Minkowski-type and Alexandrov-type theorems for polyhedral herissons [*]

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Abstract

Classical H. Minkowski theorems on existence and uniqueness of convex polyhedra with prescribed directions and areas of faces as well as the well-known generalization of H. Minkowski uniqueness theorem due to A. D. Alexandrov are extended to a class of nonconvex polyhedra which are called polyhedral herissons and may be described as polyhedra with injective spherical image.

Key words: convex polyhedron, polyhedral surface, polyhedral hedgehog, equipment, virtual polytope, polygon, Cauchy lemma, open mapping.

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1   Introduction

H. Minkowski proved that a convex polyhedron is uniquely (up to translation) determined by the areas of its faces and the unit outward normal vectors to the faces (see, for example, [1] or [5]). Moreover, H. Minkowski found natural and easily verified conditions on a finite set of unit vectors and positive numbers implying that there exists a convex polytope P such that these vectors are the outward normal vectors to the faces of P and these numbers equal the areas of the faces of P.

We extend these results (as well as a generalization of the Minkowski uniqueness theorem which is due to A. D. Alexandrov) to a class of nonconvex polyhedral surfaces. These surfaces may even admit self-intersections. The crucial point is that their spherical mapping is injective. They are called polyhedral herissons ("herisson" is a French word for "hedgehog") and were previously studied, for example, in papers by J. J. Stoker [23], L. Rodrigues and H. Rosenberg [18], P. Roitman [19] and G. Yu. Panina [16].

2 H. Minkowski and A. D. Alexandrov theorems for convex polyhedra

In this section we make some notions more precise and revise the classical H. Minkowski and A. D. Alexandrov theorems on existence and uniqueness of convex polyhedra with prescribed directions and areas of faces.

2.1. Definitions. Let L be an affine subspace of $\mathbf{R}^d$, $d \geq 2$ (the case $L = \mathbf{R}^d$ is not excluded). The convex hull of a finite set of points in L is said to be a *convex polytope* provided that it has interior points in L. A convex polytope in a two-dimensional space L is called a *convex polygon*.

Given a convex polytope P in $\mathbf{R}^d$ and a unit vector $n \in \mathbf{R}^d$, find the least real number h such that, for each $h' > h$, the hyperplane $(x, n) = h'$ does not intersect P. Here and below (x, n) stands for the scalar product of vectors x and n. The intersection of P and the above-constructed hyperplane $(x, n) = h$ is called a *face* of P with outward normal vector n, while the hyperplane $(x, n) = h$ itself is called a *supporting hyperplane* to the face. The dimension of the affine hull of a face is said to be the *dimension of the face*. Note that, under the above definition of a face, the dimension of a face can take any integer value from 0 to $d - 1$ inclusively.

Let P_1 and P_2 be convex polytopes in R^d, $d \geq 2$, and let Q_1 be a k-dimensional face of P_1, $0 \leq k \leq d - 1$. Denote by n the outward unit vector to Q_1. Let Q_2 be a face of P_2 with outward normal vector n. Then Q_2 is called *parallel* to Q_1.

Let L_j be an affine subspace of $\mathbf{R}^d$, $j = 1, 2$, and let Q_j be a convex polytope in L_j. Say that the polytope Q_1 *can be put inside* Q_2 by a translation if there exists a translation $T : \mathbf{R}^d \to \mathbf{R}^d$ such that $T(Q_1)$ is a subset of Q_2 but does not coincide with Q_2.

2.2. Theorem (A. D. Alexandrov [1]). *Let $d = 3$ and let P_1, P_2 be convex polytopes in R^d. Suppose that, given any two parallel faces of P_1 and P_2 such that at least one of the faces is of dimension $d - 1$, neither of the faces can be put inside the other face by a translation. Then P_1 and P_2 are congruent and parallel to each other (i.e. can be obtained from one another by a translation).*

2.3. Remarks. The assertion of Theorem 2.2 is trivial for $d = 2$ and, as is observed in [1], is false for $d = 4$. As a counterexample, one can consider a cube with edges of length 2 in $\mathbf{R}^4$ and a rectangular parallelepiped with edges of lengths 1, 1, 3, and 3 parallel to the edges of the cube. It is clear that none of the 3-dimensional faces of the cube can be put inside the parallel face of the parallelepiped by a translation as well as none of the 3-dimensional faces of the parallelepiped can be put inside the parallel face of the cube by a translation. Nevertheless it is clear that the cube and the parallelepiped are not congruent.

For $d > 4$, a similar counterexample can be obtained in $\mathbf{R}^d = \mathbf{R}^4 \times \mathbf{R}^{d-4}$ by considering the Cartesian product of the above-constructed 4-dimensional cube and parallelepiped with $d - 4$ unit segments.

Note that, to apply Theorem 2.2, we should have a positive answer to the question "Is it possible to put the face with outward unit normal vector n of one of the polytopes inside the parallel face of the other polytope?" for only finitely many vectors n (namely, only for the outward unit normal vectors to the 2-dimensional faces). The next theorem shows that a positive answer to the above question for all vectors $n \in \mathbf{R}^d$ implies the conclusion of Theorem 2.2 in all dimensions. The problem remains open whether it is possible to find an analog of Theorem 2.4 where a positive answer to the above question is requested for only finitely many vectors and how many vectors should be tested.

2.4. Theorem. *Let $d \geq 2$ and let P_1, P_2 be convex polytopes in $\mathbf{R}^d$. Suppose that there is no unit vector $n \in \mathbf{R}^d$ such that the face of P_1 or P_2 with outward normal vector n can be put inside the parallel face of the other of the polytopes P_1 and P_2 by a translation. Then P_1 and P_2 are congruent and parallel to each other.*

2.5. Proof. Let v be a 0-dimensional face of P_1 and let n be the unit outward normal vector to one of the supporting hyperplanes of P_1 passing through v. If the face w of P_2 with outward normal vector n has nonzero dimension, we can put v inside w by a parallel translation because v is a point. This contradicts the conditions of Theorem 2.4. Hence the dimension of w equals zero.

