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Mirror Manifolds in Higher Dimension

Brian R. Greene[1], **David R. Morrison**[2], **M. Ronen Plesser**[3]

[1] F.-R. Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, NY 14853, USA
[2] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA. On leave from: Department of Mathematics, Duke University, Box 90320, Durham, NC 27708-0320, USA
[3] Department of Physics, Yale University, New Haven, CT 06511, USA and School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA

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Abstract: We describe mirror manifolds in dimensions different from the familiar case of complex threefolds. We isolate certain simplifying features present only in dimension three, and supply alternative methods that do not rely on these special characteristics and hence can be generalized to other dimensions. Although the moduli spaces for Calabi–Yau d-folds are not "special Kähler manifolds" when $d > 3$, they still have a restricted geometry, and we indicate the new geometrical structures which arise. We formulate and apply procedures which allow for the construction of mirror maps and the calculation of order-by-order instanton corrections to Yukawa couplings. Mathematically, these corrections are expected to correspond to calculating Chern classes of various parameter spaces (Hilbert schemes) for rational curves on Calabi–Yau manifolds. Our mirror-aided calculations agree with those Chern class calculations in the limited number of cases for which the latter can be carried out with current mathematical tools. Finally, we make explicit some striking relations between instanton corrections for various Yukawa couplings, derived from the associativity of the operator product algebra.

1. Introduction

Calabi–Yau threefolds were originally introduced into string theory to provide six compact spatial dimensions which complement four Minkowski spacetime directions to yield a consistent ten dimensional background for string propagation. From a more general perspective, Calabi–Yau threefolds can be target spaces for two dimensional supersymmetric ($N = 2$) conformally invariant nonlinear sigma models with $c = 9$ – this number arising from three times the complex dimension of the target space. Such superconformal field theories have interesting applications to string backgrounds and critical systems, and have led to some striking predictions in mathematical physics. In the latter category, the recent conjectures [1], evidence from numerical studies [2], explicit construction [3], and applications [4, 5] of mirror symmetry are indications of a deep mathematical structure that, at present, is best understood from the physical viewpoint. We take that viewpoint throughout this paper.

The focus on $c = 9$, as mentioned, has its origin in the string theoretic applications of Calabi–Yau manifolds. The mathematical physics applications, however, are of interest in the more general setting of $c = 3d$ corresponding to Calabi–Yau d-folds. It is the purpose of the present paper to study mirror manifolds for more general values of d. There are a couple of motivations for this study. First, there are some aspects of mirror symmetry which are incompletely understood, even from the physical viewpoint. It is one of our hopes that by studying mirror symmetry for general dimension, the special features of dimension three can be suppressed and hence allow focus on the true (dimension independent) mathematical and physical characteristics responsible for mirror symmetry. Second, mirror manifolds in dimension three have proven themselves to be a powerful calculational tool. In particular, by making use of the mirror manifolds constructed in [3], the authors of [5] and [6, 7, 8] showed that the number of rational curves of arbitrary degree on certain Calabi–Yau threefolds – a number which cannot, in general, be effectively determined with current mathematical methods[1] – could be predicted with relative ease by making a calculation on the mirror. There are two special features of dimension three in this regard. First, rational curves (world sheet instantons) on a generic Calabi–Yau threefold are isolated whereas they arise in continuous families on higher dimensional Calabi–Yau manifolds. The analog of calculating the number of rational curves of a given degree on a Calabi–Yau threefold is the calculation of properties of Chern classes of the parameter spaces of such curves in the higher dimensional case. These parameter spaces are subspaces of the so-called *Hilbert schemes* of rational curves of a given degree. (These Hilbert schemes are analogous to the Grassmannian which parameterizes rational curves of degree one.) We will see that the integers associated with these characteristic classes, for rational curves of arbitrary degree, are easily predicted by a calculation on the mirror manifold. For the calculations associated with degree one and degree two curves, our predictions have been confirmed by Katz [9], who made explicit Chern class computations. For degree higher than two, however, the Chern class computations become very difficult and our predictions have not yet been verified.[2] A second distinction is that whereas there is one type of Yukawa coupling (for each of the (c, c) and (a, c) rings) on a threefold there are many more in the higher dimensional case. Each of these couplings probes part of the chiral and antichiral primary field ring structure and has an instanton expansion interpretable as above. We will see that associativity of the operator product algebra gives rise to striking relations amongst these instanton expansions. Mathematically, these relations can likely be established by making use of the degeneration argument invoked by Witten in [12].

The above discussion, of course, only applies to Calabi–Yau manifolds for which we have a mirror partner. General physical reasoning lends credence to the conjecture [1] that essentially all Calabi–Yau manifolds come in mirror pairs (see [13] for a review). To date, the only proven constructions of mirror manifolds[3] are those given in [3] and hence we shall focus on this subset of Calabi–Yau manifolds. For this purpose we briefly recall the main result of [3].

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[1] See note added in proof.

[2] However, very recently Ellingsrud and Strømme have generalized their earlier work [10] and have verified some of our predictions for degree three curves [11].

[3] The constructions of [3] are the only ones for which the conformal field theories are known to coincide. The many interesting proposed generalizations of this construction [14] have some supporting evidence, such as correctly predicted relations among Betti numbers, but no proof of equivalence.

Let W be a Calabi–Yau d-fold realized as a Fermat hypersurface in a weighted projective space of dimension $d+1$, which we denote by $\mathbb{WCP}^{d+1}$ (following the conventions of the physics literature). Let G be the maximal group of diagonal scaling symmetries acting on the homogeneous $\mathbb{WCP}^{d+1}$ coordinates which preserves the holomorphic d-form on W. Then W and $M = W/G$ constitute a mirror pair. Furthermore, as explained in [3, 13] a point crucial to the analysis of [4, 5] and to our study here, is that although the explicit arguments for constructing mirror pairs [3] are tied to special points in moduli space (the Fermat points), deformation arguments allow us to move away from such points via changes in either the complex structure or the Kähler structure[4]. We therefore are able to construct families of mirror pairs by deforming from the Fermat point [3].

In Sect. II we study some aspects of the moduli space of Calabi–Yau d-folds from a covariant viewpoint. Our purpose in this discussion is not to be complete, but rather to indicate the ways in which the moduli spaces for higher dimensional Calabi–Yau manifolds differ from the three dimensional case. In particular we note that these moduli spaces have certain special properties which can be viewed as a generalization of the known properties in the three dimensional case, although not all features of that case generalize. We give procedures for the derivation of the Picard–Fuchs equations governing the behavior of the period maps which involve complications that are not present in the well studied case of $d = 3$. In Sect. III we calculate the generalized "Yukawa couplings" (higher point functions) which naturally arise in this analysis (for a variety of examples) and then apply the methods of [6] to derive mirror maps and hence an instanton expansion. These higher point functions will factor into (sums of) products of three-point functions. We show that the associativity of the operator product expansion gives rise to relations amongst these three-point functions (the "conformal bootstrap equations") which translate into striking implications for the associated instanton expansions. In Sect. IV we rephrase the analysis of Sect. II in a form better suited to the incorporation of mirror symmetry and explicit calculations. This approach naturally yields the fundamental Yukawa couplings (three-point functions) which we relate to the calculations in the previous section. In Sect. V we give the mathematical interpretation of the instanton expansions found in Sect. IV (which is most easily done in the language of topological field theory). Full justification of the interpretation requires a topological field theory argument along the lines of [15] which is presented in an appendix. In Sect. VI we state our conclusions.

2. Calabi–Yau Moduli Spaces for $d > 3$

Work over the last few years [16–19] has established that the moduli spaces for Calabi–Yau threefolds are *special Kähler* manifolds. We recall the definition of these. Special Kähler manifolds are Kähler manifolds of restricted type, meaning that the Kähler class of the metric on the manifold $\mathcal{M}$ is an integral class, hence it is the first Chern class of some line bundle which we shall denote by $\mathcal{L}$. A Kähler manifold of restricted type is *special Kähler* if there exist coordinates on $\mathcal{M}$ and

[4] Technically it might be difficult to establish this statement for all but local deformations in the moduli space. We stress, however, that the results presented in [4, 5] and here all rely on our deformation reasoning applying globally in the moduli space.

a gauge choice on $\mathscr{L}$ such that the corresponding Kähler potential K is given by

$$e^{-K} = -i\left(z^a \frac{\partial\overline{\mathscr{G}}}{\partial\bar{z}^a} - \bar{z}^a \frac{\partial\mathscr{G}}{\partial z^a}\right) \tag{2.1}$$

with $\mathscr{G}$ a holomorphic function of the local complex coordinates z^a. As discussed in [5], this is equivalent to the statement that the Kähler potential has a holomorphic prepotential.

One can also think of special geometry as providing an additional restriction on the Riemann tensor beyond those implied by Kählerity. This more covariant formulation requires the existence of holomorphic sections $\kappa_{\alpha\beta\gamma}$ of $\mathscr{L}^2 \otimes \mathrm{Sym}\, T^*(\mathscr{M})^{\otimes 3}$ such that

$$R_{\bar{\delta}\alpha\bar{\beta}\gamma} = G_{\alpha\bar{\beta}}G_{\gamma\bar{\delta}} + G_{\alpha\bar{\delta}}G_{\gamma\bar{\beta}} - e^{2K}\, G^{\varepsilon\bar{\varepsilon}}\kappa_{\alpha\gamma\varepsilon}\overline{\kappa}_{\bar{\beta}\bar{\delta}\bar{\varepsilon}}\,, \tag{2.2}$$

where $G_{\alpha\bar{\beta}}$ is the Kähler metric on $\mathscr{M}$. It follows that

$$\kappa_{\alpha\gamma\varepsilon} = \partial_\alpha\partial_\gamma\partial_\varepsilon\mathscr{G}\,. \tag{2.3}$$

When $\mathscr{M}$ is the moduli space of complex structures on a Calabi–Yau threefold the structures of special geometry are realized as follows [17, 19]. A section of $\mathscr{L}$ is a choice of a holomorphic 3-form $\Omega(z)$ on the Calabi–Yau space M_z corresponding to the point $z \in \mathscr{M}$; the Kähler potential is

$$e^{-K} = \int_{M_z} \Omega \wedge \overline{\Omega}\,; \tag{2.4}$$

the sections $\kappa_{\alpha\beta\gamma}$ are the Yukawa couplings and may be written

$$\kappa_{\alpha\gamma\varepsilon}(z) = \int_{M_z} \Omega \wedge \partial_\alpha\partial_\gamma\partial_\varepsilon\Omega\,. \tag{2.5}$$

The essential point is that the Yukawa couplings and the Kähler potential on $\mathscr{M}$ are both determined by the single holomorphic function $\mathscr{G}$, and they *depend holomorphically on parameters*. (This point will be important later.) There is a similar structure on the Kähler moduli space [18].

Special geometry was first defined as a consistency requirement arising in the study of $N = 2$ supergravity [20]. String theory associates $N = 2$ supergravity models to Calabi–Yau threefolds; it thus followed that moduli spaces of Calabi–Yau threefolds must exhibit this structure. When we discuss Calabi–Yau manifolds of dimension larger than three string theory leads to no such association and indeed, as we shall see, moduli spaces will not be special Kähler. In heuristic terms, whereas special geometry implies that the Kähler potential and Yukawa couplings are determined by a single holomorphic function, a number of holomorphic functions (associated with the independent kinds of Yukawa couplings) determine these features in the higher dimensional setting.

2.1. Mathematical Preliminaries. We now turn to a study of the moduli space $\mathscr{M}$ of complex structures on a Calabi–Yau d-fold M. We denote by $\Omega(z)$ a chosen holomorphic d-form on the Calabi–Yau space corresponding to the point z in $\mathscr{M}$. As in [17, 21], Ω is naturally thought of as a section of the Hodge bundle $\mathscr{H}$ over $\mathscr{M}$ (the fibers of which are $H^d(M, \mathbb{C})$). As the parameters z vary, $\Omega(z)$ spans a holomorphic line bundle $\mathscr{L} \subset \mathscr{H}$ whose first Chern class is the Kähler form on

$\mathcal{M}$.[5] (The fiber of $\mathcal{L}$ at a point z is the corresponding subspace $H^{d,0}(M_z)$ within $H^d(M, \mathbb{C})$.) We derive differential equations for the periods of $\Omega(z)$ (over a suitable family of cycles to be discussed), called Picard–Fuchs equations, in a manner similar in spirit to [17, 22, 23].

To do so, we make use of the fact that if s is a (p, q) form valued section of $\mathcal{H}$, its covariant derivative contains $(p-1, q+1)$ valued pieces. By successively taking covariant derivatives and isolating the appropriate piece at each stage, we can construct a sequence of maps[6]

$$H^{d,0} \to H^{d-1,1} \to \cdots \to H^{1,d-1} \to H^{0,d} \to 0 . \tag{2.6}$$

In [22, 23] this sequence of maps, applied to an initially chosen section $\Omega(z)$ of $\mathcal{L}$, was used to generate the Picard–Fuchs equation for a Calabi–Yau threefold. We will find that for one-parameter families we can extend this result to $d > 3$.

The first map in (2.6), as shown in [17], is constructed from the covariant derivative $D = \nabla + \omega$, where ∇ is the flat metric-compatible connection[7] on $\mathcal{H}$, and ω is a correction term characterized by the property that D acts covariantly on sections of $\mathcal{L}$. Applying this map to a chosen section Ω, we find that the components of the one-form $D\Omega$ span $H^{d-1,1}(M)$, providing the basis for an iteration of maps as in (2.6). The connection ∇ will be discussed in greater detail in Sect. IV; here we list the following properties which we will use.

(1) ∇ is a flat holomorphic connection compatible with the metric on $\mathcal{H}$ given by

$$\langle \bar{\eta} | \psi \rangle = i^{d^2} \int_M \bar{\eta} \wedge \psi . \tag{2.7}$$

(2) In its action on $\mathcal{H}$, ∇ is constrained by the property that it maps $\mathcal{F}^p$ to $\mathcal{F}^{p-1} \otimes T^*(\mathcal{M})$, where $\mathcal{F}^p = \bigoplus_{p' \geqq p} H^{p', d-p'}$. (This property is known as Griffiths transversality.)

(3) The correction term ω can be computed using the chosen section Ω as

$$\omega = -\partial \log \langle \bar{\Omega} | \Omega \rangle . \tag{2.8}$$

Since $\mathcal{L}$ is a line bundle, the covariant derivative $D = \nabla + \omega$ admits another interpretation – it essentially coincides with the metric connection on the tensor product bundle $\mathcal{H} \otimes \mathcal{L}^{-1}$. This is seen as follows. The metric connection $\mathcal{D}_{\mathcal{L}}$ on $\mathcal{L}$ has the property

$$\mathcal{D}_{\mathcal{L}}(f\Omega) = df\,\Omega - f\omega\Omega \tag{2.9}$$

with ω as in (2.8); the dual bundle $\mathcal{L}^{-1}$ will have a connection $\mathcal{D}_{\mathcal{L}^{-1}}$ satisfying

$$\mathcal{D}_{\mathcal{L}^{-1}}(f\Omega^{-1}) = df\,\Omega^{-1} + f\omega\Omega^{-1} . \tag{2.10}$$

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[5] The bundle $\mathcal{H} \otimes \mathcal{L}^{-1}$ has fibers which can be canonically identified with $H^0(M, \Lambda^0 T) \oplus H^1(M, \Lambda^1 T) \oplus \cdots \oplus H^d(M, \Lambda^d T)$. As pointed out by Strominger [17], these fibers can also be identified with $H^d(M, \mathbb{C})$; however, *that* identification is *not* canonical. Our treatment thus differs from [17] in considering Ω as a section of $\mathcal{H}$ and not $\mathcal{H} \otimes \mathcal{L}$.

[6] We shall see later that this procedure corresponds to generating a partial basis for the chiral ring of the associated conformal field theory by successive operator products of the marginal fields.

[7] We use the term "connection" in the sense of Koszul: a connection ∇ on a bundle $\mathcal{E}$ is a map $\nabla : \mathcal{E} \to \mathcal{E} \otimes T^*(\mathcal{M})$ which satisfies the Leibniz rule $\nabla(\varphi e) = \varphi \nabla(e) + (d\varphi) e$.

