MIP models for two-dimensional non-guillotine cutting problems with usable leftovers*

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Abstract

In this study we deal with the two-dimensional non-guillotine cutting problem of how to cut a set of larger rectangular objects to a set of smaller rectangular items in exactly a demanded number of pieces. We are concerned with the special case of the problem in which the non-used material of the cutting patterns (objects leftovers) may be used in the future, for example if it is large enough to fulfill future item demands. Therefore, the problem is seen as a two-dimensional non-guillotine cutting/packing problem with usable leftovers, also known in the literature as a two-dimensional residual bin-packing problem. We use multilevel mathematical programming models to represent the problem appropriately, which basically consists of cutting the ordered items using a set of objects of minimum cost, among all possible solutions of minimum cost, choosing one that maximizes the value of the usable leftovers, and, among them, selecting one that minimizes the number of usable leftovers. Because of special characteristics of these multilevel models, they can be reformulated as one-level mixed integer programming (MIP) models. Illustrative numerical examples are presented and analysed.

Key words: Two-dimensional cutting with usable leftovers, MIP models, non-guillotine cutting and packing, multilevel mathematical programming, residual bin-packing problem.

1 Introduction

Cutting problems are closely related to packing problems and basically consist of determining the "best" way of cutting large stock objects to produce ordered small items so that one or more objectives are optimized. For surveys on cutting and packing problems and their industrial applications, readers may consult [20] and the references therein. In this study, we deal with the two-dimensional non-guillotine cutting problem of how to cut a set of larger rectangular objects to a set of smaller rectangular items in exactly demanded number of pieces. The cutting/packing is referred to as two-dimensional since it involves two relevant dimensions, namely the widths and heights of the objects and items. The term non-guillotine refers to the fact that cuts are not

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restricted to be guillotine cuts as imposed by some cutting machines or packing environments (an orthogonal guillotine cut on a rectangle is a cut from one edge of the rectangle to the opposite edge, parallel to the remaining edge). A feasible two-dimensional cutting/packing pattern for an object is one in which items are entirely placed within the object, they do not overlap each other and each item has one edge parallel to one edge of the object (i.e. an orthogonal pattern). We assume that the assortment of ordered items can be strongly heterogeneous, i.e. the set of items can be characterized by the fact that only very few items are of identical size.

We are particularly concerned with the special case of the problem in which the non-used material of the cutting patterns (object remainders or leftovers) may be used in the future, if they are large enough to fulfill future item demands. In other words, we consider that the trim loss of a cutting pattern does not necessarily represent a waste of material. If this trim loss is of a reasonable size, it can be stocked and used again as input (then called a residual piece or a retail or a leftover) in subsequent cutting processes. Otherwise, if the trim loss is considered too small to be used in the future, it represents material waste and is discarded as scrap. Therefore, the problem is seen as a two-dimensional non-guillotine cutting problem with usable leftovers. Note that the assortment of stock objects of this problem is considered heterogeneous as different leftovers of previous cutting processes are put back into stock. According to the typology in [26], this cutting problem can be characterized as a "residual bin-packing problem" because of the possibility of creating new residual pieces and the assumption of strongly heterogeneous assortment of small items. Otherwise, if the assortment of items were weakly heterogeneous, the problem would be considered as a "residual cutting-stock problem". We observe that the simple objective of minimizing the cost or trim loss of the objects cut may not be appropriate for this problem.

The use of object remainders in cutting and packing problems was apparently first discussed in [8]. However, studies dealing with this subject began mainly after the work in [13]. One-dimensional cutting/packing problems that allow the provision of residual pieces have been studied by different authors (see pioneers' work [22, 23] and some recent work such as [10, 11, 17] and the references therein). Examples of applications of one-dimensional cutting problems with usable leftovers were reported in, e.g. the textile industry [16], the agricultural light aircraft manufacturing [1], and the wood-processing industry [19]. To the best of our knowledge, all studies reported in the literature focus on one-dimensional residual bin-packing problems. We are not aware of other studies dealing with residual bin-packing problems involving two dimensions.

Besides the inherent complexity of two-dimensional residual bin-packing problems, we are also motivated by its practical relevance in different industrial settings, such as in the cutting of steel and glass stock plates into required sizes, the cutting of wood sheets and textile materials to make ordered pieces, the cutting of cardboards into boxes, among others. In this study we investigate multilevel mathematical programming models to represent appropriately two-dimensional non-guillotine cutting problems with usable object remainders. These models basically consist of cutting the ordered items using a set of objects of minimum cost, among all possible solutions of minimum cost, choosing one that maximizes the value of the usable leftovers, and, among them, selecting one that minimizes the number of usable leftovers. Because of special characteristics of these multilevel models, we show that they can be reformulated as one-level mixed integer programming (MIP) models. Numerical experiments are performed to highlight that the models represent the problem appropriately and to illustrate their performances when solving some problem instances using a commercial software.

The paper is organized as follows. In Section 2, we present a MIP model for the two-dimensional non-guillotine cutting problem without considering leftovers. In Section 3, we describe multilevel approaches for the non-guillotine cutting problem with guillotine leftovers, i.e.

the object residual pieces are restricted to be generated by two guillotine cuts and to be located to the right and top of the objects. First, a bilevel model is presented to minimize costs of the objects used and maximize the value of the generated (guillotine) leftovers. Then, the model is reformulated as a MIP model and extended to also consider the minimization of the number of valuable leftovers. In Section 4, we modify the models in order to consider the non-guillotine cutting problem with arbitrary valuable leftovers, i.e. the constraints of guillotine leftovers are relaxed and usable object remainders are free to be located anywhere within the objects. In Section 5, we report and analyse the numerical results obtained by solving the models using an optimization software. Finally, in Section 6, we present concluding remarks and discuss perspectives for future research.

2 Modeling the non-guillotine cutting problem

Some studies in the literature have proposed MIP formulations for two-dimensional non-guillotine cutting problems without considering leftovers. For example, [3] and [18] presented 0-1 linear programming formulations for the problem with a single object using 0-1 decision variables for the positions of the items cut from the object. In [25] and [9], the problem was formulated as 0-1 linear models using left, right, top, and bottom decision variables for the relative positions of each pair of pieces cut from the object (with multiple choice disjunctive constraints). Other related 0-1 linear formulations appear in [7, 14, 5, 6, 2] and 0-1 non-linear formulations were presented in [24, 21, 4].

Given a set of n demanded items with width w_i and height $h_i, i = 1, \ldots, n$, and a set of m available objects with width W_j, height H_j, and cost c_j per unit of area, $j = 1, \ldots, m$, the non-guillotine cutting problem (without residual pieces) is defined as the one of cutting the demanded items from the available objects minimizing the cost of the used objects. No rotations are allowed and there are no other constraints related to the positioning of the items within the objects, or the types of cuts of the objects (e.g. guillotine or staged cuts). We assume that the cuts of the objects are infinitely thin (otherwise we consider that the saw thickness was added to the dimensions of the objects and items). We also assume that the items' and objects' dimensions are positive integers and the objects' costs per unit of area are non-negative integers. These are not very restrictive hypotheses to deal with real instances since, due to the finite precision of the cutting and measuring tools and due to the finite precision used in any currency considered to define the objects' costs, they can be easily satisfied by a change of scale.

Let us define $u_j \in \{0, 1\}$, $j = 1, \ldots, m$, to indicate whether object j is used to cut at least a demanded item ($u_j = 1$) or not ($u_j = 0$), and $v_{ij} \in \{0, 1\}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, to assign item i to object j (in this case $v_{ij} = 1$) or not ($v_{ij} = 0$). It is clear that these variables must satisfy the relation

$$u_j \geq v_{ij},\ i = 1, \ldots, n,\ j = 1, \ldots, m, \tag{1}$$

and that variables v_{ij} must satisfy the constraint

$$\sum_{j=1}^{m} v_{ij} = 1,\ i = 1, \ldots, n, \tag{2}$$

to ensure that each item is assigned to exactly one object. Moreover, at this point, it is also possible to define the objective function to be minimized as

$$\sum_{j=1}^{m} c_j W_j H_j u_j. \tag{3}$$

If we now define (x_i, y_i), $i = 1, \dots, n$, as the Cartesian center's coordinates of item i, we can easily write constraints to model that item i must be contained within object j whenever $v_{ij} = 1$. If, without loss of generality, we arbitrarily assume that the left-bottom corner of the objects is located at the origin of the two-dimensional plane, these constraints can be written as

$$\begin{array}{rcll} x_i - w_i/2 & \geq & 0, & i = 1, \dots, n, \\ y_i - h_i/2 & \geq & 0, & i = 1, \dots, n, \\ x_i + w_i/2 & \leq & W_j + (\hat{W} - W_j)(1 - v_{ij}), & i = 1, \dots, n, j = 1, \dots, m, \\ y_i + h_i/2 & \leq & H_j + (\hat{H} - H_j)(1 - v_{ij}), & i = 1, \dots, n, j = 1, \dots, m, \end{array} \tag{4}$$

where $\hat{W} = \max_{j'=1,\dots,m}\{W_{j'}\}$ and $\hat{H} = \max_{j'=1,\dots,m}\{H_{j'}\}$. Note that the set of constraints (4) represents the big-M MIP reformulation of

$$v_{ij} = 1 \implies (x_i, y_i) \in [w_i/2, W_j - w_i/2] \times [h_i/2, H_j - h_i/2], \; i = 1, \dots, n, \; j = 1, \dots, m.$$

To avoid the overlapping between items i and i' that are being assigned to object j, constraints can be written as

$$v_{ij} = v_{i'j} = 1 \implies (|x_i - x_{i'}| \geq (w_i + w_{i'})/2 \text{ or } |y_i - y_{i'}| \geq (h_i + h_{i'})/2, \\ i = 1, \dots, n, \; i' = i + 1, \dots, n, \; j = 1, \dots, m.$$

Re-writing "$|a| \geq b$" as "$a \geq b$ or $a \leq -b$" and defining auxiliary variables $\pi_{ii'}, \tau_{ii'} \in \{0, 1\}, i = 1, \dots, n, i' = i + 1, \dots, n$, to consider the big-$M$ MIP reformulation of the resulting four-terms disjunctions, we obtain the following MIP constraints

$$\begin{array}{rcl} x_{i'} + w_{i'}/2 & \leq & x_i - w_i/2 + \hat{W}(1 - v_{ij}) + \hat{W}(1 - v_{i'j}) + \hat{W}\pi_{ii'} + \hat{W}\tau_{ii'}, \\ x_{i'} - w_{i'}/2 & \geq & x_i + w_i/2 - \hat{W}(1 - v_{ij}) - \hat{W}(1 - v_{i'j}) - \hat{W}\pi_{ii'} - \hat{W}(1 - \tau_{ii'}), \\ y_{i'} + h_{i'}/2 & \leq & y_i - h_i/2 + \hat{H}(1 - v_{ij}) + \hat{H}(1 - v_{i'j}) + \hat{H}(1 - \pi_{ii'}) + \hat{H}\tau_{ii'}, \\ y_{i'} - h_{i'}/2 & \geq & y_i + h_i/2 - \hat{H}(1 - v_{ij}) - \hat{H}(1 - v_{i'j}) - \hat{H}(1 - \pi_{ii'}) - \hat{H}(1 - \tau_{ii'}), \\ & & i = 1, \dots, n, \; i' = i + 1, \dots, n, \; j = 1, \dots, m. \end{array}$$

The first two "M-terms" of each constraint serve to neutralize the effect of the constraint whenever $v_{ij} = 0$ or $v_{i'j} = 0$, meaning that item i or item i' was not assigned to object j and, therefore, there is no overlapping to be avoided between items i and i' on object j. The other two "M-terms" of each constraint, together with the four possible combinations of values of the binary variables $\pi_{ii'}$ and $\tau_{ii'}$, serve to model the disjunction that ensures that, whenever items i' and i are assigned to the same object, item i' must be "to the left", "to the right", "below" or "above" item i (with the minimum distance to avoid the overlapping).