Let $T : \mathbf{R}^d \to \mathbf{R}^d$ be a translation such that $v = T(w)$. As far as n is arbitrary in the above reasoning, it follows that P_1 and $T(P_2)$ coincide in a neighborhood of v. In particular, for each 1-dimensional face V of P_1 that is incident to v, a small neighborhood of v in V is contained in a 1-dimensional face W of $T(P_2)$. If the lengths of V and W were different, we could put the smaller segment into the bigger one by a translation in contradiction to the conditions of Theorem 2.4. Hence, 1-dimensional faces V and W coincide and P_1 and $T(P_2)$ have one more common 0-dimensional face besides v. Denote it by $\widetilde{v}$.

Examining as before all supporting planes passing through $\widetilde{v}$, we conclude that P_1 and $T(P_2)$ coincide in a neighborhood of $\widetilde{v}$. Moreover, each 1-dimensional face of P_1 that is incident to $\widetilde{v}$ coincides with a 1-dimensional face of $T(P_2)$ and vice versa. Thus there exists one more common 0-dimensional face of P_1 and $T(P_2)$ besides v and $\widetilde{v}$.

After a finite number of iterations, this travel along 0-dimensional faces of the polytopes shows that the set of 0-dimensional faces of P_1 coincides with the set of 0-dimensional faces of $T(P_2)$. Therefore P_1 and $T(P_2)$ coincide and P_1 and P_2 are congruent and parallel to each other. $\square$

2.6. Theorem (H. Minkowski uniqueness theorem [1], [5]). *Let $d \geq 2$ and let two convex polytopes in $\mathbf{R}^d$ be such that, for every $(d-1)$-dimensional face of each of the polytopes, the parallel face of the other polytope has the same $(d-1)$-dimensional volume. Then the polytopes are congruent and parallel to each other.*

2.7. Remarks. The assertion of Theorem 2.6 is trivial for $d = 2$ and is an immediate consequence of Theorem 2.2 for $d = 3$. Nevertheless there are proofs

of Theorem 2.6 that are independent of Theorem 2.2 and valid for all $d \geq 3$.

Recall the following well-known fact: If $d \geq 2$, P is a convex polytope in $\mathbf{R}^d$, $n_1, \ldots, n_m$ are the unit outward normal vectors to $(d-1)$-dimensional faces of P, and $f_1, \ldots, f_m$ are the $(d-1)$-dimensional volumes of the corresponding faces, then $\sum_{j=1}^{m} f_j n_j = 0$.

2.8. Theorem (H. Minkowski existence theorem [1], [5]). *Let $d \geq 2$ and let $n_1, \ldots, n_m$ be unit vectors in $\mathbf{R}^d$ which do not lie in a closed half-space bounded by a hyperplane passing through the origin. Let $f_1, \ldots, f_m$ be positive real numbers such that $\sum_{j=1}^{m} f_j n_j = 0$. Then there exists a convex polytope P in $\mathbf{R}^d$ such that the vectors $n_1, \ldots, n_m$ (and only they) are the unit outward normal vectors to the $(d-1)$-dimensional faces of P and the $(d-1)$-dimensional volumes of the corresponding faces are equal to $f_1, \ldots, f_m$.*

In this paper we prove some analogs of Theorems 2.2, 2.6, and 2.8 for a class of nonconvex polyhedra.

3   Polyhedral herissons

In this section we introduce nonconvex polyhedral surfaces in $\mathbf{R}^3$ to which we are going to extend Theorems 2.2, 2.6, and 2.8.

3.1. Definitions. The image of a 2-dimensional sphere-homeomorphic cell complex under a continuous mapping into $\mathbf{R}^3$ is said to be a *polyhedral surface* if the image of each cell is a convex k-dimensional polytope in $\mathbf{R}^3$, $k = 0, 1, 2$. In this case, the image of a k-dimensional cell is called a *face*, for $k = 2$; an *edge*, for $k = 1$; and a *vertex* of the polyhedral surface, for $k = 0$.

3.2. Remark. A polyhedral surface in the sense of Definition 3.1 need not be the boundary of a convex polyhedron, may have rather complicated self-intersections, but is an orientable manifold.

3.3. Definitions. A polyhedral surface P in $\mathbf{R}^3$ is called a *polyhedral herisson* if each face Q_j of P is equipped with a unit normal vector n_j in such a way that, if faces Q_j and Q_k share an edge, then $n_j + n_k \neq 0$ and thus there is only one shortest geodesic path on the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$ joining the endpoints of n_j and n_k. Denote the shortest geodesic path by l_{jk} and suppose in addition that, for arbitrary indices j and k corresponding to adjacent faces, any two curves l_{jk} either do not intersect or have one endpoint in common. Finally, we suppose that the union of all curves l_{jk} gives rise to a partition of the sphere into convex polygons.

The set of vectors $n_1, \ldots, n_m$ involved in the above definition is called an *equipment* of the herisson. The polyhedral herisson is denoted by $(P; n_1, \ldots, n_m)$. The convex spherical polygon of the above-constructed partition of the sphere $\mathbf{S}^2$ is called the *spherical image* of the corresponding vertex.

3.4. Example. The boundary of every convex polytope in $\mathbf{R}^3$ equipped with, say, the unit outward normal vectors to the faces is a polyhedral herisson.
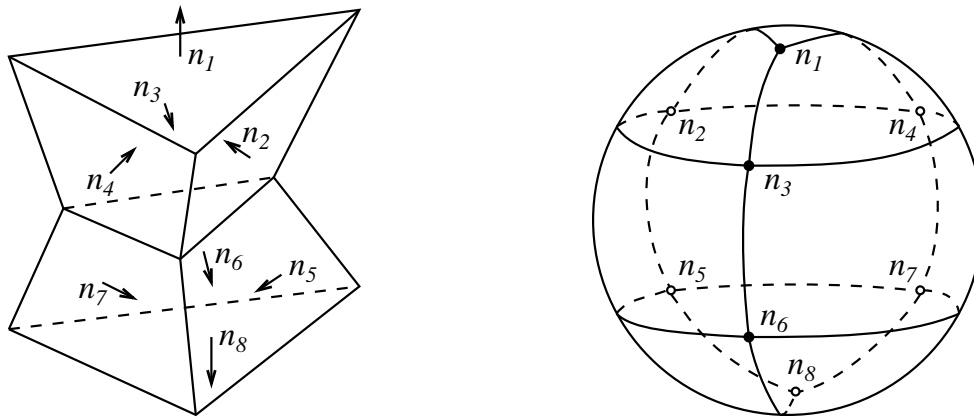
Fig. 1. *A nonconvex polyhedral surface, an equipment making it a herisson, and the corresponding sphere partition.*