If we write a section of $\mathscr{H} \otimes \mathscr{L}^{-1}$ in the form $\alpha \otimes \Omega^{-1}$ (with α a section of $\mathscr{H}$) then we find

$$\begin{aligned}\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}}(\alpha \otimes \Omega^{-1}) &= \mathscr{D}_{\mathscr{H}}(\alpha) \otimes \Omega^{-1} + \alpha \otimes \mathscr{D}_{\mathscr{L}^{-1}}(\Omega^{-1}) \\ &= \nabla(\alpha) \otimes \Omega^{-1} + \omega\alpha \otimes \Omega^{-1} = D(\alpha) \otimes \Omega^{-1}\,, \qquad (2.11)\end{aligned}$$

so that $\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}}$ is calculated in terms of the covariant derivative D. It is thus natural to consider the sequence of maps in (2.6) as coming from a corresponding sequence of maps among subbundles of $\mathscr{H} \otimes \mathscr{L}^{-1}$, and this is the route we will follow in the next subsection.

The subbundle $\mathscr{L} \subset \mathscr{H}$ gives rise (upon tensoring with $\mathscr{L}^{-1}$) to a subbundle $\mathscr{O} \subset \mathscr{H} \otimes \mathscr{L}^{-1}$ isomorphic to the trivial bundle, whose fibers are $H^0(M, \Lambda^0 T)$. This subbundle comes equipped with a canonical section $\mathbb{1}$ (the constant function 1 on the moduli space). To calculate $\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}}(\mathbb{1})$, we must write $\mathbb{1} = \Omega \otimes \Omega^{-1}$, and we then find

$$\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}}(\mathbb{1}) = D(\Omega) \otimes \Omega^{-1} = (\nabla(\Omega) + \omega\Omega) \otimes \Omega^{-1}\,. \qquad (2.12)$$

This is independent of the choice of Ω.

Notice that in this description the *a priori* arbitrariness in the choice of Ω becomes irrelevant. This gives a nice resolution to an uncomfortable apparent asymmetry in one aspect of mirror manifolds, as we now briefly mention.

The symmetry between the "vertical" cohomology $\bigoplus_p H^{p,p}_{\widetilde{M}}$ on $\widetilde{M}$ and the "horizontal" cohomology $\bigoplus_p H^{d-p,p}_M$ on its mirror M has often been emphasized. On closer inspection, though, there lurks an uncomfortable asymmetry between these two structures: there is a canonical section of $H^{0,0}_{\widetilde{M}}$ which naturally enters the discussion – namely the constant section $\mathbb{1}$. On the other hand, there is no canonical choice of section of $H^{d,0}_M$ which is the mirror cohomology group of $H^{0,0}_{\widetilde{M}}$. The resolution of this apparent asymmetry which we present here is based on two observations. First, at a more fundamental level, mirror manifolds respect a symmetry which exchanges $\bigoplus_p H^p(M, \Lambda^p T^*)$ with $\bigoplus_p H^p(\widetilde{M}, \Lambda^p T)$. The former is canonically isomorphic to $\bigoplus_p H^{p,p}_{\widetilde{M}}$ while the latter as we have discussed, is canonically isomorphic to $\mathscr{H} \otimes \mathscr{L}^{-1}$. For $p = 0$ each of these does have a canonical section, namely, $\mathbb{1}$. Second, as we have just calculated, the covariant derivative which we use to realize (2.6) is also independent of the choice of Ω. Hence, in this description everything is manifestly mirror symmetric.

2.2. Picard–Fuchs Equations. Following the basic strategy outlined above, we now consider successive derivatives of $\mathbb{1}$, attempting at each stage to isolate the component of pure type (p,q) for the smallest possible value of p.[8] An invariant way of describing this procedure is to introduce an additional correction term ω_p with the properties that (1) $\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}} + \omega_p$ acts covariantly on sections of $\mathscr{F}^p \otimes \mathscr{L}^{-1}$, and (2) $\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}} + \omega_p$ maps $\mathscr{F}^p \otimes \mathscr{L}^{-1}$ to $(\mathscr{F}^p \otimes \mathscr{L}^{-1})^{\perp}$. By Griffiths transversality, it follows that $(\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}} + \omega_p)(\chi)$ must precisely pick out the $(p-1, q+1)$ piece of $\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}}(\chi)$.

[8] Note that we are using "(p,q)" a bit loosely here; we have shifted to a calculation within $\mathscr{H} \otimes \mathscr{L}^{-1}$, and so the phrase "type (p,q)" is actually meant to indicate a section of $H^q(M, \Lambda^q T)$.

Above we observed that in the first step of (2.6) we in fact produce a basis for the cohomology group $H^{d-1,1}$. In subsequent steps this nice property will no longer hold (as pointed out in [17]). The components of successive derivatives of Ω will span a subbundle of $\mathscr{H}$ that we will term the *primary horizontal subspace*. This is most easily understood in $\mathscr{H} \otimes \mathscr{L}^{-1}$, where it comprises the subspace generated by successive cup products of the elements of $H^1(T(\mathscr{M}))$; we will restrict our attention to this subspace. Thus, to verify at each step that we have correctly projected to $(\mathscr{F}^p \otimes \mathscr{L}^{-1})^\perp$ we will check that components along the elements of a basis for the primary subspace constructed in previous steps vanish. The metric (2.7) restricts (by the Hodge–Riemann bilinear relations) to a nondegenerate pairing on the primary subspace, so we will use this to compute these projections.

To avoid cluttering the notation, we will drop the subscripts and let D denote the covariant derivative on $\mathscr{H} \otimes \mathscr{L}^{-1} \otimes \Lambda^n T^*(\mathscr{M})$ (with the appropriate value of n determined by context) derived from $\mathscr{D}_{\mathscr{H}\otimes\mathscr{L}^{-1}}$ by adding an appropriate number of Levi–Civita connection terms. The first step in (2.6) is realized quite simply as $X^{(1)} = D\mathbb{1}$; let us show that this is of pure type. To determine the $(d,0)$ component we use the metric compatibility to evaluate

$$\langle\bar{\mathbb{1}}|X^{(1)}\rangle = \partial\langle\bar{\mathbb{1}}|\mathbb{1}\rangle - \langle D\bar{\mathbb{1}}|\mathbb{1}\rangle = 0\,. \tag{2.13}$$

The inner product $\langle|\rangle$ on $\mathscr{H} \otimes \mathscr{L}^{-1}$ is the one derived from (2.7); (2.13) thus equates one-forms. The first term is trivially zero. To see that the second term vanishes, recall that $\mathbb{1}$ is a holomorphic section of a holomorphic bundle on which D is a holomorphic connection. Thus writing $X^{(1)} = \chi_\alpha^{(1)} \otimes \Omega^{-1} dz^\gamma$ we have seen that $\chi_\alpha^{(1)}$ is purely of type $(d-1,1)$. In fact, of course, the components of $X^{(1)}$ span $H^1(T)$, realizing the Kodaira–Spencer isomorphism between this cohomology group and the fibers of $T(\mathscr{M})$.

At the next stage we again set $X^{(2)} = DX^{(1)}$. A computation essentially identical to (2.13) shows that $X^{(2)}$ has no $(d,0)$ component. To determine the $(d-1,1)$ component we compute the inner products with our basis

$$\langle\overline{X_\alpha^{(1)}} \mid X_{\beta\gamma}^{(2)}\rangle = \mathscr{D}_\beta\langle\overline{X_\alpha^{(1)}} \mid X_\gamma^{(1)}\rangle - \langle D_\beta D_{\bar{\alpha}}\bar{\mathbb{1}} \mid X_\gamma^{(1)}\rangle\,. \tag{2.14}$$

In the first term, $\mathscr{D}$ is the metric connection[9] on $T^*(\mathscr{M}) \oplus \overline{T^*}(\mathscr{M})$ and the tensor on which it acts is calculated to be the Kähler metric yielding zero. In the second term we can use once more the holomorphicity of $\mathbb{1}$ to obtain a commutator term

$$\langle[D_\beta, D_{\bar{\alpha}}]\bar{\mathbb{1}} \mid X_\gamma^{(1)}\rangle\,. \tag{2.15}$$

This commutator of covariant derivatives is just the curvature of the bundle in which they act – in this case because $\mathscr{H}$ is flat we obtain the curvature of $\mathscr{L}^{-1}$. Since this does not change type, the previous calculation shows that this term also vanishes. Hence, in the sense used above, $X^{(2)}$ is of pure type $(d-2,2)$.

We note here that for $d = 3$, the above steps suffice [17,22,23] to compute the differential equation. The reason is that in this case we already have in hand a basis for $H^{1,2}$, given by the complex conjugate of our basis for $H^{2,1}$. Expressing $X^{(2)}$ in terms of $\overline{X^{(1)}}$ we can complete (2.6) by considerations similar to those above. This, of course, will not suffice for $d > 3$. This does point to one difficulty we will

[9] This connection is *not* holomorphic. For further details on manipulating connections such as this one, see Appendix A.

encounter, however. The components of the forms generated in realizing (2.6) will by definition span the primary subspace, but in general they will not be linearly independent.

We now attempt, therefore, to continue by setting $X^{(3)} = DX^{(2)}$. The above reasoning ensures that besides the desired $(d-3,3)$ part, the only other possible component is of type $(d-2,2)$. To probe for the latter we compute

$$\langle \overline{X^{(2)}_{\alpha\beta}} \mid X^{(3)}_{\gamma\delta\varepsilon} \rangle \tag{2.16}$$

which equals

$$\mathscr{D}_\varepsilon \langle \overline{X^{(2)}_{\alpha\beta}} \mid X^{(2)}_{\gamma\delta} \rangle - \langle D_\varepsilon \overline{X^{(2)}_{\alpha\beta}} \mid X^{(2)}_{\gamma\delta} \rangle \,. \tag{2.17}$$

Let us deal with the second term in (2.17) first. Quite generally, any expression of the form

$$\langle D_\alpha D_{\bar\alpha 1} \dots D_{\bar\alpha k}\, \overline{\mathbb{1}} \mid \widetilde{X}^{(k-1)}_{\alpha_1 \dots \alpha_{k-1}} \rangle \tag{2.18}$$

vanishes identically if $\widetilde{X}^{(k-1)}$ is of pure type $(d-k+1, k-1)$. To prove this, we commute the D_α successively to the right. Examination of each of the resulting terms shows that they are all of at most type $(k-2, d-k+2)$ and hence yield zero inner product. Using this result we see that

$$\langle \overline{X^{(2)}_{\alpha\beta}} \mid X^{(3)}_{\gamma\delta\varepsilon} \rangle = \mathscr{D}_\varepsilon \langle \overline{X^{(2)}_{\alpha\beta}} \mid X^{(2)}_{\gamma\delta} \rangle \,. \tag{2.19}$$

Now, this expression is generally nonzero and hence $X^{(3)}$ is not of pure type $(d-3,3)$. We can, however, search for some tensor $S \in \Gamma(T(\mathscr{M})^2 \otimes T^*(\mathscr{M})^2)$ such that

$$\widetilde{X}^{(2)}_{\alpha\beta} = S^{\gamma\delta}_{\alpha\beta} X^{(2)}_{\gamma\delta} \,, \tag{2.20}$$

with S so chosen that $D\widetilde{X}^{(2)}$ is of pure type $(d-3,3)$.[10] In general it is not clear how to construct such an S, since this requires untangling the linear dependence of the components of $X^{(2)}$ mentioned above. (In the case $d=3$ it was found [23] using the expression for $X^{(2)}$ in terms of $\overline{X^{(1)}}$.) However, in the particular case of interest to us here, for a one-parameter family, it is not difficult to proceed; we restrict attention to this case now. The Greek indices all take but one value, and will occasionally be omitted. Then the simple solution is $S = (\langle \overline{X^{(2)}} \mid X^{(2)} \rangle)^{-1}$.

One can now continue in this fashion as follows (in our special case). Reasoning as above, we need to define $X^{(k)} = D\widetilde{X}^{(k-1)} = D(S^{(k-1)}X^{(k-1)})$ with S chosen so that $\langle \overline{X^{(k-1)}} \mid \widetilde{X}^{(k-1)} \rangle$ is covariantly constant. Then the arguments of the preceding paragraphs will show that $X^{(k)}$ is of pure type. The solution for S given above extends as

$$S^{(k)} = \left(\langle \overline{X^{(k)}} \mid X^{(k)} \rangle \right)^{-1} \,. \tag{2.21}$$

This procedure thus realizes the sequence (2.6) and terminates of course at the $(d+1)^{\text{st}}$ step, yielding a differential equation of this order for $\mathbb{1}$ or equivalently for Ω and hence its periods. This is the Picard–Fuchs equation, in a form very similar to that obtained in [23] for $d=3$:

$$D(S^{(d-1)}D) \cdots (S^{(3)}D) S^{(2)} DD\mathbb{1} = 0 \,. \tag{2.22}$$

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[10] This can only determine S up to covariantly constant factors.

2.3. Analogs of Special Geometry. As mentioned earlier, for $d = 3$ it is well known that Calabi–Yau moduli spaces are special Kähler manifolds which, for example, can be characterized by (2.2). The moduli spaces for Calabi–Yau manifolds for $d > 3$ do not satisfy (2.2) but they do respect particular constraints on their respective Riemann tensors as we now briefly indicate.

For arbitrary d we have (recalling that ∇ is a flat connection on $\mathscr{H}$)

$$[D_\alpha, D_{\overline{\beta}}]X_\gamma^{(1)} = -G_{\alpha\overline{\beta}}X_\gamma^{(1)} + R^{\delta}_{\alpha\overline{\beta}\gamma}X_\delta^{(1)}\,, \tag{2.23}$$

simply expressing the curvature of D as it acts on the bundle $\mathscr{H} \otimes \mathscr{L}^{-1} \otimes T^*(\mathscr{M})$ of which $X^{(1)}$ is a section. Solving for the Riemann tensor we thus have

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = G_{\alpha\overline{\beta}}G_{\gamma\overline{\delta}} + G_{\alpha\overline{\delta}}G_{\gamma\overline{\beta}} + e^K \int \chi_{\overline{\delta}}^{(1)} D_{\overline{\beta}} D_\alpha \chi_\gamma^{(1)}\,, \tag{2.24}$$

where as earlier we have written $X^{(1)} = \chi_\alpha^{(1)} \otimes \Omega^{-1}\, dz^\alpha$. The last term on the right-hand side would, in the notation of the previous subsection, be written after integration by parts as

$$\langle \overline{X_{\beta\delta}^{(2)}} \mid X_{\alpha\gamma}^{(2)} \rangle\,, \tag{2.25}$$

in which form this equation recently appeared in [24].

We wish to find a constraint on the Riemann tensor which is written explicitly in terms of the higher dimensional analog of (2.5). This requires an explicit evaluation of the integral on the right-hand side of (2.23). In the case of $d = 3$ this is easy to do since $D_\alpha \chi_\beta^{(1)}$ is pure type $(1,2)$ and is readily expressed in terms of $\chi_{\overline{\beta}}^{(1)}$ (and similarly for the complex conjugate situation which also arises in (2.23)). When the tensor $S^{(2)}$ (as described above) exists, this can be explicitly carried out in a similar manner for $d = 4$ and leads, after some algebra, to

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = G_{\alpha\overline{\beta}}G_{\gamma\overline{\delta}} + G_{\alpha\overline{\delta}}G_{\gamma\overline{\beta}} + e^K B^{\alpha'\gamma'\overline{\beta}'\overline{\delta}'}\kappa_{\alpha'\gamma'\alpha\gamma}\overline{\kappa}_{\overline{\beta}'\overline{\delta}'\overline{\beta}\overline{\delta}}\,, \tag{2.26}$$

where $B = S^{(2)}$ and κ is the Yukawa coupling defined in any dimension d by

$$\kappa_{\alpha_1\ldots\alpha_d} = \int \Omega \wedge \partial_{\alpha_1} \cdots \partial_{\alpha_d} \Omega\,. \tag{2.27}$$

Written in this way we see the similarity to $d = 3$, the main difference being the tensor B (which is essentially the inverse of $\langle \overline{\chi_{\beta\delta}^{(2)}} | \chi_{\alpha\gamma}^{(2)} \rangle$) taking the place of $G^{\alpha\overline{\beta}}$ (which arises from the inverse of $\langle \overline{\chi}_\beta^{(1)} | \chi_\alpha^{(1)} \rangle$).