As it was pointed out in [2], it is possible to see that if two items i and i' are identical ($w_i = w_{i'}$ and $h_i = h_{i'}$) then interchanging their roles generates a symmetric solution. To avoid this situation that may slow down a branch-and-bound algorithm, constraints that avoid the overlapping of a pair of identical items can be modeled in a different way. Roughly speaking, if items i and i' with $i' > i$ are identical, instead of requesting item i' to be "to the left, to the right, below or above" item i, it is enough to request item i' to be "to the right" or "above" item i. Assume, without loss of generality, that identical items are numbered consecutively, that there are p different types of items, and that there are n_q items of the q-th type, $q = 1, \dots, p$, with $\sum_{q=1}^{p} n_q = n$. Denoting $o_q = \sum_{q'=1}^{q-1} n_{q'}$, we have that items of the q-th type are numbered from $o_q + 1$ to $o_q + n_q$. Therefore, non-overlapping constraints can be re-written as

$$\begin{array}{rcl} x_{i'} - w_{i'}/2 & \geq & x_i + w_i/2 - \hat{W}(1 - v_{ij}) - \hat{W}(1 - v_{i'j}) - \hat{W}\pi_{ii'}, \\ y_{i'} - h_{i'}/2 & \geq & y_i + h_i/2 - \hat{H}(1 - v_{ij}) - \hat{H}(1 - v_{i'j}) - \hat{H}(1 - \pi_{ii'}), \\ \multicolumn{3}{l}{q = 1, \dots, p, \; i = o_q + 1, \dots, o_q + n_q, \; i' = i + 1, \dots, o_q + n_q, \; j = 1, \dots, m,} \end{array} \tag{5}$$

and

$$
\begin{array}{rcl}
x_{i'} + w_{i'}/2 & \leq & x_i - w_i/2 + \hat{W}(1 - v_{ij}) + \hat{W}(1 - v_{i'j}) + \hat{W}\pi_{ii'} + \hat{W}\tau_{ii'}, \\
x_{i'} - w_{i'}/2 & \geq & x_i + w_i/2 - \hat{W}(1 - v_{ij}) - \hat{W}(1 - v_{i'j}) - \hat{W}\pi_{ii'} - \hat{W}(1 - \tau_{ii'}), \\
y_{i'} + h_{i'}/2 & \leq & y_i - h_i/2 + \hat{H}(1 - v_{ij}) + \hat{H}(1 - v_{i'j}) + \hat{H}(1 - \pi_{ii'}) + \hat{H}\tau_{ii'}, \\
y_{i'} - h_{i'}/2 & \geq & y_i + h_i/2 - \hat{H}(1 - v_{ij}) - \hat{H}(1 - v_{i'j}) - \hat{H}(1 - \pi_{ii'}) - \hat{H}(1 - \tau_{ii'}), \\
& & q = 1, \dots, p,\ i = o_q + 1, \dots, o_q + n_q,\ i' = o_q + n_q + 1, \dots, n,\ j = 1, \dots, m.
\end{array} \tag{6}
$$

The set of constraints (5) models the non-overlapping between identical items using a halved number of constraints and binary variables, with the extra feature of avoiding the mentioned type of symmetric solutions. The set of constraints (6) models the non-overlapping between pairs of non-identical items as previously introduced.

Summing up, the MIP model of the tackled non-guillotine cutting problem is given by minimizing (3) on $u_j \in \{0,1\}$ $(j = 1, \dots, m)$, $v_{ij} \in \{0,1\}$ $(i = 1, \dots, n,\ j = 1, \dots, m)$, $x_i,\ y_i \in \mathbb{R}$ $(i = 1, \dots, n)$, $\pi_{ii'} \in \{0,1\}$ $(i = 1, \dots, n,\ i' = i+1, \dots, n)$, and $\tau_{ii'} \in \{0,1\}$ $(q = 1, \dots, p,\ i = o_q + 1, \dots, o_q + n_q,\ i' = o_q + n_q + 1, \dots, n)$ subject to (1,2,4,5,6). There are $m + mn + n(n-1) - \sum_{q=1}^{p} n_q(n_q - 1)/2$ binary variables (which coincide with $m + mn + n(n-1)$ in the case in which there are no identical items), $2n$ continuous variables, and $3mn + 3n + 2m \sum_{q=1}^{p} n_q(n_q - 1)/2 + 4m[n(n-1)/2 - \sum_{q=1}^{p} n_q(n_q - 1)/2]$ constraints (which coincide with $3mn + 3n + n(n-1)$ in the case in which there are no identical items).

As an illustrative example, consider an instance with $m = 2$ identical objects with $W_1 = W_2 = 12$, $H_1 = H_2 = 20$, and $c_1 = c_2 = 1$, and $n = 4$ different items with $w_1 = w_2 = w_3 = w_4 = 5$ and $h_1 = 16$, $h_2 = 14$, $h_3 = 12$, and $h_4 = 8$. Figures 1(a–b) show two different feasible solutions with cost 480. Since the sum of the areas of the four demanded items is larger than the area of any of the objects, at least two objects are needed to cut the items and, therefore, the depicted feasible solutions are optimal. However, there is a feature that differentiates these two optimal solutions which is not being captured by the introduced model. Assuming that the result of a first horizontal guillotine cut could be saved as a residual piece to be used in the future, the optimal solution in Figure 1(b) has a leftover in one of the objects that can potentially be used to cut two items such as item 4 (with $w_4 = 5$ and $h_4 = 8$), while the optimal solution in Figure 1(a) does not present this property. In the following section, the notion of guillotine leftovers is introduced and added to model (1–6) in order to obtain an optimal solution to (1–6) that maximizes the value of the leftovers.

3 A multilevel approach for considering guillotine leftovers

In this section, the notion of guillotine leftovers is introduced. We arbitrarily assume that two guillotine leftovers can be produced from each used object (unused objects cannot produce residual pieces) and that each leftover is generated by a single guillotine cut, i.e. before cutting the items from an object there are two possibilities: (a) a vertical guillotine cut is made in such a way that the right-hand side of the object is a leftover and then a horizontal guillotine cut is made in such a way that the top part of the object is a leftover; or (b) the horizontal guillotine cut is made first and the vertical guillotine cut is made in second place. Figures 2(a–b) illustrate both situations, named Case A and Case B, respectively, from now on. In any case, a list containing $d \geq 1$ catalogued items with widths $\bar{w}_1, \dots, \bar{w}_d$ and heights $\bar{h}_1, \dots, \bar{h}_d$ is assumed to be given and a leftover is considered a valuable leftover if it can contain at least one item from the catalogue. Note that the list of catalogued items may coincide with the demanded items (in this case we have $d = n$ and $\bar{w}_i = w_i$ and $\bar{h}_i = h_i$ for $i = 1, \dots, n$) or it can consist of a single

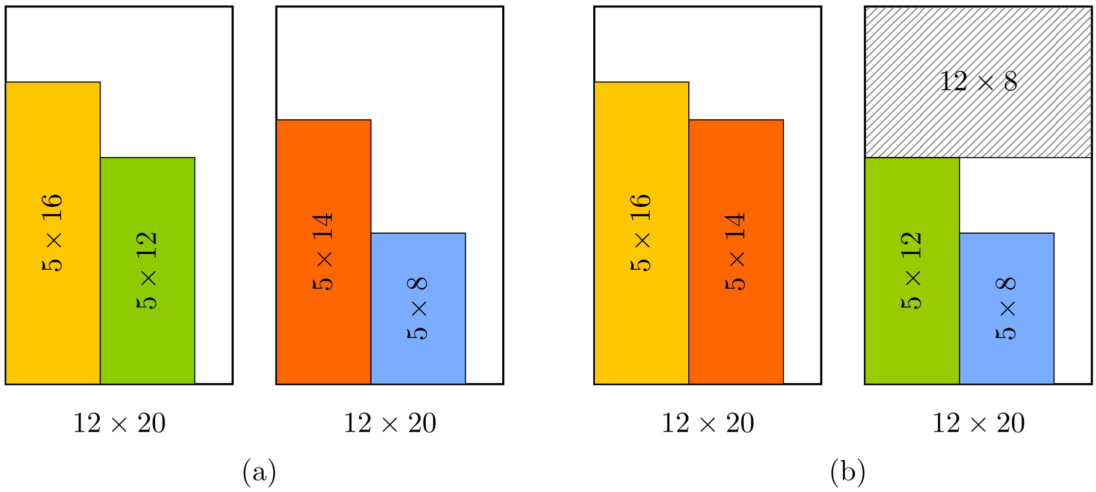


Figure 1: Graphs (a) and (b) represent two alternative optimal solutions of a simple instance with two objects and four items. If leftovers were being considered, optimal solution (b) would have the potential of producing a leftover from which two extra items like item 4 (with $w_4 = 5$ and $h_4 = 8$) might be cut in the future.

model item representing the minimum width $\bar{w}_1$ and height $\bar{h}_1$ that a leftover must have to be considered valuable. It is worth noting that the definition of leftover in the present paragraph exclude any other portion of the objects that will be considered as a trim-loss or waste of the cutting process.

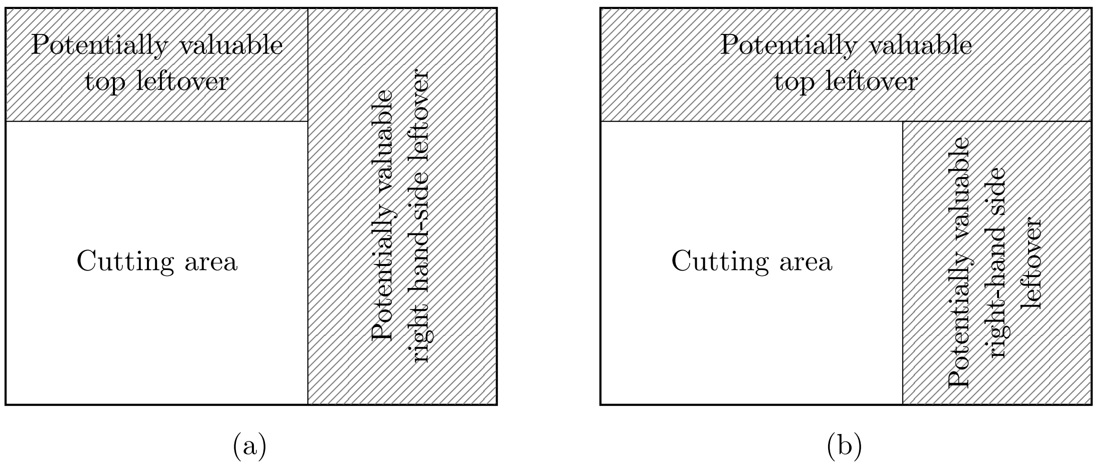


Figure 2: Graphs (a) and (b) represent the two possible ways of generating guillotine valuable leftovers by a vertical and a horizontal guillotine cut. (a) represents the case in which the vertical guillotine cut is made in the first place, while (b) represents the case in which the horizontal guillotine cut is made in the first place. It is important to notice that, although the sum of the leftovers' areas coincide in both cases, the sum of their values may differ.

Let t_j be the variable height of the top leftover of object j and let r_j be the variable width of the right-hand side leftover for $j = 1, \dots, m$. Recall that, since unused objects cannot generate leftovers, we must have $t_j = r_j = 0$ whenever $u_j = 0$. In Case A, the top leftover has a variable width and height given by $W_j - r_j$ and t_j, respectively, while the right-hand side leftover has a variable width given by r_j and a fixed height given by H_j. In Case B, the top leftover has a fixed width given by W_j and a variable height given by t_j, while the right-hand side leftover has variable width and height given by r_j and $H_j - t_j$, respectively. We define the value of a leftover as the value per unit of area of the object times the area of the leftover, whenever the leftover can hold at least one item from the catalogue, and zero otherwise. Therefore, given $t_j, r_j, j = 1, \dots, m$, the sum of the leftovers' values is given by

$$\sum_{j=1}^{m} c_j \max\{\alpha_j^t + \alpha_j^r, \beta_j^t + \beta_j^r\}, \tag{7}$$

where

$$\begin{array}{rcl}
\alpha_j^t & = & \left\{ \begin{array}{rl} (W_j - r_j)t_j, & \text{if there exits } 1 \le s \le d \text{ such that } W_j - r_j \ge \bar{w}_s \text{ and } t_j \ge \bar{h}_s, \\ 0, & \text{otherwise,} \end{array} \right. \\
\alpha_j^r & = & \left\{ \begin{array}{rl} r_j H_j, & \text{if there exits } 1 \le s \le d \text{ such that } r_j \ge \bar{w}_s \text{ and } H_j \ge \bar{h}_s, \\ 0, & \text{otherwise,} \end{array} \right. \\
\beta_j^t & = & \left\{ \begin{array}{rl} W_j t_j, & \text{if there exits } 1 \le s \le d \text{ such that } W_j \ge \bar{w}_s \text{ and } t_j \ge \bar{h}_s, \\ 0, & \text{otherwise,} \end{array} \right. \\
\beta_j^r & = & \left\{ \begin{array}{rl} r_j(H_j - t_j), & \text{if there exits } 1 \le s \le d \text{ such that } r_j \ge \bar{w}_s \text{ and } H_j - t_j \ge \bar{h}_s, \\ 0, & \text{otherwise,} \end{array} \right.
\end{array} \tag{8}$$

for $j = 1, \dots, m$.

