Moreover, we give an explicit combinatorial description how to determine these terms z_J. If we further restrict to the realisations As_{n-1}^c as described by Hohlweg and Lange in [11], we show that the coefficients y_I can be described as signed product of path-lengths of a labeled polygon.

We now give examples of Minkowski decompositions of realisations of 2-dimensional associahedra As_2^{c1} and As_2^{c2} which are contained in the affine hyperplane $x_1 + x_2 + x_3 = 6$ of $\mathbb{R}^3$. We immediately see that the Minkowski decompositions are distinct since the set of coefficients y_I differs. These associahedra are pentagons that are obtained from the classical permutahedron by making the inequality $x_1 + x_3 \geq 3$

(respectively $x_2 \geq 1$) redundant. They are described by the following complete set of tight z_I-values z_I^{c1} and z_I^{c2}:

I	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	{1, 2, 3}
z_I^{c1}	1	1	1	3	2	3	6
z_I^{c2}	1	0	1	3	3	3	6

Using Proposition 1.2, the reader may verify that

$$\mathsf{As}_2^{c1} = 1\Delta_{\{1\}} + 1\Delta_{\{2\}} + 1\Delta_{\{3\}} + 1\Delta_{\{1,2\}} + 0\Delta_{\{1,3\}} + 1\Delta_{\{2,3\}} + 1\Delta_{\{1,2,3\}}$$

and

$$\mathsf{As}_2^{c2} = 1\Delta_{\{1\}} + 0\Delta_{\{2\}} + 1\Delta_{\{3\}} + 2\Delta_{\{1,2\}} + 1\Delta_{\{1,3\}} + 2\Delta_{\{2,3\}} + (-1)\Delta_{\{1,2,3\}}.$$

Illustrations of these decompositions are given in Figs. 2 and 3.

We could stop here and be fascinated how the Möbius inversion relates the description by half spaces to the Minkowski decomposition. But we go beyond this alternating sum description for y_I and significantly simplify the formula for each y_I in Therom 4.2. In fact, each y_I can be expressed as an alternating sum of at most four non-zero values $z_{a(I)}$, $z_{b(I)}$, $z_{c(I)}$ and $z_{d(I)}$ which are tight right-hand sides for certain facet-defining inequalities as specified in the theorem. In other words, we extract combinatorial core data for the Möbius inversion of the function z_I and answer the question which subsets J of I are essential to compute y_I if the associahedron's normal fan is the normal fan of As_{n-1}^c. Figure 9 illustrates how Theorem 4.2 can be used to compute the coefficients y_I for one of the two examples shown in Fig. 1. If the associahedron coincides with some As_{n-1}^c of Hohlweg and Lange [11], Theorem 4.3 states a purely combinatorial interpretation of the values y_I. To illustrate this theorem, we recompute y_I for As_2^{c1} and As_2^{c2} in Examples 4.6 and 4.7.

The outline of the paper is as follows. Section 2 summarises necessary known facts about As_{n-1}^c and indicates some occurrences of the realisations considered here in the mathematical literature. In Sect. 3 we introduce the notion of an up and down interval

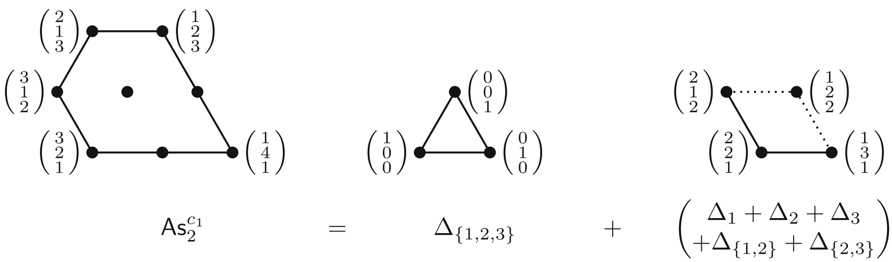


Fig. 2 The Minkowski decomposition of the 2-dimensional assiciahedron As_2^{c1} into faces of the standard simplex is actually a Minkowski sum of some faces of a standard simplex

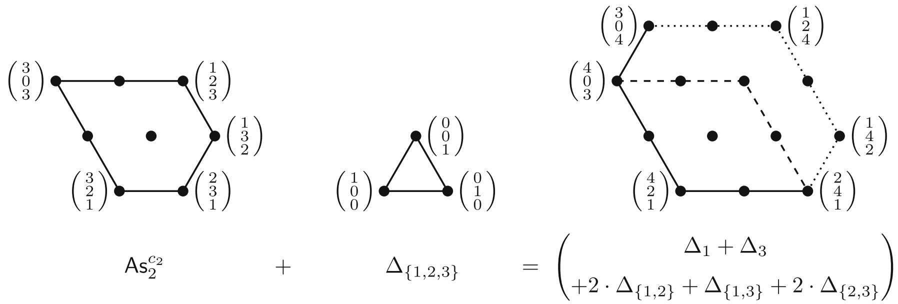


Fig. 3 The Minkowski decomposition of As_2^{c2} into dilated faces of $\Delta_{[n]}$

decomposition for subsets $I \subseteq [n]$. This decomposition depends on the choice of a Coxeter element c (or equivalently on a partition of $[n]$ induced by c) and is essential to prove Proposition 3.8. This proposition gives a combinatorial characterisation of all tight values z_I for As_{n-1}^c needed to evaluate y_I using Proposition 1.2. The main results, Theorems 4.2 and 4.3, are then stated in Sect. 4. The proof of Theorem 4.2 is long and convoluted and deferred to Sects. 6 and 7, while Theorem 4.3 is proved under the assumption of Theorem 4.2 in Sect. 4. To show that Proposition 1.2 and Theorem 4.2 do not hold for polytopes $P_n(\{z_I\})$ that are not contained in the deformation cone of the classical permutahedron, we briefly study a realisation of a 2-dimensional cyclohedron in Sect. 5.

About the same time as some of these results were achieved, Pilaud and Santos showed that the associahedra As_{n-1}^c are examples of brick polytopes [16,17]. One of their results is that any brick polytope can be expressed as a Minkowski sum of other brick polytopes. As a consequence, we have two Minkowski decompositions of As_{n-1}^c that are extremal in the following sense. The first decomposition of As_{n-1}^c has a relatively complicated structure with respect to the coefficients y_I (possibly negative numbers) but is very simple with respect to the polytopes used (faces of a standard simplex). On the other hand, the second decomposition of As_{n-1}^c has a simple structure in terms its coefficients (they are either 0 or 1) but is more complicated with respect to the polytopes used (brick polytopes). At the time of writing, the exact relationship of these two decompositions is not properly understood and remains a joint project of Pilaud with the author.

2 Associahedra as Generalised Permutahedra

Associahedra form a family of combinatorially equivalent polytopes and can be realised as generalised permutahedra. Since the combinatorics of a polytope is encoded in its face lattice, we define an associahedron as a polytope with a face lattice that is isomorphic to the lattice of sets of non-crossing proper diagonals of a convex and plane $(n+2)$-gon Q ordered by reversed inclusion.[1] This description immediately tells us

[1] A proper diagonal is a line segment connecting a pair of vertices of Q whose relative interior is contained in the interior of Q. A non-proper diagonal is a diagonal that connects vertices adjacent in ∂Q and a degenerate diagonal is a diagonal where the end-points are equal.

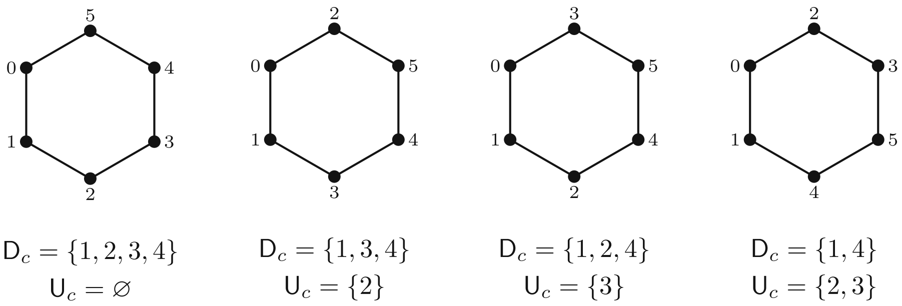


Fig. 4 The four possible c-labelings Q_c of a hexagon

that the set of k-faces is in bijection to the set of triangulations of Q with k proper diagonals removed. In particular, vertices correspond to triangulations and facets correspond to proper diagonals. Since associahedra turn out to be simple polytopes, a result of Blind and Mani-Levitska with an elegant proof due to Kalai, [3,13], guarantees that the face lattice is already determined by the 1-skeleton, so it suffices to specify the vertex-edge graph to determine the combinatorics of the face-lattice. This graph is known as the flip graph of triangulations of Q. In 2004, Loday published a beautiful combinatorial description for the vertex coordinates of associahedra constructed earlier by Shnider and Sternberg, Shnider and Stasheff [24,25,14]. Loday's description is in terms of labeled binary trees dual to the triangulations of Q. The construction of Shnider, Sternberg and Stasheff as well as Loday's vertex description was subsequently generalised by Hohlweg and Lange [11]. The latter construction explicitly describes realisations As^c_{n-1} of $(n-1)$-dimensional associahedra and exhibits them as generalised permutahedra. The construction depends on the choice of a Coxeter element c of the symmetric group Σ_n on n elements.

We now outline the construction of As^c_{n-1}. Although we use Coxeter elements in our notation to distinguish between different realisations, we do not explicitly use Coxeter elements. It is known that the Coxeter elements are in bijection to the certain partitions $\mathsf{D}_c \sqcup \mathsf{U}_c$ of $[n]$. We will use these partitions to obtain labelings Q_c of Q and refer to D_c as *down set* and to U_c as *up set*. The partitions satisfy

$$\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\} \quad \text{and} \quad \mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\},$$

so $n = \ell + m$, $|\mathsf{D}_c| = \ell \geq 2$ and $|\mathsf{U}_c| = m$. We now obtain the c-labeling Q_c of Q with label set $[n+1] \cup \{0\}$ as follows. Pick two vertices of Q which are the end-points of a path with $\ell + 2$ vertices on the boundary of Q, label the vertices of this path counter-clockwise increasing using the label set $\overline{\mathsf{D}}_c := \mathsf{D}_c \cup \{0, n+1\}$ and label the remaining path clockwise increasing using the label set U_c. The labeling Q_c has the property that the label set D_c is always on the right-hand side of the diagonal $\{0, n+1\}$ oriented from 0 to $n+1$. To illustrate these c-labelings Q_c, observe that there are four distinct partitions $\mathsf{D}_c \sqcup \mathsf{U}_c$ for $n = 4$ which yield the four labeled hexagons Q_c shown in Fig. 4. We derive values z_I for some subsets $I \subset [n]$ using oriented proper diagonals of Q_c as follows. Orient each proper diagonal δ from the smaller to the larger labeled end-point of δ, associate to δ the set R_δ that consists of all labels on

Table 1 R_δ and $\tilde{z}_I^c$ associated to the proper diagonals δ of the labelled hexagon for the associahedron on the left of Fig. 1 ($\mathsf{D}_c = 1, 3, 4$ and $\mathsf{U}_c = 2$)

δ	$\{0,3\}$	$\{0,4\}$	$\{0,5\}$	$\{1,2\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,4\}$	$\{3,5\}$
R_δ	$\{1\}$	$\{1,3\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{3\}$	$\{3,4\}$	$\{1,2\}$	$\{1,2,3\}$	$\{4\}$
$\tilde{z}_{R_\delta}^c$	1	3	6	6	1	3	3	6	1

the strict right-hand side of δ, and replace the elements 0 and $n+1$ by the smaller respectively larger label of the end-points contained in U_c if possible. For each proper diagonal δ we have $R_\delta \subseteq [n]$ but obviously not every subset of $[n]$ is of this type if $n > 2$. Now set

$$\tilde{z}_I^c := \begin{cases} \frac{|I|(|I|+1)}{2} & \text{if } I = R_\delta \text{ for some proper diagonal } \delta, \\ -\infty & \text{else,} \end{cases}$$

compare Tables 1 and 2 for the two associahedra As_3^c depicted in Fig. 1 that correspond to two different c-labelings of a hexagon. In [11] it is shown that $P_n(\{\tilde{z}_I^c\})$ is in fact an associahedron of dimension $(n-1)$ realised in $\mathbb{R}^n$ for every choice of c. In other words, to obtain these associahedra from the classical permutahedron, we make all inequalities redundant that do not correspond to a proper diagonal of Q_c. Of course, the right-hand sides $\tilde{z}_I^c = -\infty$ are not tight. Proposition 3.8 shows how we can compute the tight values for $\tilde{z}_I^c$ using finite values $\tilde{z}_I^c$ of facet-defining inequalities only. Throughout this manuscript and for any choice c, the reader may refer to this set of tight values $\{\tilde{z}_I^c\}$ to illustrate the results. But we emphasise that this specific choice $\{\tilde{z}_I^c\}$ is only assumed for Statements 4.3–4.5. All other results are valid for the larger class of z_I-coefficients where is polytope $P_n(\{z_I\})$ is an associahedron with the same normal fan as some As_{n-1}^c. Proposition 3.8 and Theorem 4.2 can be applied to this more general situation to obtain tight values for the redundant values z_I^c and to obtain the coefficients y_I of the Minkowski decomposition into faces of the standard simplex.

It is known that realisations $\mathsf{As}_{n-1}^{c_1}$ and $\mathsf{As}_{n-1}^{c_2}$ can be linear isometric for certain choices c_1 and c_2 and values z_I, [2]. While the two associahedra depicted in Fig. 1 are neither linear isometric nor do they have the same normal fan, we remark that the associahedra $\mathsf{As}_2^{c_1}$ and $\mathsf{As}_2^{c_2}$ discussed in the previous section are linear isometric and the isometry is a point reflection Φ in the hyperplane $\sum_{i\in[3]} x_i = 6$. Although the z_I- and y_I-values differ for both realisations, they transform according to this isometry. If we consider a Minkowski decomposition of $\mathsf{As}_2^{c_2}$ with respect to the faces of $\Phi(\Delta_3)$, we obtain precisely the Minkowski coefficients of $\mathsf{As}_2^{c_1}$ with respect to the faces of the standard simplex:

Table 2 R_δ and $\tilde{z}_I^c$ associated to the proper diagonals δ of the labelled hexagon for the associahedron on the right of Fig. 1 ($\mathsf{D}_c = 1, 4$ and $\mathsf{U}_c = 2, 3$)

δ	$\{0,4\}$	$\{2,4\}$	$\{3,4\}$	$\{0,5\}$	$\{0,3\}$	$\{1,2\}$	$\{2,5\}$	$\{1,3\}$	$\{1,5\}$
R_δ	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{4\}$
$\tilde{z}_{R_\delta}^c$	1	3	6	3	6	6	6	3	1

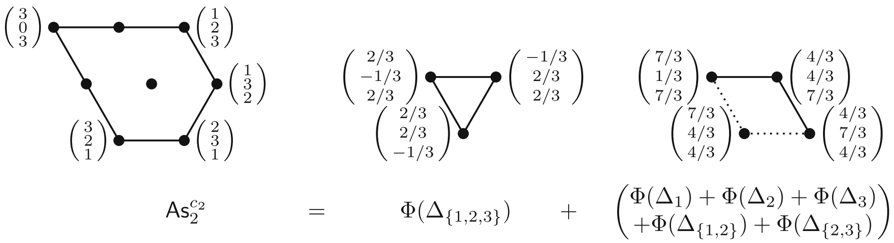


Fig. 5 The Minkowski decomposition of $\mathsf{As}_2^{c_1}$ into faces of the simplex $\Phi(\Delta_{[3]})$

$$\mathsf{As}_2^{c_2} = \Phi(\Delta_{\{1\}}) + \Phi(\Delta_{\{2\}}) + \Phi(\Delta_{\{3\}}) + \Phi(\Delta_{\{1,2\}}) \\ +\Phi(\Delta_{\{2,3\}}) + \Phi(\Delta_{\{1,2,3\}}),$$

see Fig. 5 for an illustration. We can weaken this observation a little bit to obtain a statement about realisations with linear isomorphic normal fans. Such realisations have been discussed for example by Ceballos et al. [6]. Suppose that Φ is a linear isomorphism that maps the normal fan of $\mathsf{As}_{n-1}^{c_1}$ to the normal fan of $\mathsf{As}_{n-1}^{c_2}$. Then Φ induces a transformation between the index sets of the redundant/irredundant inequalities of $\mathsf{As}_{n-1}^{c_1}$ to the redundant/irredundant inequalities of $\mathsf{As}_{n-1}^{c_2}$. Of course, the values $z_I^{c_1}$ for $\mathsf{As}_{n-1}^{c_1}$ transform only into tight right-hand sides of $\mathsf{As}_{n-1}^{c_2}$ if $\mathsf{As}_{n-1}^{c_2} = \Phi(\mathsf{As}_{n-1}^{c_1})$. Thus we obtain two Minkowski decompositions of $\mathsf{As}_{n-1}^{c_2}$: one into faces of the standard simplex Δ_n as described in Theorem 4.2 and another one into faces $\overline{\Delta}_I$ of $\Phi(\Delta_n)$. The combinatorial description of the coefficients $\overline{y}_I$ for $\mathsf{As}_{n-1}^{c_2}$ with respect to faces of $\Phi(\Delta_n)$ is the same as the description of y_I for $\mathsf{As}_{n-1}^{c_1}$ with respect to faces of Δ_n. Of course, to compute the coefficients $\overline{y}_I$, the values for the right-hand sides have to be adjusted to the right-hand sides $\overline{z}_I^{c_1}$ of $\Phi(\mathsf{As}_{n-1}^{c_1})$. As a consequence, the combinatorial data that describes the simplification of the Möbius inversion of Theorem 4.2 is already determined by the geometry of the normal fan of As_n^c up to linear isomorphism.