In the case of a one-parameter family where the tensors $S^{(k)}$ exist and the analysis above is valid, we can explicitly compute (again omitting the indices)

$$B = e^K/(2G^2 - R)\,. \tag{2.28}$$

Thus

$$(R - 2G^2)^2 = e^{2K}\kappa\overline{\kappa}\,. \tag{2.29}$$

The Hodge–Riemann bilinear identities ensure that $2G^2 - R$ is positive and hence we find (replacing the index placeholders)

$$R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} = 2G_{\alpha\overline{\alpha}}^2 - e^K |\kappa_{\alpha\alpha\alpha\alpha}|\,. \tag{2.30}$$

The approach can be pursued further. This can be done by explicitly evaluating the right-hand side of (2.24). Alternatively (and somewhat easier to calculate) one can pursue the direct analog of the three (or four) dimensional calculation and consider $[D_\alpha, D_{\overline{\beta}}]X^{(j)}$ with $j = [d-1]/2$ as before. Since $X^{(j)}$ is a section of $\mathscr{H} \otimes \mathscr{L}^{-1} \otimes T^*(\mathscr{M})^j$ the commutator involves sums of terms involving the Riemann tensor and the metric on moduli space. The advantage of operating on a section of this particular bundle is that for this value of j the action of D_α pushes us over the "half-way" point, thus allowing us to reexpress the result in terms of the complex conjugate basis (as discussed earlier). This facilitates direct calculation of the constraint on the Riemann tensor. For example, in the case of a one-parameter family with $d = 5$ we find

$$\left(\frac{R_{;\alpha}}{2G^2 - R}\right)_{;\overline{\alpha}} + \frac{|\kappa|^2}{(2G^2 - R)^2} e^{2K} = 3(G - G^{-1}R), \tag{2.31}$$

where R is the Riemann tensor and G is the metric on moduli space. In general when one attempts to evaluate the right-hand side of (2.24) in terms of the Yukawa couplings the expressions become complicated for large d.

3. Yukawa Couplings, Series Expansions and Factorization

In the previous section we described some general structural features of moduli spaces for Calabi–Yau manifolds in general dimension d. Our aim is to apply mirror symmetry to these manifolds, and to this end we will in this section introduce the physical theories related to the geometrical constructs. We will then compute the correlation functions of marginal chiral primary operators in a set of models and exhibit the series expansions predicted for these functions by mirror symmetry. Finally, we will show how these functions are predicted to factorize in terms of more fundamental correlators and extract some highly nontrivial predictions regarding this factorization. Computing the fundamental couplings will require the introduction of some additional structure and this will be the subject of the next section.

Given a Calabi–Yau space M equipped with a complex structure, Kähler metric and B-field, we can define two different topological field theories. The description of the previous section is well-suited to a discussion of the **B** model (in the terminology of [25]). In this theory the observables are naturally described by the space

$$H_B = \bigoplus_{p,q=0}^{d} H^p(\Lambda^q T), \tag{3.1}$$

where T is the holomorphic tangent bundle to M. The correlation functions of the model are computable exactly in terms of geometrical quantities. Given $\mathcal{O}_i \in H^{p_i}(\Lambda^{q_i} T)$ the correlation function vanishes unless $\sum p_i = \sum q_i = d$ and when nonzero is given by

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int_M \Omega_{i_1 \ldots i_d} \mathcal{O}_1 \wedge \cdots \wedge \mathcal{O}_s \wedge \Omega, \tag{3.2}$$

where the notation means tangent indices are contracted with Ω and the forms cupped together. Deformations of the complex structure of M are related of course to the observables corresponding to $H^1(T)$. (These are the marginal operators.) The nonzero correlators of these can be rewritten as

$$\langle X^{(1)}_{\alpha_1} \cdots X^{(1)}_{\alpha_d} \rangle \equiv \kappa_{\alpha_1 \ldots \alpha_d} = \int_M \Omega \wedge D_{\alpha_1} \ldots D_{\alpha_d} \Omega \,, \tag{3.3}$$

a quantity which was seen to play a role in the discussion of Sect. II.

There is a second topological field theory associated to a Calabi–Yau manifold M, the **A** model of [25]. For clarity below we rename the Calabi–Yau manifold to $\widetilde{M}$, but we stress that it is *not* necessary to change the manifold in order to define the **A** model. In this theory the observables naturally correspond to the de Rham cohomology of $\widetilde{M}$. The parameter space of this model is a complexification of the Kähler cone of $\widetilde{M}$; all relevant quantities are completely invariant under variations of the complex structure. The correlation functions in the **A** model are defined as sums over homotopy classes of maps from the worldsheet (which we take in all cases to be simply $\mathbb{CP}^1$, other topologies have recently been considered in [24]) to $\widetilde{M}$. In each class the contribution may be shown to localize on holomorphic maps, and the contribution of each such "instanton sector" is weighted by the exponential of the pullback of the Kähler form of $\widetilde{M}$ (evaluated on the fundamental class of the worldsheet). These series are expected to have a finite radius of convergence about a "large radius limit" point deep in the interior of the Kähler cone; the leading term is the intersection matrix of $\widetilde{M}$. The nonzero correlators of marginal operators $\widetilde{X}^{(1)}_\alpha$ can be written

$$\langle \widetilde{X}^{(1)}_{\alpha_1} \cdots \widetilde{X}^{(1)}_{\alpha_d} \rangle \equiv \widetilde{\kappa}_{\alpha_1 \ldots \alpha_d} = \int_{\widetilde{M}} \widetilde{\Omega}^{a_1 \ldots a_d} A_{a_1 \bar{a}_1} \ldots A_{a_d \bar{a}_d} \overline{\widetilde{\Omega}}^{\bar{a}_1 \ldots \bar{a}_d}$$
$$+ \text{ instanton corrections}\,, \tag{3.4}$$

where $\widetilde{\Omega}$ is a completely antisymmetric tensor field needed to normalize the topological correlation functions. In familiar applications a particularly natural choice for the latter data on $\widetilde{M}$ has been made: namely, a completely antisymmetric tensor field which is constant on the Kähler moduli space. Although not usually emphasized, we point out that, while natural, this *is* a choice and mirror symmetry predicts that there is a corresponding choice for the data on M such that (3.3) and (3.4) are equal.[11]

The instanton contributions to (3.4) from nontrivial sectors are related (as we discuss in detail in Sect. V) to certain characteristic classes of the moduli space of holomorphic maps of the appropriate homotopy type, and the extraction of explicit results on these has been one of the most successful applications of mirror symmetry. This application is based upon the following fact: If M and $\widetilde{M}$ are mirror manifolds, then the **A** model constructed from $\widetilde{M}$ is isomorphic as a topological field theory to the **B** model constructed from M. In more detail, this means that mirror symmetry implies the existence of a "mirror map" from the complexified Kähler cone of $\widetilde{M}$ to the moduli space of complex structures on M, and at each point a mapping of the spaces of observables in the two models, such that these maps preserve the correlation functions of the topological field theory.[12] In practice, to study the

[11] A similar observation has been made independently by Distler [26].

[12] In fact, as is well known, this statement is weaker than the strongest one implied by mirror symmetry – which implies in fact an isomorphism of the superconformal σ models based upon M and $\widetilde{M}$, but this version is sufficient for all of our applications here.

properties of rational curves on a manifold $\widetilde{M}$ one constructs the mirror manifold M and computes the **B** model correlation functions as we will do below. One then finds the location in moduli space of the "large complex structure limit" point (mirror to the large radius point), about which one expands the correlators. To interpret the coefficients of the expansion one must expand in coordinates related by the mirror map to the coefficients of the Kähler form on $\widetilde{M}$ in terms of a fixed basis for $H^2(\widetilde{M})$. We will find these coordinates using an *Ansatz* for the mirror map, first proposed in [5] and recently explained in [24]. These points will be discussed in more detail in the sequel.

3.1. The Computation. We now present a class of examples for which we perform the computations explicitly. All of these will be one-parameter families of Calabi–Yau manifolds (i.e. $h^{d-1,1}(M) = 1$). In particular, for simplicity, we will consider families $M_\psi^{(d)}$ of Calabi–Yau hypersurfaces constructed as follows. Let $W_\psi^{(d)}$ be a hypersurface in $\mathbb{CP}^{d+1}$ determined in terms of homogeneous coordinates z_i by the equation[13]

$$P(z;\psi) = z_1^{d+2} + \cdots + z_{d+2}^{d+2} - (d+2)\psi z_1 z_2 \cdots z_{d+2} = 0\,. \tag{3.5}$$

This defines a family of Calabi–Yau manifolds with $h^{1,1}(W) = 1$. Define $M_\psi^{(d)}$ by the quotient construction

$$M_\psi^{(d)} = W_\psi^{(d)}/(\mathbb{Z}_{d+2})^d\,. \tag{3.6}$$

By the arguments of [3], the family $M_\psi^{(d)}$ lies in the "mirror" parameter space, and indeed one verifies that $h^{d-1,1}(M) = 1$ [27]. The parameter ψ is a coordinate on the space of complex structures on $M_\psi^{(d)}$. In terms of the mirror manifold $\widetilde{M}_\psi^{(d)}$ it serves as a coordinate on the complexified Kähler cone. Note that $\widetilde{M}_\psi^{(d)}$ is a deformation of $W_\psi^{(d)}$, so computations on M will yield information about rational curves on W.

The expression (3.3) demonstrates that the numerical value of κ depends both on the choice of Ω and on the coordinate system (in the language of previous sections κ is a section of $\mathscr{L}^2 \otimes \mathrm{Sym}(T^{*\otimes d})$). As we have discussed, a necessary ingredient for the application of mirror symmetry in this context is to discover the correct map between ψ and t, where the former is our parameter on the complex structure moduli space of M and the latter denotes a coordinate on the Kähler moduli space of $\widetilde{M}$. Choosing a particularly convenient gauge for Ω

$$\Omega = \psi \frac{z_1 dz_2 \wedge \cdots \wedge dz_{d+1} + \text{cyclic permutations}}{P}\,, \tag{3.7}$$

[13] We note that one could easily extend our analysis to include cases involving weighted projective spaces (which even for the case of hypersurfaces with $h^{1,1} = 1$ become quite numerous with increasing d).

the techniques of deformation theory allow us to compute $\kappa_{\psi,\dots,\psi}$ quite simply [28]

$$\kappa = \frac{\psi^2}{H(\psi)},$$

$$H(\psi) \equiv \det\left(\frac{\partial^2 P}{\partial z_i \partial z_j}\right) = ((d+1)(d+2))^{d+2}(1-\psi^{d+2}). \tag{3.8}$$

A more natural parameter on the moduli space is $z = \psi^{-(d+2)}$, and these quantities can equally well be expressed in terms of z. In order to obtain information about rational curves on $W^{(d)}$ we need to find the correct coordinate t in terms of ψ or z.

To find this we consider the periods of the holomorphic d-form Ω along a set of n-cycles locally constant up to homology, $\varpi_i = \int_{\gamma_i} \Omega(z)$. We restrict the γ_i's to lie in the *primary horizontal subspace* of homology, which by definition is the annihilator of the orthogonal complement of the primary horizontal subspace of cohomology (introduced in Subsect. 2.2). To find the periods in terms of ψ we will make use of the fact that they satisfy – as discussed in Sect. II – a set of differential equations, the Picard–Fuchs equations. For a one-parameter family this is an ordinary differential equation with regular singular points at boundary points of the moduli space. The monodromy of the locally constant homology cycles (in the primary subspace) about these degeneration points is reflected in the monodromy of the solutions. In particular, the boundary point corresponding to large radius of $W^{(d)}$ is a singular point of "maximally unipotent monodromy" [29]. This implies [6] that a set $\varpi_0, \varpi_1, \dots, \varpi_d$ of local solutions can be found so that ϖ_0 is single-valued, and each ratio of successive solutions ϖ_{i+1}/ϖ_i has the form

$$\frac{1}{2\pi i}\log z + \text{single-valued function} \tag{3.9}$$

near the boundary point.

We then use

$$t = \frac{\varpi_1}{\varpi_0}, \tag{3.10}$$

to specify the mirror map; the coordinate q in terms of which we perform power series expansions is then represented as $q = e^{2\pi i t}$. Note that under transport about the singular point we have $t \to t+1$. This form of the mirror map was first advanced by Candelas et al. [5], formulated as described here in [29], and recently explained in [24].

For the case at hand the required Picard–Fuchs equations were derived by Lerche et al. [30]. The Picard–Fuchs equation is seen to be a generalized hypergeometric equation (recall $z = \psi^{-(d+2)}$)

$$\left[z\prod_{j=1}^{d+1}\left(z\partial_z + \frac{j}{d+1}\right) - (z\partial_z)^{d+1}\right]\varpi = 0. \tag{3.11}$$

The singular point of interest is $z = 0$. Using the method of Frobenius (see [31]) we find the following series expansions for the solutions (as the reader can verify with a straightforward computation)

$$\varpi_0 = \sum_{n\geqq 0}\prod_{j=1}^{d+1}\left(\frac{j}{d+1}\right)_n \frac{z^n}{(n!)^{d+1}},$$

$$\varpi_1 = \varpi_0 \log(z) + \frac{\partial}{\partial w}\bigg|_{w=0}\left[\sum_{n\geqq 0}\left(\prod_{j=1}^{d+1}\frac{(w+j/d+1)_n}{(w+1)_n}\right) z^n\right], \tag{3.12}$$

where

$$(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3.13}$$

is the Pochhammer symbol. Note that (3.12) yields explicit expressions for the power series coefficients using elementary properties of Gamma functions.

The required series expansion for κ is then obtained by inverting these to express z as a series in q and then inserting (3.8) (taking proper account of the change of coordinates from ψ to z). In Table 1 we give the first few terms in these series expansions of $\kappa_{tt\ldots t}$ for d in the range four to ten. Notice that, as expected, the series all involve integer coefficients. However, it is not immediately clear how to give geometrical interpretations to these integers. The key to the explanation is to recall that these d-point functions must factor into sums of products of three-point functions. As we will see in the next section, the three-point functions can be directly calculated for the **B** model, and the corresponding **A** model three-point functions have an immediate geometrical interpretation which we shall describe. The power series shown in Table 1 will factor into other power series which explicitly represent these three-point functions.

3.2. Factorization and Three-Point Functions. In the previous subsection we computed the d-point Yukawa couplings and found their series expansions. As mentioned there, the objects for which we have an immediate geometric interpretation are the three-point functions of the **A** model; this interpretation will be discussed in detail in Sect. V. In this subsection we will describe the three-point functions and relate them to the correlators computed above; Sect. IV is devoted to an algorithm for computing the three-point functions.

One of the defining properties of a topological field theory is the factorization property exhibited by its correlation functions. In the present context this means that all of the correlators can be written in terms of the nonvanishing two-point and three-point functions. Underlying this is the fact that the operators in a topological field theory form an associative, commutative graded ring on which the correlation functions determine a trace function [32, 33]. This is manifest in the **B** model; the form of (3.2) shows that multiplication in the ring is just the sheaf cup product. (This ring structure on H_B coincides with the "(c,c) ring" of the superconformal theory.) In the context of the **A** model (where we get the "(a,c) ring" or "quantum cohomology ring") this property is less obvious and will lead, after interpreting the instanton expansion coefficients in terms of rational curves, to some unsuspected properties of the latter.