3.5. Example. Consider a regular tetrahedron Δ in $\mathbf{R}^3$. Cut a smaller tetrahedron from Δ by a plane τ which is parallel to one of the faces of Δ. Consider the union of the truncated tetrahedron and its image under the reflection in τ. Denote by P the boundary of the union (Fig. 1). Equip the faces of P which are parallel to τ with the unit outward normal vectors and equip all the rest faces of P with the unit inward normal vectors. This equipment makes P a polyhedral herisson. The corresponding sphere partition is shown on the right-hand side of Fig. 1.

3.6. Remarks. Despite unhandiness of the definition, polyhedral herissons can be considered as a quite natural generalization of convex polytopes.

To clarify this point, note that every convex C^∞-surface S in $\mathbf{R}^d$ which is homeomorphic to the sphere $\mathbf{S}^{d-1}$ can be considered as the envelope of its supporting hyperplanes $(x, n) = h(n)$, $n \in \mathbf{S}^{d-1}$ (Fig. 2a). Here h is a C^∞-function on the sphere $\mathbf{S}^{d-1}$. Given a constant number t, one can expect that the C^∞-function $\mathbf{S}^{d-1} \ni n \mapsto h(n) + t$ is a support function of some surface S_t that is called parallel to S. Both smoothness and the shape of S_t strongly depend on the sign of t, i.e. on whether we build the parallel surface outside or inside of S (Fig. 2b and 2c). It may occur that the envelope of a C^∞-smooth family of hyperplanes $(x, n) = h(n) + t$, $n \in \mathbf{S}^{d-1}$, is neither convex nor regular hyperplane.
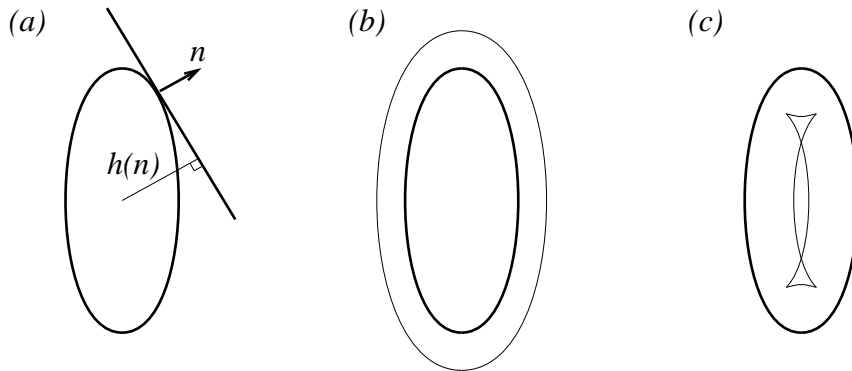
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Fig. 2. *Support function of a convex body (a) and envelopes of
support functions h(n)+const (b), (c).*

In [4], R. Langevin, G. Levitt, and H. Rosenberg began the study of such not
necessarily convex or regular hypersurfaces which admit a C^∞-parametrization
by a unit sphere. In some sense these are hypersurfaces with one-to-one Gauss
mapping that justifies the title "herisson" or hedgehog: in each direction a
unique quill is sticked out. In [4] examples are given which show how herissons
appear in the theory of minimal surfaces, algebraic hypersurfaces, singular wave
fronts, etc. In [6]–[13] it is shown that the notion of a herisson is useful for
studying various problems of convex geometry.

Strictly speaking, the notion of a polyhedral herisson is introduced in [18]
where several theorems are proven which are similar to the famous Cauchy
theorem on unique determination of a convex polytope by the intrinsic metric
of its boundary [1], [3]. G. Yu. Panina established that a polyhedral herisson
is a special case of a more general and more algebraic object called a virtual
polytope. The latter can be treated as the Minkowski difference of two convex
polytopes. It appeared naturally in the theory of polytope algebras (see [14],
[15], and [17]). Another generalization of the notion of a polyhedral herisson
(which is more geometrical in comparison with the notion of a virtual polytope)
is given in the Ph.D. thesis of P. Roitman [19].

3.7. Definitions. Let $(P; n_1, \ldots, n_m)$ be a polyhedral herisson in $\mathbf{R}^3$ and
let a set of unit normal vectors $(\nu_1, \ldots, \nu_m)$ to faces of the polyhedral surface P
be an orientation of P. By definition, put $\varepsilon_j = (n_j, \nu_j)$ for each $j = 1, \ldots, m$.
A face G_j is called *positive* if $\varepsilon_j = +1$ and *negative* if $\varepsilon_j = -1$. The product of
ε_j with a nonnegative number that is equal to the area of the convex polygon
G_j is called the *oriented area* of the face G_j.

Given two polyhedral herissons in $\mathbf{R}^3$, their faces endowed with a common
normal vector n_j are called *parallel*. The herissons are called *parallel and of the*

same orientation if the following conditions are fulfilled: (i) their equipments coincide with each other; (ii) their equipments generate the same partitions of the sphere; (iii) every pair of their parallel faces consists either of positive or of negative faces.

We say that two polyhedral herissons are *congruent and parallel* to each other if one of them (together with its equipment) is obtained from the other by a translation.

3.8. Remarks. In Definition 3.7, conditions (i)–(iii) are independent.

Indeed, convex polyhedral surfaces on Fig. 3 equipped with the unit outward normal vectors to the faces give us an example of polyhedral herissons with a common equipment but different sphere partitions. Hence (i) does not imply (ii).