We end this section relating As_n^c to earlier work. Firstly, we indicate a connection to cambrian fans, generalised associahedra and cluster algebras and secondly to convex rank texts and semigraphoids in statistics. Thirdly, we mention some earlier appearances of specific instances of As_{n-1}^c in the literature.

Fomin and Zelevinsky introduced generalised associahedra in the context of cluster algebras of finite type, [8], and it is well-known that associahedra are generalised associahedra associated to cluster algebras of type A. The construction of [11] was subsequently generalised by Hohlweg, Lange, and Thomas to generalised associahedra, [12], and depends also on a Coxeter element c. The geometry of the normal fans of these realisations is determined by combinatorial properties of c and the normal fans are c-cambrian fans (introduced by Reading and Speyer in [20]). Reading and Speyer conjectured the existence of a linear isomorphism between c-cambrian fans and g-vector fans associated to cluster algebras of finite type with acyclic initial seed (the notion of a g-vector fan for cluster algebras was introduced by Fomin and Zelevinsky [9]). In [21], Reading and Speyer describe and relate cambrian and g-vector fans in more detail and prove their conjecture up to an assumption of another conjecture of

[9]. Yang and Zelevinsky gave an alternative proof of the conjecture of Reading and Speyer in [28]. Stella recently recovered the realizations of generalized associahedra for finite type of [12] and describes the relationship to cluster algebras in detail [27].

Generalised permutahedra and therefore the associahedra As_{n-1}^c are closely related to the framework of convex rank tests and semigraphoids from statistics as discussed by Morton et al. [15]. The semigraphoid axiom characterises the collection of edges of a permutahedron that can be contracted simultaneously to obtain a generalised permutahedron. The authors also study submodular rank tests, its subclass of Minkowski sum of simplices tests and graphical rank tests. The latter one relates to graph associahedra of Carr and Devadoss [5]. Among the associahedra studied in this manuscript, Loday's realisation fits to Minkowski sum of simplices and graphical rank tests.

Some instances of As_{n-1}^c have been studied earlier. For example, the realisations of Loday, [14], and of Rote et al. [22], related to one-dimensional point configurations, are affine equivalent to As_{n-1}^c if $\mathsf{U}_c = \varnothing$ or $\mathsf{U}_c = [n]\backslash\{1, n\}$. For $\mathsf{U}_c = \varnothing$, the Minkowski decomposition into faces of a standard simplex is described by Postnikov in [18]. Moreover, Rote, Santos, and Streinu point out in Sect. 5.3 that their realisation is not affine equivalent to the realisation of Chapoton et al. [7]. It is not difficult to show that the realisation described in [7] is affine equivalent to As_3^c if $\mathsf{U}_c = \{2\}$ or $\mathsf{U}_c = \{3\}$.

3 Tight Values for all z_I^c for As_{n-1}^c

Since the facet-defining inequalities for As_{n-1}^c correspond to proper diagonals of Q_c, we know precisely the irredundant inequalities for the generalised permutahedron $P_n(\{\tilde{z}_I^c\}) = \mathsf{As}_{n-1}^c$. In this section, we determine tight values $\tilde{z}_I^c$ for all $I \subseteq [n]$ corresponding to redundant inequalities in order to be able to compute the coefficients y_I of the Minkowski decomposition of As_{n-1}^c as described by Proposition 1.2. The concept of an up and down interval decomposition induced by the partitioning $\mathsf{D}_c \sqcup \mathsf{U}_c$ (or, equivalently, induced by c) of a given interval $I \subset [n]$ is a key concept that we introduce first, it allows us to describe any $I \subseteq [n]$ in terms of unions and intersections of sets R_δ for certain proper diagonals determined by this decomposition (or, equivalently, as unions of set differences of certain sets R_δ and their complements).

Definition 3.1 (*Up and down intervals*). Let $\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$ and $\mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}$ be the partition of $[n]$ induced by a Coxeter element c.

(a) A set $S \subseteq [n]$ is a non-empty interval of $[n]$ if $S = \{r, r+1, \ldots, s\}$ for some $0 < r \le s < n$. We write S as closed interval $[r, s]$ (end-points included) or as open interval $(r-1, s+1)$ (end-points excluded). An empty interval is an open interval $(k, k+1)$ for some $1 \le k < n$.

(b) A non-empty open down interval is a set $S = \{d_r < d_{r+1} < \cdots < d_s\} \subseteq \mathsf{D}_c$ for some $1 \le r \le s \le \ell$. We write S as open down interval $(d_{r-1}, d_{s+1})_{\mathsf{D}_c}$ where we allow $d_{r-1} = 0$ and $d_{s+1} = n+1$, i.e. $d_{r-1}, d_{s+1} \in \overline{\mathsf{D}}_c$. For $1 \le r \le \ell - 1$, we also have the empty down interval $(d_r, d_{r+1})_{\mathsf{D}_c}$.

(c) A closed up interval is a non-empty set $S = \{u_r < u_{r+1} < \cdots < u_s\} \subseteq \mathsf{U}_c$ for some $1 \le r \le s \le \ell$. We write $[u_r, u_s]_{\mathsf{U}_c}$.

We often omit the words *open* and *closed* when we consider down and up intervals. There will be no ambiguity, because we are not going to deal with closed down intervals or open up intervals. Up intervals are always non-empty, while down intervals may be empty. It will be useful to distinguish the empty down intervals $(d_r, d_{r+1})_{\mathsf{D}_c}$ and $(d_s, d_{s+1})_{\mathsf{D}_c}$ if $r \neq s$ although they are equal as sets.

It might be helpful to read the following definition of the up and down interval decomposition in combination with Examples 3.3 and 3.5.

Definition 3.2 (*Up and down interval decomposition*). Let $\mathsf{D}_c \sqcup \mathsf{U}_c$ be the partition of $[n]$ induced by a Coxeter element c and $I \subset [n]$ be non-empty. The up and down interval decomposition of type (v, w) of I is a partition of I into disjoint up and down intervals $I_1^{\mathsf{U}}, \ldots, I_w^{\mathsf{U}}$ and $I_1^{\mathsf{D}}, \ldots, I_v^{\mathsf{D}}$ obtained by the following procedure.

1. Suppose there are $\tilde{v}$ non-empty inclusion maximal down intervals of I denoted by $\tilde{I}_k^{\mathsf{D}} = (\tilde{a}_k, \tilde{b}_k)_{\mathsf{D}_c}$, $1 \le k \le \tilde{v}$, with $\tilde{b}_k \le \tilde{a}_{k+1}$ for $1 \le k < \tilde{v}$. Consider also all empty down intervals $E_i^{\mathsf{D}} = (d_{r_i}, d_{r_i+1})_{\mathsf{D}_c}$ with $\tilde{b}_k \le d_{r_i} < d_{r_i+1} \le \tilde{a}_{k+1}$ for $0 \le k \le \tilde{v}$ where $\tilde{b}_0 = 1$ and $\tilde{a}_{\tilde{v}+1} = n$. Denote the open intervals $(\tilde{a}_i, \tilde{b}_i)$ and (d_{r_i}, d_{r_i+1}) of $[n]$ by $\tilde{I}_i$ and E_i respectively.
2. Consider all inclusion maximal up intervals of I contained in some interval $\tilde{I}_i$ or E_i obtained in Step 1 and denote these up intervals by
$$I_1^{\mathsf{U}} = [\alpha_1, \beta_1]_{\mathsf{U}_c}, \ldots, I_w^{\mathsf{U}} = [\alpha_w, \beta_w]_{\mathsf{U}_c}.$$
Without loss of generality, we assume $\alpha_i \le \beta_i < \alpha_{i+1}$.
3. A down interval $I_i^{\mathsf{D}} = (a_i, b_i)_{\mathsf{D}_c}$, $1 \le i \le v$, is a down interval obtained in Step 1 that is either a non-empty down interval $\tilde{I}_k^{\mathsf{D}}$ or an empty down interval E_k^{D} with the additional property that there is some up interval I_j^{U} obtained in Step 2 such that $I_j^{\mathsf{U}} \subseteq E_k$. Without loss of generality, we assume $b_i \le a_{i+1}$ for $1 \le i < v$.

Example 3.3 We describe the up and down interval decomposition for three subsets of $[4]$ which is partitioned into $\mathsf{D}_c = \{1, 3, 4\}$ and $\mathsf{U}_c = \{2\}$ and encourage the reader to sketch the steps.

(i) $J_1 = \{2, 3\}$.
The only non-empty inclusion maximal down interval of J_1 is $\tilde{I}_1^{\mathsf{D}} = (1, 4)_{\mathsf{D}_c} = \{3\}$; there are no empty down intervals E_i^{D} to be considered. As inclusion maximal up intervals of J_1 contained in $\tilde{I}_1 = (1, 4) = \{2, 3\}$, we identify $I_1^{\mathsf{U}} = [2, 2]_{\mathsf{U}_c} = \{2\}$. The up and down interval decomposition of J_1 is $(1, 4)_{\mathsf{D}_c} \sqcup [2, 2]_{\mathsf{U}_c}$. Its type is $(1, 1)$.

(ii) $J_2 = \{2\}$.
There is no non-empty inclusion maximal down interval of J_2 to be considered, but there is one empty down interval $E_1^{\mathsf{D}} = (1, 3)_{\mathsf{D}_c}$ such that $E_1 = (1, 3) = \{2\}$

contains one inclusion maximal up interval $I_1^{\mathsf{U}} = [2, 2]_{\mathsf{U}_c} = \{2\}$ of J_2. It follows that the up and down interval decomposition of J_2 is $(1, 3)_{\mathsf{D}_c} \sqcup [2, 2]_{\mathsf{U}_c}$. Its type is $(1, 1)$.

(iii) Consider $J_3 = \{2, 4\}$.
The only non-empty inclusion maximal down interval of J_3 is $\tilde{I}_1^{\mathsf{D}} = (3, 5)_{\mathsf{D}_c} = \{4\}$; there is one empty down interval $E_1^{\mathsf{D}} = (1, 3)_{\mathsf{D}_c}$ such that E_1 contains an inclusion maximal up interval of J_3, this is the up interval $I_1^{\mathsf{U}} = [2, 2]_{\mathsf{U}_c} = \{2\}$. There is no non-empty inclusion maximal up interval contained in $\tilde{I}_1^{\mathsf{D}}$. It follows that the up and down interval decomposition of J_3 is $\big((1, 3)_{\mathsf{D}_c} \sqcup [2, 2]_{\mathsf{U}_c}\big) \sqcup \big((3, 5)_{\mathsf{D}_c}\big)$. Its type is $(2, 1)$.

Definition 3.4 (*Nested up and down interval decomposition, nested components*) Let $\mathsf{D}_c \sqcup \mathsf{U}_c$ be the partition of $[n]$ induced by a Coxeter element c and $I \subset [n]$ be non-empty.

(a) The up and down interval decomposition of I is nested if its type is $(1, w)$.
(b) A nested component of I is an inclusion-maximal subset J of I such that the up and down decomposition of J is nested.

The definition of a nested up and down interval decomposition can be rephrased as follows: all up intervals are contained in the interval (a_1, b_1) of $[n]$ obtained from the unique (empty or non-empty) down interval $I_1^{\mathsf{D}} = (a_1, b_1)_{\mathsf{D}_c}$. The following example describes the up and down interval decompositions of $I = R_\delta$ for all proper diagonals δ of Q_c. The situation is illustrated in Fig. 6. As a consequence, we observe that the up and down interval decomposition for R_δ is always nested if δ is a proper diagonal.

Example 3.5 Let $\mathsf{D}_c \sqcup \mathsf{U}_c$ be the partition of $[n]$ induced by a Coxeter element c. The proper diagonals $\delta = \{a, b\}$, $a < b$, of the c-labeled polygon Q_c are in bijection to certain non-empty proper subsets $R_\delta \subset [n]$ that have an up and down interval decomposition of type $(1, 0)$, $(1, 1)$, or $(1, 2)$. More precisely, we have

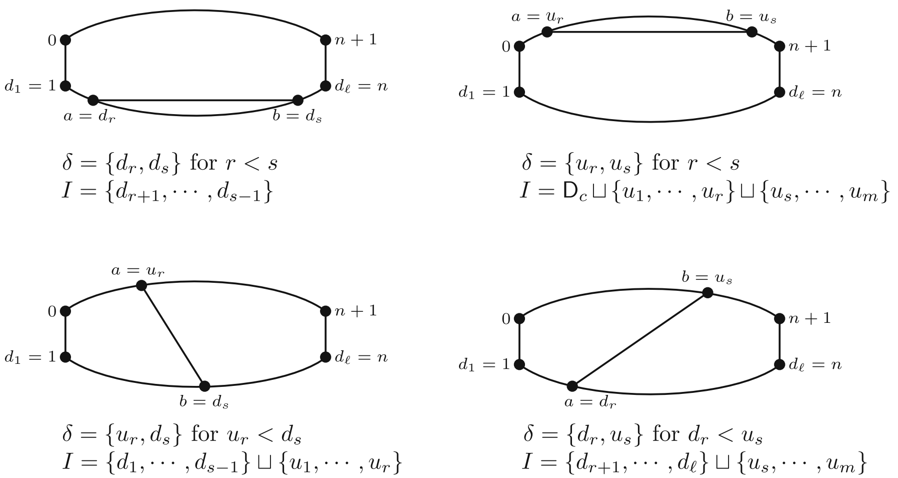


Fig. 6 The four possible situations for a diagonal $\delta = \{a, b\}$ of Example 3.5

(i) $R_\delta = (a, b)_{\mathsf{D}_c}$ iff R_δ has an up and down decomposition of type $(1, 0)$.
(ii) $R_\delta = (0, b)_{\mathsf{D}_c} \cup [u_1, a]_{\mathsf{U}_c}$ or $R_\delta = (a, n+1)_{\mathsf{D}_c} \cup [b, u_m]_{\mathsf{U}_c}$ iff R_δ has a decomposition of type $(1, 1)$.
(iii) $R_\delta = (0, n+1)_{\mathsf{D}_c} \cup [u_1, a]_{\mathsf{U}_c} \cup [b, u_m]_{Up_c}$ iff R_δ has an up and down decomposition of type $(1, 2)$.

To simplify notation, we extend the definition of R_δ to the non-proper diagonals $\delta = \{0, u_1\}$ and $\delta = \{u_m, n+1\}$ by defining $R_{\{0,u_1\}} = R_{\{u_m,n+1\}} = [n]$. An example of the diagonals $\delta_{i,j}$ associated to an up and down interval decomposition defined in the next Lemma is discussed and illustrated in Example 3.7 and Fig. 7.

Lemma 3.6 *Given the partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c. Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type (v, w) and nested components of type $(1, w_1), \cdots, (1, w_v)$. For $1 \le i \le v$ and $1 \le j \le w_i$, denote by $[\alpha_{i,j}, \beta_{i,j}]_{\mathsf{U}_c}$ the inclusion maximal up intervals contained in the down interval $(a_i, b_i)_{\mathsf{D}_c}$ where $\beta_{i,j} < \alpha_{i,j+1}$ and $b_i \le a_{i+1}$.*

Associate to the nested component $(1, w_i)$ the diagonal $\delta_{i,1} = \{a_i, b_i\}$ if $w_i = 0$. If $w_i > 0$ then associate to the nested component $(1, w_i)$ the diagonals

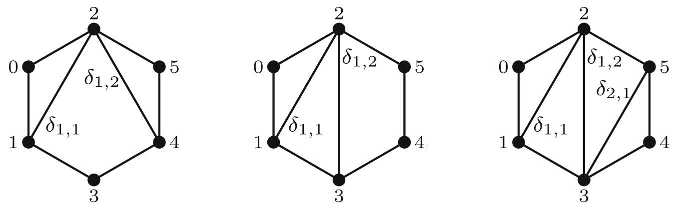
$$\begin{aligned} \delta_{i,1} &:= \{a_i, \alpha_{i,1}\}, \\ \delta_{i,j} &:= \{\beta_{i,j-1}, \alpha_{i,j}\} \textit{ for } 1 < j \le w_i, \quad \textit{and} \\ \delta_{i,w_i+1} &:= \{\beta_{i,w_i}, b\}. \end{aligned}$$

Then the diagonals $\delta_{i,j}$ are non-crossing and

$$I = \bigcup_{i\in[v]} \bigcap_{j\in[w_i+1]} R_{\delta_{i,j}} = \bigcup_{i\in[v]} \big(R_{\delta_{i,w_i+1}} \backslash \big(\bigcup_{j\in[w_i]} [n] \backslash R_{\delta_{i,j}}\big)\big).$$

Proof It follows from the definition of nested components that $\delta_{i,j}$ and $\delta_{i',j'}$ are non-crossing if $i \ne i'$. That $\delta_{i,j}$ and $\delta_{i,j'}$ are non-crossing within a nested component is implied by $\beta_{i,j} < \alpha_{i,j+1}$.