To combine the guillotine leftovers with the non-guillotine cutting problem of the previous section, it is enough to replace the set of constraints (4), that ensures that each items must be placed within the object to which it was assigned, by a modified set of constraints that ensures that each item must be placed within the "Cutting area" of the object to which it was assigned (see Figure 2). The set of modified constraints is given by

$$\begin{array}{rcll}
x_i - w_i/2 & \ge & 0, & i = 1, \dots, n, \\
y_i - h_i/2 & \ge & 0, & i = 1, \dots, n, \\
x_i + w_i/2 & \le & W_j - r_j + (\hat{W} - W_j)(1 - v_{ij}), & i = 1, \dots, n, j = 1, \dots, m, \\
y_i + h_i/2 & \le & H_j - t_j + (\hat{H} - H_j)(1 - v_{ij}), & i = 1, \dots, n, j = 1, \dots, m,
\end{array} \tag{9}$$

plus

$$0 \le t_j \le H_j u_j \text{ and } 0 \le r_j \le W_j u_j, \quad j = 1, \dots, m. \tag{10}$$

Constraint (10) reflects the fact that the height of the top leftovers and the width of the right-hand side leftovers are non-negative quantities and that they must be zero when the object is not being used.

Now, we are able to present our first model of the non-guillotine cutting problem with guillotine leftovers. It consists of cutting the demanded items using a set of objects of minimum cost and, among all possible solutions of minimum cost, to choose one that maximizes the value of the guillotine leftovers. Variables of the problem are: α_j^t, α_j^r, β_j^t, β_j^r, t_j, $r_j \in \mathbb{R}$ $(j = 1, \dots, m)$ to characterize the leftovers; $u_j \in \{0, 1\}$ $(j = 1, \dots, m)$ to determine the used objects; $v_{ij} \in \{0, 1\}$ $(i = 1, \dots, n,\ j = 1, \dots, m)$ to assign items to objects; and $\pi_{ii'} \in \{0, 1\}$ $(i = 1, \dots, n,$ $i' = i + 1, \dots, n)$ and $\tau_{ii'} \in \{0, 1\}$ $(t = 1, \dots, p,\ i = o_q + 1, \dots, o_q + n_q,\ i' = i + 1, \dots, n_q)$ to model the items' overlapping. The model is given by maximizing (7) on α_j^t, α_j^r, β_j^t, β_j^r, t_j, r_j, u_j, v_{ij}, $\pi_{ii'}$, and $\tau_{ii'}$ subject to (8) and subject to t_j, r_j, u_j, v_{ij}, $\pi_{ii'}$, and $\tau_{ii'}$ being a solution of minimizing (3) subject to (1,2,9,10,5,6). This kind of optimization problem in which there is a constraint that requires that a subset of the variables must be a solution of another optimization problem is known as the bilevel programming problem [12]. The problem that appears as a constraint is called the *lower level* problem while the main problem is termed the *upper level* problem.

Figures 1(a–b) represent two different feasible solutions of an instance of the bilevel programming problem defined above (they are feasible because they are both optimal solutions to the non-guillotine cutting problem of minimizing the cost of the used objects). If we define the list of catalogued items as the list of demanded items, we have that the objects of the feasible solution on Figure 1(a) has no valuable leftovers, while the right-hand side object in Figure 1(b) has a valuable leftover. In fact, it may not be hard to realise that the solution in Figure 1(b) is an optimal solution.

The remainder of the present section focuses on a MIP reformulation of the problem. Moreover, a new feature is added to the model: among the solutions with minimum cost, we would like to find one with maximum value of the leftovers, and, among them, one with the smallest possible number of leftovers (in order to include a concern related to stocking the leftovers for future usage into the model). This addition would give rise to a three-level optimization problem, but its inclusion is delayed to be presented after the MIP reformulation of the problem to simplify the presentation. Therefore, in the rest of the present section, we first reformulate the presented bilevel model into a MIP bilevel problem. Then, a one-level MIP reformulation of the bilevel MIP problem is given and, finally, the one-level MIP model is modified to include the concern with the number of leftovers.

3.1 Reformulation as a MIP bilevel problem

To construct a MIP bilevel model of the problem described above we need to eliminate: (i) the term $\max\{\alpha_j^t + \alpha_j^r, \beta_j^t + \beta_j^r\}$ that appears in the objective function (7), (ii) the "if statements" that appear in the definition of α_j^t, α_j^r, β_j^t, and β_j^r in (8), and (iii) the nonlinear term $r_j t_j$ that appears in the definition of α_j^t and β_j^r in (8).

By defining continuous variables γ_j and binary variables $\eta_j \in \{0, 1\}$, for $j = 1, \ldots, m$, and adding constraints

$$\begin{array}{rcll} \gamma_j & \leq & \alpha_j^t + \alpha_j^r + W_j H_j \eta_j, & j = 1, \ldots, m, \\ \gamma_j & \leq & \beta_j^t + \beta_j^r + W_j H_j (1 - \eta_j), & j = 1, \ldots, m, \end{array} \tag{11}$$

the objective function (7) can be re-written as the linear function

$$\sum_{j=1}^{m} c_j \gamma_j. \tag{12}$$

It is easy to see that the binary variable η_j forces γ_j to be less than or equal to $\alpha_j^t + \alpha_j^r$ or less than or equal to $\beta_j^t + \beta_j^r$, and that the quantity $W_j H_j$ in (12) plays the role of the large positive constant of the big-M MIP formulation of this disjunction. Since the objective function (12) is maximized, we have that $\gamma_j = \max\{\alpha_j^t + \alpha_j^r, \beta_j^t + \beta_j^r\}$ for $j = 1, \ldots, m$, at a solution, as desired.

Considering the assumption that the item and object dimensions are positive integers, it can be proved that r_j and t_j, for $j = 1, \ldots, m$, assume integer values at a solution (see, for example, [5, 6] and the references therein). To model the product $r_j t_j$ using MIP constraints, we consider the well-known trick of replacing one of the variables by its binary representation reducing the product to a sum of products of integer and binary variables, for which a simple MIP representation exists (see, for example, [15], [27]). Let $\theta_{\ell j} \in \{0, 1\}$, $j = 1, \ldots, m$, $\ell =$

$1, \dots, \lfloor \log_2(H_j) \rfloor + 1$. Note that H_j is an upper limit for the possible values that t_j may assume and that $\lfloor \log_2(H_j) \rfloor + 1$ is an upper limit on the number of bits needed for the binary representation of t_j, $j = 1, \dots, m$. Therefore, we have that each t_j can be expressed as

$$t_j = \sum_{\ell=1}^{\lfloor \log_2(H_j) \rfloor + 1} 2^{\ell-1} \theta_{\ell j}, \ j = 1, \dots, m, \tag{13}$$

and, hence, each product $r_j t_j$ can be expressed as

$$r_j t_j = \sum_{\ell=1}^{\lfloor \log_2(H_j) \rfloor + 1} 2^{\ell-1} r_j \theta_{\ell j}, \ j = 1, \dots, m.$$

Therefore, the product of two integer variables was reduced to the sum of products of an integer and a binary variable. The value of each of these products is equal to zero when the binary variable is equal to zero and coincides with the value of the integer variable when the binary variable is equal to one. This situation can be modeled using MIP constraints considering continuous variables $\omega_{\ell j}$, $j = 1, \dots, m$, $\ell = 1, \dots, \lfloor \log_2(H_j) \rfloor + 1$, such that

$$\omega_{\ell j} = \begin{cases} r_j, & \text{whenever } \theta_{\ell j} = 1, \\ 0, & \text{otherwise,} \end{cases}$$

which can be achieved by adding the MIP constraints

$$0 \le \omega_{\ell j} \le r_j \text{ and } r_j - (1 - \theta_{\ell j}) W_j \le \omega_{\ell j} \le \theta_{\ell j} W_j, \ j = 1, \dots, m, \ \ell = 1, \dots, \lfloor \log_2(H_j) \rfloor + 1. \tag{14}$$

Then, we have that variable t_j may be dismissed, substituting every appearance of it by its binary representation (13) and every appearance of the product $r_j t_j$ (in the definitions (8) of α_j^t and β_j^r) may be replaced by its MIP reformulation given by

$$r_j t_j = \sum_{\ell=1}^{\lfloor \log_2(H_j) \rfloor + 1} 2^{\ell-1} \omega_{\ell j}, \ j = 1, \dots, m. \tag{15}$$

To keep the number of auxiliary binary variables $\theta_{\ell j}$ (and continuous variables $\omega_{\ell j}$) to its minimum, for each j, t_j should be replaced by its binary expansion whenever $H_j \le W_j$ and r_j should be replaced by its binary expansion, otherwise. We will ignore this possibility to simplify the presentation.

To model (8) with MIP constraints, let $\bar{\alpha}_{sj}^t, \bar{\alpha}_{sj}^r, \bar{\beta}_{sj}^t, \bar{\beta}_{sj}^r \in \{0, 1\}$, $s = 1, \dots, d$, $j = 1, \dots, m$, and consider the constraints:

$$\begin{array}{lllllll} \bar{w}_s & \le & W_j - r_j + \hat{W}(1 - \bar{\alpha}_{sj}^t) & \text{and} & \bar{h}_s & \le & t_j + \hat{H}(1 - \bar{\alpha}_{sj}^t), \\ \bar{w}_s & \le & r_j + \hat{W}(1 - \bar{\alpha}_{sj}^r) & \text{and} & \bar{h}_s & \le & H_j + \hat{H}(1 - \bar{\alpha}_{sj}^r), \\ \bar{w}_s & \le & W_j + \hat{W}(1 - \bar{\beta}_{sj}^t) & \text{and} & \bar{h}_s & \le & t_j + \hat{H}(1 - \bar{\beta}_{sj}^t), \\ \bar{w}_s & \le & r_j + \hat{W}(1 - \bar{\beta}_{sj}^r) & \text{and} & \bar{h}_s & \le & H_j - t_j + \hat{H}(1 - \bar{\beta}_{sj}^r), \\ & & & & & & s = 1, \dots, d, \ j = 1, \dots, m, \end{array} \tag{16}$$

and

$$\begin{array}{lllllllll} 0 & \le & \alpha_j^t & \le & W_j t_j - r_j t_j & \text{and} & \alpha_j^t & \le & W_j H_j \sum_{s=1}^d \bar{\alpha}_{sj}^t, \\ 0 & \le & \alpha_j^r & \le & r_j H_j & \text{and} & \alpha_j^r & \le & W_j H_j \sum_{s=1}^d \bar{\alpha}_{sj}^r, \\ 0 & \le & \beta_j^t & \le & W_j t_j & \text{and} & \beta_j^t & \le & W_j H_j \sum_{s=1}^d \bar{\beta}_{sj}^t, \\ 0 & \le & \beta_j^r & \le & r_j H_j - r_j t_j & \text{and} & \beta_j^r & \le & W_j H_j \sum_{s=1}^d \bar{\beta}_{sj}^r, \\ & & & & & & & & j = 1, \dots, m. \end{array} \tag{17}$$

Note that, once again, in (16–17), t_j should be replaced by its binary representation (13) and $r_j t_j$ should be replaced by its MIP reformulation (15) (their appearances are being preserved with the only purpose of simplifying the presentation). The set of constraints (16) ensures that variables $\bar{\alpha}^t_{sj}$, $\bar{\alpha}^r_{sj}$, $\bar{\beta}^t_{sj}$, and $\bar{\beta}^r_{sj}$ assume value zero when the s-th catalogued item cannot be held within the top leftover of Case A, the right-hand side leftover of Case A, the top leftover of Case B, and the right-hand side leftover of Case B, respectively. When the item can be held within the leftover, the variable is free to assume value zero or one. Therefore, the set of constraints (17) ensures that variables α^t_j, α^r_j, β^t_j, and β^r_j must be null, when the corresponding leftover cannot hold any item in the catalogue. Otherwise, they are free to assume any value between zero and the area of the corresponding leftover. Since these variables limit the growth of γ_j in (11), the objective function (12) is a non-negative linear combination of the γ_j, and the upper level problem is a maximization problem, at a solution, variables α^t_j, α^r_j, β^t_j, and β^r_j will achieve their maximum possible values that will coincide with the area of the valuable leftovers.