Given a polyhedral herisson, alter the orientation of the corresponding polyhedral surface without altering the equipment. The initial and modified herissons possess properties (i) and (ii) but do not possess property (iii).
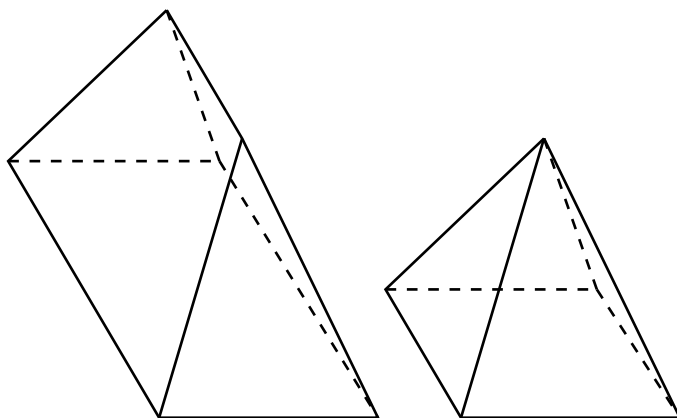

Fig. 3. *Two convex polyhedral surfaces which, being equipped
with outward normals, produce two herissons
with a common equipment but different sphere partitions.*

4 Uniqueness theorems for polyhedral herissons

4.1. Theorem. *Given two polyhedral herissons in $\mathbf{R}^3$ which are parallel and of the same orientation, suppose that they have no two parallel faces such that one of the faces can be put inside the other face by a translation. Then the herissons are congruent and parallel to each other.*

4.2. Remarks. Theorem 4.1 may be considered as an analog of Theorem 2.2 though does not include it as a particular case (Theorem 2.2 works even if the dimension of one of the parallel faces is 0 or 1). Moreover, the proof of Theorem 4.1 presented below is similar to that of Theorem 2.2 given in [1]. Both proofs are based on Lemmas 4.3 and 4.4 which are given below without proofs (the reader can find the proofs in [1] and [23]).

4.3. Lemma (A. Cauchy [1, Chap. II, §1]). *Let a sphere-homeomorphic cell complex be given such that no 2-cell is bounded by two edges. Let the edges of the complex be arbitrarily labeled with the numbers +1, 0, or −1 and let the index j be assigned to each vertex of the complex that is equal to the number of changes of signs of edges from +1 to −1 and from −1 to +1 counted in making one circuit around the vertex (edges labeled with 0 are ignored). The conclusion is that either all the edges of the complex are labeled with 0 or there is a vertex which is incident to an edge labeled with +1 or −1 and for which $j \leq 2$.*

An example is given in Fig. 4 where the edges of a cell complex isomorphic to the boundary of a tetrahedron are labeled with the numbers +1, 0, or −1 and the index j is given for each vertex.
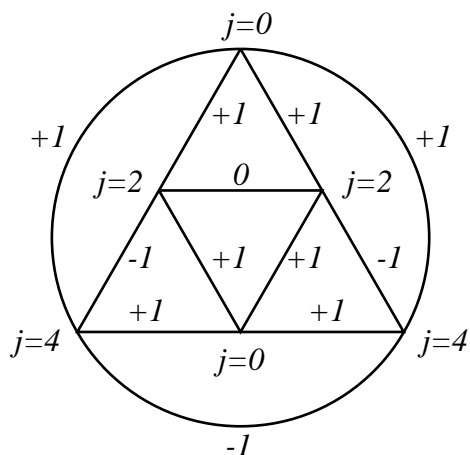

Fig. 4. *j-Indices of vertices of a labeled tetrahedron.*

4.4. Lemma (A. D. Alexandrov [1, Chap. VI, §1]). *Let two convex polygons be given such that neither of them can be put inside the other by a translation. Let the vertices and edges of the polygons be labeled with the numbers +1, 0, or −1 according to the following rules:*

(i) *if a vertex of one of the polygons is such that, for every outward normal vector to the polygon at the point, the parallel face of the other polygon is a vertex, then both vertices are labeled with 0;*

(ii) *if a vertex of one of the polygons is such that, for some outward normal vector to the polygon at the point, the parallel face of the other polygon is an*

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edge, then the vertex is labeled with −1 and the edge is labeled with +1;

(iii) if an edge of one of the polygons is such that the parallel face of the other polygon is a vertex, then the edge is labeled with +1 and the vertex is labeled with −1;

(iv) finally, if an edge of one of the polygons is such that the parallel face of the other polygon is an edge, then the longer edge is labeled with +1 and the shorter edge is labeled with −1; if the edges in question have equal lengths, both of them are labeled with 0.

Let each of the polygons be assign with the index i that is equal to the number of changes of signs of edges and vertices from +1 to −1 and from −1 to +1 counted in making one circuit around the polygon (edges and vertices labeled with 0 are ignored). The conclusion is that either the polygons are congruent and parallel (and, hence, i = 0) or i ≥ 4 for each of the polygons.

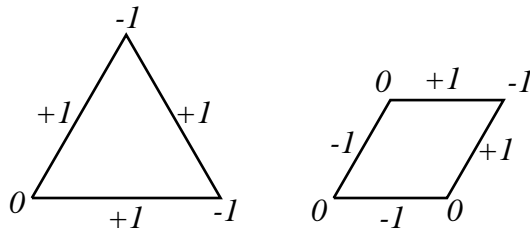An example of calculation of the index i is shown in Fig. 5.

Fig. 5. $i=4$.

4.5. Proof of Theorem 4.1. Label the edges of the two given polyhedral herissons with +1, 0, or −1 by applying the rules described in Lemma 4.4 to each pair of their parallel faces. Since the herissons are parallel and of the same orientation, it follows that parallel faces have mutually parallel edges. Therefore each vertex is labeled with 0. Lemma 4.4 implies that either parallel faces are congruent and thus all their edges are labeled with 0 or some of their edges are labeled with +1 or −1 and in the latter case $i ≥ 4$ for these faces.

Transfer the above-constructed distribution of the numbers +1, 0, and −1 from the edges of the polyhedral surface Q of one of the herissons to the cell complex K whose image is Q. Then for every 2-cell of K either all 1-cells bounding the 2-cell are labeled with 0 or some of the 1-cells are labeled with +1 or −1 and in the latter case $i ≥ 4$.