The ring structure implies the existence of a topological version of the operator product expansion, in the form

$$\mathcal{O}_\alpha^{(i)}\mathcal{O}_\beta^{(j)} = \sum_\rho C_{\alpha\beta}^{(i,j)\rho}\mathcal{O}_\rho^{(i+j)}. \tag{3.14}$$

Table 1. *d*-point functions in dimension *d*.

d	d-point functions
4	$6 + 120960\,q + 4136832000\,q^2 + 148146924602880\,q^3 + 5420219848911544320\,q^4 + 200623934537137119778560\,q^5$ $+ 7478994517395643259712737280\,q^6 + 280135301818357004749298146851840\,q^7 + 10528167289356385699173014219946393600\,q^8$ $+ 396658819202496234945300681212382224722560\,q^9 + 14972930462574202465673643937107499992165427200\,q^{10}$ $+ 566037069767251121484562070892662863943365345190400\,q^{11} + \cdots$
5	$7 + 3727381\,q + 2637885990187\,q^2 + 1927092954108108787\,q^3 + 1425153551321014327663291\,q^4$ $+ 1060347883438857662557634869906\,q^5 + 791661306374088776109692880989252173\,q^6$ $+ 592348256908461616176898022359492565546566\,q^7 + 443865568545713063761643598030194801299861575595\,q^8$ $+ 332947403131697202086626568381790256001850741509664373\,q^9 + \cdots$
6	$8 + 106975232\,q + 1672023727001600\,q^2 + 26611692333081695092736\,q^3 + 426129121674687823674948571136\,q^4$ $+ 6842148599241293047857339542861643776\,q^5 + 110018992594692024449889564415904439556898816\,q^6$ $+ 1770551943055574073245974844490813198478975912902656\,q^7$ $+ 28508925683951911989843155602330000507452539542539447947264\,q^8 + \cdots$
7	$9 + 3103936929\,q + 1165013014173543657\,q^2 + 441297815019235844688286425\,q^3$ $+ 167606183678231435989321323352019097\,q^4 + 63725266235392545772891574625466284314384997\,q^5$ $+ 24241288949516830588590880856230885330230589276883697\,q^6$ $+ 9224015119655326077755094748604809293729079769166286338124125\,q^7 + \cdots$
8	$10 + 94327552000\,q + 930496455109619200000\,q^2 + 9217712440694086335170560000000\,q^3$ $+ 91408512461766539388538224090616000000000\,q^4$ $+ 906798958458710503048638459436511928645436567552000\,q^5$ $+ 8997152189131293850735805864983772379107049616519528407040000\,q^6$ $+ 89275777693040346906449878258202162124048469440553887673238192128000000\,q^7 + \cdots$
9	$11 + 3049747360561\,q + 865196274264724937872931\,q^2 + 245891784376657937170481797615461001\,q^3$ $+ 69909581514948393506313481730554975387628730971\,q^4$ $+ 19878335509713026535802529940943471736182876279710737890681\,q^5$ $+ 5652532179569595758241560307343719407665962471280448209489098916126883\,q^6 + \cdots$
10	$12 + 105530993897472\,q + 938751865652732974917414912\,q^2 + 835663701195207462849842436375312020 2752\,q^3$ $+ 7439893683444139256621493691690662854027188247481 1392\,q^4$ $+ 66239445284774070849837962307659206004255394437729345309680022 3232\,q^5$ $+ 589755241941434343638761082760255175714657880143896940423903673544310628 0767488\,q^6 + \cdots$

Our notation here is that a superscript (j) indicates that the corresponding operator is in $H^j(M, \Lambda^j T)$ (for a **B** model computation; the grading property is universal but not always as obvious) and the subscripts are labels. Using (3.14) it is possible to express a correlator in terms of correlators with fewer fields. In turn, the expansion coefficients $C^{(i)}$ themselves may be expressed in terms of the two-point and three-point correlators.

This comes about as follows. The two-point function determines a nondegenerate metric on H_B (of (3.1))

$$\eta_{\alpha\beta} = \langle \mathcal{O}_\alpha^{(i)} \mathcal{O}_\beta^{(j)} \rangle \,. \tag{3.15}$$

The properties of (3.2) guarantee that for arguments $\mathcal{O}$ of pure type, $\eta_{\alpha\beta}$ is nonzero only between complimentary types $i + j = d$.[14] The metric η depends holomorphically on the parameters and is flat. In fact, it is possible to choose a basis which varies so that $\eta_{\alpha\beta}$ is constant. For a one-parameter family, we can restrict attention to the primary horizontal subspace which is one-dimensional in each graded piece. We can then certainly choose our normalizations so that $\eta^{(i,j)} = c\delta_{i+j,d}$, where c is the degree of the variety.[15]

In a similar manner, the three-point functions determine maps

$$Y_i^j : H^i(M, \Lambda^i T) \times H^j(M, \Lambda^j T) \times H^{d-i-j}(M, \Lambda^{d-i-j} T) \to \mathbb{C} \tag{3.16}$$

given in terms of some basis for H_B by[16]

$$Y_i^j = \langle \mathcal{O}^{(i)} \mathcal{O}^{(j)} \mathcal{O}^{(d-i-j)} \rangle = C^{(i,j)} \eta^{(i+j, d-i-j)} = cC^{(i,j)} \,. \tag{3.17}$$

Because η is invertible, we can use this to express $C^{(i,j)}$ in terms of Y. This is the sense in which all correlators are determined by the two-point and three-point functions. There is an obvious symmetry $Y_i^j = Y_j^i = Y_i^{d-i-j}$ among these functions. The associativity of the ring of local operators leads to some less obvious relations which we now discuss.

We now turn to the final goal of this section: to show that a complete set of three-point functions is provided by those which involve at least one element in $H^1(M, T)$, i.e. the Y_i^1. The essential idea here is that a four-point function can be factored into (sums of) products of pairs of three-point functions in up to three distinct ways by using the associativity of the operator product expansion. To illustrate this point, consider, for example, a four-point function $\langle \mathcal{O}_\alpha^{(1)} \mathcal{O}_\beta^{(1)} \mathcal{O}_\gamma^{(2)} \mathcal{O}_\delta^{(2)} \rangle$ on a Calabi–Yau sixfold. By factoring this four-point function in the two distinct possible ways we have

$$\sum_{\rho,\sigma} C_{\alpha\beta}^{(1,1)\rho} C_{\delta\gamma}^{(2,2)\sigma} \mathcal{O}_\rho^{(2)} \mathcal{O}_\sigma^{(4)} = \sum_{\rho,\sigma} C_{\alpha\delta}^{(1,2)\rho} C_{\beta\gamma}^{(1,2)\sigma} \mathcal{O}_\rho^{(3)} \mathcal{O}_\sigma^{(3)} \,. \tag{3.18}$$

---

[14] The metric η differs from the metric G discussed in Sect. II, even when restricted to the subspace of H_B corresponding to marginal deformations; the relation between these two was the subject of [34].

[15] If we try to suppress this degree c by a change of basis, then for d even, in the middle cohomology group $H^{d/2,d/2}$ we would have to leave the realm of integral cohomology and allow a square root as a coefficient. For this reason, we stick with this almost-standard normalization.

[16] For the special case of one-parameter families in which we focus only on a single element in each $H^p(M, \Lambda^p T)$ we use the same symbol for the map and its image in $\mathbb{C}$ (for specially chosen normalization of the arguments).

Thus we have

$$\sum_{\rho,\sigma} C_{\alpha\beta}^{(1,1)\rho} C_{\delta\gamma}^{(2,2)\sigma} \eta_{\rho\sigma}^{(2,4)} = \sum_{\rho,\sigma} C_{\alpha\delta}^{(1,2)\rho} C^{(1,2)\sigma}{}_{\beta\gamma} \eta_{\rho\sigma}^{(3,3)} . \tag{3.19}$$

We see from this equality that if we know the metric η and the operator product coefficients $C^{(1,1)}$ and $C^{(1,2)}$, then associativity gives us a set of linear equations for the coefficients $C^{(2,2)}$. In the normalization discussed above we can make this more explicit and find

$$C^{(1,1)} C^{(2,2)} = (C^{(1,2)})^2 . \tag{3.20}$$

Using (3.17), (3.20) gives a relation

$$Y_2^2 = (Y_2^1)^2 / Y_1^1 . \tag{3.21}$$

This same reasoning is readily used to show that for arbitrary d (in the primary horizontal subspace) we have

$$Y_i^j = \prod_{k=0}^{j-1} Y_{i+k}^1 \Big/ \prod_{k=1}^{j-1} Y_k^1 . \tag{3.22}$$

We thus see that all Yukawa couplings Y_i^j are determined in terms of those which contain at least one member of $H^1(M,T)$.

As discussed above, the Y_i^j are interpretable as three-point functions on the mirror $\widetilde{M}$ involving elements of $H^i(\widetilde{M}, \Lambda^i T^*)$, $H^j(\widetilde{M}, \Lambda^j T^*)$, and $H^{d-i-j}(\widetilde{M}, \Lambda^{d-i-j} T^*)$. These three-point functions have instanton expansions whose coefficients depend on the rational curves on $\widetilde{M}$. The identities in (3.21) (and their straightforward generalizations to higher dimensional moduli spaces) thus provide various relations among the numbers associated to rational curves. These relations provide a sensitive consistency check on our methods (as we shall see).

4. The Mirror Map and Three-Point Functions

As discussed earlier, the arguments of [3] establish an abstract isomorphism between the moduli spaces of complex structures on M and Kähler structures on $\widetilde{M}$ and between the associated Hilbert spaces which preserves the correlation functions.[17] A full understanding of mirror symmetry, and certainly its application to computing properties of rational curves, involves the explicit form of these isomorphisms. As mentioned in the previous section, an *Ansatz* for the so-called "mirror map" between the moduli spaces was proposed (and verified in an example) in [5]; this has since been checked in many other examples and has recently been explained in [24]. This map provides naturally a part of the required isomorphism between the Hilbert spaces $H^p(M, \wedge^p T)$ and $H^p(\widetilde{M}, \wedge^p T^*)$, since the tangent directions to moduli space are related to the subspace of marginal operators (recall this is simply the subspace at the first nonzero grading). This isomorphism was used in the previous section to relate correlators of these operators in the two models, and the fact that the series of Table 1 yield integer coefficients is a signal that we have performed the mapping correctly. As we have seen, however, these correlators are in some sense

[17] The argument in [3] establishes this up to possible global considerations.

secondary objects derived from the more fundamental three-point couplings; it is to these fundamental objects that a geometrical interpretation (in the **A** model) may be given. These however necessarily involve non-marginal operators, so we will need to extend the mirror map to a complete isomorphism between the Hilbert spaces. In the special case of Calabi–Yau threefolds, mapping the space of marginal operators in fact suffices to extract all of the required information. As mentioned in Sect. II, complex conjugation generates from a basis of these a basis for the entire space; performing this operation in both spaces leads to two bases related by mirror symmetry. In other cases, however, constructing bases (as sections over moduli space) which are mapped to each other by mirror symmetry requires more structure. We will supply this structure and give a systematic method for finding such bases and hence exploiting mirror symmetry. We will focus our attention on one-parameter families of Calabi–Yau d-folds although the extension to higher dimensional moduli spaces should be relatively straightforward.

As a brief summary for the rest of this section, we note that our approach is, roughly, as follows. The mirror map, as discussed in Sect. III, determines a coordinate (the ratio of periods) on the moduli space of complex structures on M related by mirror symmetry to the natural coordinate on the space of Kähler structures on $\widetilde{M}$. Our goal is to use this information, which essentially gives us mirror symmetric bases of $H^1(M,T^*)$ and $H^1(\widetilde{M},T)$ to construct mirror symmetric bases of $H^p(M,\wedge^p T)$ and $H^p(\widetilde{M},\wedge^p T^*)$. In essence, we construct such bases by beginning with elements in the H^1 cohomology groups and generating the primary subspace by successive operator products of these. On $\widetilde{M}$ we will relate this to an integral basis of $H^{p,p}(\widetilde{M},\mathbb{Z})$. On M, we find a systematic approach using the Gauss–Manin connection.

4.1. The Gauss–Manin Connection and the Choice of Basis. The Gauss–Manin connection ∇ was introduced in Sect. II as the flat holomorphic connection compatible with the metric on $\mathscr{H}$. This connection can also be defined by the following important property. As we move around in the parameter space of complex structures of M, the decomposition $H^d(M,\mathbb{C}) = \bigoplus_p H^{p,d-p}(M,\mathbb{C})$ varies since the meaning of a (p,q) form depends upon the complex structure. We can, however, also consider a topological basis of $H^d(M,\mathbb{C})$ (for example, the duals of topological homology cycles in $H_d(M,\mathbb{C})$) which does not vary with the complex structure. The Gauss–Manin connection measures the variance of the former basis with respect to the latter. To see this explicitly, let $\gamma_1,\ldots,\gamma_k$ be a topological basis of $H_d(M,\mathbb{C})$ and consider $\alpha(z)$ to be a holomorphically varying element in $\mathscr{F}^p$. Then, we can write

$$\alpha(z) = \sum_\mu \left(\int_{\gamma_\mu} \alpha(z) \right) \gamma_\mu^* \tag{4.1}$$

with $\{\gamma_\mu^*\}$ being the dual basis of $\{\gamma_\mu\}$ in $H^d(M,\mathbb{C})$. We define the action of ∇ to be

$$\nabla\alpha = \sum_{i,\mu} \left(\int_{\gamma_\mu} \partial_{z_i}\alpha(z) \right) \gamma_\mu^* dz^i \, . \tag{4.2}$$

In other words, the Gauss–Manin connection is defined by demanding that the topological sections γ_μ are flat sections. Then, covariant differentiation turns into ordinary differentiation with respect to the parameters of the complex structure moduli space.

We will momentarily see that the Gauss–Manin connection plays a crucial role in finding and implementing the extended mirror map.

We now, once again, specialize our discussion to the case $h_M^{d-1,1} = h_{\widetilde{M}}^{1,1} = 1$ and to the primary horizontal and vertical subspaces of $H^{j,d-j}(M,\mathbb{C})$ and $H^{j,j}(\widetilde{M},\mathbb{C})$ generated by these one-dimensional spaces. Our goals are to

1) find a map from the moduli space $\mathcal{M}_M^{c.s.}$ of complex structures on M parameterized by the complex coordinate z to the "Kähler" moduli space $\mathcal{M}_{\widetilde{M}}^K$ parameterized by the complex coordinate t and to

2) find the explicit isomorphism between $\bigoplus_p H^{p,p}(\widetilde{M},\mathbb{C})$ and $\bigoplus_p H^{p,d-p}(M,\mathbb{C})$ such that the **A** model Yukawa couplings $\langle\widetilde{\mathcal{O}}^{(i)}\widetilde{\mathcal{O}}^{(j)}\widetilde{\mathcal{O}}^{(k)}\rangle$ as functions of t are equal to the **B** model Yukawa couplings $\langle\mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(k)}\rangle$ as functions of z (for corresponding basis elements) once we express t in terms of z using the mirror map.

To this end, we first note that there is an especially convenient basis for the primary vertical subspace of $\bigoplus_p H^{p,p}(\widetilde{M},\mathbb{C})$. It can be described as $e_0, e_1, \ldots, e_d$, where each e_p is the integral generator of $H^{p,p}(\widetilde{M},\mathbb{C})$ which is the Poincaré dual of a submanifold of complex codimension p. (We in fact take e_p of the form $e_1 \cup \cdots \cup e_1$ (with p terms).) As discussed earlier, it is this basis which gives rise to the simplest geometrical interpretation of three-point correlation functions. Goal (2) will be achieved if we can find the mirror image of this basis in $H^d(M_z,\mathbb{C})$. Moreover, since the Kähler moduli space of $\widetilde{M}$ is locally isomorphic to $H^1(T^*_{\widetilde{M}}) = H^{1,1}(\widetilde{M})$, the generator e_1 of $H^{1,1}(\widetilde{M})$ determines a natural coordinate t on the Kähler moduli space. (The Kähler form will be written as te_1.) So we can actually achieve both goals (1) and (2) by finding the appropriate analogous basis in $H^d(M_z,\mathbb{C})$, since the analog of e_1 can be used to specify a coordinate.