Now, we are ready to present the MIP bilevel model of our problem. Variables of the problem are: $v_{ij} \in \{0,1\}$ ($i = 1,\dots,n$, $j = 1,\dots,m$) to assign items to object; $u_j \in \{0,1\}$ ($j = 1,\dots,m$) to distinguish used and unused objects; $(x_i, y_i) \in \mathbb{R}^2$ ($i = 1,\dots,n$) to define the position of the items within the objects; $\pi_{ii'} \in \{0,1\}$ ($i = 1,\dots,n$, $i' = i+1,\dots,n$) and $\tau_{ii'} \in \{0,1\}$ ($q = 1,\dots,p$, $i = o_q+1,\dots,o_q+n_q$, $i' = o_q+n_q+1,\dots,n$) to model the overlapping between the items; $r_j \in \mathbb{R}$ ($j = 1,\dots,m$) to define the unused right-hand side margin of the cutting area of the objects that limit the maximum width of the leftovers; $\theta_{\ell j} \in \{0,1\}$ ($\ell = 1,\dots,\lfloor \log_2(H_j) \rfloor + 1$, $j = 1,\dots,m$) to consider the binary representation of the top margin of the cutting area of the objects that limit the maximum height of the leftovers; $\omega_{\ell j} \in \mathbb{R}$ ($\ell = 1,\dots,\lfloor \log_2(H_j) \rfloor + 1$, $j = 1,\dots,m$) to model the product that appears when computing the area of the leftovers; $\gamma_j \in \mathbb{R}$ and $\eta_j \in \{0,1\}$ ($j = 1,\dots,m$) to select between Case A or B of the guillotine cuts that generate the leftovers; α^t_j, α^r_j, β^t_j, and $\beta^r_j \in \mathbb{R}$ ($j = 1,\dots,m$) to represent the area of the valuable leftovers; and $\bar{\alpha}^t_{sj}$, $\bar{\alpha}^r_{sj}$, $\bar{\beta}^t_{sj}$, and $\bar{\beta}^r_{sj} \in \{0,1\}$ ($s = 1,\dots,d$, $j = 1,\dots,m$) to represent whether an item from the catalogue can be placed within a leftover. The problem consists of maximizing (12) on v_{ij}, u_j, x_i, y_i, $\pi_{ii'}$, $\tau_{ii'}$, r_j, $\theta_{\ell j}$, $\omega_{\ell j}$, γ_j, η_j, α^t_j, α^r_j, β^t_j, β^r_j, $\bar{\alpha}^t_{sj}$, $\bar{\alpha}^r_{sj}$, $\bar{\beta}^t_{sj}$, and $\bar{\beta}^r_{sj}$ subject to (11,14,16,17) and subject to v_{ij}, u_j, x_i, y_i, $\pi_{ii'}$, $\tau_{ii'}$, r_j and $\theta_{\ell j}$ being a solution of minimizing (3) subject to (1,2,9,10,5,6).

3.2 Reformulating as a MIP problem and adding the minimization of the number of valuable leftovers

The reformulation of the MIP bilevel problem introduced in the previous subsection as a MIP problem is based on the fact that the objective functions (12) and (3) of the lower and the upper level, respectively, assume integer values only and have trivial lower and upper bounds. Therefore, the objective function of the upper level problem can be used, with the proper scaling, as a tie break between all possible optimal solutions of the lower level problem. This reformulation is possible for the presented bilevel problem because, by nature, it is a simple and particular bilevel programming problem in which the solutions of the lower level problem do not depend on the values of the variables of the upper level problem.

It is easy to see that the objective function (12) assumes integer values that are greater than or equal to zero and strictly smaller than $\sum_{j=1}^m c_j W_j H_j$. Therefore, the combination of the objective functions (3) and (12) given by

$$\left(\sum_{j=1}^m c_j W_j H_j \right) \left(\sum_{j=1}^m c_j W_j H_j u_j \right) - \sum_{j=1}^m c_j \gamma_j, \tag{18}$$

plays the desired role. Note that the term related to the objective function (3) appears multiplied by a strict upper limit of the objective function (12). It implies that each unit in the value of the term related to the objective function (3) is more important than the whole value of the term related to objective function (12). Hence, the term related to objective function (12) takes part in the play-off between the solutions of minimum cost, choosing one with the maximum value of usable leftovers. Therefore, the MIP reformulation being sought is given by minimizing (18) on v_{ij}, u_j, x_i, y_i, $\pi_{ii'}$, $\tau_{ii'}$, r_j, $\theta_{\ell j}$, $\omega_{\ell j}$, γ_j, η_j, α_j^t, α_j^r, β_j^t, β_j^r, $\bar{\alpha}_{sj}^t$, $\bar{\alpha}_{sj}^r$, $\bar{\beta}_{sj}^t$, and $\bar{\beta}_{sj}^r$ subject to (11,14,16,17,1,2,9,10,5,6). (Constraints related to some of the variables being binary are omitted and are the same as in the previous subsection.)

The total number of valuable leftovers is always less than or equal to twice the number of objects, therefore, strictly smaller than $2m+1$. The idea for considering the number of leftovers simply consists of adding binary variables to indicate whether a leftover is valuable or not, summing them up in order to count the number of leftovers, and then using it as a tie break to choose a solution with the smallest number of leftovers. Naturally, we are talking about determining the solutions with a minimum cost of the objects and, among them, selecting one with the maximum value of the valuable leftovers and, among them, choosing one with the minimum number of valuable leftovers.

To count the number of leftovers, let ξ_j^t, $\xi_j^r \in \{0,1\}$, $j=1,\dots,m$, be auxiliary binary variables corresponding to the top and the right-hand side leftovers of object j, respectively. Recall that $\eta_j = 0$ means that leftovers of Case A are being considered on object j and that, in case the top and right-hand side leftovers are valuable, their areas are given by α_j^t and α_j^r, respectively. Analogously, $\eta_j = 1$ means that Case B is considered in object j and, in case the top and right-hand side leftovers are valuable, their areas are given by β_j^t and β_j^r, respectively. Moreover, note that, due to the integrality of t_j (given by its binary representation) and r_j, areas of valuable leftovers are strictly positive integer values. Consider the following constraints

$$\begin{array}{lllllll} \alpha_j^t & \leq & W_jH_j\xi_j^t + W_jH_j\eta_j, & \text{and} & \beta_j^t & \leq & W_jH_j\xi_j^t + W_jH_j(1-\eta_j), \quad j=1,\dots,m, \\ \alpha_j^r & \leq & W_jH_j\xi_j^r + W_jH_j\eta_j, & \text{and} & \beta_j^r & \leq & W_jH_j\xi_j^r + W_jH_j(1-\eta_j), \quad j=1,\dots,m. \end{array} \tag{19}$$

If Case A is chosen ($\eta_j = 0$) then $\xi_j^t = 1$ whenever $\alpha_j^t > 0$, while if Case B is chosen ($\eta_j = 1$) then $\xi_j^t > 0$ whenever $\beta_j^t > 0$. It means that ξ_j^t is forced to be 1 whenever the top leftover is valuable, and it can assume values 0 or 1, otherwise. The same reasoning is also true for ξ_j^r which is related to the right-hand side leftover. Hence $\sum_{j=1}^m(\xi_j^t + \xi_j^r)$ is free to vary from the number of valuable leftovers to $2m$, which is strictly smaller than $2m+1$. Therefore, by the same reasoning applied to the development of the objective function (18), the optimal solutions to minimizing

$$(2m+1)\left[\left(\sum_{j=1}^m c_jW_jH_j\right)\left(\sum_{j=1}^m c_jW_jH_ju_j\right) - \sum_{j=1}^m c_j\gamma_j\right] + \sum_{j=1}^m(\xi_j^t + \xi_j^r), \tag{20}$$

are also optimal solutions to minimizing (18), and, in case of multiple optimal solutions of minimizing (18), (20) is such that the smaller its value, the smaller the number of leftovers.

We now summarize the MIP model introduced in the present section for the non-guillotine cutting problem with guillotined leftovers. It consists of satisfying the items' demand with minimum objects' cost and, among the solutions with minimum cost, it chooses one with the maximum value of the valuable leftovers and, among them, it chooses one with the smallest possible number of valuable leftovers. Input data of an instance is given by: the number of items n; the widths w_i and heights h_i ($i=1,\dots,n$) of the items (it is assumed that identical items are numbered consecutively, so that there are $p \le n$ types of items, n_q items of each type $q = 1, \dots, p$, $\sum_{q=1}^{p} n_q = n$, and $o_q = \sum_{q'=1}^{q-1} n_{q'}$ $(q = 1, \dots, p)$, i.e. items of the q-th type are numbered from o_q+1 to o_q+n_q); a catalogue consisting of d items with widths $\bar{w}_s$ and heights $\bar{h}_s$ $(s = 1, \dots, d)$; the number of objects m; the widths W_j and heights H_j $(j = 1, \dots, m)$ of the objects; and the costs c_j $(j = 1, \dots, m)$ per unit of area of the objects. All dimensions are assumed to be positive integers and costs are assumed to be non-negative integers.

Variables of the model are: $v_{ij} \in \{0,1\}$ $(i = 1, \dots, n,\ j = 1, \dots, m)$ to assign items to objects; $u_j \in \{0,1\}$ $(j = 1, \dots, m)$ to distinguish used and unused objects; $(x_i, y_i) \in \mathbb{R}^2$ $(i = 1, \dots, n)$ to define the position of the items within the objects; $\pi_{ii'} \in \{0,1\}$ $(i = 1, \dots, n,\ i' = i+1, \dots, n)$ and $\tau_{ii'} \in \{0,1\}$ $(q = 1, \dots, p,\ i = o_q+1, \dots, o_q+n_q,\ i' = i+1, \dots, o_q+n_q)$ to model the overlapping between the items; $r_j \in \mathbb{R}$ $(j = 1, \dots, m)$ to define the unused right-hand side margin of the cutting area of the objects; $\theta_{\ell j} \in \{0,1\}$ $(\ell = 1, \dots, \lfloor \log_2(H_j) \rfloor + 1,\ j = 1, \dots, m)$ to construct the binary expansion of the unused top margin of the cutting area of the objects; $\omega_{\ell j} \in \mathbb{R}$ $(\ell = 1, \dots, \lfloor \log_2(H_j) \rfloor + 1,\ j = 1, \dots, m)$ to model the product of the top and the right-hand side margins of the cutting area of the objects; $\eta_j \in \{0,1\}$ $(j = 1, \dots, m)$ to select between Case A or B of the guillotine cuts that generate the leftovers; $\gamma_j \in \mathbb{R}$ $(j = 1, \dots, m)$ to represent the sum of the areas of the top and the right-hand side valuable leftovers of an object; α_j^t, α_j^r, β_j^t, and $\beta_j^r \in \mathbb{R}$ $(j = 1, \dots, m)$ to represent the area of the possible valuable leftovers of an object; $\bar{\alpha}_{sj}^t$, $\bar{\alpha}_{sj}^r$, $\bar{\beta}_{sj}^t$, and $\bar{\beta}_{sj}^r \in \{0,1\}$ $(s = 1, \dots, d,\ j = 1, \dots, m)$ to represent whether an item from the catalogue can be placed within a leftover; and ξ_j^t and $\xi_j^r \in \{0,1\}$ $(j = 1, \dots, m)$ to count the number of valuable leftovers.

The model consists of minimizing (20) subject to (11,14,16,17,1,2,9,10,5,6,19), noting that appearances of t_j must be replaced by its binary expansion (13) and appearances of the product $r_j t_j$ must be replaced by its MIP formulation (15). For future references (in the numerical experiments), this model is simply named $\mathcal{M}^{\rm GL}$ that stands for two-dimensional non-guillotine cutting with *guillotine* leftovers. There are $nm + 4m + dm + n(n-1) - \sum_{q=1}^{p} n_q(n_q-1)/2 + \sum_{j=1}^{m}(\lfloor \log_2(H_j) \rfloor + 1)$ binary variables, $2n + 6m + \sum_{j=1}^{m}(\lfloor \log_2(H_j) \rfloor + 1)$ continuous variables, and $22m + 8dm + 3nm + 3n + 3\sum_{j=1}^{m}(\lfloor \log_2(H_j) \rfloor + 1) + 2m\sum_{q=1}^{p} n_q(n_q-1)/2 + 4m[n(n-1)/2 - \sum_{q=1}^{p} n_q(n_q-1)/2]$ constraints. Recall that, every time the quantity $\sum_{j=1}^{m}(\lfloor \log_2(H_j) \rfloor + 1)$ appears in the number of variables or constraints, it would be replaced by $\sum_{j=1}^{m}(\lfloor \log_2(\min\{W_j, H_j\}) \rfloor + 1)$.

We conclude this section with an illustrative example of the different models introduced in the present section. Consider a simple instance with $n = 2$ identical items with $w_1 = w_2 = 5$ and $h_1 = h_2 = 8$ and a single object $(m = 1)$ with $W_1 = 10$ and $H_1 = 24$. (The object's cost per unit of area is not relevant in this case.) Assume that the catalogue coincides with the demanded items without repetitions, i.e. $d = 1$, $\bar{w}_1 = w_1$, and $\bar{h}_1 = h_1$. Figures 3(a–c) represent three different optimal solutions to the cutting problem, without considering leftovers, since all of them use only one object. The solutions of Figures 3(b–c) are also optimal when guillotine leftovers are added to the model, since both have leftovers with an identical value (identical sum of areas) and the whole area of the object is used with items or valuable leftovers. However, only the solution of Figure 3(c) is optimal when the concern regarding the number of leftovers is added to the model, since the leftovers are concentrated into a single piece.