To see the identities on I, we first remark that $I = \bigcap_{j\in[w_1+1]} R_{\delta_{1,j}}$ follows directly from the the up and down interval decomposition of I and the definition of R_δ if I has only one nested component. If I consists of more than one nested component,



$J_1 = \{2, 3\}$ $J_2 = \{2\}$ $J_3 = \{2, 4\}$

Fig. 7 The associated diagonals $\delta_{i,j}$ for the three examples considered in Example 3.7

we obtain the claim since it holds for each nested component separately. The second identity is a simple reformulation of the first. This is easily seen in case of just one nested component: instead of intersecting the sets R_δ, we choose $\delta = \delta_{1,w_1+1}$ and remove the complements $[n]\setminus R_{\delta_{1,j}}$, $1 \le j \le w_1$ from R_δ. This yields $\bigcap_{j\in[w_1+1]} R_{\delta_{i,j}}$. □

Example 3.7 We briefly discuss the diagonals associated to the up and down interval decomposition for the three subsets $J_1 = \{2,3\}$, $J_2 = \{2\}$ and $J_3 = \{2,4\}$ of $[4]$ partitioned by $\mathsf{D}_c = \{1,3,4\}$ and $\mathsf{U}_c = \{2\}$. These examples are illustrated in Fig. 7.

(i) $J_1 = (1,4)_{\mathsf{D}_c} \sqcup [2,2]_{\mathsf{U}_c}$ and the associated diagonals are $\delta_{1,1} = \{1,2\}$ and $\delta_{1,2} = \{2,4\}$.
(ii) $J_2 = (1,3)_{\mathsf{D}_c} \sqcup [2,2]_{\mathsf{U}_c}$ and the associated diagonals are $\delta_{1,1} = \{1,2\}$ and $\delta_{1,2} = \{2,3\}$.
(iii) $J_3 = \big((1,3)_{\mathsf{D}_c} \sqcup [2,2]_{\mathsf{U}_c}\big) \sqcup \big((3,5)_{\mathsf{D}_c}\big)$ and the associated diagonals are $\delta_{1,1} = \{1,2\}$, $\delta_{1,2} = \{2,3\}$ and $\delta_{2,1} = \{3,5\}$.

The final proposition of this section resolves the quest for tight values z_I^c of all redundant inequalities of an associahedron that has the normal fan of As_{n-1}^c. If we denote this associahedron by $P_n(\{\tilde{z}_I^c\})$, then the inequalities that correspond to an index set $I = R_\delta$ for some proper diagonal of Q_c are precisely the facet defining inequalities and all other inequalities are redundant.

Proposition 3.8 *Given the partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c. Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type (v,w) and nested components of type $(1,w_1),\dots,(1,w_v)$. For $1 \le i \le v$ and $1 \le j \le w_i$, denote by $[\alpha_{i,j},\beta_{i,j}]_{\mathsf{U}_c}$ the inclusion maximal up intervals contained in the down interval $(a_i,b_i)_{\mathsf{D}_c}$ where $\beta_{i,j} < \alpha_{i,j+1}$ and $b_i \le a_{i+1}$. The diagonals $\delta_{i,j}$ are defined as in Lemma 3.6. For non-empty $I \subseteq [n]$ we set*

$$z_I^c := \sum_{i\in[v]} \Big(\sum_{j\in[w_i+1]} \tilde{z}^c_{R_{\delta_{i,j}}} - w_i \tilde{z}^c_{[n]} \Big).$$

Then $P(\{z_I^c\}) = P(\{\tilde{z}_I^c\})$ and all z_I^c are tight.

Proof The verification of the inequality is a straightforward calculation:

$$\begin{aligned}
\sum_{i\in I} x_i &= \sum_{i\in[v]} \sum_{k\in\bigcap_{j\in[w_i+1]} R_{\delta_{i,j}}} x_k \\
&= \sum_{i\in[v]} \Big(\sum_{k\in[v]} x_k - \sum_{j\in[w_i+1]} \sum_{k\in[n]\setminus R_{\delta_{i,j}}} x_k \Big) \\
&\ge \sum_{i\in[v]} \Big(\tilde{z}^c_{R_{[n]}} + \sum_{j\in[w_i+1]} \big(\tilde{z}^c_{R_{\delta_{i,j}}} - \tilde{z}^c_{[n]} \big) \Big).
\end{aligned}$$

The first equality is an application of Lemma 3.6 and the second equality is a simple reformulation. The inequality holds, since $\sum_{i\in R_\delta} x_i \ge \tilde{z}^c_{R_\delta}$ is equivalent to $-\sum_{i\in[n]\setminus R_\delta} x_i \ge \tilde{z}^c_{R_\delta} - z_{[n]}$ for every proper diagonal δ. □

Definition 3.9 Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type (v, w) and nested components of type $(1, w_1), \cdots, (1, w_v)$. As in Lemma 3.6, we associate diagonals $\delta_{i,j}$ for $1 \le i \le v$ and $1 \le j \le w_i$. The subset $\mathcal{D}_I$ of proper diagonals of $\{\delta_{i,j} | 1 \le i \le v \text{ and } 1 \le j\}$ is called set of proper diagonals associated to I. Similarly, we say that $\delta \in \mathcal{D}_I$ is a proper diagonal associated to I.

We end this section with some remarks. First, if a non-proper diagonal $\delta = \{0, u_1\}$ or $\delta = \{u_m, n+1\}$ occurs as a diagonal associated to the first or last nested component, the formula for z_I^c in Proposition 3.8 can be simplified by cancelation of the corresponding terms $\tilde{z}_{[n]}^c$. Second, for any proper diagonal δ of Q_c, we obtain $z_{R_\delta}^c = \tilde{z}_{R_\delta}^c$. And finally, we can characterise the face of $P(\{\tilde{z}_I^c\})$ that minimises the linear functional $\sum_{i \in I} x_i$ for a given non-empty and proper subset $I \subset [n]$.

Corollary 3.10 *Associate the linear functional $\varphi_I(x) = \sum_{i \in I} x_i$ to a non-empty proper subset $I \subset [n]$ and denote the facet of $P(\{\tilde{z}_I^c\})$ that is supported by $\sum_{i \in R_\delta} x_i = \tilde{z}_{R_\delta}^c$ for the proper diagonal δ by F_{R_δ}. Then the intersection $\bigcap_{\delta \in \mathcal{D}_I} F_{R_\delta}$ is the minimizing face of $P(\{\tilde{z}_I^c\})$ for φ_I.*

4 Main Results and Examples

Substitution of Proposition 3.8 into Proposition 1.2 provides a way to compute all Minkowski coefficients y_I since all tight values z_I^c for $\mathsf{As}_{n-1}^c = P_n(\{z_I^c\})$ are known:

$$y_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} z_J^c = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \sum_{i \in [v_J]} \Big(\sum_{j \in [w_i + 1]} \tilde{z}^c_{R_{\delta^J_{i,j}}} - w_i \tilde{z}^c_{[n]} \Big). \tag{1}$$

The goal of this section is to provide two simpler formulae for y_I. The first one, given in Theorem 4.2, simplifies Formula (1) to at most four non-zero summands for each $I \subseteq [n]$. The second one, stated in Theorem 4.3, is only valid if the right-hand sides of the facet-defining inequalities satisfy $z_I^c = \frac{|I|(|I|+1)}{2}$. The values y_I are then described as a (signed) product of two numbers that measure certain paths of Q_c. Theorem 4.3 can be seen as a new aspect to relate combinatorics of the labeled n-gon Q_c to a construction of As_{n-1}^c: the coefficients for the Minkowski decomposition into faces of the standard simplex can be obtained from the combinatorics of Q_c. Two other relations of the combinatorics of Q_c to the geometry of As_{n-1}^c were known before. It is possible to extract the coordinates of the vertices [14, 11], but it is also possible to determine the facet normals and the right-hand sides for their inequalities [11].

From now on, we use the following notation and make some general assumptions unless explicitly mentioned otherwise. Let $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ be the partition of $[n]$ induced by some fixed Coxeter element c with

$$\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\} \quad \text{and} \quad \mathsf{U}_c = \{u_1 < \cdots < u_m\}.$$

A non-empty subset $I \subseteq [n]$ with up and down interval decomposition of type (v, w) has nested components $(1, w_i)$, $1 \le i \le v$, such that the inclusion maximal up intervals $[\alpha_{i,j}, \beta_{i,j}]_{\mathsf{U}_c}$ contained in the down interval $(a_i, b_i)_{\mathsf{D}_c}$ satisfy $\beta_{i,j} < \alpha_{i,j+1}$ and $b_i \leq a_{i+1}$. For nested I, that is, if $v = 1$, we simplify notation and drop one subscript: we write $(a, b)_{\mathsf{D}_c} \cup \bigcup_{j=1}^{w} [\alpha_j, \beta_j]_{\mathsf{U}_c}$ for the up and down interval decomposition where $\alpha_j < \beta_j \leq \alpha_{j+1}$ as before. Nevertheless, we do not drop an index for the associated diagonals δ_{ij} introduced in Lemma 3.6, we continue to denote them by $\delta_{i,j}$ or $\delta_{1,j}$ to avoid a conflict with the diagonals δ_1, δ_2, δ_3 and δ_4 defined next. To that respect, we define γ (respectively Γ) to denote the smallest (respectively largest) element of a nested set I and associate the following four diagonals of the c-labeled $(n+2)$-gon Q_c to this nested set I:

$$\delta_1 = \{a, b\}, \qquad \delta_2 = \{a, \Gamma\}, \qquad \delta_3 = \{\gamma, b\}, \qquad \text{and} \qquad \delta_4 = \{\gamma, \Gamma\}.$$

In general, not all diagonals δ_i will be proper diagonals, but it will be useful to consider the subset $\mathscr{D}_I$ of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ that consists of proper diagonals only. We emphasize that the diagonals δ_i should be distinguished from the diagonals $\delta_{i,j}$ defined in Lemma 3.6 and the set $\mathcal{D}_I$ should be distinguished from $\mathscr{D}_I$.

Example 4.1 We discuss the four diagonals δ_1, δ_2, δ_3 and δ_4 associated to three subsets $J_1, J_2, J_3 \subseteq [4]$ which is partitioned into $\mathsf{D}_c = \{1, 3, 4\}$ and $\mathsf{U}_c = \{2\}$. These associated set $\mathscr{D}_I$ are illustrated in Fig. 8.

(i) Since $J_1 = \{2, 3\} = (1, 4)_{\mathsf{D}_c} \sqcup [2, 2]_{\mathsf{U}_c}$ is nested, we have $\gamma = 2$ and $\Gamma = 3$. It follows that

$$\delta_1 = \{1, 4\}, \qquad \delta_2 = \{1, 3\}, \qquad \delta_3 = \{2, 4\} \qquad \text{and} \qquad \delta_4 = \{2, 3\}.$$

In this situation, all diagonals δ_i except diagonal $\delta_2 = \{1, 3\}$ are proper diagonals. Therefore, $\mathscr{D}_I = \{\delta_1, \delta_3, \delta_4\}$

(ii) Since $J_2 = \{2\} = (1, 3)_{\mathsf{D}_c} \sqcup [2, 2]_{\mathsf{U}_c}$ is nested, we have $\gamma = \Gamma = 2$. This implies

$$\delta_1 = \{1, 3\}, \qquad \delta_2 = \{1, 2\}, \qquad \delta_3 = \{2, 3\} \qquad \text{and} \qquad \delta_4 = \{2, 2\}.$$

In this situation, the diagonals δ_1 and δ_4 are not proper while the diagonals δ_2 and δ_3 are proper. Hence, $\mathscr{D}_I = \{\delta_2, \delta_3\}$.

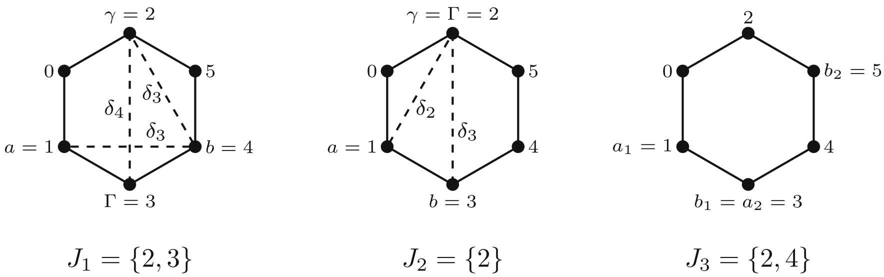


Fig. 8 The diagonals of $\mathscr{D}_J$ (the proper diagonals among the associated diagonals δ_i) for the three examples of Example 4.1

(iii) The set $J_3 = \{2, 4\}$ is not nested since its up and down interval decomposition is of type $(2, 1)$. We do not associate diagonals δ_i to J_3, the set $\mathscr{D}_I$ is empty.

We now extend our definition of R_δ and $z^c_{R_\delta}$ to all non-proper and degenerate diagonals δ. If $\delta = \{0, n+1\}$ and $\mathsf{U}_c = \varnothing$ we set $R_\delta := [n]$ and $z^c_{R_\delta} = z^c_{[n]}$. Otherwise, if $\delta = \{x, y\}$ is not a proper diagonal (different from $\delta = \{0, n+1\}$ if $\mathsf{U}_c = \varnothing$), we set:

$$R_\delta := \begin{cases} \varnothing & \text{if } x, y \in \overline{\mathsf{D}}_c, \\ [n] & \text{otherwise,} \end{cases} \qquad \text{and} \qquad z^c_{R_\delta} := \begin{cases} 0 & \text{if } R_\delta = \varnothing, \\ z^c_{[n]} & \text{if } R_\delta = [n]. \end{cases}$$

The main result, Theorem 4.2, actually combines two statements. Firstly, there is a more efficient way to compute the coefficients of the Minkowski decomposition of an associahedron $\mathsf{As}^c_{n-1} = P(\{z^c_I\})$ compared to the alternating sum proposed by Proposition 1.2. Secondly, the terms z^c_I for redundant inequalities that are needed to compute y_I are combinatorially characterised and depend on the choice of c or equivalently on the normal fan of As^c_{n-1}. Of course, their precise values depend on the values z^c_I of inequalities that are facet-defining.

Theorem 4.2 *Let I be non-empty subset of $[n]$. Then the Minkowski coefficient y_I of $\mathsf{As}^c_{n-1} = P(\{z^c_I\})$ is*

$$y_I = \begin{cases} (-1)^{|I \setminus R_{\delta_1}|}\left(z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + z^c_{R_{\delta_4}}\right) & \textit{if } v = 1, \\ 0 & \textit{otherwise.} \end{cases}$$

We prove Theorem 4.2 in Sect. 6. An example illustrating the theorem for the left associahedron As^c_3 of Fig. 1 ($\mathsf{D}_c = \{1, 3, 4\}$ and $\mathsf{U}_c = \{2\}$) is given in Fig. 9 where we also explicitly compute the y_I-values for this realisation with $z^c_I = \frac{|I|(|I|+1)}{2}$ for the facet-defining inequalities.

For the rest of this section, we specialise to realisations with this specific choice of z_I-values. We obtain a nice combinatorial interpretation of the coefficients y_I in Theorem 4.3 and characterise the vanishing y_I-values in Corollary 4.5.