To motivate our solution to this question, let's look more closely at the primary vertical sub-basis $e_0, e_1, \ldots, e_d$ of $\bigoplus_p H^{p,p}(\widetilde{M},\mathbb{Z})$. We have, in this basis,

$$\eta^{(i,j)} = \langle e_i, e_j\rangle = c\delta_{i+j,d}\ , \tag{4.3}$$

where c is a specific constant, the *degree* of $\widetilde{M}$, calculated by integrating $e_1 \cup \cdots \cup e_1$ (with d terms) over $\widetilde{M}$.[18] Also note that we clearly have $e_1 e_d = 0$. Our basis, therefore, satisfies the following three features:

1) $e_1 e_{j-1} = c^{-1} A^1_{j-1}(t) e_j$,

2) $\eta^{(i,j)} = \langle e_i, e_j\rangle = c\delta_{i+j,d}$,

3) $e_1 e_d = 0$,

where we have used $A^1_{j-1}(t)$ to denote the **A** model Yukawa coupling $\langle e_1 e_{j-1} e_j\rangle$ as a function of t.[19]

[18] In odd dimension we could change basis to get rid of this constant, but in even dimension doing so would introduce the square root of the degree as a coefficient, which could provide a good basis for *real* cohomology but not for *integral* cohomology.

[19] These functions coincide with the function Y^1_{j-1} of Subsect. 3.2; we introduce the notation A^1_{j-1} here and B^1_{j-1} below in order to emphasize when these functions are being calculated on the **A** model of $\widetilde{M}$, and when on the **B** model of M.

Of course, property (1) follows from (2), but writing it in this manner will be useful shortly. In particular, we interpret (1) as follows: the operator product of e_1 and e_{j-1} is a functional multiple of e_j, with the multiplier depending on the parameter in the Kähler moduli space. We note that on the **A** model side these properties do not uniquely single out a basis; rather, they are properties characteristic of a set of bases, amongst which is the basis of integral generators.

We now mimic these properties on the **B** model side; we will see that a slightly stronger version of these properties, when combined with an analysis of the monodromy, does determine an essentially unique basis.

We will formulate our basis for the **B** model bundle in such a way that both the basis and the correlation functions manifestly have a holomorphic dependence on moduli. As has been recognized since the work of Griffiths [35], there is an inherent conflict between choosing bases of pure (p,q) type, and choosing bases which vary holomorphically with moduli. (This is why we introduced the bundles $\mathscr{F}^p$ rather than working directly with $H^{p,q}(M)$ in our discussion of the Gauss–Manin connection, since the $\mathscr{F}^p$'s are the holomorphically varying objects.) Although the first choice might appear at first sight to be better adapted to a study of mirror symmetry (since we usually work on the **A** side with bases of pure type), the holomorphic dependence of **B** model correlation functions is difficult to see if calculations are made in a non-holomorphic gauge. So we adopt the second strategy, and abandon pure type in favor of holomorphically varying bases. At the end of the analysis, we can obtain a basis of pure type by simply projecting to the appropriate (p,q) pieces.

At a single point in the moduli space, the **B** model three-point functions

$$B^1_{j-1} : H^1(M,T) \times H^{j-1,d-j+1} \times H^{d-j,j} \to \mathbb{C} \tag{4.4}$$

have a natural description in algebraic geometry coming from "variation of Hodge structure": they describe what is called the "differential of the period map." In fact, identifying $H^1(M,T)$ with the tangent space to the complex structure moduli space at the point corresponding to M, this three-point function describes the $(p-1,q+1)$ part of a derivative (with respect to parameters) of a family of (p,q) forms (taking $p=j-1$ and $q=d-j+1$, say). The Gauss–Manin connection introduced above reproduces this derivative information while preserving holomorphic dependence. The result of a "pure type" differentiation may differ from the Gauss–Manin answer by some terms of lower type, but all such terms vanish after wedging with a $(d-p+1,d-q-1)$ form and integrating (the prescription for calculating the three-point function).

Let us fix a holomorphic vector field ξ in the moduli space in such a way that the directional Gauss–Manin derivative $\nabla_\xi \Omega$ produces a chosen initial basis vector $\alpha_1(z)$ in $H^{d-1,1}$. Then one can show that the following two operations are identical:

i) taking the directional Gauss–Manin derivative ∇_ξ and projecting onto the (p,q) term in the result, for largest q,

ii) taking the operator product with the (chiral, chiral) field α_1 of charge $(1,1)$ corresponding to ξ.

This fact can be established in conformal field theory by the methods of [19] along with the nonrenormalization theorem of [36] which establishes the equality between operator products amongst (chiral, chiral) fields and standard mathematical wedge products.

The operation (i), however, does not respect holomorphicity (as a function of the moduli space coordinate), as we have noted. Holomorphicity requires that we do not project the result onto the term of highest antiholomorphic degree. On the other hand, agreement with the conformal field theory operator product demands that we do. It appears that essentially all correlation functions, though, are insensitive to these additional lower order terms which are responsible for holomorphicity. Hence, by including these terms we gain the benefit of holomorphically varying elements (as we do on the **A** side) without altering the values of correlation functions. Thus, the central assumption of our analysis is that we construct the basis on the **B** side by imposing the same three conditions as on the **A** side *with the replacement of the operator product by the action of the* (*unprojected*) *Gauss–Manin connection.*[20]

That is, we build our basis of the horizontal subspace of $\bigoplus_p H^{p,d-p}(M,\mathbb{C})$ by beginning with $\alpha_0 = \Omega$, and then seeking $\{\alpha_j\}$ such that

1′) $\nabla_{\alpha_1}\alpha_{j-1} = c^{-1}B^1_{j-1}\alpha_j$,

2′) $\langle\alpha_i,\alpha_j\rangle = c\delta_{i+j,d}$, and

3′) $\nabla_{\alpha_1}\alpha_d = 0$,

where c is an appropriately chosen constant (which will correspond to the degree of the mirror variety). Note that B^1_0 is constant, and equal to the degree c, so that $\nabla_{\alpha_1}\alpha_0 = \alpha_1$ provides the link between the directional derivative ∇_{α_1} and the form α_1. We also note that were we to use the projected Gauss–Manin derivative, condition 3′ would be trivial (as is its counterpart condition 3). However, because we use the (unprojected) Gauss–Manin derivative, which does not yield results of pure type, our **B** model conditions are somewhat more stringent than their **A** model counterparts. This will manifest itself in the solutions to these conditions being essentially unique, unlike the case on the **A** side. (At first sight it might appear asymmetric to begin with $\alpha_0 = \Omega$ since on the **A** side we begin with $e_0 = 1$. This is just an artifact of our working in $\mathscr{H}$ rather than in $\mathscr{H}\otimes\mathscr{L}^{-1}$, as we have discussed earlier, in which Ω can be thought of as $\mathbb{1} = \Omega\otimes\Omega^{-1}$. In fact, we will shortly find it convenient to essentially divide by Ω in a similar manner).

To find a basis meeting these conditions it proves convenient to introduce a set of topological homology cycles $\gamma_0,\gamma_1,\ldots,\gamma_d$ spanning the primary horizontal subspace, such that the cup product pairing on the dual cohomology cycles γ^*_μ satisfies

$$(\gamma^*_\mu,\gamma^*_\nu) = \begin{cases} 0 & \text{if } \mu+\nu > d \\ c & \text{if } \mu+\nu = d \end{cases} \tag{4.5}$$

(there is no constraint on the values when $\mu+\nu < d$). We can then express our basis α_i in terms of the γ_μ by writing the "period matrix"

$$P = (P_{i\mu}) = \left(\int_{\gamma_\mu}\alpha_i\right). \tag{4.6}$$

(Indices on matrix elements run from 0 through d). We claim that we can achieve constraints (2′) and (3′) by performing row operations to put this matrix in

[20] A basis consisting of forms of pure type can then be obtained from the basis we construct by simply projecting each basis element to the appropriate (p,q) piece.

upper triangular form with the diagonal entries being all one, that is, achieving the conditions

$$\int_{\gamma_\mu} \alpha_i = \begin{cases} 0 & \text{if } i > \mu \\ 1 & \text{if } i = \mu \end{cases} . \tag{4.7}$$

The row operations we allow include adding one row to a later row, and multiplying a row by an arbitrary holomorphic function of z. (It is clearly necessary to allow this last step, if we are to achieve $\int_{\gamma_i} \alpha_i = 1$.) These row operations effectively alter the basis $\{\alpha_j\}$, but they do preserve the property $\alpha_j \in \mathscr{F}^{d-j}$. Note that the use of holomorphic bundles $\mathscr{F}^{d-j}$ was crucial here, since we must allow arbitrary holomorphic functions as multipliers.

To see that (1′) holds for this new basis is straightforward. Writing

$$\alpha_{j-1} = \gamma_{j-1}^* + \sum_{l=j}^{d} \left(\int_{\gamma_l} \alpha_{j-1} \right) \gamma_l^* , \tag{4.8}$$

we find

$$\nabla_{\alpha_1} \alpha_{j-1} = \sum_{l=j}^{d} \frac{d}{d\alpha_1} \left(\int_{\gamma_l} \alpha_{j-1} \right) \gamma_l^* . \tag{4.9}$$

This is an element of $\mathscr{F}^{d-j}$, and so must be a linear combination of $\alpha_0, \ldots, \alpha_j$. It follows from (4.7) that the coefficient of α_0 in the linear combination should agree with the coefficient of γ_0^* in (4.9), but this is zero. That being the case, the coefficient of α_1 in the linear combination should agree with the coefficient of γ_1^* in (4.9), but this too is zero. Continuing to argue in this way we find that $\nabla_{\alpha_1} \alpha_{j-1}$ must simply be a multiple $f_{j-1}(z) \cdot \alpha_j$. In fact, the multiplier is easily seen to be

$$f_{j-1}(z) = \frac{d}{d\alpha_1} \int_{\gamma_j} \alpha_{j-1} . \tag{4.10}$$

Condition (3′) holds as it translates into the covariant derivative of the last row of the matrix vanishing – this is clearly true as the last row of the matrix is constant. To check condition (2′), first note that because we have preserved the condition $\alpha_j \in \mathscr{F}^{d-j}$, by considering types in the wedge product we find

$$\langle \alpha_i, \alpha_j \rangle = 0 \quad \text{if } i + j < d . \tag{4.11}$$

Thus, we may assume $i + j \geqq d$. We then calculate

$$\langle \alpha_i, \alpha_j \rangle = \sum_{\mu,\nu} \left(\int_{\gamma_\mu} \alpha_i \right) \left(\int_{\gamma_\nu} \alpha_j \right) (\gamma_\mu^*, \gamma_\nu^*) . \tag{4.12}$$

For any term in this last sum which is non-zero, we must have

$$d \leqq i + j \leqq \mu + \nu \leqq d \tag{4.13}$$

(using (4.7) and (4.5)). Thus, all inequalities are equalities, and we find

$$\langle \alpha_i, \alpha_j \rangle = \sum_{\mu,\nu} \delta_{\mu i} \delta_{\nu j} (c \delta_{\mu+\nu,d}) = c \delta_{i+j,d} , \tag{4.14}$$

as required.

(As one final check, we can evaluate the three-point function

$$B^1_{j-1} = \langle \alpha_1 \alpha_{j-1} \alpha_{d-j} \rangle = f_{j-1}(z) \langle \alpha_j, \alpha_{d-j} \rangle = c \cdot f_{j-1}(z) , \tag{4.15}$$

so $f_{j-1}(z) = c^{-1} B^1_{j-1}$ as asserted in (1′).)

Notice that in performing row operations to make the matrix P upper triangular with 1's on the diagonal, the only manipulation which affected the top row divided it by $\int_{\gamma_0} \Omega(z)$, thereby making the $(0,0)$ entry in the new matrix equal to 1 and the $(0,1)$ entry equal to $(\int_{\gamma_1} \Omega(z))/(\int_{\gamma_0} \Omega(z))$. As the derivative of the top row with respect to t is the second row, and since the $(1,1)$ entry is 1, we directly see that in our new basis, $\nabla_{\alpha_1} = \partial_t$ with

$$t = \frac{\int_{\gamma_1} \Omega(z)}{\int_{\gamma_0} \Omega(z)} . \tag{4.16}$$

This is precisely the same coordinate *Ansatz* used in [5] and established in [24] as being mirror to the *integral* generator of $H^2(\widetilde{M})$; we see here that this form of the mirror map *emerges* from our three conditions. Although our conditions on the **A** side do not uniquely specify a basis, as we discuss below, our slightly stronger conditions on the **B** side, combined with monodromy properties, make the basis essentially unique. Since our procedure on the **B** side has picked out the first element of this basis to be the known mirror of an integral generator, we expect that the same is true for the other elements of the **B**-basis, as desired.

Having now satisfied the characteristics of the **A** model basis for the primary vertical subspace of $\bigoplus_p H^{p,p}(\widetilde{M}, \mathbb{Z})$ with the **B** model basis of the primary horizontal subspace of $\bigoplus_p H^{p,d-p}(M, \mathbb{C})$ (under our central assumption discussed above), we now must ask ourselves about the uniqueness of this procedure. The first point to make about uniqueness is this: any basis which satisfies our conditions (1′), (2′) and (3′) must also satisfy (4.7) for *some* choice of homology cycles γ_μ. This can be seen as follows. We start with an arbitrary basis $\gamma_0, \dots, \gamma_d$ of the primary horizontal subspace and form the period matrix (4.6) with respect to that basis. We then perform column operations on this matrix to put it into upper triangular form with 1's on the diagonal, but this time we restrict ourselves to using *constants* as multipliers for the columns. (This has the effect of changing the basis γ_μ, using linear combinations with constant (complex) coefficients. Under such a change, the γ's will remain a basis of the primary subspace of $H^d(M, \mathbb{C})$.) We are aiming for the condition (4.7), but since we have restricted our allowed multipliers it would seem problematic to achieve $\int_{\gamma_i} \alpha_i = 1$.

However, conditions (1′) and (3′) come to our rescue. First, (3′) implies that the bottom row of P is *constant*. Therefore, by suitable constant-coefficient column operations we can put the bottom row in the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix} . \tag{4.17}$$

It then follows from (1′) with $j = d$ that every entry but the last one in the penultimate row is constant. Again applying constant-coefficient column operations (which do not involve the last column) we can achieve for the bottom two rows:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & \star \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} , \tag{4.18}$$

where ★ is an unknown quantity. Continuing in this way row by row produces (4.7).

Although this argument eliminates the apparent arbitrariness of using the condition (4.7) to achieve (1′),(2′) and (3′), it still leaves us with a procedure that is not unique – the starting set of cycles $\{\gamma_\mu\}$ used to produce the basis $\{\alpha_j\}$ is not unique. We can, however, make this choice essentially unique by going to a boundary point in the moduli space of the **B** model. As discussed in [5, 29], the cycles γ_μ have nontrivial monodromy about boundary points in this moduli space. We also know, from the **A** model, that at a large radius boundary point we have the identification of t and $t+1$. Thus, consistency of the mirror map will follow if the monodromy of the γ_μ is ensured to yield $(\int_{\gamma_1}\Omega(z))/(\int_{\gamma_0}\Omega(z)) \to (\int_{\gamma_1}\Omega(z))/(\int_{\gamma_0}\Omega(z))+1$. This is sufficient to almost uniquely fix the cycles and hence our procedure for generating the mirror map, as we shall now show.

On the **A** model side, the physics is the same at $t+1$ as it is at t, and the quantity $q=e^{2\pi i t}$ serves as the natural parameter (near the boundary) on the true moduli space of physical theories. On the **B** model side, our monodromy property effectively means

$$\frac{\int_{\gamma_1}\Omega(z)}{\int_{\gamma_0}\Omega(z)} = \frac{1}{2\pi i}\log z + f(z) \tag{4.19}$$

for some single-valued function $f(z)$: the "t" type parameter is $\int_{\gamma_1}\Omega(z)/\int_{\gamma_0}\Omega(z)$ while the "q" type parameter is the exponential of this. Our directional derivative ∇_{α_1} (which is being identified with the mirror of the operator product with e_1) behaves like $\frac{d}{dt}=q\frac{d}{dq}$ near the large complex structure limit. In particular, since the three-point functions

$$\int q\frac{d}{dq}(\alpha_{j-1}\wedge\alpha_{d-j}) = c^{-1}\cdot f_{j-1}(q) = c^{-1}\cdot q\frac{d}{dq}\int_{\gamma_j}\alpha_{j-1} \tag{4.20}$$

have expansions of the form

$$a_{j-1,0}+a_{j-1,1}\,q+a_{j-1,2}\,q^2+\cdots \tag{4.21}$$

(consisting of a topological term plus quantum corrections), we see that whenever $a_{j-1,0}\neq 0$, the quantity $\frac{d}{dq}\int_{\gamma_j}\alpha_{j-1}$ must have a pole at $q=0$: the leading term in its Laurent expansion will be $c\cdot a_{j-1,0}q^{-1}$. Thus the period $\int_{\gamma_j}\alpha_{j-1}$ will have the form

$$\int_{\gamma_j}\alpha_{j-1} = c\cdot a_{j-1,0}\log q + \text{single-valued function}\,. \tag{4.22}$$

Now we know that the topological terms in these three-point functions cannot vanish, since they give the degree of the variety, which is nonzero. Thus, every entry in the first superdiagonal of the period matrix has a $\log q$ type monodromy. This is a very strong property, called *maximally unipotent* in [29].