4 An extension for considering arbitrary valuable leftovers

In the previous section, valuable leftovers were arbitrary restricted to be generated by two guillotine cuts and were located on the right and the top of the objects. In the present section,

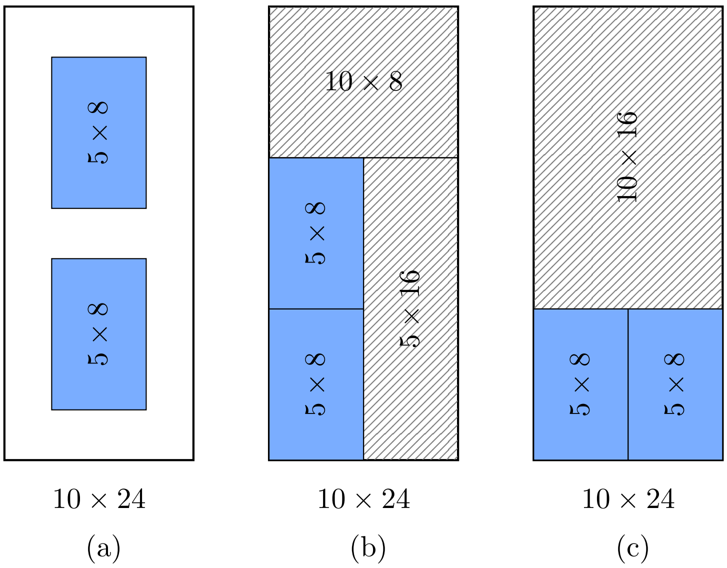


Figure 3: Three different optimal solutions to the cutting problem without considering leftovers. Solutions represented by (b) and (c) are also optimal when the value of the leftovers is added to the model, while only (c) is optimal when the number of leftovers is also taken into consideration.

those constraints are relaxed and valuable leftovers are free to be situated anywhere within the object. The arbitrary constraint of being two rectangular-shaped pieces per object remains. Of course, this new definition of leftovers includes the previous one as a particular case. In this section, we present the necessary modifications to the model presented in the previous section, in order to incorporate this more general definition of leftovers.

Let $\dot{w}_j^1$ and $\dot{h}_j^1 \in \mathbb{R}$ be the width and height of leftover 1 of object j and let $\dot{w}_j^2$ and $\dot{h}_j^2 \in \mathbb{R}$ be the width and height of leftover 2 of object j, for $j = 1, \ldots, m$. Constraints

$$0 \le \dot{w}_j^k \le W_j u_j \;\text{ and }\; 0 \le \dot{h}_j^k \le H_j u_j, \quad k = 1,2, \quad j = 1, \ldots, m, \tag{21}$$

ensure that the dimensions of the leftovers must be within the limits of the corresponding object and must be null whenever the object is not being used. As before, we consider variables $\dot{\alpha}_j^1$ and $\dot{\alpha}_j^2 \in \mathbb{R}$ to represent the area of the valuable leftovers, i.e.

$$\dot{\alpha}_j^k = \begin{cases} \dot{w}_j^k \, \dot{h}_j^k, & \text{if there exits } 1 \le s \le d \text{ such that } \dot{w}_j^k \ge \bar{w}_s \text{ and } \dot{h}_j^k \ge \bar{h}_s, \\ 0, & \text{otherwise,} \end{cases} \qquad k = 1,2, \; j = 1, \ldots, m. \tag{22}$$

To model (22) with MIP constraints, consider variables $\dot{\theta}_{\ell j}^1, \dot{\theta}_{\ell j}^2 \in \{0,1\}$, $\ell = 1, \ldots, \lfloor log_2(H_j) \rfloor + 1$, $j = 1, \ldots, m$, for the binary representation of $\dot{h}_j^1$ and $\dot{h}_j^2$, respectively, and $\dot{\omega}_{\ell j}^1, \dot{\omega}_{\ell j}^2 \in \mathbb{R}$, with the same indices range, to represent the product between $\dot{w}_j^1$ and the binary representation of $\dot{h}_j^1$, and the product between $\dot{w}_j^2$ and the binary representation of $\dot{h}_j^2$, respectively. Therefore, the binary representations of $\dot{h}_j^1$ and $\dot{h}_j^2$ are given by

$$\dot{h}_j^k = \sum_{\ell=1}^{\lfloor log_2(H_j) \rfloor + 1} 2^{\ell-1} \dot{\theta}_{\ell j}^k \quad k = 1,2, \quad j = 1, \ldots, m. \tag{23}$$

As in the previous section, $\dot{h}_j^1$ and $\dot{h}_j^2$ are not variables of the model. They are kept in the formulation to simplify the presentation, but any appearance of them should be replaced by the corresponding binary representation given by (23). The following constraints

$$0 \le \dot{\omega}_{\ell j}^k \le \dot{w}_j^k \text{ and } \dot{w}_j^k-(1-\dot{\theta}_{\ell j}^k)W_j \le \dot{\omega}_{\ell j}^k \le \dot{\theta}_{\ell j}^k W_j, \;\; k=1,2, \;\; j=1,\dots,m, \;\; \ell=1,\dots,\lfloor \log_2(H_j)\rfloor+1, \tag{24}$$

allow us to write the MIP representation of the products $\dot{w}_j^1\,\dot{h}_j^1$ and $\dot{w}_j^2\,\dot{h}_j^2$ as

$$\dot{w}_j^k\,\dot{h}_j^k = \sum_{\ell=1}^{\lfloor \log_2(H_j)\rfloor+1} 2^{\ell-1}\dot{\omega}_{\ell j}^k \;\; k=1,2, \;\; j=1,\dots,m. \tag{25}$$

Once again, the products $\dot{w}_j^1\,\dot{h}_j^1$ and $\dot{w}_j^2\,\dot{h}_j^2$ are preserved in the model to simplify the presentation, but any appearance of them should be replaced by the corresponding MIP representation in (25).

Consider variables $\dot{\bar{\alpha}}_{sj}^1, \dot{\bar{\alpha}}_{sj}^2 \in \{0,1\}$, $s=1,\dots,d$, $j=1,\dots,m$, to represent whether the s-th catalogued item can be held by leftovers 1 and 2 of object j, respectively. The MIP constraints to achieve this goal are given by

$$\bar{w}_s \le \dot{w}_j^k + \hat{W}(1-\dot{\bar{\alpha}}_{sj}^k) \text{ and } \bar{h}_s \le \dot{h}_j^k + \hat{H}(1-\dot{\bar{\alpha}}_{sj}^k), \;\; k=1,2, \;\; s=1,\dots,d, \;\; j=1,\dots,m, \tag{26}$$

and, using those variables, the following constraints ensure that $\dot{\alpha}_j^1$ and $\dot{\alpha}_j^2$ represent the area of the valuable leftovers 1 and 2 of object j, respectively,

$$0 \le \dot{\alpha}_j^k \le \dot{w}_j^k\dot{h}_j^k \text{ and } \dot{\alpha}_j^k \le W_j H_j \sum_{s=1}^{d} \dot{\bar{\alpha}}_{sj}^k, \;\; j=1,\dots,m. \tag{27}$$

To avoid symmetric solutions related to interchanging the roles of leftovers 1 and 2 of any object, the overlapping between the leftovers themselves is modeled requiring leftover 2 to be to the right or above leftover 1, with the necessary distance to avoid the overlapping. Defining variables $(\dot{x}_j^k, \dot{y}_j^k)$ to represent the center's coordinates of the leftover k of object j, $k=1,2$, $j=1,\dots,m$, and variables $\dot{p}_j \in \{0,1\}$, $j=1,\dots,m$, the non-overlapping constraints are given by

$$\dot{x}_j^2 - \dot{w}_j^2/2 \ge \dot{x}_j^1 + \dot{w}_j^1/2 - \hat{W}\dot{p}_j \text{ and } \dot{y}_j^2 - \dot{h}_j^2/2 \ge \dot{y}_j^1 + \dot{h}_j^1/2 - \hat{H}(1-\dot{p}_j), \;\; j=1,\dots,m, \tag{28}$$

while the constraints

$$\dot{w}_j^k/2 \le \dot{x}_j^k \le W_j - \dot{w}_j^k/2 \text{ and } \dot{h}_j^k/2 \le \dot{y}_j^k \le H_j - \dot{h}_j^k/2, \;\; k=1,2, \;\; j=1,\dots,m, \tag{29}$$

ensure that each leftover must be located within its corresponding object. The non-overlapping between the leftovers and the items requires variables $\dot{p}_{ij}^k$, $\dot{q}_{ij}^k \in \{0,1\}$, $k=1,2$, $i=1,\dots,n$, $j=1,\dots,m$, and is given by

$$\begin{array}{rcl}
\dot{x}_j^k + \dot{w}_j^k/2 & \le & x_i - w_i/2 + \hat{W}(1-v_{ij}) + \hat{W}\dot{p}_{ij}^k + \hat{W}\dot{q}_{ij}^k, \\
\dot{x}_j^k - \dot{w}_j^k/2 & \ge & x_i + w_i/2 - \hat{W}(1-v_{ij}) - \hat{W}\dot{p}_{ij}^k - \hat{W}(1-\dot{q}_{ij}^k), \\
\dot{y}_j^k + \dot{h}_j^k/2 & \le & y_i - h_i/2 + \hat{H}(1-v_{ij}) + \hat{H}(1-\dot{p}_{ij}^k) + \hat{H}\dot{q}_{ij}^k, \\
\dot{y}_j^k - \dot{h}_j^k/2 & \ge & y_i + h_i/2 - \hat{H}(1-v_{ij}) - \hat{H}(1-\dot{p}_{ij}^k) - \hat{H}(1-\dot{q}_{ij}^k), \\
 & & k=1,2, \;\; i=1,\dots,n, \;\; j=1,\dots,m.
\end{array} \tag{30}$$

Defining variables $\dot{\xi}_j^1$, $\dot{\xi}_j^2 \in \{0,1\}$, $j = 1,\dots,m$, to count the number of valuable leftovers, and including the constraints

$$\dot{\alpha}_j^k \le W_j H_j \dot{\xi}_j^k, \quad k = 1,2, \quad j = 1,\dots,m, \tag{31}$$

we are able to define the objective function of the model as

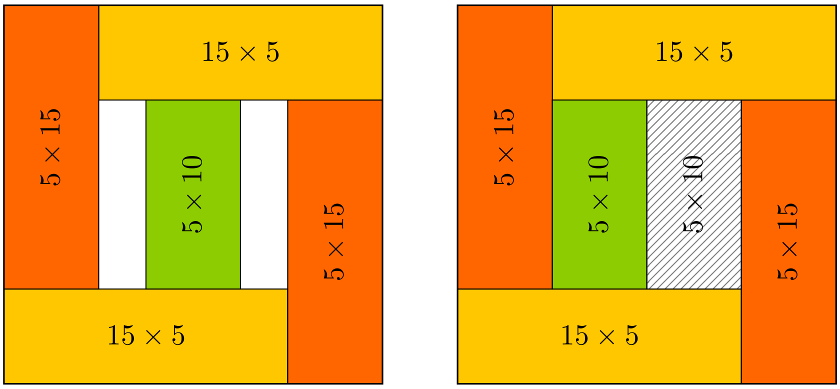
$$(2m+1)\left[\left(\sum_{j=1}^{m} c_j W_j H_j\right)\left(\sum_{j=1}^{m} c_j W_j H_j u_j\right) - \sum_{j=1}^{m} c_j(\dot{\alpha}_j^1 + \dot{\alpha}_j^2)\right] + \sum_{j=1}^{m}(\dot{\xi}_j^1 + \dot{\xi}_j^2). \tag{32}$$

Therefore, the model is given by minimizing (32) on $u_j \in \{0,1\}$ ($j = 1,\dots,m$), $v_{ij} \in \{0,1\}$ ($i = 1,\dots,n$, $j = 1,\dots,m$), x_i, $y_i \in \mathbb{R}$ ($i = 1,\dots,n$), $\pi_{ii'} \in \{0,1\}$ ($i = 1,\dots,n$, $i' = i+1,\dots,n$), $\tau_{ii'} \in \{0,1\}$ ($q = 1,\dots,p$, $i = o_q,\dots,o_q+n_q$, $i' = i+1,\dots,o_q+n_q$), $\dot{w}_j^k \in \mathbb{R}$ ($k = 1,2$, $j = 1,\dots,m$), $\dot{\theta}_{\ell j}^k \in \{0,1\}$, $\dot{\omega}_{\ell j}^k \in \mathbb{R}$ ($k = 1,2$, $j = 1,\dots,m$, $\ell = 1,\dots,\lfloor \log_2(H_j)\rfloor + 1$), $\dot{\alpha}_j^k \in \mathbb{R}$ ($k = 1,2$, $j = 1,\dots,m$), $\dot{\bar{\alpha}}_{sj}^k$ ($k = 1,2$, $j = 1,\dots,m$, $s = 1,\dots,d$), $\dot{x}_j^k$, $\dot{y}_j^k \in \mathbb{R}$, and $\dot{\xi}_j^k \in \{0,1\}$ ($k = 1,2$, $j = 1,\dots,m$) subject to (1,2,4,5,6,21,24,26,27,28,29,30,31). For future references (in the numerical experiments), this model is simply named $\mathcal{M}^{\text{NGL}}$ that stands for two-dimensional non-guillotine cutting with *non-guillotine* leftovers. We finish this section with an illustrative simple example that highlights the advantages of defining arbitrary leftovers, instead of guillotine leftovers as before. Consider an instance with $n = 5$ items (of $p = 3$ different types), $w_1 = w_2 = 15$, $h_1 = h_2 = 5$, $w_3 = w_4 = 5$, $h_3 = h_4 = 15$, $w_5 = 5$ and $h_5 = 10$, and a single object ($m = 1$) with $W_1 = H_1 = 20$. Once again, since there is only one object, its value per unit of area is not relevant. Assume that the items' catalogue used to define valuable leftovers coincides with the list of demanded items without repetitions (i.e. $d = p = 3$, $\bar{w}_1 = w_1$, $\bar{h}_1 = h_1$, $\bar{w}_2 = w_3$, $\bar{h}_2 = h_3$, $\bar{w}_3 = w_5$, and $\bar{h}_3 = h_5$). Figures 4(a–b) illustrate two different optimal solutions to the non-guillotine cutting problem with guillotine leftovers. In both cases the value of the leftovers is zero. However, when arbitrary leftovers are considered, Figure 4(b) shows an optimal solution with a non-null valuable leftover. Clearly, since the total area of the object is occupied by items or the leftover, it is an optimal solution, while the solution of Figure 4(a) with no valuable leftovers is not.