If I has a nested up and down interval decomposition, the *signed lengths* K_γ and K_Γ of I are integers defined as follows. Let $|K_\Gamma|$ be the number of edges of the path in ∂Q connecting b and Γ that does not use the vertex labeled a. The sign of K_Γ is negative if and only if $\Gamma \in \mathsf{D}_c$. Similarly, $|K_\gamma|$ is the length of path in ∂Q connecting a and γ not using label b and K_γ is negative if and only if $\gamma \in \mathsf{D}_c$. Equivalently, we have that K_γ (respectively K_Γ) is a positive integer if and only if $\gamma \in \mathsf{U}_c$ (respectively $\Gamma \in \mathsf{U}_c$) and that $K_\gamma = -1$ (respectively $K_\Gamma = -1$) if and only if $\gamma \in \mathsf{D}_c$ (respectively $\Gamma \in \mathsf{D}_c$). We can now express the coefficients y_I of As^c_{n-1} in terms of K_γ and K_Γ. The following theorem is an easy consequence of Theorem 4.2.

Theorem 4.3 *Let K_Γ and K_γ be the signed lengths of I as defined above if $I \subseteq [n]$ has a nested up and down interval decomposition of type $(1, k)$. Then the Minkowski*

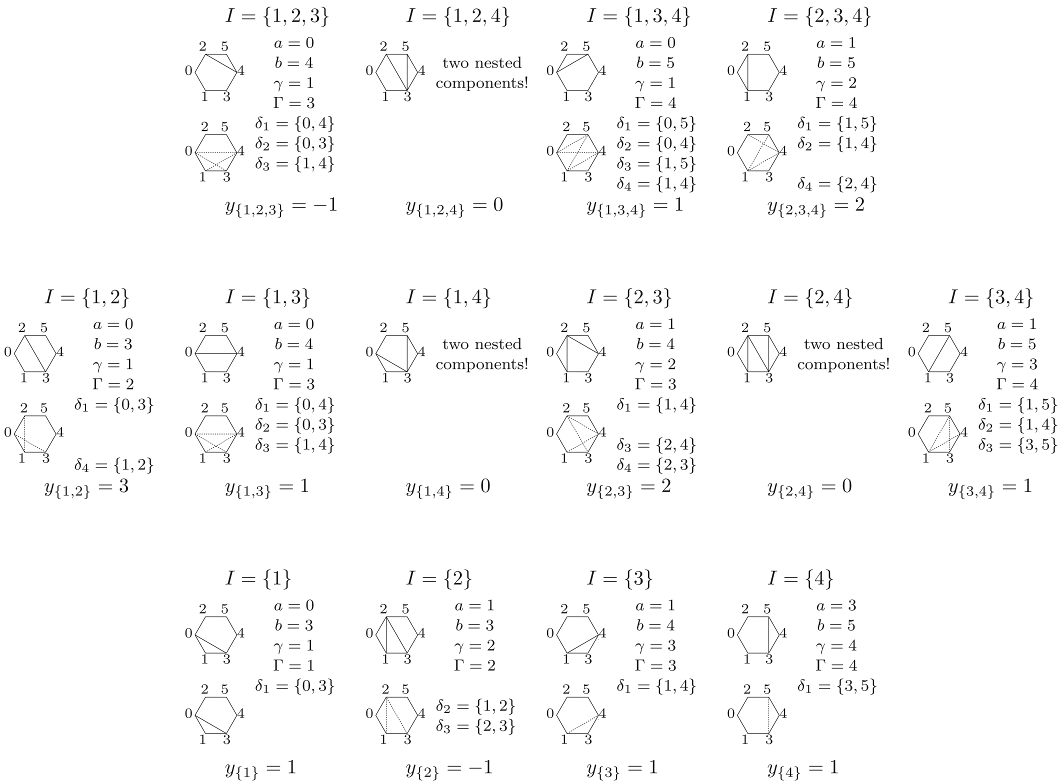


Fig. 9 Details for the computation of the Minkowski coefficients y_I of As_3^c in case $\mathsf{D}_c = \{1, 3, 4\}$ and $\mathsf{U}_c = \{2\}$. The first line states I, the second line's first column pictures the non-crossing proper diagonals $\delta_{i,j}$ associated to I, while the second column gives the values for a, b, γ, and Γ if I is of type $(1, w)$. The third line's first column illustrates the proper diagonals of $\mathscr{D}_I \subseteq \{\delta_1, \delta_2, \delta_3, \delta_4\}$, the second column specifies their end-points. We finally state the value for y_I

coefficient y_I *of* As^c_{n-1} *is*

$$y_I = \begin{cases} (-1)^{|I\setminus(a,b)\mathsf{D}|} K_\gamma K_\Gamma & \textit{if } I \neq \{u_s\} \subseteq \mathsf{U}_c \textit{ and } v = 1, \\ (n+1) - K_\gamma K_\Gamma & \textit{if } I = \{u_s\} \subseteq \mathsf{U}_c, \\ 0 & \textit{if } v \geq 2. \end{cases}$$

Proof By Theorem 4.2, the claim is trivial if I has up and down interval decomposition of type $v > 1$. We therefore assume $v = 1$, set $K := |R_{\delta_1}|$ and observe

$$K_\Gamma = |R_{\delta_2}| - |R_{\delta_1}| \text{ and } K_\gamma = |R_{\delta_3}| - |R_{\delta_1}|.$$

Thus

$$|R_{\delta_4}| = \begin{cases} K + K_\gamma + K_\Gamma & \text{if } I \neq \{u_s\}, \\ K + K_\gamma + K_\Gamma - 1 & \text{if } I = \{u_s\}, \end{cases}$$

as well as

$$\begin{aligned} z^c_{R_{\delta_1}} &= \frac{K(K+1)}{2}, \\ z^c_{R_{\delta_2}} &= \frac{(K+K_\Gamma)(K+K_\Gamma+1)}{2}, \\ z^c_{R_{\delta_3}} &= \frac{(K+K_\gamma)(K+K_\gamma+1)}{2}, \text{ and} \\ z^c_{R_{\delta_4}} &= \begin{cases} \frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2} & \text{if } I \neq \{u_s\}, \\ \frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2} - (n+1) & \text{if } I = \{u_s\}. \end{cases} \end{aligned}$$

A direct computation shows

$$z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + \frac{(K+K_\Gamma+K_\gamma)(K+K_\Gamma+K_\gamma+1)}{2} = K_\Gamma K_\gamma.$$

The claim is now an immediate consequence of Theorem 4.2. □

Corollary 4.4 *For* $n \geq 2$ *and any choice* $\mathsf{D}_c \sqcup \mathsf{U}_c$*, we have* $y_{[n]} = (-1)^{|\mathsf{U}_c|}$.

Proof The claim follows directly either from Theorem 4.2 or from Theorem 4.3. To obtain the claim from Theorem 4.2, observe that $[n]\setminus R_{\delta_1} = \mathsf{U}_c$ and

$$z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + z^c_{R_{\delta_4}} = 1.$$

To obtain the claim from Theorem 4.3, we remark that $[n]\setminus R_{\delta_1} = I\setminus(a,b)_{\mathsf{D}}$ and $K_\gamma = K_\Gamma = -1$ since $a = 0,\ b = n+1,\ \gamma = 1$, and $\Gamma = n$. □

Corollary 4.5 *Let* $n \geq 2$ *and* $\mathsf{D}_c \sqcup \mathsf{U}_c$ *be a partition induced by some Coxeter element* c*. Then* $y_I = 0$ *if and only if* I *has an up and down decomposition of type* (v_I, w_I) *with* $v_I > 1$ *or* $n = 3$ *and* $I = \mathsf{U}_c = \{2\}$.

Proof Since K_γ and K_Γ are non-zero, Theorem 4.3 implies $y_I \neq 0$ if $I \neq \{u_s\}$. So we assume $I = \{u_s\}$. It now suffices to prove that $y_I = 0$ if and only if $n = 3$.

If $n = 2$ then $I = \{u_s\} \subseteq \mathsf{U}_c$ is impossible, so we have $n \geq 3$. From $R_{\delta_2} \cup R_{\delta_3} = [n]$ and $R_{\delta_2} \cap R_{\delta_3} = \{u_s\}$ we conclude $K_\gamma + K_\Gamma = n+1$. On the other hand, Theorem 4.3 implies that $y_I = 0$ is equivalent to $K_\Gamma K_\gamma = n+1$. By substitution we have

$$K_\Gamma^2 - (n+1)K_\Gamma + (n+1) = 0$$

and solving for K_Γ gives

$$K_{\Gamma,1/2} = -\frac{-(n+1)}{2} \pm \sqrt{\frac{(n+1)^2}{4} - (n+1)} = \frac{(n+1) \pm \sqrt{n^2 - 2n - 3}}{2}.$$

Since K_Γ is a positive integer, we conclude that $\sqrt{n^2 - 2n - 3}$ is a positive integer. In particular, $n^2 - 2n - 3 = (n+1)(n-3)$ must be a square. For $n = 3$, we conclude $K_\Gamma = 2$, that is $I = \mathsf{U}_c = \{2\}$. For $n > 3$ we derive the contradiction $(n+1) = r^2(n-3)$ or $(n-3) = r^2(n-1)$ for some positive integer r. □

We now illustrate Theorem 4.3 by recomputing the y_I-values for $\mathsf{As}_2^{c_1}$ and $\mathsf{As}_2^{c_2}$ mentioned in the introduction. For $n = 3$, there are two possible partitions of $\{1, 2, 3\}$ that correspond to the two Coxeter elements of Σ_3: either $\mathsf{D}_{c_1} = \{1, 2, 3\}$ and $\mathsf{U}_{c_1} = \varnothing$ or $\mathsf{D}_{c_2} = \{1, 3\}$ and $\mathsf{U}_{c_2} = \{2\}$.

Example 4.6 Consider $\mathsf{D}_{c_1} = \{1, 2, 3\}$ and $\mathsf{U}_{c_1} = \varnothing$ which yields Loday's realisation.

(i) We have $y_I = 1$ for $I = \{i\}$ and $1 \leq i \leq 3$.
The up and down interval decomposition of $\{i\}$ is $(i-1, i+1)_{\mathsf{D}}$ and $\gamma = \Gamma = i$. It follows that $K_\gamma = K_\Gamma = -1$ and $I \backslash (a, b)_{\mathsf{D}} = \varnothing$. Thus $y_I = 1$.
(ii) We have $y_I = 1$ for $I = \{i, i+1\}$ and $1 \leq i \leq 2$.
Then $I = (i-1, i+2)_{\mathsf{D}}$, $\gamma = i$, and $\Gamma = i+1$. It follows that $K_\gamma = K_\Gamma = -1$ and $I \backslash (a, b)_{\mathsf{D}} = \varnothing$. Thus $y_I = 1$.
(iii) We have $y_I = 0$ for $I = \{1, 3\}$.
Then $I = (0, 2)_{\mathsf{D}} \sqcup (2, 4)_{\mathsf{D}}$, so I is of type $(2, 0)$ and $y_I = 0$ by Corollary 4.5.
(iv) We have $y_I = 1$ for $I = \{1, 2, 3\}$.
Then $I = (0, 4)_{\mathsf{D}}$, $\gamma = 1$ and $\Gamma = 3$ implies $K_\gamma = K_\Gamma = -1$ and $I \backslash (a, b)_{\mathsf{D}} = \varnothing$. Thus $y_I = 1$. Of course, we could also use Corollary 4.4 instead.

Altogether we have $y_I \in \{0, 1\}$ and $\mathsf{As}_2^{c_1}$ is a Minkowski sum of faces of the standard simplex:

$$\mathsf{As}_2^{c_1} = 1\Delta_{\{1\}} + 1\Delta_{\{2\}} + 1\Delta_{\{3\}} + 1\Delta_{\{1,2\}} + 1\Delta_{\{2,3\}} + 1\Delta_{\{1,2,3\}}.$$

Recall Fig. 2 for a visualisation of this equation of polytopes.

Example 4.7 Consider $\mathsf{D}_{c_2} = \{1, 3\}$ and $\mathsf{U}_{c_2} = \{2\}$. The associahedron $\mathsf{As}_2^{c_2}$ is isometric to $\mathsf{As}_2^{c_1}$, [2], but it is not the Minkowski sum of faces of a standard simplex as we show now.

(i) We have $y_I = 1$ for $I = \{1\}$ and $I = \{3\}$.
The up and down interval decomposition is $(0,3)_{\mathsf{D}}$ and $(1,4)_{\mathsf{D}}$ respectively. Therefore we have $\gamma = \Gamma = 1$ and $\gamma = \Gamma = 3$ respectively. It follows $K_\gamma = K_\Gamma = -1$ and $I \backslash (a,b)_{\mathsf{D}} = \varnothing$.

(ii) We have $y_I = 0$ for $I = \{2\}$.
The up and down interval decomposition is $(1,3)_{\mathsf{D}} \sqcup [2,2]_{\mathsf{U}}$, so I is of type $(1,1)$. We have $\gamma = \Gamma = 2$ which implies $K_\gamma = K_\Gamma = 2$. Since $n = 3$, we conclude $y_I = (3+1) - 2 \cdot 2 = 0$. Of course, we could have used Corollary 4.5 instead.

(iii) We have $y_I = 2$ for $I = \{i, i+1\}$ and $1 \le i \le 2$.
Then $I = (i-1, i+2)_{\mathsf{D}}$, $\gamma = i$, and $\Gamma = i+1$, that is, $K_\gamma = -1$, $K_\Gamma = 2$. Moreover, $I \backslash (a,b)_{\mathsf{D}} = \{2\}$ if $I = \{1,2\}$ and $K_\gamma = 2$, $K_\Gamma = -1$, and $I \backslash (a,b)_{\mathsf{D}} = \{2\}$ if $I = \{2,3\}$.

(iv) We have $y_I = 1$ for $I = \{1,3\}$.
Then $I = (0,4)_{\mathsf{D}}$, $\gamma = 1$, and $\Gamma = 3$. It follows that $K_\gamma = K_\Gamma = -1$ and $I \backslash (a,b)_{\mathsf{D}} = \varnothing$.

(v) We have $y_I = -1$ for $I = \{1,2,3\}$.
Then $I = (0,4)_{\mathsf{D}} \sqcup [2,2]_{\mathsf{U}}$ with $\gamma = 1$ and $\Gamma = 3$. It follows that $K_\gamma = K_\Gamma = -1$ and $I \backslash (a,b)_{\mathsf{D}} = \{2\}$. Again, we could have used Corollary 4.4 instead.

Thus, we obtain the following Minkowski decomposition into dilated faces of the standard simplex:

$$\mathsf{As}_2^{c_2} = 1\Delta_{\{1\}} + 1\Delta_{\{3\}} + 2\Delta_{\{1,2\}} + 1\Delta_{\{1,3\}} + 2\Delta_{\{2,3\}} + (-1)\Delta_{\{1,2,3\}}.$$

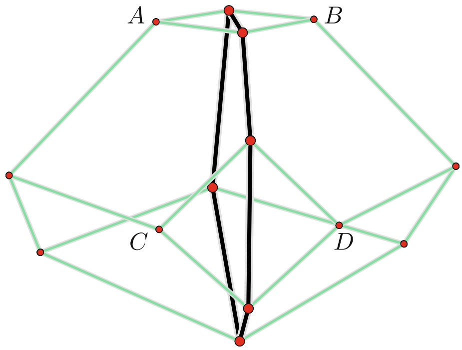
Recall that an illustration of this decomposition is given in Fig. 3.

5 A Remark on Cyclohedra

We now show that Proposition 1.2 does not hold if we consider a polytope obtained by 'moving some inequalities of the permutahedron past vertices'. The example is a cyclohedron which also known as Bott-Taubes polytope or type B generalised permutahedron [4,7,26]. A Minkowski decomposition of 'generalised permutahedra of type B' (similar to Proposition 1.2 for generalised permutahedra) is not known.