By assumption, K is sphere homeomorphic and thus there exists a dual cell complex C for it. We can treat C as follows: In each face of Q choose a point and join two such points by a simple curve on Q if and only if the corresponding 2-cells share an edge on Q. Here we assume that any two of these curves can intersect each other at endpoints only. A set of 1-cells in C bounds a 2-cell of C if and only if the corresponding 1-cells of K are incident to a vertex and no other 1-cell of K is incident to the vertex. All we need from the definition of the

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dual graph is that the edges of K and C are in one-to-one correspondence. This makes it possible to transfer the above-constructed distribution of the numbers $+1$, 0, and -1 from the edges of K to the edges of C.

Note that no 2-cell of C is bounded by only two 1-cells of C. Otherwise there would be a 0-cell v of K incident to only two 1-cells of K. In this case there would be a vertex on Q incident to only two edges and thus to only two faces. Denote the edges by e_1 and e_2 and the faces by f_1 and f_2. If e_1 and e_2 do not lie on one straight line then f_1 and f_2 lie on one plane. It follows that the corresponding vectors n_1 and n_2 of the equipment of the herisson either coincide or are opposite. Both possibilities contradict with Definition 3.3 of a polyhedral herisson. If e_1 and e_2 lie on one straight line then we can modify K in such a way as to exclude v and replace the two edges e_1 and e_2 by a single edge which joins together the vertices of e_1 and e_2 that are different from v. Obviously, Q can be treated as a piecewise affine image of the modified complex. On the other hand, the complex C corresponding to the modified complex K contains no 2-cell which is bounded by only two 1-cells.

Note that the index j of a vertex of K is equal to the index i of the corresponding 2-cell of K. Comparing the restrictions imposed on the indices i and j by Lemmas 4.3 and 4.4 we conclude that both indices identically equal zero. Moreover, we conclude that every pair of parallel faces is a pair of congruent and parallel convex polygons.

Now prove that polyhedral herissons satisfying the conditions of Theorem 4.1 can be obtained (together with the equipments) from one another by a translation. To this end superpose some two parallel faces of the herissons by a translation in such a way that the two corresponding vectors of the equipments coincide. Denote the coincident faces by Q_1. Let Q_2 be a face of the first herisson which shares a common edge with Q_1. The vector of the equipment attached to Q_2 defines a point on the sphere $\mathbf{S}^2$ which is joint with the vector of the equipment attached to Q_1. By definition, the equipments of the herissons generate the same sphere partition. Hence there is a face $\widetilde{Q}_2$ of the second herisson that is parallel to Q_2 and shares an edge with Q_1. Obviously, the dihedral angle between Q_1 and Q_2 is equal to the dihedral angle between Q_1 and $\widetilde{Q}_2$. Besides, we already know that the parallel faces Q_2 and $\widetilde{Q}_2$ can be superposed by a translation. Thus Q_2 and $\widetilde{Q}_2$ coincide; moreover, the attached vectors of the equipments also coincide.

If we continue this process of checking that next adjacent faces automatically coincide, we see that, as soon as we have superposed the faces Q_1 of the herissons, we have superposed the herissons in total (including the equipments).

$\square$

4.6. Theorem. *Given two polyhedral herissons in $\mathbf{R}^3$ which are parallel and of the same orientation, suppose that their parallel faces have equal oriented areas. Then the herissons are congruent and parallel to each other.*

4.7. Remarks. Theorem 4.6 is similar to Theorem 2.6 but is valid for nonconvex polyhedral surfaces.

Theorem 4.6 is an immediate consequence of Theorem 4.1.

In the proof of Theorem 4.1 (and thus of Theorem 4.6) we do not use that the equipment of a herisson generates a sphere partition into just convex polygons (see Definition 3.3).

5 Existence theorems for polyhedral herissons

In this section we prove the main result of the paper. Namely, we prove Theorem 5.7 that generalizes the Minkowski existence theorem for convex polytopes with prescribed directions of faces and areas of faces to the case of polyhedral herissons. We begin with auxiliary results.

5.1. Lemma. *Let $(P; n_1, \ldots, n_m)$ be a polyhedral herisson in $\mathbf{R}^3$, let $(\nu_1, \ldots, \nu_m)$ be an orientation of the polyhedral surface P, and let $f_1, \ldots, f_m$ be the oriented areas of the faces of the herisson. Then $\sum_{j=1}^m f_j n_j = 0$.*

5.2. The **proof** is obtained by the direct calculation

$$\sum_{j=1}^m f_j n_j = \sum_{j=1}^m |f_j| \varepsilon_j n_j = \sum_{j=1}^m |f_j| \nu_j = 0,$$

where we use the notations of Definition 3.7 and the last equality is written according to the well-known observation due to Minkowski which was mentioned in Remark 2.7. $\square$

5.3. Lemma. *Let P_0 be a convex polygon on the Euclidean plane such that no two sides of P_0 lie on parallel straight lines. Let $\mathcal{S}$ be a family of convex polygons P satisfying the following conditions:*

(a) *for every side of every $P \in \mathcal{S}$ there is a parallel side of P_0;*

(b) *for every $P \in \mathcal{S}$ and every side of P_0 there is a parallel side of P;*

(c) *the areas of all polygons $P \in \mathcal{S}$ are uniformly bounded from above.*

Then the perimeters of all polygons $P \in \mathcal{S}$ are also uniformly bounded from above.

5.4. Remark. The assertion of Lemma 5.2 is not true without the assumption that P_0 has no parallel sides. To make sure of this, we can consider the square with unit-length sides as P_0 and the sequence of rectangles P_k with sides parallel to the sides of P_0 and side lengths k and $1/k$, $k \in \mathbf{N}$, as $\mathcal{S}$.