In the presence of maximally unipotent monodromy, we need a basis $\gamma_0,\ldots,\gamma_d$ such that the monodromy action takes the form

$$\gamma_\mu \mapsto \gamma_\mu + \sum_{\nu<\mu} m_{\mu\nu}\gamma_\nu \tag{4.23}$$

for some constants $m_{\mu\nu}$. Moreover, our basis should satisfy (4.5); these two properties together fix the γ_μ's up to scalar multiples.

Notice that although our procedure for generating the mirror map and the appropriate basis in the **B** model required that we start with α_0 equal to some holomorphic three form $\Omega(z)$, in reality the particular initial choice of Ω is irrelevant, as we quickly indicated earlier. Directly we see this as our three conditions lead us to rescale α_0 by $1/\int_{\gamma_0}\Omega(z)$. Alternatively, we could rephrase all of our analysis along the lines of Sect. II in which we work in the context of $\mathscr{H}\otimes\mathscr{L}^{-1}$ rather than $\mathscr{H}$. As discussed in that section, the analysis can be phrased as starting with the canonical section $\mathbb{1}$ of $\mathcal{O}\subset\mathscr{H}\otimes\mathscr{L}^{-1}$, thus ensuring that the results do not depend on any initial choice of Ω. This approach is closer, in fact, to our **A** model description because in that setting we choose $e_0=\mathbb{1}$ and, furthermore, because the fibers of $\mathscr{H}\otimes\mathscr{L}^{-1}$ are canonically isomorphic to $H^p(M,\Lambda^pT)$. The latter, as we have discussed, is the precise geometrical description of the (c,c) ring, just as $H^p(M,\Lambda^pT^*)$ is that for the (a,c) ring.

It is worthwhile reemphasizing that the basis elements α_i which we have derived here are generally of mixed type. This is due to our implicit requirement that the basis be holomorphically varying over moduli space. It is straightforward to see that it is only the $(p,d-p)$ part with largest p contained in each α_i that contributes to correlation functions. Thus, if we are willing to sacrifice holomorphic variation we can eliminate the lower order pieces. Such a **B** model basis would more closely match the **A** model analysis. Alternatively, we could modify the **A** model basis to behave more like the holomorphically varying **B** model basis.

There is an added bonus to our procedure beyond naturally generating the mirror map and mirror bases. The fundamental three-point functions Y_j^1 (and their associated instanton expansions) can be directly extracted from the matrix (4.6). This is easily seen by noting that the three-point function Y_j^1 can be expressed as

$$Y_j^1(\alpha_1,\alpha_j,\alpha_{d-j-1})=\int\limits_{M_z}\alpha_{d-j-1}\wedge\nabla_{\alpha_1}\alpha_j\,. \tag{4.24}$$

Substituting in the basis which puts P into upper triangular form, we directly calculate that

$$Y_j^1=c\cdot\partial_t(P_{j,j+1})\,. \tag{4.25}$$

Let us reemphasize that these three-point functions, although calculated on M_z, are now to be thought of as three-point functions on $\widetilde{M}_{t(z)}$. Since we have carefully extracted the mirror map and identified the bases of cohomology on both sides, (4.25) can directly be interpreted as an instanton sum as in (4.6).

We apply this formalism to specific examples in the next subsection.

4.2. Holomorphic Picard–Fuchs Equation and Three-Point Functions. We now employ the discussion of the last subsection to calculate all of the three-point functions and their associated instanton sums for the independent set of Yukawa couplings Y_j^1 for the mirror manifolds built on the $M_\psi^{(d)}$ introduced in Sect. III.

In practice, we carry out the procedure of the last subsection as follows. We have $M_\psi^{(d)}$described by Eq. (3.5), where we are using the coordinate $z=\psi^{-(d+2)}$ on the moduli space. One can directly check from the Picard–Fuchs equation (3.11) that the only point in the moduli space with maximally unipotent monodromy is the point $z = 0$. We adapt the methods of [6] to do our calculation at that point.

We take as our initial basis of the horizontal subspace of $\mathscr{F}^0$ the d-forms

$$\alpha_0 = \Omega, \qquad \alpha_1 = z\partial_z\Omega, \ldots, \qquad \alpha_d = (z\partial_z)^d\Omega\,. \tag{4.26}$$

The differentiation operator $z\partial_z$ acts on this basis via a matrix of the form

$$A(z) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ B_0(z) & B_1(z) & \ldots & \ldots & B_d(z) \end{pmatrix}, \tag{4.27}$$

where the $B_j(z)$ are determined from the Picard–Fuchs equation (3.11) as follows [6]. Write the Picard–Fuchs operator in the form

$$z\prod_{j=1}^{d+1}\left(z\partial_z + \frac{j}{d+1}\right) - (z\partial_z)^{d+1} = (z-1)(z\partial_z)^{d+1} + z\sum_{j=0}^{d} c_j(z\partial_z)^j\,, \tag{4.28}$$

and divide by $z-1$ to produce the operator

$$(z\partial_z)^{d+1} - \sum_{j=0}^{d} c_j\frac{z}{1-z}(z\partial_z)^j\,. \tag{4.29}$$

Then the entries in the bottom row of the matrix $A(z)$ are the quantities $B_j(z) = c_j\frac{z}{1-z}$. Note that $B_j(0) = 0$, so that the methods of [6] can be directly used to solve the equation.

For any homology cycle γ in the primary horizontal subspace, the vector

$$\varpi(z) = \begin{pmatrix} \int_\gamma \Omega \\ \int_\gamma z\partial_z\Omega \\ \vdots \\ \int_\gamma (z\partial_z)^d\Omega \end{pmatrix} \tag{4.30}$$

is a solution to the matrix equation

$$z\partial_z\varpi(z) = A(z)\varpi(z)\,. \tag{4.31}$$

Most of these solutions are multiple-valued; the multiple-valuedness can be accounted for in advance as follows. Our desired basis of homology cycles $\gamma_0, \ldots, \gamma_d$ will have the property that $\int_{\gamma_0}\Omega$ is single-valued, and

$$2\pi i\frac{\int_{\gamma_j}\Omega}{\int_{\gamma_{j-1}}\Omega} = \log z + \text{single valued function}\,. \tag{4.32}$$

We take the corresponding vectors $\varpi_j(z)$ which are solutions to (4.31) and arrange them as columns in a matrix $\Phi(z)$. This matrix of multiple-valued functions satisfies the equation $z\partial_z\Phi(z) = A(z)\Phi(z)$. In addition, there is a matrix $S(z)$ with single-valued entries such that

$$\Phi(z) = S(z) \cdot z^{A(0)}, \tag{4.33}$$

where $z^{A(0)}$ denotes $e^{(\log z)A(0)} = I + (\log z)A(0) + \frac{1}{2!}(\log z)^2 A(0)^2 + \cdots$. The equation satisfied by $S(z)$ is

$$z\partial_z S(z) + S(z) \cdot A(0) = A(z)S(z) \tag{4.34}$$

(see [6]).

In our case, the matrix $A(0)$ takes a particularly simple form

$$A(0) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}, \tag{4.35}$$

which leads immediately to

$$z^{A(0)} = \begin{pmatrix} 1 & \log z & \frac{1}{2!}(\log z)^2 & \cdots & \frac{1}{d!}(\log z)^d \\ & 1 & \log z & \cdots & \frac{1}{(d-1)!}(\log z)^{d-1} \\ & & 1 & \ddots & \vdots \\ & & & \ddots & \log z \\ & & & & 1 \end{pmatrix}. \tag{4.36}$$

Also thanks to the special form of $A(0)$, Eq. (4.34) can be written as

$$z\partial_z\sigma_j(z) + \sigma_{j-1}(z) = A(z)\sigma_j(z), \tag{4.37}$$

where $\sigma_0(z), \ldots, \sigma_d(z)$ are the columns of $S(z)$ (setting $\sigma_{-1}(z) = 0$). Solutions to Eq. (4.37) can then be found by power series techniques.

The next step is to put the solution matrix $\Phi(z)$ into upper triangular form with 1's on the diagonal by means of row operations. Since $z^{A(0)}$ is upper triangular with 1's on the diagonal, it suffices to put $S(z)$ into upper triangular form with 1's on the diagonal. This is a straightforward manipulation with power series, and produces a matrix $\widetilde{S}(z)$. We then have

$$P = \widetilde{S}(z)z^{A(0)}, \tag{4.38}$$

where P is the period matrix (4.6).

Using (4.36) and (4.38), we deduce that $P_{j,j+1} = \log z + \widetilde{S}_{j,j+1}$. (Our matrix indices still run from 0 through d.) Thus, the Yukawa coupling is given by (4.25),

$$Y_j^1 = c \cdot (1 + z\partial_z \widetilde{S}_{j,j+1}) \cdot z\partial_z t\,, \tag{4.39}$$

where c is the degree. (The factor of $z\partial_z t$ is present to change from $z\partial_z$ gauge to ∂_t gauge.) Since $Y_0^1 = c$, we can solve for the change of gauge

$$z\partial_z t = \frac{1}{1 + z\partial_z \widetilde{S}_{0,1}} \tag{4.40}$$

and find that

$$Y_j^1 = c \cdot \frac{1 + z\partial_z \widetilde{S}_{j,j+1}}{1 + z\partial_z \widetilde{S}_{0,1}}\,. \tag{4.41}$$

This is then expressed as a power series in q; the results of these computations are displayed[21] in Tables 2 and 3 (which cover the cases $4 \leqq d \leqq 10$).

4.3. Factorization and the Other Yukawa Couplings. Armed with the Yukawa couplings Y_j^1, we can give a second expression for the d-point functions which were calculated in Sect. III, by using the factorization rules. We first calculate

$$\begin{aligned}\kappa_{\alpha\alpha\cdots\alpha} &= \int \Omega \wedge D_\alpha \cdots D_\alpha \Omega = \langle \mathcal{O}^{(1)} \cdots \mathcal{O}^{(1)} \rangle \\ &= C^{(1,1)} \langle \mathcal{O}^{(2)} \mathcal{O}^{(1)} \cdots \mathcal{O}^{(1)} \rangle = C^{(1,1)} C^{(2,1)} \langle \mathcal{O}^{(3)} \mathcal{O}^{(1)} \cdots \mathcal{O}^{(1)} \rangle \\ &= \cdots = C^{(1,1)} \cdots C^{(d-2,1)} \langle \mathcal{O}^{(d-1)} \mathcal{O}^{(1)} \rangle\,. \end{aligned} \tag{4.42}$$

As pointed out in Eq. (3.17), we have $C^{(i,j)} = c^{-1} Y_j^i$. Using this, and the relation $\langle \mathcal{O}^{(d-1)} \mathcal{O}^{(1)} \rangle = c$ we find

$$\kappa_{\alpha\alpha\cdots\alpha} = (c^{-1})^{d-2} Y_1^1 Y_2^1 \cdots Y_{d-2}^1\,. \tag{4.43}$$

The d-point function can then be calculated from the three-point functions given in Tables 2 and 3; when one does so, one finds precisely the same series for d-point functions as given in Table 1. This remarkably delicate factorization property of the d-point function power series provides strong evidence that we have not only correctly found the coordinates to use in the mirror map, but we have found the correct bases for the entire horizontal subspace which maps to the integral, topological basis of the vertical subspace under mirror symmetry.

The three-point functions Y_j^1 can also be used to generate other Yukawa couplings Y_j^i with $j \neq 1$, using formula (3.22). We have explicitly calculated these, and displayed the answers in Tables 2 and 3 (for $4 \leqq d \leqq 10$) along with the Y_j^1's calculated previously.

[21] The tables express the couplings as series in $q^l/(1-q^l)$, from which the power series expansions themselves are easily derived.

Table 2. Three-point functions in dimension d, $4 \leqq d \leqq 8$.

d	Three-point functions
4	$Y_1^1 = 6 + 60480\frac{q}{1-q} + 1763536320\frac{q^2}{1-q^2} + 56296406496960\frac{q^3}{1-q^3} + 1883452671845660160\frac{q^4}{1-q^4} + 6477940390922054964000\frac{q^5}{1-q^5} \cdots$
5	$Y_1^1 = 7 + 1009792\frac{q}{1-q} + 488959144352\frac{q^2}{1-q^2} + 274758045709320936\frac{q^3}{1-q^3} + 166051192150334178451456\frac{q^4}{1-q^4} + \cdots$
5	$Y_2^1 = 7 + 1707797\frac{q}{1-q} + 1021575491286\frac{q^2}{1-q^2} + 667645611326779470\frac{q^3}{1-q^3} + 454542525929947966588896\frac{q^4}{1-q^4} + \cdots$
6	$Y_1^1 = 8 + 15984640\frac{q}{1-q} + 133588638826496\frac{q^2}{1-q^2} + 1386812286427872761856\frac{q^3}{1-q^3} + 16010798260253954110394728448\frac{q^4}{1-q^4} + \cdots$
6	$Y_2^1 = 8 + 37502976\frac{q}{1-q} + 448681408315392\frac{q^2}{1-q^2} + 6002251230562024144896\frac{q^3}{1-q^3} + 84488476029298653829523177472\frac{q^4}{1-q^4} + \cdots$
6	$Y_2^2 = 8 + 59021312\frac{q}{1-q} + 821654025830400\frac{q^2}{1-q^2} + 12197109744970010814464\frac{q^3}{1-q^3} + 186083410628492378226388631552\frac{q^4}{1-q^4} + \cdots$
7	$Y_1^1 = 9 + 253490796\frac{q}{1-q} + 39031273362637440\frac{q^2}{1-q^2} + 8078045888048061054330324\frac{q^3}{1-q^3} + 1935263133090844196814494099500032\frac{q^4}{1-q^4} + \cdots$
7	$Y_2^1 = 9 + 763954092\frac{q}{1-q} + 187554590257349088\frac{q^2}{1-q^2} + 53621695689211084188650940\frac{q^3}{1-q^3} + 16467077347090342394985001860453504\frac{q^4}{1-q^4} + \cdots$
7	$Y_3^1 = 9 + 1069047153\frac{q}{1-q} + 312074852318965368\frac{q^2}{1-q^2} + 101447807418804760300649304\frac{q^3}{1-q^3} + 34554976338508396442154650827251216\frac{q^4}{1-q^4} + \cdots$
7	$Y_2^2 = 9 + 1579510449\frac{q}{1-q} + 506855012110118424\frac{q^2}{1-q^2} + 174633921378662035929052320\frac{q^3}{1-q^3} + 62036347648424671947435351078921912\frac{q^4}{1-q^4} + \cdots$
8	$Y_1^1 = 10 + 4120776000\frac{q}{1-q} + 12607965435718224000\frac{q^2}{1-q^2} + 56689974104916623862439224000\frac{q^3}{1-q^3} + \cdots$
8	$Y_2^1 = 10 + 15274952000\frac{q}{1-q} + 80684596772238448000\frac{q^2}{1-q^2} + 524473167338866432254165048000\frac{q^3}{1-q^3} + \cdots$
8	$Y_3^1 = 10 + 27768048000\frac{q}{1-q} + 200581960800610752000\frac{q^2}{1-q^2} + 1639883435802047356497671952000\frac{q^3}{1-q^3} + \cdots$
8	$Y_2^2 = 10 + 38922224000\frac{q}{1-q} + 295035175517918176000\frac{q^2}{1-q^2} + 2467449594491156931046837776000\frac{q^3}{1-q^3} + \cdots$
8	$Y_3^2 = 10 + 51415320000\frac{q}{1-q} + 444475303469701680000\frac{q^2}{1-q^2} + 4089048226644406809222184680000\frac{q^3}{1-q^3} + \cdots$