The canonical embedding of the hyperoctahedral group W_n in the symmetric group S_{2n} induces realisations Cy_n^c of cyclohedra using realisations As_{2n-1}^c for certain *symmetric* choices c. To obtain these realisations of cyclohedra, we follow [11] and intersect As_{2n-1}^c with 'type B hyperplanes' $x_i + x_{2n+1-i} = 2n - 1$ for $1 \le i < n$. A 2-dimensional cyclohedron Cy_2^c obtained from As_3^c (with up set $\mathsf{U}_c = \{2\}$) by intersection with $x_1 + x_4 = 5$ is shown in Fig. 10 (the hyperplane $x_2 + x_3 = 5$ is implicitly used since As_3^c is contained in $x_1 + x_2 + x_3 + x_4 = 10$). A similar construction does not yield a cyclohedron if one starts with the other associahedron of Fig. 1 where $\mathsf{U}_c = \{2,3\}$. The tight right-hand sides of this realisation of the cyclohedron are obviously the tight right-hand sides of As_3^c except $z_{\{1,4\}}^c = z_{\{2,3\}}^c = 5$. The inequalities $x_1 + x_4 \ge 2$ and $x_2 + x_3 \ge 2$ are redundant for As_3^c and altering the level sets for these inequalities from 2 (for As_3^c) to 5 (for Cy_2^c) means that we move past the four vertices A, B, C,

Fig. 10 A 2-dimensional cyclohedron Cy_2 (*black*) obtained from As_3^c



and D, so the realisation of the cyclohedron is not in the deformation cone of the classical permutahedron. We now show by example that Proposition 1.2 does not hold in this situation. To this respect, we list the function z_I of tight right hand-sides for all inequalities of the permutahedron (that is, facet-defining or not for the cyclohedron) and its Möbius inverse y_I, both defined on the boolean lattice:

$$\begin{array}{c} z_{\{1,2,3,4\}} = 10 \\ y_{\{1,2,3,4\}} = 5 \end{array}$$

$$\begin{array}{cccc} z_{\{1,2,3\}} = 6 & z_{\{1,2,4\}} = 4 & z_{\{1,3,4\}} = 6 & z_{\{2,3,4\}} = 6 \\ y_{\{1,2,3\}} = -4 & y_{\{1,2,4\}} = -3 & y_{\{1,3,4\}} = -2 & y_{\{2,3,4\}} = -1 \end{array}$$

$$\begin{array}{cccccc} z_{\{1,2\}} = 3 & z_{\{1,3\}} = 3 & z_{\{1,4\}} = 5 & z_{\{2,3\}} = 5 & z_{\{2,4\}} = 0 & z_{\{3,4\}} = 3 \\ y_{\{1,2\}} = 3 & y_{\{1,3\}} = 1 & y_{\{1,4\}} = 3 & y_{\{2,3\}} = 5 & y_{\{2,4\}} = 0 & y_{\{3,4\}} = 1 \end{array}$$

$$\begin{array}{cccc} z_{\{1\}} = 1 & z_{\{2\}} = -1 & z_{\{3\}} = 1 & z_{\{4\}} = 1 \\ y_{\{1\}} = 1 & y_{\{2\}} = -1 & y_{\{3\}} = 1 & y_{\{4\}} = 1 \end{array}.$$

In other words, if Proposition 1.2 were true for 'generalised permutahedra not in the deformation cone of the classical permutahedron', then the following equation of polytopes has to hold:

$$\begin{aligned} &\mathsf{Cy}_2^c + (\Delta_2 + 4\Delta_{123} + 3\Delta_{124} + 2\Delta_{134} + \Delta_{234}) \\ &\quad = \Delta_1 + \Delta_3 + \Delta_4 + 3\Delta_{12} + \Delta_{13} + 3\Delta_{14} + 5\Delta_{23} + \Delta_{34} + 5\Delta_{1234}. \end{aligned}$$

One way to see that this equation does not hold is to compute the number of vertices of the polytope on the left-hand side (27 vertices) and on the right-hand side (20 vertices) using for example `polymake` [10].

6 A Proof of Theorem 4.2

This section is devoted to the proof of Theorem 4.2 under the assumption that Lemma 6.3 holds; Lemma 6.3 is proved in Sect. 7. We begin with an outline of the strategy to prove Theorem 4.2.

First, we prove Proposition 6.2 which weakens Theorem 4.2 in two senses: we restrict to $I \subset [n]$ with a nested decomposition and we restrict to the situation where $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, that is, where all four diagonals δ_i are proper. That the statement of Proposition 6.2 is actually the statement of Theorem 4.2 weakened by these additional assumptions follows from Corollary 6.7.

Lemma 6.3 states precisely which subsets of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ are sets $\mathscr{D}_I$ for some $I \subset [n]$ with a nested up and down interval decomposition. Lemma 6.4 then expresses the Minkowski coefficients y_I using these sets $\mathscr{D}_I$ if $I \subset [n]$ has a nested up and down interval decomposition and $|\mathscr{D}_I| < 4$. Lemmas 6.5 and 6.6 then imply the claim of Theorem 4.2 when $I \subset [n]$ has a nested decomposition and not all δ_i are proper. Finally, Lemma 6.8 covers the cases $I \subset [n]$ where I does not have a nested decomposition and Lemma 6.9 settles $I = [n]$.

It will be convenient to rewrite Eq. (1) that was obtained at the beginning of Sect. 4 by combination of Propositions 1.2 and 3.8:

$$\begin{aligned} y_I &= \sum_{J\subseteq I}(-1)^{|I\setminus J|}\sum_{i\in[v_J]}\Big(\sum_{j\in[w_i+1]}\tilde{z}^c_{R_{\delta^J_{i,j}}} - w_i\tilde{z}^c_{[n]}\Big) \\ &= \sum_{J\subseteq I}(-1)^{|I\setminus J|}\sum_{i\in[v_J]}\Big(\tilde{z}^c_{R_{\delta^J_{i,m_{J,i}}}} + \sum_{j\in[m_{J,i}-1]}\big(\tilde{z}^c_{R_{\delta^J_{i,j}}} - \tilde{z}^c_{[n]}\big)\Big) \end{aligned}$$

where $m_{J,i}$ is either w_i^J or $w_i^J + 1$ in order to simplify the involved sum.

Suppose now that the proper diagonal δ occurs on the right-hand side of this rewritten formula for y_I, that is, δ is one of the associated diagonals $\delta^J_{i,j}$ for some $J \subseteq I$. We now distinguish whether δ occurs as a single summand $\tilde{z}^c_{R_{\delta^J_{i,m_{J,i}}}}$ or as a compound summand $(\tilde{z}^c_{R_{\delta^J_{i,j}}} - \tilde{z}^c_{[n]})$ and make the following definition.

Definition 6.1 Let $I \subset [n]$ be non-empty and $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$.

(a) A proper diagonal δ (associated to $J \subseteq I$) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I), if there exists an index $i \in [v_J]$ such that $\delta = \delta^J_{i,m_{J,i}}$.
(b) A proper diagonal δ (associated to $J \subseteq I$) is of type $\big(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]}\big)$ (in the expression for y_I), if there exist indices $i \in [v_J]$ and $j \in [m_{J,i} - 1]$ such that $\delta = \delta^J_{i,j}$

A geometric interpretation of these notions is the following. The proper diagonal δ (associated to $J \subseteq I$) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I), if δ is the 'rightmost' proper diagonal associated to a nested component of J. Similarly, the proper diagonal δ (associated to $J \subseteq I$) is of type $\big(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]}\big)$ (in the expression for y_I), if δ is a proper diagonal associated to a nested component of J, but it is not the rightmost one.

Proposition 6.2 *Let I be a non-empty proper subset of $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ with up and down interval decomposition of type $(1, w)$ and $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_3, \delta_4\}$. Then the Minkowski coefficient y_I of the generalised permutahedron $P(\{\tilde{z}^c_I\})$ with normal fan of As^c_{n-1} is given by*

$$y_I = \sum_{\delta\in\mathscr{D}_I}(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}.$$

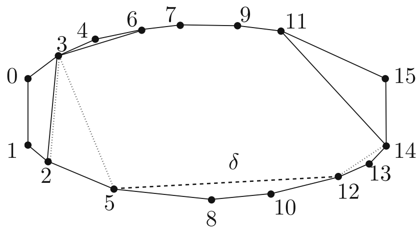


Fig. 11 Let $I = \{3, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ with $\gamma = 3 \in \mathsf{U}_c$ and $\Gamma = 13 \in \mathsf{D}_c$. Its up and down decomposition is $I = (2, 14)_{\mathsf{D}_c} \cup [3, 3]_{\mathsf{U}_c} \cup [6, 11]_{\mathsf{U}_c}$. The diagonal $\delta = \{5, 12\}$ appears in the right hand side for y_I since $(5, 12)_{\mathsf{D}_c} \subseteq (2, 14)_{\mathsf{D}_c}$ and the up and down interval decomposition of R_δ has type $(1, 0)$. Since $(2, 5) \cap I = \{3\}$ and $(12, 14) \cap I = \{13\}$, δ is associated to $S \in \{\{8, 10\}, \{3, 8, 10\}, \{8, 10, 13\}, \{3, 8, 10, 13\}\}$, the diagonals associated to the up and down interval decompositions of S form a subset of the dashed diagonals. The contribution of δ to y_I vanishes. The only diagonals in this example associated to some $J \subseteq I$ with up and down interval decomposition of type $(1, 0)$ and non-vanishing contribution to y_I are $\delta_1 = \{2, 14\}$ and $\delta_2 = \{2, 13\}$

The proof is not difficult but long and convoluted, so we first outline the proof. The goal is to simplify the rewritten Eq. (1) for y_I stated above. To that respect, we first study the potential contribution of a proper diagonal δ that occurs in the sum on the right-hand side. Given such a diagonal δ, we study which sets $S \subseteq I$ satisfy $\delta \in \mathcal{D}_S$ in order to collect all terms that involve $z^c_{R_\delta}$. We will show that the corresponding sum vanishes often. This result is obtained by a case study that depends on the type of the up and down interval decomposition of R_δ. Since the up and down interval decomposition of R_δ is of type $(1, 0)$, $(1, 1)$ or $(1, 2)$ for any proper diagonal δ, we study these cases in detail. After the necessary information is deduced for every possible diagonal δ, we further simplify the formula for y_I by another case study that distinguishes whether γ or Γ is element of D_c or U_c.

Proof By assumption, the set $I \subset [n]$ has an up and down interval decomposition of type $(1, w)$, that is, $I = (a, b)_{\mathsf{D}_c} \sqcup \bigsqcup_{j=1}^{w} [\alpha_j, \beta_j]_{\mathsf{U}_c}$. Let δ be some diagonal $\delta^J_{i,j}$ that occurs on the right-hand side of the equation for y_I. In other words, δ is a proper and non-degenerate diagonal $\delta^J_{i,j}$ associated to the up and down interval decomposition of type (v^J, w^J) for some $J \subseteq I$. By Example 3.5, the up and down interval decomposition of R_δ is either of type $(1, 0)$, $(1, 1)$ or $(1, 2)$. A good understanding which sets $S \subseteq I$ (besides J) satisfy $\delta \in \mathcal{D}_S$ is essential for the simplification. The complete proof is basically a case study of these three cases for R_δ.

1. R_δ has up and down decomposition of type $(1, 0)$, see Fig. 11.
   Then $R_\delta = (\tilde{a}, \tilde{b})_{\mathsf{D}_c} \subseteq (a, b)_{\mathsf{D}_c}$ and we may consider $J = R_\delta \subseteq I$ as witness for the occurrence of δ in the right-hand side of (1). Let $S \subseteq I$ be a set with $\delta \in \mathcal{D}_S$. Then $J = (\tilde{a}, \tilde{b})_{\mathsf{D}_c}$ is necessarily a nested component of type $(1, 0)$ of S and all other nested components are subsets of $(a, \tilde{a}) \cap I$ and $(\tilde{b}, b) \cap I$. It follows that $S \subseteq I$ satisfies $\delta \in \mathcal{D}_S$ if and only if

$$R_\delta \subseteq S \subseteq R_\delta \cup \big((a, \tilde{a}) \cap I\big) \cup \big((\tilde{b}, b) \cap I\big).$$

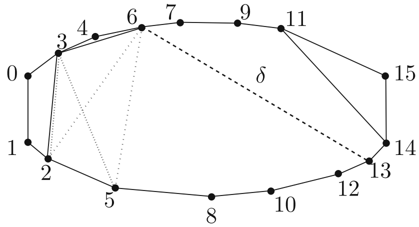


Fig. 12 Consider I as in Fig. 11. For $\delta=\{6,13\}$, the up and down interval decomposition of R_δ is of the required sub-type of $(1,1)$. δ is associated to $S\in\{\{3,5,6,8,10,12\},\{5,6,8,10,12\},\{3,6,8,10,12\},\{6,8,10,12\}\}$ since $(2,6)\cap I=\{3,5\}$ and $(13,14)\cap I=\varnothing$ and some of the diagonals are associated to the interval decomposition of S. The contribution of δ to y_I vanishes. Diagonals of the required sub-type of $(1,1)$ and non-vanishing contribution to y_I are the diagonals $\delta_3=\{3,14\}$ and $\delta_4=\{3,13\}$

We now collect all terms for $\tilde{z}^c_{R_\delta}$ in the expression for y_I. Since δ is a proper diagonal, we have $\tilde{z}^c_{R_\delta}\neq 0$ and the resulting alternating sum vanishes if and only if there is more than one term of this type, that is, if and only if

$$\big((a,\tilde{a})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)\neq\varnothing.$$

If $\big((a,\tilde{a})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)=\varnothing$, we obtain $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ as contribution for y_I. For later use in this proof, we note that $\big((a,\tilde{a})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)=\varnothing$ guarantees $\delta\in\mathscr{D}_I$. Note that R_{δ_1} is always of type $(1,0)$ if the up and down decomposition of R_δ is of type $(1,0)$. Similarly, we have $\delta_2\in\mathscr{D}_I$ with R_{δ_2} of type $(1,0)$ if additionally $\Gamma\in\mathsf{D}_c$, $\delta_3\in\mathscr{D}_I$ with R_{δ_3} of type $(1,0)$ if additionally $\gamma\in\mathsf{D}_c$, and $\delta_4\in\mathscr{D}_I$ with R_{δ_4} of type $(1,0)$ if additionally $\gamma,\Gamma\in\mathsf{D}_c$.

2. R_δ has up and down decomposition of type $(1,1)$.
   In contrast to Case 1, $R_\delta\subseteq I$ is not true in general any more. We distinguish two cases, either $\delta=\{\tilde{\beta},\tilde{b}\}$ with $\tilde{\beta}<\tilde{b}$, $\tilde{\beta}\in\mathsf{U}_c$ and $\tilde{b}\in\mathsf{D}_c$ or $\delta=\{\tilde{a},\tilde{\alpha}\}$ with $\tilde{a}<\tilde{\alpha}$, $\tilde{a}\in\mathsf{D}_c$ and $\tilde{\alpha}\in\mathsf{U}_c$.
   a. $\delta=\{\tilde{\beta},\tilde{b}\}$, see Fig. 12
      Observe first that $R_\delta=(0,\tilde{b})_{\mathsf{D}_c}\cup[u_1,\tilde{\beta}]_{\mathsf{U}_c}$ with $\tilde{\beta}<\tilde{b}\leq b$. Since we assume that δ appears in the right-hand side of (1), we have $\tilde{\beta}\in I$ and may consider $J=R_\delta\cap I$.
      If $S\subseteq I$ is a subset with $\delta\in\mathcal{D}_S$ then δ must be the 'rightmost' diagonal of one nested component for S. This means that the diagonal δ associated to S is never of type $(\tilde{z}^c_{R_\delta}-\tilde{z}^c_{[n]})$ in the expression for y_I. Similarly to Case 1, we conclude that the terms $\tilde{z}^c_{R_\delta}$ cancel if and only if

$$\big((a,\tilde{\beta})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)\neq\varnothing\qquad\text{or}\qquad\tilde{z}^c_{R_\delta}=0.$$

      Again, $\tilde{z}^c_{R_\delta}\neq 0$ since δ is a proper diagonal and the terms for $\tilde{z}^c_{R_\delta}$ do not cancel if and only if there is only one subset $S\subseteq I$ with $\delta\in\mathcal{D}_S$, that is, if $((a,\tilde{\beta})\cap I)\cup((\tilde{b},b)\cap I)=\varnothing$.

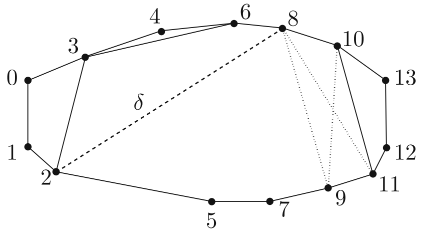


Fig. 13 Let $I = \{3, 5, 6, 7, 8, 9, 10\}$ with $\gamma = 3 \in \mathsf{U}_c$ and $\Gamma = 10 \in \mathsf{D}_c$. Its up and down decomposition is $I = (2, 11)_{\mathsf{D}_c} \cup [3, 3]_{\mathsf{U}_c} \cup [6, 10]_{\mathsf{U}_c}$. For $\delta = \{2, 8\}$, the up and down interval decomposition of R_δ is of the required sub-type of (1, 1). Since $(2, 2) \cap I = \varnothing$ and $(8, 11) \cap I = \{9, 10\}$, δ is associated to $S \in \{\{5, 7, 8\}, \{5, 7, 8, 9\}, \{5, 7, 8, 10\}, \{5, 7, 8, 9, 10\}\}$. Thus δ does not contribute to y_I. In this figure, only $\delta_2 = \{2, 10\}$ contributes $-(\tilde{z}^c_{R_{\delta_2}} - \tilde{z}_{[n]})$ to y_I

For later use in this proof, we mention the two possible scenarios if

$$((a, \tilde{\beta}) \cap I) \cup ((\tilde{b}, b) \cap I) = \varnothing.$$

Firstly, if $\gamma \in \mathsf{U}_c$ and $\Gamma \in \mathsf{D}_c$, then $\delta \in \{\delta_3, \delta_4\}$ and the contribution of δ_3 and δ_4 to y_I is

$$(-1)^{|I \setminus R_{\delta_3}|} \tilde{z}^c_{R_{\delta_3}} \qquad \text{and} \qquad (-1)^{|I \setminus R_{\delta_4}|} \tilde{z}^c_{R_{\delta_4}}.$$

Secondly, if $\gamma, \Gamma \in \mathsf{U}_c$, then $\delta = \delta_3$ and the contribution to y_I is $(-1)^{|I \setminus R_{\delta_3}|} \tilde{z}^c_{R_{\delta_3}}$.

b. $\delta = \{\tilde{a}, \tilde{\alpha}\}$

Observe first that $R_\delta = (\tilde{a}, n+1)_{\mathsf{D}_c} \cup [\tilde{\alpha}, u_m]_{\mathsf{U}_c}$ with $a \le \tilde{a} < \tilde{\alpha}$. Since we assume that δ appears in the right-hand side of (1), we have $\tilde{\alpha} \in I$ and may consider $J = R_\delta \cap I$.