5 Illustrative numerical examples

In this section, we illustrate the introduced MIP models $\mathcal{M}^{\text{GL}}$ and $\mathcal{M}^{\text{NGL}}$ with numerical experiments. Ten arbitrary small instances were considered to illustrate both models as described in Table 1. In all instances, we considered $c_j = 1$ for all j and assumed that the catalogue of items that defines the leftovers corresponds to the list of demanded items. There is no need to include duplicated items in this catalogue, as there is also no need to include an item that is at least as wide and at least as high as another item in the catalogue. Items in the catalogue are the underlined items in Table 1. Recall that m is the number of objects, and n, p, and d are the number of items, the number of types of items, and the number of items in the catalogue, respectively. Inequality $n \ge p$ always holds by definition and inequality $p \ge d$ holds by the way the catalogue was generated in our instances test set.

MIP models $\mathcal{M}^{\text{GL}}$ and $\mathcal{M}^{\text{NGL}}$ were implemented in C/C++ using the ILOG Concert Technology 2.9 and compiled with g++ from gcc version 4.6.1 (GNU compiler collection). Numerical experiments were conducted using a machine with two 2.67GHz Intel Xeon CPU X5650 processors, 8GB of RAM memory, and running GNU/Linux operating system (Ubuntu 12.04 LTS,



20 × 20 (a) 20 × 20 (b)

Figure 4: Two different optimal solutions, with no leftovers, to the non-guillotine cutting problem with guillotine leftovers. Only (b) represents also an optimal solution to the non-guillotine cutting problem with arbitrary leftovers. Moreover, when arbitrary leftovers are considered, the solution in (b) presents a non-null leftover.

Instance	Objects		Items			
	m	$W_j \times H_j$	n	p	d	$w_i \times h_i$
1	2	22×17, 14×30	5	2	2	$3(\underline{2 \times 11})$, $2(\underline{5 \times 5})$
2	2	17×29, 24×10	2	1	1	$2(\underline{4 \times 10})$
3	2	18×19, 26×22	3	1	1	$3(\underline{5 \times 4})$
4	3	24×12, 15×18, 17×13	7	3	3	$4(\underline{3 \times 3})$, $\underline{4 \times 2}$, $2(\underline{7 \times 1})$
5	2	20×10, 29×12	5	2	1	$4(\underline{7 \times 1})$, 11×1
6	2	22×17, 14×30	7	3	2	$2(11 \times 11)$, $3(\underline{2 \times 11})$, $2(\underline{5 \times 5})$
7	5	27×23, $2(19 \times 17)$, $2(19 \times 19)$	9	2	1	$5(9 \times 6)$, $4(\underline{5 \times 3})$
8	2	19×17, 16×11	9	2	1	$5(3 \times 2)$, $4(\underline{3 \times 1})$
9	2	18×20, 13×10	10	2	1	$4(3 \times 4)$, $6(\underline{1 \times 1})$
10	2	22×14, 18×22	10	2	2	$7(\underline{4 \times 5})$, $3(\underline{5 \times 2})$

Table 1: Description of the problem instances. Dimensions are given in the format width × height. Notation $a(b \times c)$ means that there are a objects or items with dimension $b \times c$. When a is omitted it means that there is a single copy of the object or item.

kernel 3.2.0-33). Instances were solved using IBM ILOG CPLEX 12.1.0. By default, a solution is reported as optimal by the solver when

$$\text{absolute gap} = \text{best feasible solution} - \text{best lower bound} \leq \varepsilon_{\text{abs}}$$

or

$$\text{relative gap} = \frac{|\text{ best feasible solution} - \text{best lower bound }|}{1\text{e}{-10} + |\text{ best feasible solution }|} \leq \varepsilon_{\text{rel}},$$

with $\varepsilon_{\text{abs}} = 10^{-6}$ and $\varepsilon_{\text{rel}} = 10^{-4}$, where "best feasible solution" means the smallest value of the objective function related to a feasible solution generated by the method. As mentioned before, the objective functions (20) and (32) of the two models have the particular property of assuming integer values at feasible points. Moreover, since they represent a weighted combination of the used object costs, the leftover values and the number of leftovers, they assume relatively "large" integer values (between 10^5 and 10^7 for the considered instances). Hence, a stopping criterion based on a relative error less than or equal to 10^{-4} may have the undesired effect of stopping the method prematurely. On the other hand, due to the integrality of the objective function values, an absolute error strictly smaller than 1 is enough to prove the optimality of the incumbent solution. Therefore, in the numerical experiments we considered $\varepsilon_{\text{abs}} = 1 - 10^{-6}$ and $\varepsilon_{\text{rel}} = 0$. All other parameters of the solver were used with their default values unless otherwise stated.

Table 2 describes the solutions to the ten instances associated with model $\mathcal{M}^{\text{GL}}$. For each solution, the table shows the cost of the used objects, the value of the leftovers, and the number of leftovers. The objective function value shown in the table corresponds to the objective function (20) and it is the composition of the other three values. Figures in the table show that all instances were solved to optimality in a few seconds. Figures 5–7 complete the description of the solutions by presenting their graphical representation. It is easy to see that, in instances 2, 3, 8, and 9, the leftover value cannot be improved by considering *non*-guillotine leftovers, since the whole area of the used objects is covered by items or valuable leftovers. In the remaining six instances, there is space for potential improvement of the leftovers value when considering model $\mathcal{M}^{\text{NGL}}$ instead of model $\mathcal{M}^{\text{GL}}$. Instance 7 is the only one in which the solution contains no usable leftovers.

Inst.	Optimal value	Solutions description			Effort measurements		
		Objects cost	Leftovers value	# leftovers	MIP Iterations	B&B Nodes	CPU Time
1	1,483,517	374	253	2	1,788	459	0.14
2	878,801	240	160	1	55	3	0.04
3	1,561,532	342	282	2	187	29	0.04
4	1,203,988	221	161	2	17,467	4,021	0.64
5	623,912	220	178	2	1,213	355	0.12
6	3,150,029	794	431	4	1,246,789	380,775	26.14
7	7,898,319	361	0	0	83,286	7,319	1.84
8	438,452	176	134	2	333,791	77,952	12.35
9	318,122	130	76	2	68,051	13,318	6.22
10	1,083,557	308	121	2	654,578	163,069	16.98

Table 2: Numerical results for the ten instances of the two-dimensional non-guillotine cutting problem with guillotine leftovers named $\mathcal{M}^{\text{GL}}$.

To solve instances 1–10 associated with model $\mathcal{M}^{\text{NGL}}$, in addition to $\varepsilon_{\text{abs}} = 1 - 10^{-6}$ and $\varepsilon_{\text{rel}} = 0$, a CPU time limit of 2 hours was also imposed. Table 3 describes the solutions to the ten instances associated with model $\mathcal{M}^{\text{NGL}}$. Figures in the table show that instances 1–5 were rapidly solved to optimality and that instances 8–10 required, approximately, 0:10, 0:46, and 1:43 hours of CPU time, respectively, to be solved to optimality. The stopping criterion related to the absolute gap was not satisfied within the imposed CPU time limit in instances 6 and 7. In these two cases, the best lower bound and the value of the best feasible solution found are reported in the table. Figures 8–10 complete the description of the solutions by presenting their cutting patterns' representation. As expected, the cost of the used objects in a solution to an instance of model $\mathcal{M}^{\text{NGL}}$ coincides with the cost of the used objects in a solution to the corresponding instance of model $\mathcal{M}^{\text{GL}}$. Improvements in the value of the leftovers were obtained for instances 4 and 10, that were increased in 2 and 15 units, respectively.

It can be shown that the best feasible solution to instance 7 of model $\mathcal{M}^{\text{NGL}}$ reported in Table 3 is an optimal solution. Optimality comes from the facts that: (a) the cost of the used objects coincides with the one obtained for instance 7 of model $\mathcal{M}^{\text{GL}}$ (see Table 2), and (b) the problem of packing an additional 5×3 items within the used object is infeasible (the solver was able to show it in only 3.44 seconds). Since this problem is infeasible, the problem with an additional 9×6 item (wider and higher than a 5×3 item) is also infeasible. Hence, as a leftover must hold at least an item, no valuable leftover can be placed and the solution with no leftovers is optimal. On the other hand, the best feasible solution for instance 6 reported in Table 3 is not optimal and a better solution is reported below.

Inst.	Optimal value	Solutions description			Effort measurements		
		Objects cost	Leftovers value	# leftovers	MIP Iterations	B&B Nodes	CPU Time
1	1,483,517	374	253	2	97,191	27,695	6.67
2	878,801	240	160	1	484	83	0.05
3	1,561,532	342	282	2	6,962	2,051	0.49
4	1,203,974	221	163	2	690,507	225,627	21.15
5	623,912	220	178	2	79,450	22,927	8.60
6	3,150,028*	794	431	3	542,446,442	75,274,696	≥7,200.00
7	7,898,319*	361	0	0	261,653,326	38,044,049	≥7,200.00
8	438,452	176	134	2	27,208,549	7,474,203	584.47
9	318,122	130	76	2	104,199,205	34,030,526	2,746.57
10	1,083,482	308	136	2	74,299,550	20,064,829	6,154.87

Table 3: Numerical results for the ten instances of the two-dimensional non-guillotine cutting problem with non-guillotine leftovers named $\mathcal{M}^{\mathrm{NGL}}$. *Optimal values reported for instances 6 and 7 are in fact the best values found associated with a feasible solution reported by the solver. The best lower bounds found are $3,147,296.22$ and $7,896,471.47$, respectively.

The analysis of the reported optimal solutions calls the attention to a redundant linear constraint that may enhance the quality of the lower bounds obtained from the linear relaxation of the introduced models and, therefore, improve the overall performance of an exact MIP solver based on branch-and-bound when applied to both models. This constraint ensures that, given a used object, "the sum of the areas of the usable leftovers of the object plus the sum of the areas of the items assigned to the object cannot be greater than the area of the object". For model $\mathcal{M}^{\mathrm{GL}}$, constraints are given by

$$\gamma_j + \left(\sum_{i=1}^{n} w_i h_i v_{ij} \right) \leq W_j H_j u_j, \; j = 1, \ldots, m, \tag{33}$$

while, for model $\mathcal{M}^{\mathrm{NGL}}$, constraints are given by

$$\dot{\alpha}_j^1 + \dot{\alpha}_j^2 + \left(\sum_{i=1}^{n} w_i h_i v_{ij} \right) \leq W_j H_j u_j, \; j = 1, \ldots, m. \tag{34}$$

We called model $\mathcal{M}^{\mathrm{GL}}$ with the addition of the redundant constraint (33) as $\mathcal{M}_{+}^{\mathrm{GL}}$, and model $\mathcal{M}^{\mathrm{NGL}}$ with the addition of the redundant constraint (34) as $\mathcal{M}_{+}^{\mathrm{NGL}}$. Since models $\mathcal{M}_{+}^{\mathrm{GL}}$ and $\mathcal{M}_{+}^{\mathrm{NGL}}$ are equivalent to models $\mathcal{M}^{\mathrm{GL}}$ and $\mathcal{M}^{\mathrm{NGL}}$, respectively, only an improvement in the efficiency of the considered solver would be expected. However, numerical experiments revealed a similar performance of the solver when applied to models $\mathcal{M}_{+}^{\mathrm{GL}}$ and $\mathcal{M}_{+}^{\mathrm{NGL}}$ in comparison to its performance when applied to models $\mathcal{M}^{\mathrm{GL}}$ and $\mathcal{M}^{\mathrm{NGL}}$. The highlight was that a better solution was found, within the time limit of 2 hours, for instance 6 of model $\mathcal{M}_{+}^{\mathrm{NGL}}$ (or $\mathcal{M}^{\mathrm{NGL}}$). The new best lower bound is $3,150,001.05$ (the former was $3,147,296.22$) and the new best objective function value associated with a feasible solution is $3,150,004$ (the former was $3,150,028$), which corresponds to a cost of the used objects equal to 794, a value of leftovers equal to 436, and 4 residual pieces. The solver achieved the maximum allowed CPU time performing 262, 355, 954 MIP iterations and generating 15, 561, 377 branch-and-bound nodes. Figure 11 illustrates this new solution. Optimality of the solution cannot be easily proved. Optimality of the cost of the used objects comes from the fact that it coincides with the one obtained for instance 6 of model $\mathcal{M}^{\mathrm{GL}}$ that was solved to optimality (see Table 2). Optimality of the leftover value comes from the fact that the whole area of the used objects is covered by items or valuable leftovers (see Figure 11). These two facts imply that the reported solution is a feasible solution of the implicitly introduced three-level MIP problem. However, there is no trivial way to determine whether a solution with (the same cost of the used objects, the same leftovers value, and) a smaller number of leftovers exists. The reported feasible solution has four leftovers. The value of the reported lower bound prevents the existence of a solution with a single leftover, but solutions with exactly two or three leftovers may exist.