Table 3. Three-point functions in dimension d, $d = 9, 10$.

d	Three-point functions
9	$Y_1^1 = 11 + 69407571816\frac{q}{1-q} + 4565325719860021608624\frac{q^2}{1-q^2} + 486831084305817727642305709925784\frac{q^3}{1-q^3} + \cdots$
9	$Y_2^1 = 11 + 307393401172\frac{q}{1-q} + 37005001823802188657624\frac{q^2}{1-q^2} + 5868069805933786797109659387704124\frac{q^3}{1-q^3} + \cdots$
9	$Y_3^1 = 11 + 695221679878\frac{q}{1-q} + 127922335050535174614916\frac{q^2}{1-q^2} + 27500598436953801920237040452936322\frac{q^3}{1-q^3} + \cdots$
9	$Y_4^1 = 11 + 905702054829\frac{q}{1-q} + 193693669320390878077186\frac{q^2}{1-q^2} + 46631251296278131748940806631976872\frac{q^3}{1-q^3} + \cdots$
9	$Y_2^2 = 11 + 933207509234\frac{q}{1-q} + 173901546566279203106468\frac{q^2}{1-q^2} + 37310808146135703367927046095148086\frac{q^3}{1-q^3} + \cdots$
9	$Y_3^2 = 11 + 1531516162891\frac{q}{1-q} + 364629304647788940660824\frac{q^2}{1-q^2} + 93483169112936030274354586771249766\frac{q^3}{1-q^3} + \cdots$
9	$Y_3^3 = 11 + 1919344441597\frac{q}{1-q} + 498705676383823268404990\frac{q^2}{1-q^2} + 135578347091808508663450919122287332\frac{q^3}{1-q^3} + \cdots$
10	$Y_1^1 = 12 + 1217507106816\frac{q}{1-q} + 1861791822397620935737344\frac{q^2}{1-q^2} + 5128660247833325056966281364761206784\frac{q^3}{1-q^3} + \cdots$
10	$Y_2^1 = 12 + 6306655500288\frac{q}{1-q} + 18415607624138339954786304\frac{q^2}{1-q^2} + 76684904282498644296301812327894878208\frac{q^3}{1-q^3} + \cdots$
10	$Y_3^1 = 12 + 17225362851840\frac{q}{1-q} + 83885220561474498867757056\frac{q^2}{1-q^2} + 499293309557937087326375598615454660608\frac{q^3}{1-q^3} + \cdots$
10	$Y_4^1 = 12 + 28015971489792\frac{q}{1-q} + 179982840924749584358866944\frac{q^2}{1-q^2} + 1316838352593364835173763931348327770112\frac{q^3}{1-q^3} + \cdots$
10	$Y_2^2 = 12 + 22314511245312\frac{q}{1-q} + 107227899142191919158312960\frac{q^2}{1-q^2} + 627029137534107622315490005905787576320\frac{q^3}{1-q^3} + \cdots$
10	$Y_3^2 = 12 + 44023827234816\frac{q}{1-q} + 297755098999730079369412608\frac{q^2}{1-q^2} + 2228162463490181234075299799539491962880\frac{q^3}{1-q^3} + \cdots$
10	$Y_4^2 = 12 + 54814435872768\frac{q}{1-q} + 417950364467570984815214592\frac{q^2}{1-q^2} + 3418037850981202953204910537008894799872\frac{q^3}{1-q^3} + \cdots$
10	$Y_3^3 = 12 + 65733143224320\frac{q}{1-q} + 527556832251612742800359424\frac{q^2}{1-q^2} + 4445632319073239456054488536360314019840\frac{q^3}{1-q^3} + \cdots$

5. Mathematical Interpretation and Comparison of Instanton Sums

In the case of $d=3$, the use of the series expansion of $\kappa_{\gamma_1\gamma_2\gamma_3}$ to predict the numbers of rational curves on the mirror is by now well known [37, 33]. For $d>3$, in addition to the existence of more than one kind of Yukawa coupling, there is one other important new consideration. Holomorphic curves are no longer generically isolated as they are for $d=3$, but rather come in continuous families. Thus, the integers which arise in the series expansions of Yukawa couplings no longer count numbers of curves *per se*. In this section we will give a mathematical interpretation of these integers, describe what can be calculated by directly using that mathematical interpretation, and compare with our predictions from the mirror analysis.

We need to describe the instanton contributions to our three-point functions, regarded as correlation functions in the **A** model of [25]. We start with three of our chosen basis vectors $e_i \in H^{i,i}$, $e_j \in H^{j,j}$ and $e_{d-i-j} \in H^{d-i-j,d-i-j}$, and fix three points $P_1=0$, $P_2=1$ and $P_3=\infty$ on the worldsheet $\Sigma = \mathbb{CP}^1$. We choose explicit complex submanifolds H_i, H_j and H_{d-i-j} of complex codimension i, j, and $d-i-j$, respectively, which are Poincaré dual to the cohomology classes. We form local operators $\mathcal{O}^{(i)}(P_1)$, $\mathcal{O}^{(j)}(P_2)$, and $\mathcal{O}^{(d-i-j)}(P_3)$ which have delta function support on maps $\Phi : \Sigma \to \widetilde{M}$ for which $\Phi(P_1)\in H_i$ (or $\Phi(P_2)\in H_j$, or $\Phi(P_3)\in H_{d-i-j}$ in the other cases). The three-point function $\langle \mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(d-i-j)}\rangle$ can be written as a sum over cohomology classes of maps Φ. We index those classes by specifying η, the class of the image of the map, and m, the degree of the map. We perturb the complex structure on $\widetilde{M}$ to a generic almost-complex structure, and we let $\Phi_{m,\eta}$ be a typical map in its class. Then the three-point function should be written as [37]:

$$\langle \mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(d-i-j)}\rangle = \sum_{m,\eta} e^{\int_\Sigma \Phi^*_{m,\eta}(K)}\,\#(\mathcal{G}^{(i,j)}_{m,\eta})\,, \tag{5.1}$$

where

$$\mathcal{G}^{(i,j)}_{m,\eta} = \{\text{pseudo-holomorphic maps } \Phi : \Sigma \to \widetilde{M} \text{ of degree } m \text{ and class } \eta \\ \text{such that } \Phi(P_1)\in H_i,\ \Phi(P_2)\in H_j,\ \Phi(P_3)\in H_{d-i-j}\}\,. \tag{5.2}$$

(The coefficients in (5.1) are closely related to the "Gromov–Witten" invariants of $\widetilde{M}$.) As it stands, formula (5.1) is somewhat problematic, since the moduli spaces of holomorphic maps $\Sigma\to\widetilde{M}$ which are not one-to-one (i.e., those with $m\neq 1$) fail to have the expected dimension; thus, the set $\mathcal{G}^{(i,j)}_{m,\eta}$ of maps satisfying the stated conditions is not finite when $m>1$. There is a cure for this, however, in the form of a "multiple cover formula" which for threefolds was conjectured in [5] and proven at the level of physical rigor in [15]. We extend this formula to the present context in Appendix B. Using it, we can rewrite our expression using degree 1 maps only:

$$\langle \mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(d-i-j)}\rangle = \langle e_i e_j e_{d-i-j}\rangle + \sum_\eta \frac{e^{\int_\Sigma \Phi^*_{1,\eta}(K)}}{1-e^{\int_\Sigma \Phi^*_{1,\eta}(K)}}\,\#(\mathcal{G}^{(i,j)}_{1,\eta})\,. \tag{5.3}$$

In the one-parameter case the homology classes will be labeled by $\mathbb{Z}_+$ and we can rewrite (5.3) as

$$\langle \mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(d-i-j)}\rangle = \langle e_i e_j e_{d-i-j}\rangle + \sum_{l>0} \frac{q^l}{1-q^l}\, n^i_j(l)\,, \tag{5.4}$$

where $q = e^K$ is the single parameter, l is the degree of a homology class η_l, and[22]

$$n^l_j(l) = \#(\mathscr{G}^{(i,j)}_{1,\eta_l}) . \tag{5.5}$$

(It is necessary to separate out the degree 0 "constant" maps when writing (5.3), since they are not included in the multiple cover analysis, but lead rather to the "topological" term $\langle e_i e_j e_{d-i-j} \rangle$.)

Equation (5.4) provides the basis of comparison between the instanton expansions of **A** model correlation functions, and the series expansions we have found for the **B** model correlation functions on the mirror manifold. In particular, if we write the **B** model series expansions from the previous section in the form of (5.4) (as was done in Tables 2 and 3), we can read off the predicted values for the invariants $n^l_j(l)$. It is gratifying to observe that all calculated coefficients are in fact integers.

The actual calculation of the numbers $n^l_j(l)$ using classical techniques in algebraic geometry – necessary if we wish to check the predictions – is a challenging task. There are two principal difficulties. First, the moduli spaces $\mathscr{G}^{(i,j)}_{1,\eta_l}$ may fail to have dimension zero (even though $m = 1$) for a particular choice of complex structure on $\widetilde{M}$. Zero-dimensional moduli spaces can sometimes be obtained by perturbing the original complex structure, but in general it is necessary to pass to a nearby almost-complex structure in order to guarantee the correct dimension [38, 39, 40]. Doing so allows the number $n^l_j(l)$ to be calculated in principle, but in practice it is not known how to carry out the calculation in terms of the almost-complex structure. Techniques for calculating $n^l_j(l)$ directly on $\widetilde{M}$ (even when $\mathscr{G}^{(i,j)}_{1,\eta_l}$ has the wrong dimension) have been pioneered by Katz [41], but these techniques do not yet apply in complete generality. In fact, satisfactory *definitions* of the numbers $n^l_j(l)$ (which stay purely within algebraic geometry) are not yet known.

The second difficulty occurs even when no perturbation of complex structure is necessary. Simply put, the evaluation of the numbers $n^l_j(l)$ using the classical tools of algebraic geometry is a very hard task, and effective methods are not known except in the simplest cases.[23] To calculate $n^l_j(l)$, one first describes $\mathscr{G}^{(i,j)}_{1,\eta_l}$ as an intersection of certain subvarieties in a moduli space of curves. (This is the translation of (5.2) into algebraic geometry.) The number of points in the space should then be found using the standard techniques of algebraic intersection theory. However, those techniques require a compact moduli space, and the moduli space at hand is not compact. It can be compactified by adjoining points corresponding to certain "limiting" curves of other types – the resulting compact space is known as a Hilbert scheme. The delicate part of the computation is to properly account for the portion of the answer which comes from the limiting curves, and this requires knowing the structure of those curves in detail. As l increases, the types of limiting curves which must be considered grow more and more complex.

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[22] The notation $n^l_j(l)$ is chosen to match that of [9].

[23] See note added.

For $l = 1$ and 2, these difficulties can be overcome, and Katz [9] has checked the predictions in Tables 2 and 3 for $l = 1$ and 2 (that is, the coefficients of $\frac{q}{1-q}$ and $\frac{q^2}{1-q^2}$), obtaining agreement in each case.[24]

The associativity relations (3.22) now imply some relations among the numbers $n^i_j(l)$ which had not been observed in the mathematics literature. It is likely that the geometric explanation of these relations in terms of four-point functions which has been put forward by Witten [12] can be used to give a complete mathematical proof of these new relations. (The subtleties in that proof would again involve issues of compactifying moduli spaces appropriately.) Katz [9] has directly proved these relations in the case $l = 1$.

6. Conclusions

Our focus in this paper has been an analysis of some aspects of mirror symmetry for Calabi–Yau manifolds whose complex dimension is greater than three, the previously studied case. We have found that a number of new issues arise. First, the geometric constraints characterizing the associated complex structure and Kähler moduli spaces differ from the threefold case, in which they have usually been referred to as the "constraints of special geometry." The analogue of special geometry in the higher dimensional case (for one-parameter families) can be summarized by a general constraint valid for all dimensions including three – Eq. (2.24) – but the explicit evaluation of this constraint in terms of the Riemann curvature tensor and the Yukawa couplings is certainly sensitive to the dimension. We have explicitly worked this out for one-parameter examples in the case of dimension four and five. Second, whereas there is one type of Yukawa coupling (in each of the **A** and **B** models) in the case of dimension three, the number of Yukawa couplings rapidly grows as a function of the dimension. By making use of the associativity of the operator product algebra, we identified a fundamental subset of couplings on which all others are functionally dependent. Third, whereas the exploitation of mirror symmetry in the case of threefolds only requires understanding a preferred set of moduli space coordinates ("special coordinates"), in higher dimension we require more structure: a preferred basis of (part of) the cohomology ring. We have presented an efficient algorithm for generating such bases (in one-parameter models), making use of the Gauss–Manin connection. Furthermore, we have shown that our procedure naturally reproduces the special coordinates discussed in the three dimensional setting as well as giving a calculationally tractable procedure for generating the independent set of Yukawa couplings. Fourth, in dimension three, rational curves on a Calabi–Yau manifold are generically isolated whereas in higher dimension they come in families. This requires a reinterpretation of the instanton expansion of Yukawa couplings in higher dimension in terms of the characteristic classes of the parameter spaces of rational curves. We have done this and explicitly carried out such calculations for one-parameter Calabi–Yau manifolds of complex dimension at most ten. In the limited number of cases in which such characteristic classes can be effectively calculated using conventional mathematical methods, we find agreement. The calculational power of mirror symmetry is thereby once again affirmed.

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[24] Very recently Ellingsrud and Strømme have also verified some of our predictions for $l = 3$ [11].

Appendix A. Some Remarks on Covariant Derivatives

The analysis of Sect. II involved several times a need to differentiate sections of (non-holomorphic) bundles of the form $V = \mathscr{H} \otimes \mathscr{L}^{-1} \otimes (T^*)^p \otimes (\overline{T}^*)^q \otimes \overline{\mathscr{H}} \otimes \overline{\mathscr{L}}^{-1}$, where T is the holomorphic tangent bundle of the moduli space $\mathscr{M}$. Since each factor occurring in this bundle is itself either holomorphic or antiholomorphic there is a natural covariant derivative we can define on the tensor product. Namely, on each component factor we define the complex metric connection and we extend this to a covariant derivative on the product by the Leibniz rule. More specifically, if Q is a holomorphic bundle with Hermitian fiber metric $h_{a\bar{b}}$, there is a unique connection which is compatible with the metric, i.e.

$$d\langle\bar{s}|t\rangle = \langle\overline{Ds}|t\rangle + \langle\bar{s}|Dt\rangle\,, \tag{A.1}$$

where s and t are local smooth sections of Q and the inner product $\langle\,|\,\rangle$ is that given by h, and which agrees with ordinary $\bar{\partial}$ differentiation in the $(0,1)$ direction. The connection ω satisfying these conditions can be written

$$\omega = (\partial h)h^{-1}\,. \tag{A.2}$$

Clearly this construction also works for an antiholomorphic bundle by demanding agreement with partial differentiation in the $(1,0)$ direction. (One must take the complex conjugate of the formulas.) Quite generally, if we have connections on each of n bundles $A_1,\ldots,A_n$, then the sum of these connections provides a connection on the product bundle $A_1 \otimes \cdots \otimes A_n$. Hence, by using the complex metric connections or their complex conjugates on each individual factor, their sum is a connection on V. Of course, this connection, while compatible with the metric, no longer agrees with partial differentiation in either the $(1,0)$ or $(0,1)$ directions.