If $S \subseteq I$ with $\delta \in \mathcal{D}_S$, then δ (associated to S) can be of type $\tilde{z}^c_{R_\delta}$ or $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ in the expression for y_I. The diagonal δ is of type $\tilde{z}^c_{R_\delta}$ if and only if $R_\delta = R_\delta \cap I$ and $S = R_\delta \cup M$ for some subset $M \subseteq (a, \tilde{a}) \cap I$. The diagonal δ is of type $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ for all other subsets $S \subseteq I$ with $\delta \in \mathcal{D}_S$, in particular, we conclude $R_\delta \supset R_\delta \cap S$.

We distinguish two sub-cases: either δ (associated to S) is of type $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ (in the expression for y_I) for all $S \subseteq I$ with $\delta \in \mathcal{D}_S$ or there is an $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that δ (associated to S) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I).

i. δ is of type $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ for all $S \subseteq I$ with $\delta \in \mathcal{D}_S$, see Fig. 13.

As mentioned, we have $R_\delta \supset R_\delta \cap S$ for all sets $S \subseteq I$ with $\delta \in \mathcal{S}$. Moreover, these sets are in bijection to the subsets of $\big((a, \tilde{a}) \cap I\big) \cup \big((\tilde{\alpha}, b) \cap I\big)$:

$$S = \big(R_\delta \cap (I \setminus B)\big) \cup A \qquad \text{for } A \subseteq (a, \tilde{a}) \cap I \text{ and } B \subseteq (\tilde{\alpha}, b) \cap I.$$

If there is more than one set $S \subseteq I$ with $\delta \in \mathcal{D}_S$, then collecting all summands $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ in the expression for y_I yields a vanishing alternating sum. If there is only one set $S \subseteq I$ with $\delta \in \mathcal{D}_S$ as associated diagonal then $\big((a, \tilde{a}) \cap I\big) \cup \big((\tilde{\alpha}, b) \cap I\big) = \varnothing$ and it follows that $\Gamma = \tilde{\alpha} \in \mathsf{U}_c$ and $\tilde{a} \in \{a, \gamma\} \cap \mathsf{D}_c$.

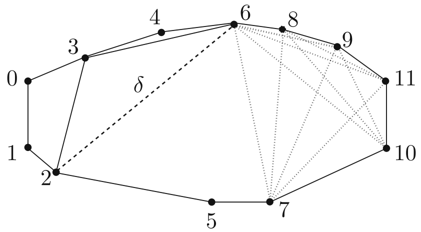


Fig. 14 Consider $I = \{3, 5, 6, 7, 8, 9, 10\}$ with $\gamma = 3 \in \mathsf{U}_c$ and $\Gamma = 10 \in \mathsf{D}_c$. The up and down interval decomposition is $I = (2, 11)_{\mathsf{D}_c} \cup [3, 3]_{\mathsf{U}_c} \cup [6, 9]_{\mathsf{U}_c}$. For $\delta = \{2, 6\}$, the up and down interval decomposition of R_δ has the required sub-type of $(1, 1)$. We have $R_\delta = R_\delta \cap I$, so $J = \{5, 6, 7, 8, 9, 10\}$ is the unique $J \subseteq I$ such that $\delta \in \mathcal{D}_J$ is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I). For all other $S \subseteq I$ with $\delta \in \mathcal{D}_S$, δ is of type $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ in the expression of y_I. Since $\gamma \in \mathsf{U}_c$, δ contributes $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{[n]} = -\tilde{z}^c_{[n]}$ to y_I. In this example $\delta = \{2, 8\}$ and $\delta = \{2, 9\}$, contribute $-\tilde{z}^c_{[n]}$ and $\tilde{z}^c_{[n]}$ to y_I

For later use in this proof, we note that $\gamma \in \mathsf{D}_c$ implies $\delta \in \{\delta_2, \delta_4\}$. The only possible contributions of δ in the expression for y_I are therefore

$$(-1)^{|I\setminus R_{\delta_2}|}(\tilde{z}^c_{R_{\delta_2}} - \tilde{z}^c_{[n]}) \qquad \text{and} \qquad (-1)^{|I\setminus R_{\delta_4}|}(\tilde{z}^c_{R_{\delta_4}} - \tilde{z}^c_{[n]}).$$

But since the corresponding subsets $R_{\delta_2} \cap I$ and $R_{\delta_4} \cap I$ differ by γ, the contribution to y_I can be simplified to

$$(-1)^{|I\setminus R_{\delta_2}|}\tilde{z}^c_{R_{\delta_2}} + (-1)^{|I\setminus R_{\delta_4}|}\tilde{z}^c_{R_{\delta_4}}.$$

If $\gamma \in \mathsf{U}_c$, then $\delta = \delta_2$ and we obtain

$$(-1)^{|I\setminus R_{\delta_2}|}(\tilde{z}^c_{R_{\delta_2}} - \tilde{z}^c_{[n]})$$

as contribution for y_I.

ii. There is an $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that δ is of type $\tilde{z}^c_{R_\delta}$, see Fig. 14. Since δ must be the 'rightmost' diagonal associated to S if δ (associated to S) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I), we conclude $R_\delta = R_\delta \cap I$. In particular, we have $\Gamma = n$ and $b = n + 1$ and thus $(\tilde{\alpha}, b) \cap I \neq \varnothing$ and $(\tilde{\alpha}, b) \cap I = (\tilde{\alpha}, b)$. If $(a, \tilde{a}) \cap I \neq \varnothing$, then collecting terms for $\tilde{z}^c_{R_\delta}$ and $\tilde{z}^c_{[n]}$ in the expression for y_I again yields no contribution. We may therefore assume $(a, \tilde{a}) \cap I = \varnothing$, that is $\tilde{a} \in \{a, \gamma\} \cap \mathsf{D}_c$. First suppose that $\gamma \in \mathsf{D}_c$. Then δ is either $\delta_a = \{a, \tilde{\alpha}\}$ or $\delta_\gamma = \{\gamma, \tilde{\alpha}\}$. Now δ is of type $\tilde{z}^c_{R_\delta}$ in the expression of y_I if and and only if δ is associated to R_{δ_a} or R_{δ_γ}. In all other situations, δ is of type $(\tilde{z}^c_{R_\delta} - \tilde{z}^c_{[n]})$ in the expression of y_I and is associated to a set $R_{\delta_a}\setminus M$ or $R_{\delta_\gamma}\setminus M$ with non-empty $M \subseteq (\tilde{\alpha}, n+1)$. Collecting terms for $\tilde{z}^c_{R_{\delta_a}}$, $\tilde{z}^c_{R_{\delta_\gamma}}$, and $\tilde{z}^c_{[n]}$ yields a vanishing contribution as desired (collecting the terms for $\tilde{z}^c_{[n]}$ for fixed δ does not yield a vanishing contribution, but the terms from δ_a and δ_γ cancel). If $\gamma \in \mathsf{U}_c$ then a similar argument gives

$$(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{[n]} \qquad \text{for } \delta = \{a, \tilde{\alpha}\} \text{ with } \tilde{\alpha} \in \mathsf{U}_c \text{ and } R_\delta = R_\delta \cap I$$

as contribution for y_I.

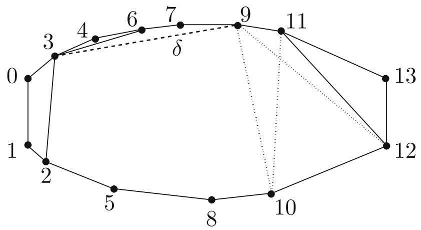


Fig. 15 Consider $I = \{3, 5, 6, 7, 8, 9, 10, 11\}$ with $\gamma = 3 \in \mathsf{U}_c$ and $\Gamma = 11 \in \mathsf{U}_c$. The up and down interval decomposition is $I = (2, 12)_{\mathsf{D}_c} \cup [3, 3]_{\mathsf{U}_c} \cup [6, 11]_{\mathsf{U}_c}$. For $\delta = \{3, 9\}$, the up and down interval decomposition of R_δ is of the required sub-type of $(1, 2)$ and δ does not contribute to y_I since $(\beta, b) \cap I = (9, 12) = \{10, 11\}$. In this example, only $\delta_4 = \{3, 11\}$ contributes to y_I

3. R_δ has up and down decomposition of type $(1, 2)$.
   If R_δ is of type $(1, 2)$ then $\delta = \{\alpha, \beta\}$ with $\alpha, \beta \in \mathsf{U}_c$ and there is $u \in \mathsf{U}_c$ such that $a < \alpha < u < \beta < b$. This in turn gives

$$R_\delta = (0, n + 1)_{\mathsf{D}_c} \cup [u_1, \alpha]_{\mathsf{U}_c} \cup [\beta, u_m]_{\mathsf{U}_c}$$

   as up and down interval decomposition for R_δ. By arguments as before, we conclude that collecting terms for $\tilde{z}^c_{R_\delta}$ and $\tilde{z}^c_{[n]}$ yields a vanishing contribution to y_I if $(a, \alpha) \cap I \neq \varnothing$. We therefore assume that $(a, \alpha) \cap I = \varnothing$ which is equivalent to $\gamma = \alpha \in \mathsf{U}_c$. As a consequence, δ is an associated diagonal of $S \subseteq I$ if and only if $S = (R_\delta \cap I)\backslash M$ for some $M \subseteq (\beta, b) \cap I$.
   We now distinguish two cases: either there is an $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that δ (associated to S) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I) or not.
   a. There is no $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that δ (associated to J) is of type $\tilde{z}^c_{R_\delta}$, see Fig. 15.
      If $(\beta, b) \cap I \neq \varnothing$ then collecting the terms $\tilde{z}^c_{R_\delta}$ and $\tilde{z}^c_{[n]}$ cancel respectively. If $(\beta, b) \cap I = \varnothing$ then we have $\Gamma = \beta \in \mathsf{U}_c$ and $\delta = \delta_4$. In this situation, δ has a unique contribution to y_I which equals $(-1)^{|I\backslash R_{\delta_4}|}(\tilde{z}^c_{R_{\delta_4}} - \tilde{z}^c_{[n]})$.
   b. There is a set $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that δ is of type $\tilde{z}^c_{R_\delta}$, see Fig. 16.
      Since δ is the 'rightmost' diagonal associated to $S \subseteq I$ and since $(a, \alpha) \cap I = \varnothing$, we conclude that $b = n+1$ and $\Gamma = n \in \mathsf{D}_c$ (recall that we also have $\alpha = \gamma \in \mathsf{U}_c$). Now observe that the set $S \subseteq I$ with $\delta \in \mathcal{D}_S$ such that δ (associated to S) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I) is unique: it is $R_\delta \cap I$. In particular, we have $[\beta, n] \cap I = [\beta, n]$. Collecting terms $\tilde{z}^c_{R_\delta}$ for all subsets $S \subseteq I$ with $\delta \in \mathcal{D}_S$ cancel, but collecting the terms $\tilde{z}^c_{[n]}$ does not vanish: we have a contribution of $(-1)^{|I\backslash R_\delta|}\tilde{z}^c_{[n]}$ to y_I. We conclude that every diagonal $\delta = \{\gamma, \beta\}$ with $\beta \in \mathsf{U}_c$, $[\beta, n] \cap I = [\beta, n]$ and $\{\gamma, \beta\} \neq \{u_r, u_{r+1}\}$ contributes $(-1)^{|I\backslash R_\delta|}\tilde{z}^c_{[n]}$ to y_I.

After this analysis of possible contributions to y_I induced by proper diagonals, we now prove

$$y_I = \sum_{\delta \in \mathscr{D}_I} (-1)^{|I\backslash R_\delta|}\tilde{z}^c_{R_\delta},$$

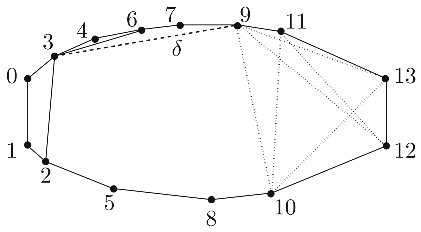


Fig. 16 Consider $I = \{3, 5, 6, 7, 8, 9, 10, 11, 12\}$ with $\gamma = 3 \in \mathsf{U}_c$ and $\Gamma = 12 \in \mathsf{D}_c$. The up and down interval decomposition is $I = (2, 13)_{\mathsf{D}_c} \cup [3, 3]_{\mathsf{U}_c} \cup [6, 11]_{\mathsf{U}_c}$. For $\delta = \{3, 9\}$, the up and down interval decomposition of R_δ is of the required sub-type of $(1, 2)$ and since $(\beta, b)\cap = I(9, 13) \cap I = \{10, 11, 12\} \neq \varnothing$, the diagonal δ is associated to eight sets. The contribution of δ to y_I is $-\tilde{z}^c_{[n]}$. In this example, the four diagonals $\delta' \in \{\{3, 6\}, \{3, 7\}, \{3, 9\}, \{3, 11\}\}$ are of the required sub-type of $(1, 2)$ each contributes $(-1)^{|I\setminus R_{\delta'}|}\tilde{z}^c_{R_{\delta'}}$ to y_I

where we assume that I is a non-empty proper subset of $[n]$ with a nested up and down decomposition and $|\mathscr{D}_I| = 4$. We distinguish the following four cases:

1. $\gamma, \Gamma \in \mathsf{D}_c$.
   Then δ_1, δ_2, δ_3, and δ_4 contribute $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ to y_I according to Case 1 and no other diagonal contributes according to the previous analysis. The claim follows immediately.
2. $\gamma \in \mathsf{D}_c$ and $\Gamma \in \mathsf{U}_c$.
   Then δ_1 and δ_3 contribute $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ to y_I according to Case 1, while δ_2 and δ_4 contribute $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ to y_I according to Case 2(b)i. No other diagonal contributes to y_I. The claim follows immediately.
3. $\gamma, \Gamma \in \mathsf{U}_c$.
   The only diagonals with a contribution to y_I are δ_1 (Case 1), δ_2 (Case 2(b)i), δ_3 (Case 2a) and δ_4 (Case 3a). Taking their contribution into account, we obtain

$$y_I = (-1)^{|I\setminus R_{\delta_1}|}\tilde{z}^c_{R_{\delta_1}} + (-1)^{|I\setminus R_{\delta_2}|}(\tilde{z}^c_{R_{\delta_2}} - \tilde{z}^c_{[n]}) + (-1)^{|I\setminus R_{\delta_3}|}\tilde{z}^c_{r_{\delta_3}} + (-1)^{|I\setminus R_{\delta_4}|}(\tilde{z}^c_{R_{\delta_4}} - \tilde{z}^c_{[n]}).$$

   The claim follows since $I\setminus R_{\delta_2}$ and $I\setminus R_{\delta_4}$ differ by γ.
4. $\gamma \in \mathsf{U}_c$ and $\Gamma \in \mathsf{D}_c$.
   We distinguish the two sub-cases $\Gamma \neq n$ and $\Gamma = n$.
   (a) $\Gamma \neq n$ implies that there is no $u \in \mathsf{U}_c$ such that $[u, n] = [u, n] \cap I$.
   In this situation, δ_1 and δ_3 contribute $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ to y_I according to Case 1 and δ_2 and δ_4 contribute $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ according to Case 2a. No other diagonal contributes, so the claim follows immediately.
   (b) $\Gamma = n$.
   If there is no $u \in \mathsf{U}_c$ such that $[u, n] = [u, n] \cap I$ then δ_1 and δ_2 contribute according to Case 1 and δ_3 and δ_4 contribute according to Case 2a. No other diagonal contributes, so the claim follows immediately.
   If there exists $u \in \mathsf{U}_c$ such that $[u, n] = [u, n] \cap I$ then denote by $u_{\min}$ the smallest element of U_c such that $[u_{\min}, n] = [u_{min}, n] \cap I$. Now diagonals δ_1 and δ_2 contribute to y_I according to Case 1 and diagonals δ_3, δ_4 according to

Case 2a. But in this situation, according to Cases 2(b)ii and 3b, we also have contributions of diagonals $\{a, u\}$ and $\{\gamma, u\}$ for $u \in [u_{min}, u_m]_{\mathsf{U}_c}$. This yields

$$\sum_{\delta\in\mathscr{D}_I} (-1)^{|I\setminus R_\delta|}\tilde{z}_{R_\delta} + \sum_{\substack{\delta=\{a,\alpha\}\text{ with}\\ \alpha\in[u_{min},u_m]_{\mathsf{U}_c}}} (-1)^{|I\setminus R_\delta|}\tilde{z}^c_{[n]} + \sum_{\substack{\delta=\{\gamma,\alpha\}\notin\partial Q\text{ with}\\ \alpha\in[u_{min},u_m]_{\mathsf{U}_c}}} (-1)^{|I\setminus R_\delta|}\tilde{z}^c_{[n]}.$$

But the second and third sum cancel, so we end up with the claim. □

In fact, the methods used in the proof of Proposition 6.2 suffice to prove the degenerate cases $\mathscr{D}_I \neq \{\delta_1, \delta_2, \delta_3, \delta_4\}$ as well. But before we try to analyse these cases, we remark that some subsets of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ never form a set $\mathscr{D}_I$ associated to $I \subseteq [n]$ and $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$.