6 Concluding remarks

In this study, we introduced MIP models for two-dimensional non-guillotine cutting/packing problems with usable object remainders. We considered non-guillotine cutting patterns with both guillotine and non-guillotine types of leftovers. The MIP models are based on bilevel and three-level mathematical formulations for the problems, but because of special characteristics of these multilevel models, they can be reformulated as one-level MIP models. To the best of our knowledge, there are no other studies in the literature dealing with non-guillotine cutting/packing problems with usable leftovers involving two or more dimensions. Therefore, the presented models state members of a class of residual bin-packing problems that were not formally defined in the literature up to now. Numerical experiments illustrate the models and their scope and limitations. The contribution of the presented approach is mainly in methodology. The scale of the test instances is much smaller than those of practical instances. It is not wise to try the model in applications to obtain optimal solutions for practical bin packing instances, at least in the near future. However, the formal definition of these variants of two-dimensional cutting problems with residual pieces opens up interesting possibilities to develop dedicated exact and heuristic methods for the resolution and practical application of these and other cutting/packing problems of two and more dimensions. Another interesting line of research would be to develop optimization procedures based on the multilevel MIP models to analyse the trade-offs between the cost of the used objects and the value and number of the usable leftovers generated in cutting/packing with residual pieces.

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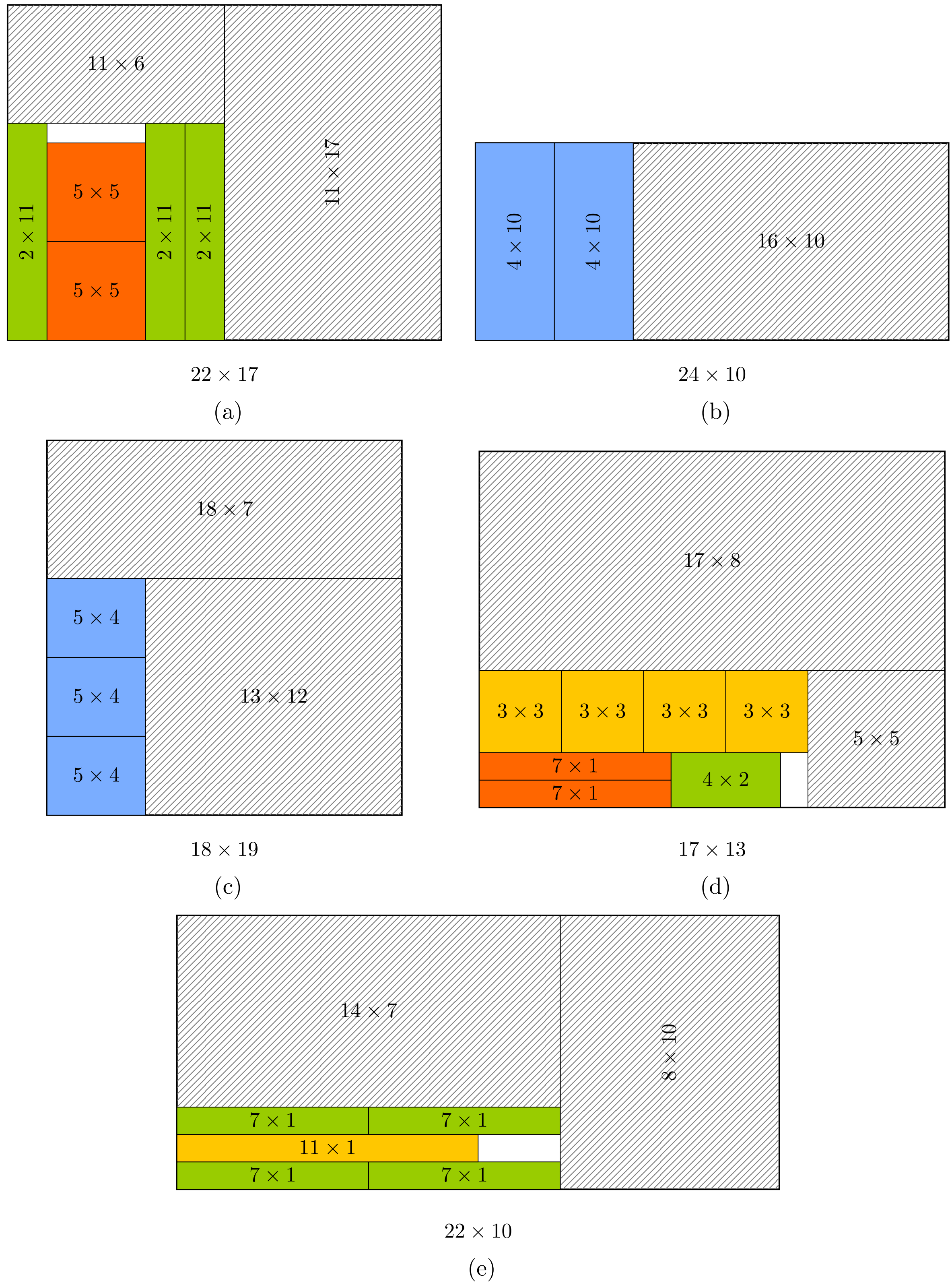

Figure 5: Graphs (a–e) illustrate the solutions to instances 1–5 of the two-dimensional non-guillotine cutting problem with guillotine leftovers (named $\mathcal{M}^{\rm GL}$), respectively (each solution uses a single object).

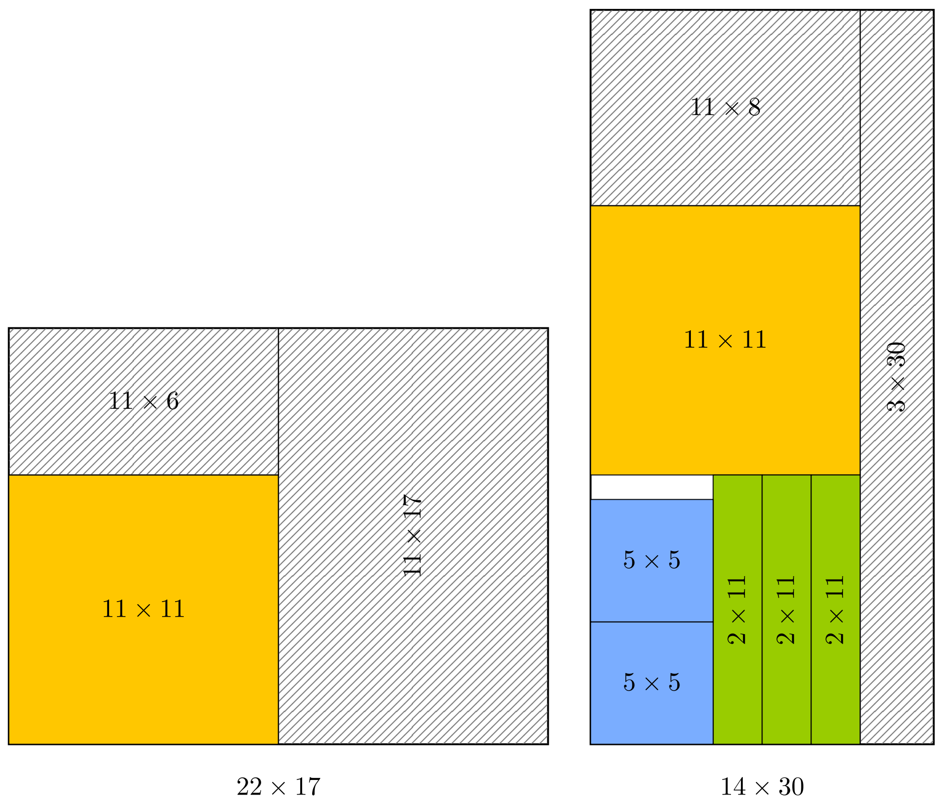

Figure 6: Graphical representation of the solution for instances 6 of the two-dimensional non-guillotine cutting problem with guillotine leftovers (named $\mathcal{M}^{\mathrm{GL}}$). The solution uses two objects.

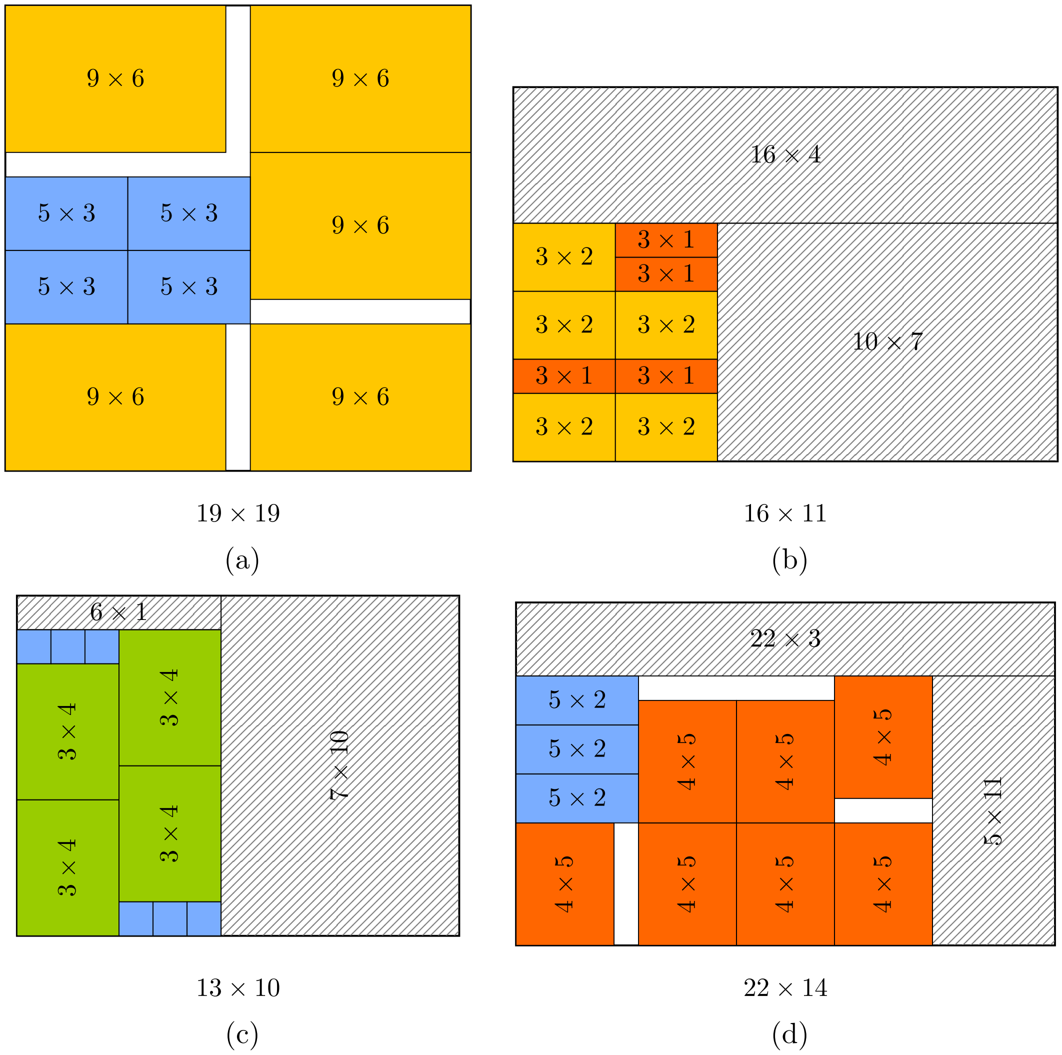

Figure 7: Graphs (a–d) illustrate the solutions to instances 7–10 of the two-dimensional non-guillotine cutting problem with guillotine leftovers (named $\mathcal{M}^{\text{GL}}$), respectively (each solution uses a single object). Small items in graph (c) with no explicit dimensions correspond to 1×1 items.