5.5. The **proof** of Lemma 5.3 is obtained by reductio ad absurdum. Suppose the perimeters of polygons of $\mathcal{S}$ are not bounded from above. Conditions (a) and (b) imply that every polygon $P \in \mathcal{S}$ has the same number of sides as P_0. Therefore there is a sequence of polygons P_k in $\mathcal{S}$ such that the length l_k of a maximal side of P_k tends to infinity as $k \to \infty$. Denote by α the least angle between straight lines containing two different sides of P_0. Obviously, $\alpha > 0$, since P_0 has no parallel sides. For each $k \in \mathbf{N}$, construct an isosceles triangle Δ_k such that the side of P_k of length l_k is the base of Δ_k; the angles attached

to the base of Δ_k equal α; Δ_k and P_k lie on the same side from their common side of length l_k (Fig. 6).
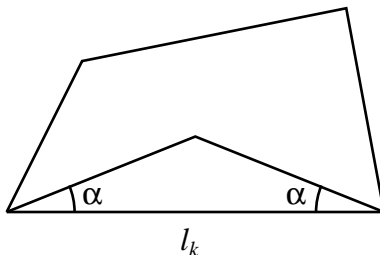

Fig. 6. *Triangle* Δ_k.

Because of the choice of α, Δ_k is contained in P_k. The latter fact leads to a contradiction. Namely, on the one hand, the area of Δ_k equals $\frac{1}{4}l_k^2 \sin^2 \alpha$ and thus tends to infinity as $k \to \infty$. On the other hand, the area is less than or equal to the area of P_k and hence is uniformly bounded from above. This contradiction concludes the proof. $\square$

5.6. Definition. A polyhedral herisson is said to be in a *general position* if no three vectors of its equipment are coplanar.

5.7. Theorem. *Let $(P; n_1, \ldots, n_m)$ be a polyhedral herisson in $\mathbf{R}^3$ in a general position. Fix an orientation of the polyhedral surface P and denote the oriented areas of the faces of $(P; n_1, \ldots, n_m)$ by $f_1, \ldots, f_m$ (in particular, it is assumed that none of the numbers $f_1, \ldots, f_m$ equals zero). Suppose $g_1, \ldots, g_m$ are real numbers such that (1) $f_j \cdot g_j > 0$ for every $j = 1, \ldots, m$ and (2) $\sum_{j=1}^{m} g_j n_j = 0$. Then there exists a polyhedral herisson in $\mathbf{R}^3$ which is parallel to and of the same orientation as $(P; n_1, \ldots, n_m)$ and is such that $g_1, \ldots, g_m$ are the oriented areas of the faces.*

5.8. Remark. Theorem 5.7 is similar to Theorem 2.8 but, even in the 3-space, the former does not contain the latter as a particular case. The main difference is that in Theorem 2.8 we start from abstract sets of vectors $n_1, \ldots, n_m$ and real numbers $f_1, \ldots, f_m$ which satisfy some relation. In contrast to this, to apply Theorem 5.7 we should know from the very beginning that the sets of vectors $n_1, \ldots, n_m$ and numbers $f_1, \ldots, f_m$ correspond to some really existing polyhedral herisson in $\mathbf{R}^3$. Only after this, Theorem 5.7 shows to which extent we can vary the areas of faces of the polyhedral herisson.

5.9. The **proof** of Theorem 5.7 is obtained below by methods close to those used in [1] for proving Theorem 2.8.

By definition, put $g_j(t) = (1 - t)f_j + tg_j$ for $0 \leq t \leq 1$ and $1 \leq j \leq m$. Let T be the subset of the segment $[0, 1] \subset \mathbf{R}$ that consists of real numbers t for each of which there exists a polyhedral herisson in $\mathbf{R}^3$ which is parallel to and of the same orientation as $(P; n_1, \ldots, n_m)$ and for which the real numbers $g_1(t), \ldots, g_m(t)$ are the oriented areas of the faces.

Show that T is nonempty, closed, and open. Clearly, in this case T coincides with $[0, 1]$ and, in particular, contains the point $t = 1$. Obviously, the latter statement is equivalent to the conclusion of Theorem 5.7.

Trivially, T contains the point $t = 0$ and thus is nonempty.

Now prove that T is closed. Suppose a sequence of points $t_1, \ldots, t_k, \ldots$ is contained in T and converges to some point t_0. It is sufficient to prove that $t_0 \in T$. Since $t_k \in T$ there exists a polyhedral herisson $(P_k; n_1, \ldots, n_m)$ in $\mathbf{R}^3$ which is parallel to and of the same orientation as $(P; n_1, \ldots, n_m)$ and for which the numbers $g_1(t), \ldots, g_m(t)$ are the oriented areas of the faces.

Let v_1 be a vertex of $(P_1; n_1, \ldots, n_m)$. Denote by v_k, $k \in \mathbf{N}$, the vertex of $(P_k; n_1, \ldots, n_m)$ that has the same spherical image as v_1. Denote by $(\widetilde{P_k}; n_1, \ldots, n_m)$ the image of $(P_k; n_1, \ldots, n_m)$ under a translation that superposes v_k and v_1.

Note that the numbers $g_1(t_k), \ldots, g_m(t_k)$ are uniformly bounded from above for $k \in \mathbf{N}$. Besides, $(P; n_1, \ldots, n_m)$ is in a general position. Hence none of the faces of $(P_k; n_1, \ldots, n_m)$, $k \in \mathbf{N}$, has mutually parallel sides. It follows from Lemma 5.3 that the perimeters of all faces of all $(P_k; n_1, \ldots, n_m)$, $k \in \mathbf{N}$, are uniformly bounded from above. Thus all $(\widetilde{P_k}; n_1, \ldots, n_m)$, $k \in \mathbf{N}$, are contained in a ball in $\mathbf{R}^3$.

Fixing each of the vectors $n_1, \ldots, n_m$ in turn and applying Blaschke's selection theorem [2] to the sequence of faces (i.e. convex polygons) of $(\widetilde{P_k}; n_1, \ldots, n_m)$, $k \in \mathbf{N}$, equipped with the above-chosen normal vector, we conclude that there exists a subsequence of polyhedral herissons $(\widetilde{P_{k_j}}; n_1, \ldots, n_m)$, $j \in \mathbf{N}$, such that every sequence of faces equipped with a common normal vector converges. It is clear that the limit of each of these sequences of faces is a convex polygon and the limit of the sequence $(\widetilde{P_{k_j}}; n_1, \ldots, n_m)$, $j \in \mathbf{N}$, is a polyhedral herisson $(P_0; n_1, \ldots, n_m)$, for which the numbers $g_1(t_0), \ldots, g_m(t_0)$ are the oriented areas of the faces. This yields $t_0 \in T$ and thus T is closed.