Lemma 6.3 *Let $n \geq 3$ and $I \subset [n]$ be non-empty with up and down interval decomposition of type $(1, w)$. Then*

(a) *There is no partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c and no non-empty $I \subset [n]$ such that $\mathscr{D}_I$ is one of the following sets:*

$$\varnothing,\quad \{\delta_2\},\quad \{\delta_3\},\quad \{\delta_4\},\quad \{\delta_1,\delta_2\},\quad \{\delta_1,\delta_3\},\quad \{\delta_2,\delta_4\},\quad \textit{or}\quad \{\delta_3,\delta_4\}.$$

(b) *There is a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c and a non-empty $I \subset [n]$ such that $\mathscr{D}_I$ is one of the following sets:*

$$\{\delta_1\},\ \{\delta_1,\delta_4\},\ \{\delta_2,\delta_3\},\ \{\delta_1,\delta_2,\delta_3\},\ \{\delta_1,\delta_2,\delta_4\},\ \{\delta_1,\delta_3,\delta_4\},\ \{\delta_2,\delta_3,\delta_4\},$$
$$\textit{or } \{\delta_1,\delta_2,\delta_3,\delta_4\}.$$

The proof of Part (a) is left to the reader, while the situation of Part (b) is carefully discussed in Sect. 7.

Lemma 6.4 *Let $n \geq 3$ and $I \subset [n]$ be non-empty with up and down interval decomposition of type $(1, w)$ and $|\mathscr{D}_I| < 4$. Then*

(a) *Suppose that I satisfies one of the following conditions*
  (i) $\mathscr{D}_I = \{\delta_1\}$ *(Lemma 7.1),*
  (ii) $\mathscr{D}_I = \{\delta_1, \delta_3, \delta_4\}$, $(a, b)_{\mathsf{D}} = \{\Gamma\}$*, and* $\gamma \in \mathsf{U}_c$ *(Lemma* 7.6 (b) *and* (c)),
  (iii) $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_4\}$, $(a, b)_{\mathsf{D}} = \{\gamma\}$*, and* $\Gamma \in \mathsf{U}_c$ *(Lemma* 7.5 (b) *and* (c)),
  (iv) $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_3\}$ *and* $(a, b)_{\mathsf{D}} = \{\gamma, \Gamma\}$ (Lemma 7.4 (a))
  (v) $\mathscr{D}_I = \{\delta_2, \delta_3, \delta_4\}$ *and* $(a, b)_{\mathsf{D}} = \varnothing$*(Lemma* 7.7).
  Then the Minkowski coefficient y_I of As^c_{n-1} is

$$y_I = \sum_{\delta\in\mathscr{D}_I} (-1)^{|I\setminus R_\delta|} z_{R_\delta}.$$

(b) *Suppose that* I *satisfies one of the following conditions*
  (i) $\mathscr{D}_I = \{\delta_1, \delta_4\}$ (*Lemma* 7.2),
  (ii) $\mathscr{D}_I = \{\delta_2, \delta_3\}$ *(Lemma* 7.3),
  (iii) $\mathscr{D}_I = \{\delta_1, \delta_3, \delta_4\}$ *and* $\bigcup_{i=1}^k [\alpha_i, \beta_i]_{\mathsf{U}_c} = \{\Gamma\}$ *(Lemma* 7.6 (a)),
  (iv) $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_4\}$ *and* $\bigcup_{i=1}^k [\alpha_i, \beta_i]_{\mathsf{U}_c} = \{\gamma\}$ (Lemma 7.5 (a)),
  (v) $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_3\}$ *and* $\bigcup_{i=1}^k [\alpha_i, \beta_i]_{\mathsf{U}_c} = \{\gamma, \Gamma\}$ (*Lemma* 7.4 (b)).

Then the Minkowski coefficient y_I *of* As_{n-1}^c *is*

$$y_I = (-1)^{|\{\gamma,\Gamma\}|} z_{[n]} + \sum_{\delta \in \mathscr{D}_I} (-1)^{|I \setminus R_\delta|} z_{R_\delta}.$$

Proof The proof of the claim is a study of the 14 mentioned cases that characterise the non-empty proper subsets $I \subset [n]$ with $\mathscr{D}_I \neq \{\delta_1, \delta_2, \delta_3, \delta_4\}$. These 14 cases are described in detail in Sect. 7, the proofs are along the lines of the proof of Proposition 6.2. □

Lemma 6.5 *For* $n \geq 3$*, let* I *be non-empty proper subset of* $[n]$ *with up and down interval decomposition of type* $(1, w)$ *and* $|\mathscr{D}_I| < 4$.

(a) *In all cases of Lemma* 6.4 *(a) we have* $R_\delta = \varnothing$ *if* $\delta \in \{\delta_1, \delta_2, \delta_3, \delta_4\} \setminus \mathscr{D}_I$. *Thus*

$$y_I = \sum_{i=1}^{4} (-1)^{|I \setminus R_{\delta_i}|} z_{R_{\delta_i}}.$$

(b) *In all cases of Lemma* 6.4 *(b) there is precisely one* $\delta \in \{\delta_1, \delta_2, \delta_3, \delta_4\} \setminus \mathscr{D}_I$ *with* $R_\delta \neq \varnothing$:
  (i) $R_{\delta_2} = [n]$ (*Lemma* 7.2 (a) *and Lemma* 7.6 (a)) *and we have*

$$y_I = (-1)^{|I \setminus R_{\delta_2}|} z_{R_{\delta_2}} + \sum_{\delta \in \mathscr{D}_I} (-1)^{|I \setminus R_\delta|} z_{R_\delta} = \sum_{i=1}^{4} (-1)^{|I \setminus R_{\delta_i}|} z_{R_{\delta_i}}.$$

  (ii) $R_{\delta_3} = [n]$ (*Lemma* 7.2 (b) *and Lemma* 7.5 (a)) *and we have*

$$y_I = (-1)^{|I \setminus R_{\delta_3}|} z_{R_{\delta_3}} + \sum_{\delta \in \mathscr{D}_I} (-1)^{|I \setminus R_\delta|} z_{R_\delta} = \sum_{i=1}^{4} (-1)^{|I \setminus R_{\delta_i}|} z_{R_{\delta_i}}.$$

  (iii) $R_{\delta_4} = [n]$ (*Lemma* 7.3 *and Lemma* 7.4 (b)) *and we have*

$$y_I = (-1)^{|I \setminus R_{\delta_4}| + |\{\gamma,\Gamma\}|} z_{R_{\delta_4}} + \sum_{\delta \in \mathscr{D}_I} (-1)^{|I \setminus R_\delta|} z_{R_\delta}.$$

Moreover, we have $\gamma \neq \Gamma$ *except for Lemma* 7.3 *(a) where* $\gamma = \Gamma \in \mathsf{U}_c$.

Proof The first case is trivial, since we only add vanishing terms to

$$\sum_{\delta\in\mathscr{D}_I}(-1)^{|I\setminus R_\delta|}z_{R_\delta}.$$

The second case is a bit more involved. First observe that $\gamma\neq\Gamma$ except for Case (a) of Lemma 7.3 when $\gamma=\Gamma\in\mathsf{U}_c$. Now, using the description given in Sect. 7, it is straighforward to check $(-1)^{|\{\gamma,\Gamma\}|}=(-1)^{|I\setminus R_{\delta_2}|}$ for the first subcase, $(-1)^{|\{\gamma,\Gamma\}|}=(-1)^{|I\setminus R_{\delta_3}|}$ for the second subcase and $(-1)^{|\{\gamma,\Gamma\}|}=(-1)^{|I\setminus R_{\delta_4}|+|\{\gamma,\Gamma\}|}$ for the third subcase. □

Lemma 6.6 *Let I be non-empty subset of $[n]$ with up and down interval decomposition of type $(1,w)$. Then*

$$(-1)^{|I\setminus R_{\delta_1}|}=(-1)^{|I\setminus R_{\delta_2}|+1}=(-1)^{|I\setminus R_{\delta_3}|+1}=(-1)^{|I\setminus R_{\delta_4}|+|\{\gamma,\Gamma\}|}.$$

Proof The claim for δ_2 follows from

$$R_{\delta_2}\cap I=\begin{cases}(R_{\delta_1}\cap I)\sqcup\{\Gamma\}, & \Gamma\in\mathsf{U}_c,\\ (R_{\delta_1}\cap I)\setminus\{\Gamma\}, & \Gamma\in\mathsf{D}_c.\end{cases}$$

The case for δ_3 is similar. For δ_4 we have to consider

$$R_{\delta_4}\cap I=\begin{cases}(R_{\delta_1}\cap I)\sqcup\{\Gamma,\gamma\} & \gamma,\Gamma\in\mathsf{U}_c,\\ \big((R_{\delta_1}\cap I)\sqcup\{\gamma\}\big)\setminus\{\Gamma\} & \gamma\in\mathsf{U}_c,\ \Gamma\in\mathsf{D}_c,\\ \big((R_{\delta_1}\cap I)\sqcup\{\Gamma\}\big)\setminus\{\gamma\} & \Gamma\in\mathsf{U}_c,\ \gamma\in\mathsf{D}_c,\\ (R_{\delta_1}\cap I)\setminus\{\Gamma,\gamma\} & \gamma,\Gamma\in\mathsf{D}_c.\end{cases}$$

□

We combine Proposition 6.2, Lemmas 6.5 and 6.6 to obtain Theorem 4.2 if $I\subset[n]$ has an up and down interval decomposition of type $(1,w)$:

Corollary 6.7 *Let I be non-empty proper subset of $[n]$ with up and down interval decomposition of type $(1,w)$ and $\mathscr{D}_I\subseteq\{\delta_1,\delta_2,\delta_3,\delta_4\}$. Then*

$$y_I=(-1)^{|I\setminus R_{\delta_1}|}\big(z^c_{R_{\delta_1}}-z^c_{R_{\delta_2}}-z^c_{R_{\delta_3}}+z^c_{R_{\delta_4}}\big).$$

The techniques to prove Proposition 6.2 also enable us to compute the Minkowski coefficient y_I of As^c_{n-1} if the up and down interval decomposition of I is of type (v,w), $v>1$, and $I\neq[n]$.

Lemma 6.8 *Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type (v,w) with $v>1$. Then $y_I=0$ for the Minkowski coefficient of As^c_{n-1}.*

Proof For every proper diagonal $\delta = \{d_1, d_2\}$ with $d_1 < d_2$ that appears in the expression for y_I, there is a nested component $N = (a_i, b_i)_{\mathsf{D}} \sqcup \bigsqcup_{j=1}^{w_i} [\alpha_{i,j}, \beta_{i,j}]_{\mathsf{U}}$ of I such that $a_i \le d_1 < d_2 \le b_i$. Now δ appears in the expression for y_I for every set S where $R_\delta \cap N \subseteq S \subseteq I$. Since $v > 1$, the diagonal δ never contributes to y_I. □

We now analyse the remaining case $I = [n]$ and consider $(0, n+1)_{\mathsf{D}} \sqcup [u_1, u_m]_{\mathsf{U}}$ as up and down interval decomposition of I.

Lemma 6.9 *For any partition $\mathsf{D}_c \sqcup \mathsf{U}_c = [n]$ induced by some Coxeter element c, the Minkowski coefficient $y_{[n]}$ satisfies*

$$y_{[n]} = (-1)^{|[n]\setminus R_{\delta_1}|}\big(z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + z^c_{R_{\delta_4}}\big).$$

Proof For $I = [n]$, we have $a = 0$, $\gamma = 1$, $\Gamma = n$, and $b = n+1$. We associate to the up and down interval decomposition of $[n]$ precisely one diagonal that is not proper and rewrite the formula for $y_{[n]}$ as

$$y_{[n]} = z_{[n]} + \sum_{J\subset[n]} (-1)^{|[n]\setminus J|} z_J.$$

We are now interested in the contribution of proper diagonals that are associated to $J \subset [n]$ and distinguish four cases. To find the contributions in each case, we proceed along the lines of the proof of Proposition 6.2.

(1) $\mathsf{U}_c \ne \varnothing$ and $\mathsf{D}_c \ne \{1, n\}$.
Then $\mathscr{D}_{[n]} = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ and each diagonal of $\mathscr{D}_{[n]}$ contributes to $y_{[n]}$ as well as all proper diagonals $\{0, u\}$ and $\{1, u\}$ with $u \in \mathsf{U}_c$ since $a = 0$ and $\gamma = 1$. Hence we have

$$\sum_{\delta\in\mathscr{D}_{[n]}} (-1)^{|[n]\setminus R_\delta|} z^c_{R_\delta} + \sum_{\substack{\delta=\{0,\alpha\}\text{ with}\\ \alpha\in[u_2,u_m]_{\mathsf{U}_c}}} (-1)^{|[n]\setminus R_\delta|} z^c_{[n]} + \sum_{\substack{\delta=\{1,\alpha\}\text{ with}\\ \alpha\in[u_1,u_m]_{\mathsf{U}_c}}} (-1)^{|[n]\setminus R_\delta|} z^c_{[n]}$$

for $\sum_{J\subset[n]}(-1)^{|[n]\setminus J|} z_J$. Since $\{0, u_1\}$ is not a proper diagonal, the second and third sum do not cancel and the term $(-1)^{|[n]\setminus R_{\{1,u_1\}}|} z^c_{[n]}$ remains. Now $|[n]\setminus R_{\{1,u_1\}}| = 1$ and

$$\sum_{\delta\in\mathscr{D}_{[n]}} (-1)^{|[n]\setminus R_\delta|} z^c_{R_\delta} = (-1)^{|[n]\setminus R_{\delta_1}|}\big(z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + z^c_{R_{\delta_4}}\big)$$

imply the claim.

(2) $\mathsf{U}_c = \varnothing$ and $\mathsf{D}_c \ne \{1, n\}$.
Then $\mathscr{D}_{[n]} = \{\delta_2, \delta_3, \delta_4\}$ and we have

$$\sum_{J\subset[n]} (-1)^{|[n]\setminus J|} z_J = \sum_{\delta\in\mathscr{D}_{[n]}} (-1)^{|[n]\setminus R_\delta|} z^c_{R_\delta}.$$

The claim follows now from $R_{\delta_1} = [n]$ and Lemma 6.6.

(3) $\mathsf{U}_c \neq \varnothing$ and $\mathsf{D}_c = \{1, n\}$.
We have $\mathscr{D}_{[n]} = \{\delta_1, \delta_2, \delta_3\}$. Now each diagonal of $\mathscr{D}_{[n]}$ and all proper diagonals $\delta_{0,u} = \{0, u\}$ and $\delta_{1,u} = \{1, u\}$ with $u \in \mathsf{U}_c$ contribute to y_I since $a = 0$ and $\gamma = 1$. Similar to the first case, a term $(-1)^{|[n]\setminus R_{\{1,u_1\}}|} z^c_{[n]}$ is not canceled and we obtain

$$y_{[n]} = (-1)^{|[n]\setminus R_{\delta_1}|}\big(z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}}\big).$$

Since $z_{R_{\delta_4}} = z_\varnothing = 0$, the claim follows.