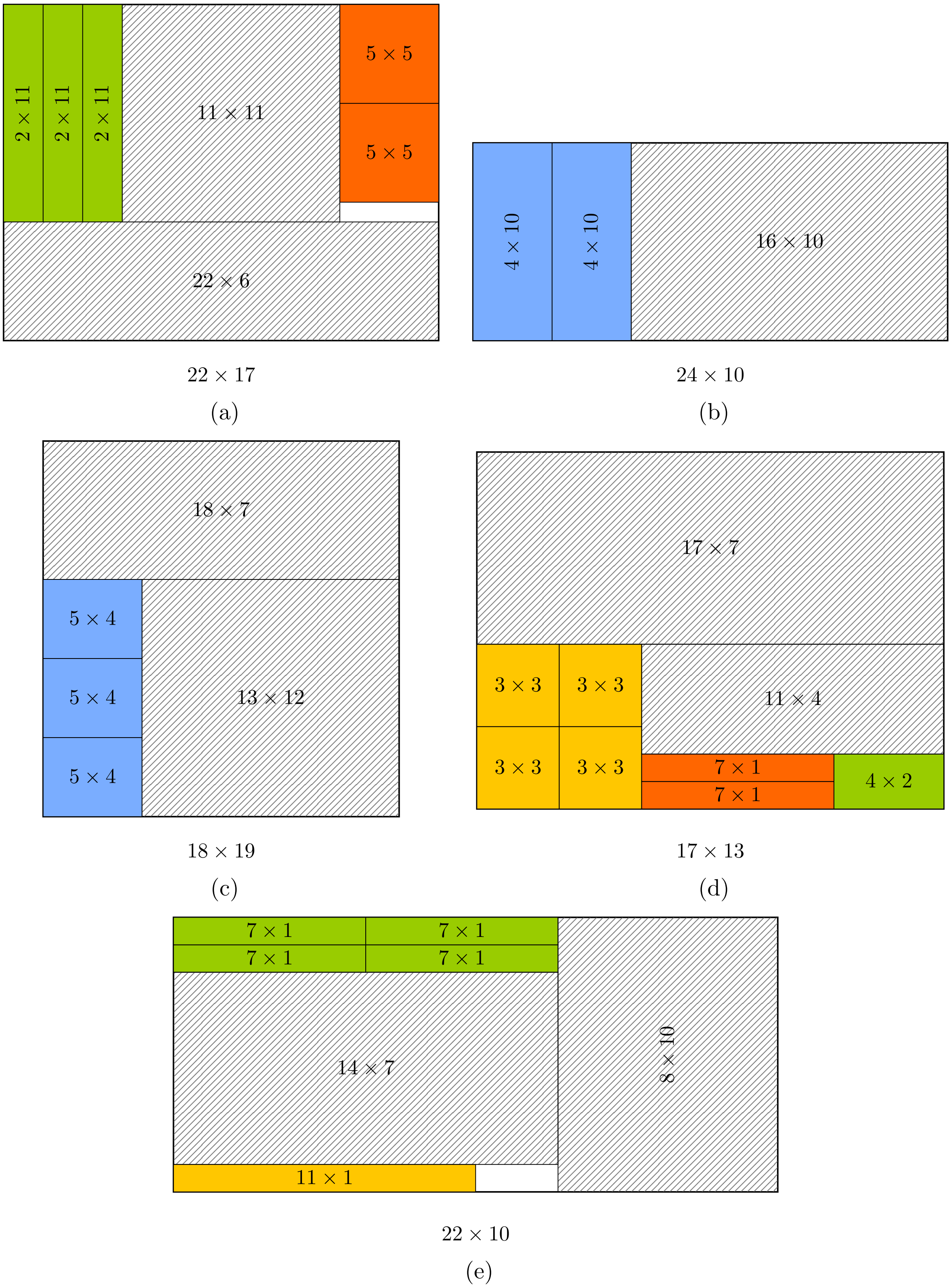

Figure 8: Graphs (a–e) illustrate, respectively, the solutions (cutting patterns) to instances 1–5 of the two-dimensional non-guillotine cutting problem with non-guillotine leftovers (named $\mathcal{M}^{\text{NGL}}$), respectively.

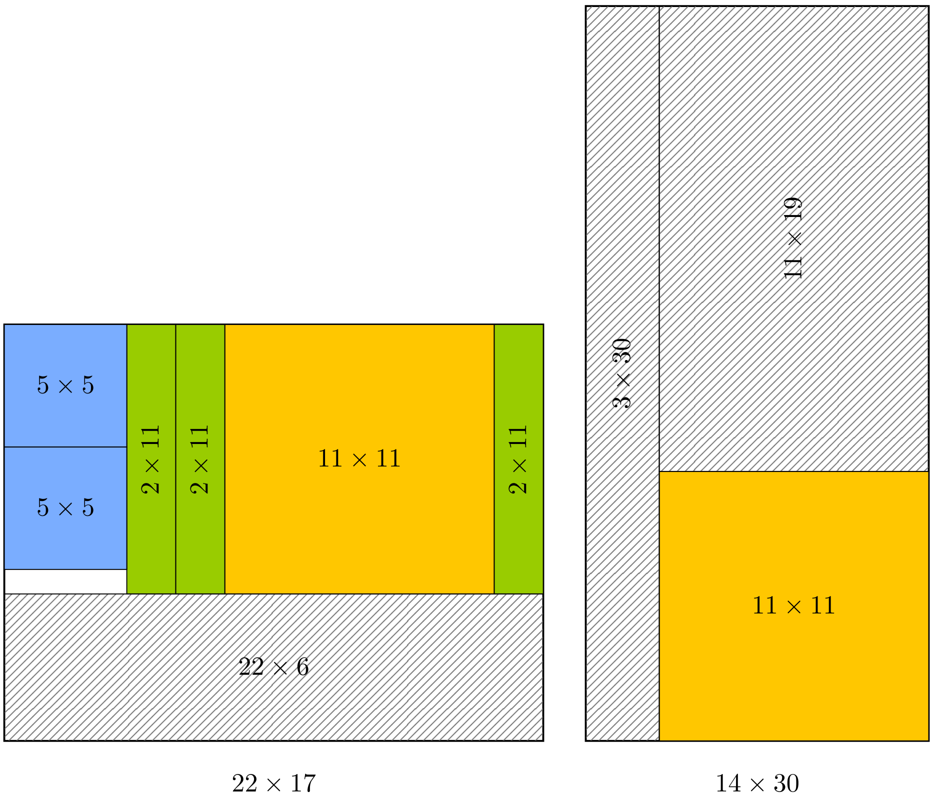

Figure 9: Graphical representation of the solution (cutting patterns) to instance 6 of the two-dimensional non-guillotine cutting problem with non-guillotine leftovers (named $\mathcal{M}^{\rm NGL}$).

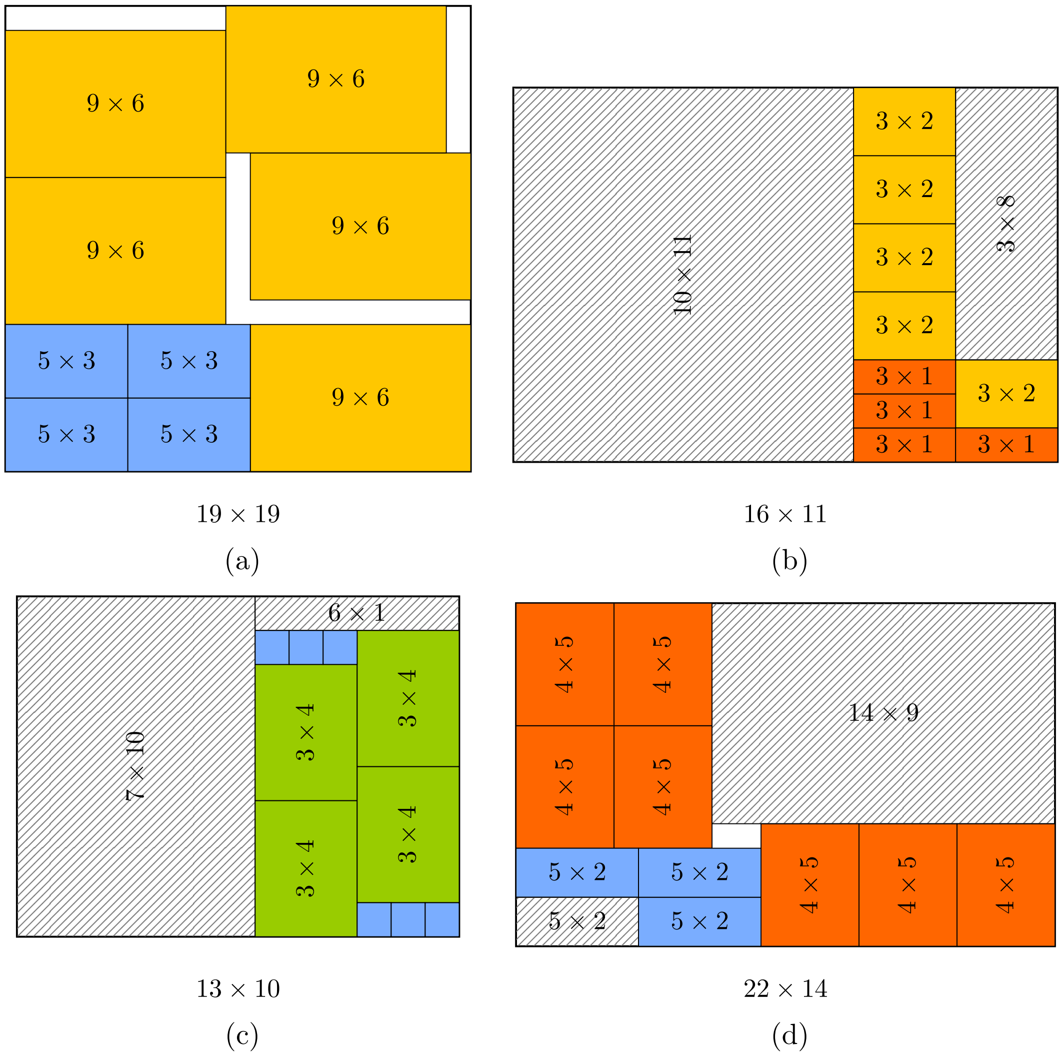

Figure 10: Graphs (a–d) illustrate the solutions to instances 7–10 of the two-dimensional non-guillotine cutting problem with non-guillotine leftovers (named $\mathcal{M}^{\mathrm{NGL}}$), respectively. Small items in graph (c) with no explicit dimensions correspond to 1×1 items.

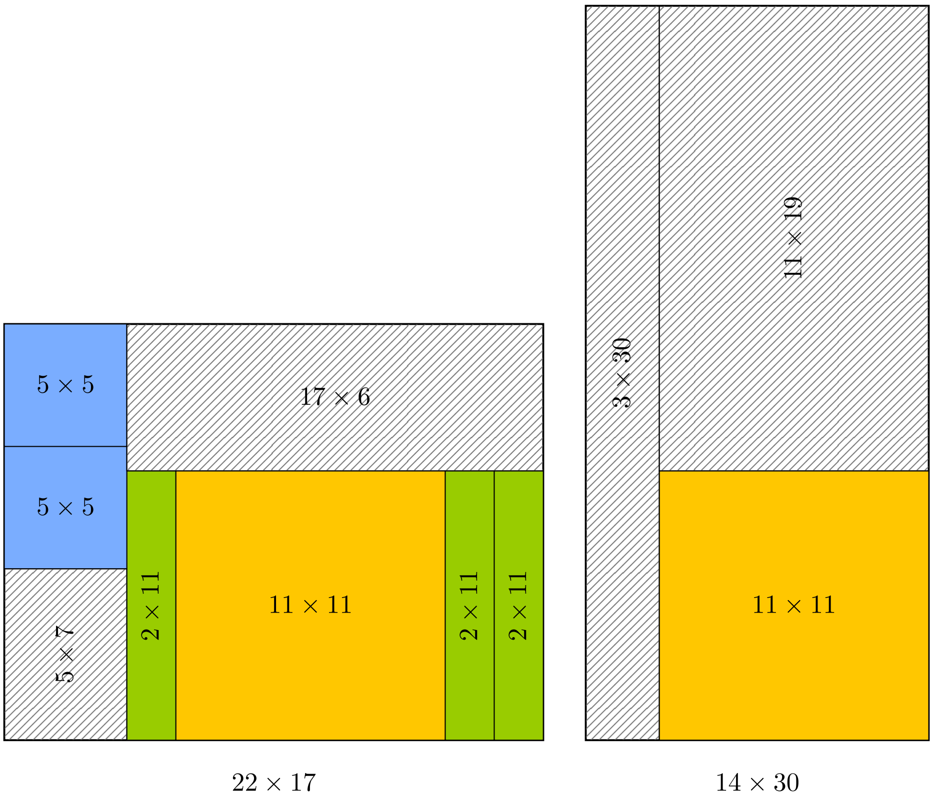

Figure 11: Graphical representation of the best feasible solution found for instance 6 of model $\mathcal{M}_{+}^{\text{NGL}}$.