It proves instructive to explicitly write out one consequence of metric compatibility. Let s and t be sections of V. Metric compatibility implies

$$\begin{aligned} d\langle\bar{s}|t\rangle &= \partial_\alpha\langle\bar{s}|t\rangle dz^\alpha + \bar{\partial}_{\bar{\alpha}}\langle\bar{s}|t\rangle d\bar{z}^{\bar{\alpha}} = \langle\overline{Ds}|t\rangle + \langle\bar{s}|Dt\rangle \\ &= \langle\overline{(D^{1,0}+D^{0,1})s}|t\rangle + \langle\bar{s}|(D^{1,0}+D^{0,1})\,t\rangle\,. \end{aligned} \tag{A.3}$$

Now, we can decompose the equality above by type to get

$$\partial_\alpha\langle\bar{s}|t\rangle = \langle\overline{D^{0,1}_{\bar{\alpha}}s}|t\rangle + \langle\bar{s}|D^{1,0}_\alpha t\rangle \tag{A.4}$$

and its complex conjugate. From (A.4) we then have

$$\partial_\alpha\langle\bar{s}|t\rangle = \langle D^{1,0}_\alpha\bar{s}|t\rangle + \langle\bar{s}|D^{1,0}_\alpha t\rangle\,, \tag{A.5}$$

where $D^{1,0}$ in the first term on the right-hand side is a covariant derivative acting on sections of $\overline{V}$.

Implicit in the above discussion is that the symbol $\langle\,|\,\rangle$ is the inner product on V. More generally, we can replace this inner product on V by an inner product just on $\mathscr{H}\otimes\mathscr{L}$, $\langle\,|\,\rangle_{\mathscr{H}\otimes\mathscr{L}}$. Then, $\langle\bar{s}|t\rangle_{\mathscr{H}\otimes\mathscr{L}}$ is a section of $(T^*)^p\otimes\overline{(T^*)^q}$ and we similarly have

$$D^{1,0}_\alpha\langle\bar{s}|t\rangle_{\mathscr{H}\otimes\mathscr{L}} = \langle D^{1,0}_\alpha\bar{s}|t\rangle + \langle\bar{s}|D^{1,0}_\alpha t\rangle\,. \tag{A.6}$$

It is important to bear in mind that in (A.6) the meaning of the derivative is determined by the object on which it acts. Explicitly, the $D^{1,0}$ on the left-hand side acts on sections of $(T^*)^p \otimes \overline{(T^*)^q}$; the first on the right-hand side acts on sections of $\overline{V}$ while the last acts on sections of V. We have repeatedly made use of (A.6) in Sect. II.

Appendix B. The Multiple Cover Formula in Higher Dimension

Let X be a Calabi–Yau d-fold. Our derivation of the multiple cover formula roughly follows Sect. 4 of [15], but there are some new twists in higher dimension. We will make some general position assumptions during the derivation which seem to us quite reasonable, but which lack a complete justification at present. As with the earlier formula [15], though, the success of the predictions made with this method gives us confidence that the formula is indeed correct.

Let $\mathscr{H}$ be a fixed component of the Hilbert scheme of X, which parametrizes a family $f : \mathscr{C} \to \mathscr{H}$ of rational curves on X with the property that $T_X|_C = \mathcal{O}_C(2) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C^{\oplus(d-3)}$ for the general curve C in the family. Let $M_m^{\mathscr{H}}$ be the moduli space for holomorphic maps $\Phi : \mathbb{CP}^1 \to X$ which are degree m covers of the rational curves parametrized by $\mathscr{H}$. We wish to evaluate the contributions to three-point functions $\langle \mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(d-i-j)}\rangle$ made by $M_m^{\mathscr{H}}$. Since $M_m^{\mathscr{H}}$ has the wrong dimension, we will need to calculate the top Chern class of a certain bundle.

We would like to describe the family $\mathscr{C}$ in terms of the sheaf $f_*\mathcal{O}_{\mathscr{C}}(1)$. We cannot do so directly since the map $f : \mathscr{C} \to \mathscr{H}$ may fail to have a section. However, we can always find a generically finite map $g : \widetilde{\mathscr{H}} \to \mathscr{H}$ such that the pulled-back family $\widetilde{\mathscr{C}}$ has a section. (Take $\widetilde{\mathscr{H}}$ to be the zero-locus of a multi-section of f, for example.) We shall do so, and eventually arrive at the conclusion that the contribution to the three-point functions are independent of m; this latter statement will then hold for $\mathscr{H}$ as well as $\widetilde{\mathscr{H}}$.

We thus replace $\mathscr{H}$ by $\widetilde{\mathscr{H}}$ and assume that $f : \mathscr{C} \to \mathscr{H}$ has a section so that $\mathscr{C}$ can be described in terms of $\mathscr{V} := f_*\mathcal{O}_{\mathscr{C}}(1)$ as follows: over a Zariski-open subset $\mathscr{H}_0 \subset \mathscr{H}$, this sheaf restricts to a locally free sheaf $\mathscr{V}_0$ of rank two, and the $\mathbb{CP}^1$-bundle $\mathbb{P}(\mathscr{V}_0)$ is birational to $\mathscr{C}$. In fact, by blowing up $\mathscr{H}$, we may assume that $\mathscr{V}_0$ has a locally free extension over all of $\mathscr{H}$. We shall do this, and shall also replace $\mathscr{C}$ by the projectivization of that locally free extension. After making those birational modifications of our data, we arrive at the situation in which $\mathscr{V} = f_*\mathcal{O}_{\mathscr{C}}(1)$ is locally free, and $\mathscr{C} = \mathbb{P}(\mathscr{V})$. The modifications we have made can be expected to be located outside of the subspace in which the calculation of the three-point functions is localized. There is a natural map $\imath : \mathscr{C} \to X$, and we treat the pullback $\imath^*(T_X)$ of T_X to $\mathscr{C}$ as coinciding with $\mathcal{O}_{\mathscr{C}}(2) \oplus \mathcal{O}_{\mathscr{C}}(-1) \oplus \mathcal{O}_{\mathscr{C}}(-1) \oplus \mathcal{O}_{\mathscr{C}}^{\oplus(d-3)}$. This also holds generically, and the places where it fails can be expected to be located outside of the crucial subspace. (These are our general position assumptions.)

To describe a point in $M_m^{\mathscr{H}}$, we must specify the image curve, and specify a ratio of two relatively prime degree m polynomials to define the map. We compactify the moduli space using graphs of maps, motivated by the work of Gromov [38] (cf. also [15]). To construct the graph compactification, we first extend from pairs of relatively prime polynomials to arbitrary pairs of polynomials, obtaining the space $\overline{M} := \mathbb{P}(\mathrm{Sym}^m\mathscr{V} \oplus \mathrm{Sym}^m\mathscr{V})$. The graphs of the maps can then be naturally taken in the space

$$Z := \mathbb{CP}^1 \times (\mathscr{C} \times_{\mathscr{H}} \overline{M}) \tag{B.1}$$

with the closure $\overline{\Gamma}$ of the universal graph Γ described by the equation

$$\frac{s}{t} = \frac{\sum a_i x^i y^{m-i}}{\sum b_i x^i y^{m-i}}, \tag{B.2}$$

or equivalently

$$t \sum a_i x^i y^{m-i} - s \sum b_i x^i y^{m-i} = 0, \tag{B.3}$$

where $[x, y]$ are homogeneous coordinates on $\mathbb{CP}^1$, $[s, t]$ are homogeneous coordinates on a fiber C of $\mathscr{C}$, and $[a_0, \ldots, a_m, b_0, \ldots, b_m]$ give the coordinates in a fiber of $\overline{M} \to \mathscr{H}$. Counting degrees in (B.3), it follows that the line bundle associated to $\overline{\Gamma}$ can be written as

$$\mathcal{O}(\overline{\Gamma}) = \mu^*(\mathcal{O}_{\mathbb{CP}^1}(m)) \otimes \nu^*(\mathcal{O}_{\mathscr{C}}(1)) \otimes \pi^*(\mathcal{O}_{\overline{M}}(1)), \tag{B.4}$$

where $\mu : Z \to \mathbb{CP}^1$, $\nu : Z \to \mathscr{C}$ and $\pi : Z \to \overline{M}$ are the natural projection maps.

The tangent bundle T_X determines a bundle $\mathscr{E} := (\iota \circ \nu)^*(T_X)$ on Z, which restricts to the bundle $\mathscr{E}|_{\overline{\Gamma}}$ on the graph-closure $\overline{\Gamma}$. Following the methods developed in [42] and [15], we must calculate the top Chern class of the bundle $R^1\pi_*(\mathscr{E}|_{\overline{\Gamma}})$ whose fibers are the obstruction groups for the moduli problem. We will do this by using the short exact sequence

$$0 \to \mathscr{E}(-\overline{\Gamma}) \to \mathscr{E} \to \mathscr{E}|_{\overline{\Gamma}} \to 0. \tag{B.5}$$

It is convenient to write $\mathscr{E}(-\overline{\Gamma}) = \mathscr{F} \otimes \pi^*(\mathcal{O}_{\overline{M}}(-1))$. Then we have

$$\begin{aligned} \mathscr{E} &= \nu^*(\mathcal{O}_{\mathscr{C}}(2) \oplus \mathcal{O}_{\mathscr{C}}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathscr{C}}^{\oplus(d-3)}), \\ \mathscr{F} &= \mu^*(\mathcal{O}_{\mathbb{CP}^1}(-m)) \otimes \nu^*(\mathcal{O}_{\mathscr{C}}(1) \oplus \mathcal{O}_{\mathscr{C}}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathscr{C}}(-1)^{\oplus(d-3)}). \end{aligned} \tag{B.6}$$

We compute the cohomology of these bundles on a fiber S of π. Such a fiber can be written in the form $S = \mathbb{CP}^1 \times C$, with C the image of the corresponding map (one of the curves in the family $\mathscr{C}$). When restricted to S, our bundles become

$$\begin{aligned} \mathscr{E}|_S &= (\nu|_S)^*(\mathcal{O}_C(2) \oplus \mathcal{O}_C(-1)^{\oplus 2} \oplus \mathcal{O}_C^{\oplus(d-3)}), \\ \mathscr{F}|_S &= (\mu|_S)^*(\mathcal{O}_{\mathbb{CP}^1}(-m)) \otimes (\nu|_S)^*(\mathcal{O}_C(1) \oplus \mathcal{O}_C(-2)^{\oplus 2} \oplus \mathcal{O}_C(-1)^{\oplus(d-3)}). \end{aligned} \tag{B.7}$$

It is easy to calculate the spaces of global sections:

$$\begin{aligned} H^0(S, \mathscr{E}|_S) &= H^0(C, \mathcal{O}(2) \oplus \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}^{\oplus(d-3)}) \cong \mathbb{C}^d, \\ H^0(S, \mathscr{F}|_S) &= \{0\}. \end{aligned} \tag{B.8}$$

We can also compute H^2's using Serre duality and the canonical bundle formula

$$K_S = (\mu|_S)^*(\mathcal{O}_{\mathbb{CP}^1}(-2)) \otimes (\nu|_S)^*(\mathcal{O}_C(-2)). \tag{B.9}$$

The results are that $H^2(S, \mathscr{E}|_S)^*$ is isomorphic to

$$\begin{aligned} &H^0(S, (\mu|_S)^*(\mathcal{O}_{\mathbb{CP}^1}(-2)) \otimes (\nu|_S)^*(\mathcal{O}_C(-4) \oplus \mathcal{O}_C(-1)^{\oplus 2} \oplus \mathcal{O}_C(-2)^{\oplus(d-3)})) \\ &= \{0\}, \end{aligned} \tag{B.10}$$

and that $H^2(S,\mathscr{F}|_S)^*$ is isomorphic to

$$H^0(S,(\mu|_S)^*(\mathcal{O}_{\mathbb{CP}^1}(m-2))\otimes(v|_S)^*(\mathcal{O}_C(-3)\oplus\mathcal{O}_C^{\oplus 2}\oplus\mathcal{O}_C(-1)^{\oplus(d-3)})$$
$$\cong H^0(S,(\mu|_S)^*(\mathcal{O}_{\mathbb{CP}^1}(m-2)\oplus\mathcal{O}_{\mathbb{CP}^1}(m-2)))\cong\mathbb{C}^{2m-2}\,. \tag{B.11}$$

This last calculation can be done as a bundle calculation, not just fiber by fiber. Doing so gives a natural isomorphism between $(R^2\pi_*\mathscr{F})^*$, and $R^0\pi_*\mathscr{G}$, where

$$\mathscr{G}:=\mu^*(\mathcal{O}_{\mathbb{CP}^1}(m-2)\oplus\mathcal{O}_{\mathbb{CP}^1}(m-2))\,. \tag{B.12}$$

Because Z is a product of $\mathbb{CP}^1$ and $\mathscr{C}\times_{\mathscr{H}}\overline{M}$, the bundle $R^0\pi_*\mathscr{G}$ is trivial, being canonically isomorphic to

$$\mathcal{O}_{\overline{M}}\otimes H^0(\mathbb{CP}^1,\mathcal{O}_{\mathbb{CP}^1}(m-2)\oplus\mathcal{O}_{\mathbb{CP}^1}(m-2))\,. \tag{B.13}$$

To complete our calculation, we note that Riemann–Roch tells us that $\chi(S,\mathscr{E}|_S)=d$ and therefore that $h^1(S,\mathscr{E}|_S)=0$. As a consequence, we find that $R^1\pi_*\mathscr{E}=R^2\pi_*\mathscr{E}=0$, and that $R^0\pi_*\mathscr{E}$ is locally free of rank d. We also find that the short exact sequence (B.5) gives rise to a long exact sequence whose nonzero terms split into two exact sequences:

$$0\to R^0\pi_*\mathscr{E}\to R^0\pi_*(\mathscr{E}|_{\overline{\Gamma}})\to R^1\pi_*(\mathscr{E}(-\overline{\Gamma}))\to 0\,,$$
$$0\to R^1\pi_*(\mathscr{E}|_{\overline{\Gamma}})\to R^2\pi_*(\mathscr{E}(-\overline{\Gamma}))\to 0\,. \tag{B.14}$$

It then follows from the projection formula that

$$R^1\pi_*(\mathscr{E}|_{\overline{\Gamma}})\cong R^2\pi_*(\mathscr{E}(-\overline{\Gamma}))\cong(R^2\pi_*\mathscr{F})\otimes\mathcal{O}_{\overline{M}}(-1)\cong(R^0\pi_*\mathscr{G})^*\otimes\mathcal{O}_{\overline{M}}(-1)$$
$$\cong H^0(\mathbb{CP}^1,\mathcal{O}_{\mathbb{CP}^1}(m-2)\oplus\mathcal{O}_{\mathbb{CP}^1}(m-2))^*\otimes\mathcal{O}_{\overline{M}}(-1)\,. \tag{B.15}$$

We now see that the top Chern class $c_{2m-2}(R^1\pi_*(\mathscr{E}|_{\overline{\Gamma}}))$ coincides with

$$c_1(\mathcal{O}_{\overline{M}}(-1))^{2m-2}=c_1(\mathcal{O}_{\overline{M}}(1))^{2m-2}\,, \tag{B.16}$$

and so is a class whose intersection with every fiber of $\overline{M}\to\mathscr{H}$ is a linear space of dimension 3.

The contribution of $M_n^{\mathscr{H}}$ to the three-point function $\langle\mathcal{O}^{(i)}\mathcal{O}^{(j)}\mathcal{O}^{(d-i-j)}\rangle$ is calculated by an integral

$$\int_{\overline{M}}\overline{e}_i\wedge\overline{e}_j\wedge\overline{e}_{d-i-j}\wedge c_{2m-2}(R^1\pi_*(\mathscr{E}|_{\overline{\Gamma}}))\,, \tag{B.17}$$

where the $\overline{e}$'s are the induced classes on $\overline{M}$, with delta-function support on those maps which take a fixed basepoint P to a fixed cycle H. These integrals localize on a finite number of fibers of $\overline{M}\to\mathscr{H}$, and in each such fiber the last term $c_{2m-2}(R^1\pi_*(\mathscr{E}|_{\overline{\Gamma}}))$ serves to reduce the integral to an integral over $\mathbb{CP}^3$. Each delta-function support condition has the same cohomological effect on $\mathbb{CP}^3$ regardless of the value of m, so we recover the *same* instanton contribution for $m>1$ as for $m=1$, namely, the number of points in $\mathscr{H}$ whose corresponding rational curve meets the stated conditions. Summing over m, we get a term of the form $q^l/(1-q^l)$ times the $m=1$ instanton number, as asserted in (5.4).

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