(4) $\mathsf{U}_c = \varnothing$ and $\mathsf{D}_c = \{1, n\}$.
We have $\mathscr{D}_{[n]} = \{\delta_2, \delta_3\}$, $R_{\delta_1} = [n]$ and $R_{\delta_4} = \varnothing$. Hence

$$y_{[n]} = z_{[n]} + \sum_{\delta\in\mathscr{D}_{[n]}} (-1)^{|[n]\setminus R_\delta|} z^c_{R_\delta} = (-1)^{|[n]\setminus R_{\delta_1}} \big(z^c_{R_{\delta_1}} - z^c_{R_{\delta_2}} - z^c_{R_{\delta_3}} + z^c_{R_{\delta_4}}\big).$$

□

7 Characterisation of $\mathscr{D}_I \neq \{\delta_1, \delta_2, \delta_3, \delta_4\}$ for $I \subset [n]$

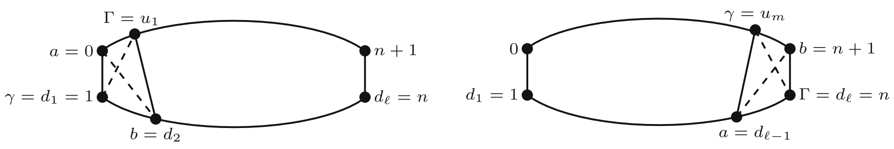
As stated in Lemma 6.3, not all 15 proper subsets of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ appear as set of proper diagonals $\mathscr{D}_I$ for $I \subset [n]$ with up and down decomposition of type $(1, w)$ and some Coxeter element c. The proof that a subset does not appear is not difficult, for example, we can show that if $\mathscr{D}_I$ contains certain diagonal(s) then $\mathscr{D}_I$ is forced to contain certain others. In this section we discuss Lemma 6.3(b) in detail and study the sets $\mathscr{D}_I$ with $|\mathscr{D}_I| < 4$. The seven proper subset of $\{\delta_1, \delta_2, \delta_3, \delta_4\}$ that are possible are characterised in Lemmas 7.1–7.7. We identified 14 conditions that characterise these seven subsets.

Lemma 7.1 *If $\mathscr{D}_I = \{\delta_1\}$, then $I = \{d_r\}$ with $1 \le r \le \ell$.*

Proof $\delta_1 \in \mathscr{D}_I$ implies $(a, b)_{\mathsf{D}_c} \neq \varnothing$ and $\delta_2, \delta_3, \delta_4 \notin \mathscr{D}_I$ imply $\gamma = \Gamma \in \mathsf{D}_c$. □

Lemma 7.2 (compare Fig. 17). *If $\mathscr{D}_I = \{\delta_1, \delta_4\}$, then either*

(a) $I = \{d_1, u_1\}$ *and* $u_1 < d_2$*, or*
(b) $I = \{u_m, d_\ell\}$ *and* $d_{\ell-1} < u_m$.



(a) $I = \{1, u_1\}$ and $u_1 < d_2$ **(b)** $I = \{u_m, n\}$ and $d_{\ell-1} < u_m$

Fig. 17 Schematic illustrations: the two cases of $\mathscr{D}_I = \{\delta_1, \delta_4\}$ (Lemma 7.2)

Proof $\delta_2, \delta_3 \notin \mathscr{D}_I$ imply that $\{a, \Gamma\}$ and $\{\gamma, b\}$ are (non-degenerate) edges of Q_c. In particular, neither $\gamma, \Gamma \in \mathsf{D}_c$ nor $\gamma, \Gamma \in \mathsf{U}_c$ is possible.

Firstly, suppose $\gamma \in \mathsf{D}_c$ and $\Gamma \in \mathsf{U}_c$. Then $\delta_2 \notin \mathscr{D}_I$ implies $a = 0$, $\Gamma = u_1$, and $\gamma = d_1 = 1$. Now $\delta_3 \notin \mathscr{D}_I$ yields $b = d_2$ and $\Gamma = u_1$ requires $u_1 < d_2$ and we have shown (a).

Secondly, suppose $\gamma \in \mathsf{U}_c$ and $\Gamma \in \mathsf{D}_c$. Then $\delta_3 \notin \mathscr{D}_I$ implies $b = n+1$, $\gamma = u_m$, and $\Gamma = d_\ell = n$. Now $\delta_2 \notin \mathscr{D}_I$ yields $a = d_{\ell-1}$ and $\gamma = u_m$ requires $d_{\ell-1} < u_m$. This gives (b). □

Lemma 7.3 (compare Fig. 18). *If $\mathscr{D}_I = \{\delta_2, \delta_3\}$ then either*

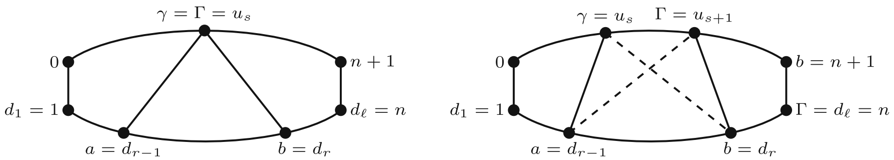
(a) $I = \{u_s\}$ *with* $1 \le s \le m$, *or*
(b) $I = \{u_s, u_{s+1}\}$ *with* $1 \le s < m$.

Proof From $\delta_1 \notin \mathscr{D}_I$, we obtain $(a, b)_{\mathsf{D}_c} = \varnothing$, thus $a < \gamma \le \Gamma < b$ and $\gamma, \Gamma \in \mathsf{U}_c$. Now $\delta_4 \notin \mathscr{D}_I$ implies that $\{\gamma, \Gamma\}$ is either degenerate or an edge of Q_c. This proves the claim.

Lemma 7.4 (Compare Fig. 19) *If $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_3\}$, then either*

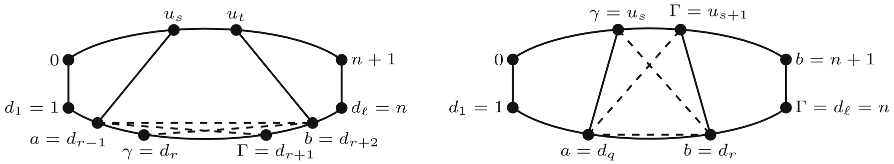
(a) $I = \{d_r, d_{r+1}\} \sqcup M$ *with* $1 \le r < \ell$ *and* $M \subseteq [d_r, d_{r+1}] \cap \mathsf{U}_c$ *or*
(b) $I = M \sqcup \{u_s, u_{s+1}\}$ *with* $1 \le s < m$ *and* $M = [u_s, u_{s+1}] \cap \mathsf{D}_c \neq \varnothing$.

Proof $\delta_1 \in \mathscr{D}_I$ implies $(a, b)_{\mathsf{D}_c} \neq \varnothing$, while $\delta_4 \notin \mathscr{D}_I$ implies that $\{\gamma, \Gamma\}$ is either an edge of Q_c or $\gamma = \Gamma$. Suppose first $\gamma = \Gamma$. Then $\gamma = \Gamma \in \mathsf{D}_c$ implies the contradiction $\mathscr{D}_I = \{\delta_1\}$, while $\gamma = \Gamma \in \mathsf{U}_c$ implies $(a, b)_{\mathsf{D}_c} = \varnothing$, contradicting $\delta_1 \in \mathscr{D}_I$. We therefore assume $\gamma \neq \Gamma$ and only have to distinguish the cases $\gamma, \Gamma \in \mathsf{D}_c$ and



(a) $I = \{u_s\}$ and $1 \le s < m$ **(b)** $I = \{u_s, u_{s+1}\}$ and $1 \le s < m$

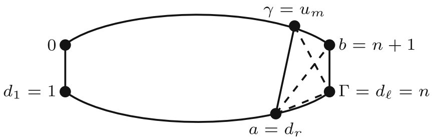
Fig. 18 Schematic illustrations: the two cases of $\mathscr{D}_I = \{\delta_2, \delta_3\}$ (Lemma 7.3)



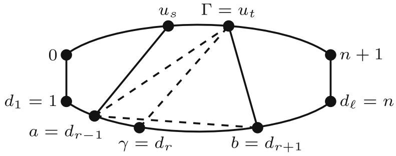
(a) $I = \{d_r, d_{r+1}\} \cup M$ with $1 \le r < \ell$ and $M \subseteq [d_r, d_{r+1}] \cap \mathsf{U}_c$

(b) $I = M \cup \{u_s, u_{s+1}\}$ with $1 \le s < m$ and $M = [u_s, u_{s+1}] \cap \mathsf{D}_c \neq \varnothing$

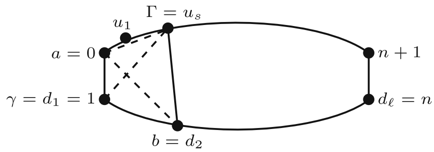
Fig. 19 Schematic illustrations: the two cases of $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_3\}$ (Lemma 7.4)



(a) $I = \{d_{r+1}, \ldots, d_\ell\} \cup \{u_m\}$ with $d_r < u_m < d_{r+1} < d_\ell$



(b) $I = \{d_r\} \cup M$ with $1 < r < \ell$, and $M \subseteq [d_r, d_{r+1}] \cap \mathsf{U}_c$ and $M \neq \varnothing$



(c) $I = \{d_1\} \cup M$ with $M \subseteq [d_1, d_2] \cap \mathsf{U}_c$ and $M \setminus \{u_1\} \neq \varnothing$

Fig. 20 Schematic illustrations: the three cases of $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_4\}$ (Lemma 7.5)

$\gamma, \Gamma \in \mathsf{U}_c$, the other cases $\gamma \in \mathsf{D}_c$, $\Gamma \in \mathsf{U}_c$ and $\gamma \in \mathsf{U}_c$ and $\Gamma \in \mathsf{D}_c$ are not possible since $\delta_4 \notin \mathscr{D}_I$.

Firstly, suppose $\gamma, \Gamma \in \mathsf{D}_c$. Then $\gamma = d_r$ and $\Gamma = d_{r+1}$ for some $1 \le r \le \ell - 1$, since $\delta_4 \notin \mathscr{D}_I$. This implies $I = \{d_r, d_{r+1}\} \cup \big([d_r, d_{r+1}] \cap \mathsf{U}_c\big)$, which proves claim (a). Secondly, suppose $\gamma, \Gamma \in \mathsf{U}_c$. Then $\gamma = u_s$ and $\Gamma = u_{s+1}$ for some $1 \le s \le m - 1$. But this implies $[u_s, u_{s+1}] \cap \mathsf{D}_c = (d_q, d_r)_{\mathsf{D}_c} \neq \varnothing$ and

$$I = \big([u_s, u_{s+1}] \cap \mathsf{D}_c\big) \cup [u_s, u_{s+1}]_{\mathsf{U}_c},$$

which proves (b). □

Lemma 7.5 is symmetric to Lemma 7.6, the proofs are along the same lines.

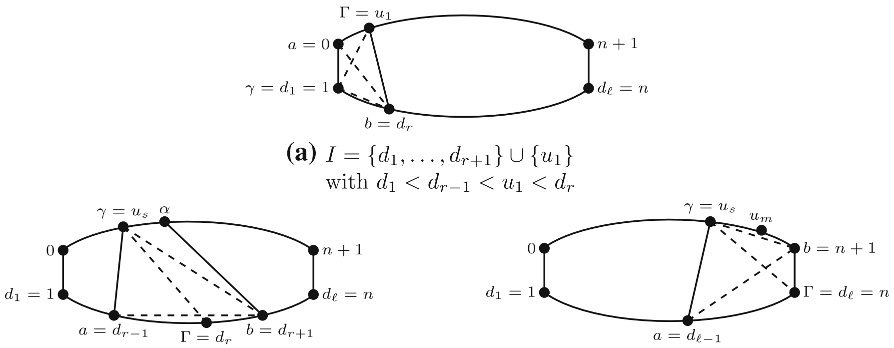
Lemma 7.5 (Compare Fig. 20). *If $\mathscr{D}_I = \{\delta_1, \delta_2, \delta_4\}$, then either*

(a) $I = \{d_{r+1}, \ldots, d_\ell\} \cup \{u_m\}$ *with* $d_r < u_m < d_{r+1} < d_\ell$, *or*
(b) $I = \{d_r\} \cup M$ *with* $1 < r < \ell$ *and* $\varnothing \neq M \subseteq [d_r, d_{r+1}] \cap \mathsf{U}_c$ *or*
(c) $I = \{d_1\} \cup M$ *with* $M \subseteq [d_1, d_2] \cap \mathsf{U}_c$ *and* $M \setminus \{u_1\} \neq \varnothing$.

Proof Since $\delta_1 \in \mathscr{D}_I$, we have $(a, b)_{\mathsf{D}_c} \neq \varnothing$, that is, a, b are not consecutive numbers in D_c. From $\delta_3 \notin \mathscr{D}_I$, we deduce that $\{\gamma, b\}$ is an edge of Q_c and $\gamma, \Gamma \in \mathsf{U}_c$ is therefore impossible unless $\gamma = \Gamma$. Moreover, $\delta_4 \in \mathscr{D}_I$ implies that $\gamma = \Gamma$ is impossible. We now have two cases to distinguish.

Firstly, suppose $\gamma = u_m$ and $b = n + 1$. Then $\Gamma = d_\ell = n$ and $\delta_2 \in \mathscr{D}_I$ implies $(a, \Gamma)_{\mathsf{D}_c} \neq \varnothing$. Together with $a = \max\{d \in \mathsf{D}_c | d < u_m\}$ we have $a = d_r$ for some $1 \le r \le \ell - 2$ with $u_m < d_{r+1}$ and $I = (d_r, n + 1)_{\mathsf{D}_c} \cup [u_m, u_m]_{\mathsf{U}_c}$, this shows (a).

Secondly, we suppose $\gamma = d_r$ and $b = d_{r+1}$ for some $1 \le r \le \ell - 1$ and $\Gamma \in (\gamma, b) \cap \mathsf{U}_c$. If $\gamma = 1$ then $\delta_2 \in \mathscr{D}_I$ implies $\Gamma \neq u_1$, so we distinguish the cases $\gamma = 1$ and $\gamma \neq 1$. Suppose first that $\gamma = d_r$ with $r > 1$. If $[d_r, d_{r+1}] \cap \mathsf{U}_c \neq \varnothing$



(a) $I = \{d_1, \dots, d_{r+1}\} \cup \{u_1\}$ with $d_1 < d_{r-1} < u_1 < d_r$

(b) $I = \{d_r\} \cup M$ with $1 < r < \ell$, and $M \subseteq [d_{r-1}, d_r] \cap \mathsf{U}_c$ and $M \neq \varnothing$

(c) $I = \{d_\ell\} \cup M$ with $M \subseteq [d_{\ell-1}, d_\ell] \cap \mathsf{U}_c$ and $M \setminus \{u_m\} \neq \varnothing$

Fig. 21 Schematic illustrations: the three cases of $\mathscr{D}_I = \{\delta_1, \delta_3, \delta_4\}$ (Lemma 7.6)

then we immediately have the claim for every non-empty $M \subseteq [d_r, d_{r+1}] \cap \mathsf{U}_c$. If $[d_r, d_{r+1}] \cap \mathsf{U}_c = \varnothing$ then $\gamma = \Gamma \in \mathsf{D}_c$ which is impossible. Thus we have shown (b). Suppose now that $\gamma = d_1 = 1$. Then $a = 0$, $b = d_2$, and $\delta_2 \in \mathscr{D}_I$ implies $\Gamma \in \mathsf{U}_c \backslash \{u_1\}$. This proves (c). ◻

Lemma 7.6 (Compare Fig. 21). *If* $\mathscr{D}_I = \{\delta_1, \delta_3, \delta_4\}$, *then either*

(a) $I = \{d_1, \dots, d_{r-1}\} \cup \{u_1\}$ *with* $d_1 < d_{r-1} < u_1 < d_r$, *or*
(b) $I = \{d_r\} \cup M$ *with* $1 < r < \ell$ *and* $\varnothing \neq M \subseteq [d_{r-1}, d_r] \cap \mathsf{U}_c$ *or*
(c) $I = \{d_\ell\} \cup M$ *with* $M \subseteq [d_{\ell-1}, d_\ell] \cap \mathsf{U}_c$ *and* $M \backslash \{u_m\} \neq \varnothing$.

Lemma 7.7 *If* $\mathscr{D}_I = \{\delta_2, \delta_3, \delta_4\}$, *then*

$$I = \{u_s, \dots, u_t\} \text{ with } s+1 < t \text{ and } (u_s, u_t) \cap \mathsf{D}_c = \varnothing.$$

Proof From $\delta_1 \notin \mathscr{D}_I$, we obtain $(a, b)_{\mathsf{D}_c} = \varnothing$, in particular, $a = d_r$ and $b = d_{r+1}$ for some $1 \leq r \leq \ell - 1$. Thus $\gamma, \Gamma \in \mathsf{U}_c$ and because of $\delta_4 \in \mathscr{D}_I$ we have $\gamma = u_s$ and $\Gamma = u_t$ for some $1 \leq s < s+1 < t \leq u_m$. But then $I = M$ for some $M \subseteq [u_s, u_t]_{\mathsf{U}_c}$ with $u_s, u_t \in M$. ◻

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