Finally, prove that T is open. We need the following auxiliary constructions.

Each polyhedral herisson parallel to a given polyhedral herisson $(P; n_1, \ldots, n_m)$ can be characterized by its support numbers $h_1, \ldots, h_m$. It follows that the set of all polyhedral herissons parallel to $(P; n_1, \ldots, n_m)$ can be considered as a subset of $\mathbf{R}^m$. This subset is open because small displacements of planes of the faces do not destroy the faces. Unite all the parallel and congruent polyhedral herissons as a single object called a class. Since a translation in $\mathbf{R}^3$ is determined by three parameters, each class is determined by $m - 3$ parameters. Thus the set of all classes can be treated as an $(m - 3)$-dimensional manifold denoted by $\mathbf{A}$.

Denote by $\mathbf{B}$ the set of all m-tuples of nonzero real numbers $f_1, \ldots, f_m$ satisfying the condition $\sum_{j=1}^{m} f_j n_j = 0$, where the vectors $n_1, \ldots, n_m$ are supposed to be fixed. The set $\mathbf{B}$ can be considered as a subset of $\mathbf{R}^m$, namely as the $(m - 3)$-dimensional plane $\sum_{j=1}^{m} f_j n_j = 0$ from which the points are deleted that have at least one zero coordinate.

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Define a mapping φ on **A** that takes each polyhedral herisson parallel to and of the same orientation as $(P; n_1, \ldots, n_m)$ (or the set of its support numbers $h_1, \ldots, h_m$) to the set $f_1, \ldots, f_m$ of the oriented areas of its faces. Lemma 5.1 implies that φ maps **A** into **B**. We have seen above that **A** and **B** are of the same dimensions. Since the areas of faces depend continuously on location of planes containing the faces and thus on the support numbers, it follows that φ is continuous. Uniqueness Theorem 4.6 implies that φ is injective.

Recall the well-known Brouwer domain invariance theorem: If U is an open set in $\mathbf{R}^k$ and $\psi : U \to \mathbf{R}^k$ is a continuous one-to-one mapping, then $\psi(U)$ is an open subset of $\mathbf{R}^k$. Applying this theorem to φ we conclude that $\varphi(\mathbf{A})$ is an open subset of **B**.

Interpreting the set of numbers $g_1(t), \ldots, g_m(t)$ as a point of **B**, we see that, as $t \to t_0$, these points tend to the point $g_1(t_0), \ldots, g_m(t_0)$ of $\varphi(\mathbf{A})$. Since $\varphi(\mathbf{A})$ is open, this implies that, for all $t \in [0, 1]$ sufficiently close to t_0, the point $g_1(t), \ldots, g_m(t)$ lies in $\varphi(\mathbf{A})$. Thus all $t \in [0, 1]$ sufficiently close to t_0 lie in T, i.e., T is open. $\square$

5.10. Example. We show that Theorem 5.7 cannot be directly extended to polyhedral herissons which are not in a general position.

The nonconvex 11-hedron presented in Fig. 7 is the union of two regular tetrahedra and a regular triangular prism (all the three are treated as bodies here) some of whose axes and planes of symmetry mutually coincide. Denote by P the boundary of this 11-hedron that is a nonconvex sphere-homeomorphic surface. The faces of P that do not intersect the triangular prism are called the *bases* of P while the common faces of P and the triangular prism are called the *waist* of P.
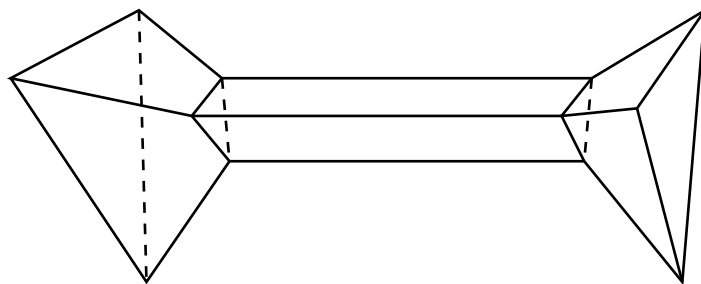

Fig. 7. *A nonconvex polyhedral surface.*

Make P a herisson $H = (P; n_1, \ldots, n_{11})$ by assigning the unit outward normal vectors n_1 and n_{11} to the bases of P and the unit inward normal vectors $n_2, \ldots, n_{10}$ to all other faces. Orient P by unit outward normal vectors. Then the bases of P are its positive faces while all the other faces are negative.

To be more precise, assume that P is constructed from regular tetrahedra with unit edge length and the area of the waist is equal to 1.

To each of the eleven vectors n_j, $j = 1, \ldots, 11$, assign a number g_j, $j = 1, \ldots, 11$, according to the following rule: assign the number $\sqrt{3}/4$ (i.e. the area of an isosceles triangle with side length 1) both to n_1 and n_{11} (i.e. to the vectors attached to the bases of P); assign the number $-1/3$ to each of those vectors n_j that correspond to the waist of P; and assign the number $-\sqrt{3}/4$ to the rest vectors n_j.

We treat the numbers $g_1, \ldots, g_{11}$ as the limit values of the oriented areas of the faces of polyhedral herissons $H_k = (P_k; n_1, \ldots, n_{11})$ constructed as follows. Given $k \in \mathbf{N}$, P_k is combinatorially equivalent to P, the equipment of P_k coincides with that of P, the length of an orthogonal cross-section of the waist of P_k equals $1/k$, and the length of the waist of P_k is equal to k. The oriented areas of the rest faces of P_k either equal $\sqrt{3}/4$ or tend to $-\sqrt{3}/4$ as $k \to \infty$. This implies that conditions (1) and (2) of Theorem 5.7 hold true for the set of numbers $g_1, \ldots, g_{11}$.

However prove that there is no polyhedral herisson that is parallel to and of the same orientation as H and is such that the numbers $g_1, \ldots, g_{11}$ are the oriented areas of its faces.

Assume the contrary. Then there exists a polyhedral herisson H_0 that is parallel to and of the same orientation as H and is such that the numbers $g_1, \ldots, g_{11}$ are the oriented areas of its faces. Using the notations introduced in Proof 5.9 of Theorem 5.7, denote by $H_{\mathbf{A}}$ the point of $\mathbf{A}$ that corresponds to the polyhedral herisson H_0 (i.e. the set of its support numbers) and denote by $H_{\mathbf{B}}$ the point of $\mathbf{B}$ that corresponds to H_0 (i.e. the set of the oriented areas of the faces of H_0). Let U be a neighborhood of $H_{\mathbf{A}}$ in $\mathbf{A}$ such that the sum of the absolute values of the support numbers of any polyhedral herisson from U is less than twice the sum of the absolute values of the support numbers of H_0. From Proof 5.9 it follows that $\varphi(U)$ is an open neighborhood of the point $H_{\mathbf{B}}$. In particular, $\varphi(U)$ contains infinitely many polyhedral herissons (more precisely, the corresponding tuples of the oriented areas of their faces) from the sequence H_k, $k \in \mathbf{N}$. Uniqueness Theorem 4.6 implies that only one point of the manifold $\mathbf{A}$ (or, in terms of Proof 5.9, only one class) corresponds to the polyhedral herisson $H_k \in \mathbf{B}$ and, obviously, that a single point lies in U. Consequently there is an infinite subsequence of H_k, $k \in \mathbf{N}$, with the sum of the absolute values of the support numbers being uniformly bounded from above. The latter contradicts the fact that the length of the waist of H_k equals k. This contradiction shows that a polyhedral herisson H_0 with the above-indicated properties does not exist.

6 Concluding remarks

In this section we briefly discuss several problems on polyhedral herissons that are beyond the scope of the paper.

6.1. Remarks. Recall that a convex polytope is called a *parallelotope* if the 3-dimensional space $\mathbf{R}^3$ can be filled with its parallel copies in such a way that every two copies either do not intersect, or have a common vertex, or a common edge (a whole edge rather than a part of an edge), or a common face (an entire face again) [1]. Parallelotopes play an important role in such different fields of sciences as number theory, crystallography, stochastic geometry, theory of vector measures, etc. We only mention that H. Minkowski proved his famous uniqueness theorem (see Theorem 2.6 above) with a view to prove that every parallelotope has a center of symmetry.

There are a complete classification of parallelotopes in $\mathbf{R}^3$ (see, for example, [1]) and several isolated attempts to study nonconvex space-filling bodies (see, for example, [20]). It turns out that there are space-filling nonconvex polyhedral herissons. A simple example can be constructed as follows. Consider an equilateral trapezium (as a convex body in the plane) such that the angle between the bigger base and each of the lateral sides equals $\pi/4$, and the smaller base is 3 times shorter than the bigger base. Denote by P the nonconvex body that is the union of the trapezium and its image under the symmetry with respect to the straight line passing through the smaller base of the trapezium (Fig. 8). Denote by Q the Cartesian product of P with a straight line segment not contained in the plane passing through P. The reader will have no difficulty in showing that the boundary of Q can be made a polyhedral herisson and parallel copies of the herisson can fill the space.
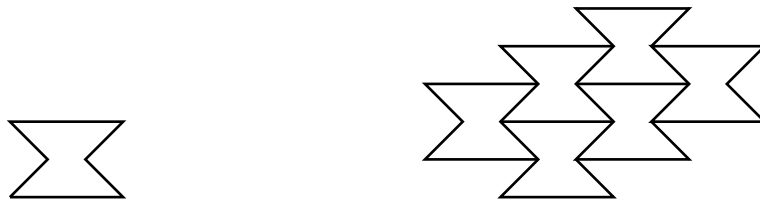

Fig. 8. *The bottom face of a space-filling nonconvex polyhedral herisson.*

6.2. Problem. Does there exist a classification of space-filling polyhedral herissons in $\mathbf{R}^3$?

6.3. Remarks. To prove his famous uniqueness theorem (see Theorem 2.6 above), H. Minkowski developed a machinery based on an operation that is now known as the Minkowski sum of convex bodies and made an enormous advance

in the theory of isoperimetric inequality by proving, for example, the so-called Minkowski inequality for mixed volumes (see, for example, [1], [2], or [21]).

Let P and Q be convex polytopes and let R be their Minkowski sum. It is well known that R is a convex polytope too. Moreover, a face of R with a prescribed outward normal vector n can be obtained (up to translation) as the Minkowski sum of the faces (which may degenerate into edges or vertices) of P and Q with outward normal vector n.

This property leads naturally to a definition of the Minkowski sum of non-convex polyhedral herissons. Actually, faces of polyhedral herissons are convex polygons and the Minkowski sum is already defined for them. On the other hand, from the definition of a polyhedral herisson it follows that the notion of a face with a given outward normal vector is defined properly.

6.4. Problem. For polyhedral herissons, do there exist inequalities similar to those for mixed volumes of convex bodies?

6.5. Remarks. In [16], the notion of the mixed volume is introduced for virtual polytopes. As we have mentioned above, the notion of a virtual polytope generalizes that of a polyhedral herisson. Nevertheless, no inequality for mixed volumes is given in [16]. Isoperimetric-type inequalities for "smooth" herissons are established in [9].

6.6. Problem. Does there exist a many-dimensional generalization of the notion of a polyhedral herisson in $\mathbf{R}^3$ that allows us to prove a rigidity theorem similar to theorems proven in [18] and to show that for every cross-polytope in $\mathbf{R}^5$, there is an equipment which makes it a herisson?

6.7. Remark. An affirmative answer to Problem 6.6 will imply that there are no flexible cross-polytopes in the Euclidean 5-space. This result will support the hypothesis that there are no flexible polyhedra in $\mathbf{R}^n$ for $n \geq 5$. Note that the largest collection of examples of flexible cross-polytopes in the Euclidean 4-space may be found in [22].

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