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Mirror Principle for Flag Manifolds

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Abstract

In this paper, using mirror principle developped by Lian, Liu and Yau [8, 9, 10, 11, 12, 13] we obtained the A and B series for the equivariant tangent bundles over homogenous spaces using Chern polynomial. This is necessary to obtain related cohomology valued series for given arbitrary vector bundle and multiplicative characteristic class. Moreover, this can be used as a valuable testing ground for the theories which associates quantum cohomologies and J functions of non-abelian quotient to abelian quotients via quantization.*

1. Introduction

It is an interesting question to obtain A series for equivarant tangent bundles and Chern Polynomials since this will be necessary to obtain A series for a general vector bundle and multiplicative characteristic class. Now assume $\mathbb{T}$ is an algebraic torus and X be a $\mathbb{T}$-manifold with a $\mathbb{T}$ equivariant embedding in $Y := \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l}$ such that pull backs of hyperplane classes $H = (H_1, \ldots, H_l)$ generate $H^2(X, \mathbb{Q})$. We will use the same notations for equivariant classes and their restriction to X. Let $\check{K} \subset H_2(X)$ be the set of points in $H_2(X, \mathbb{Z})_{\text{free}}$ in the dual of the closure of the Kähler cone of X. $\check{K}$ is a semi-group and defines a partial ordering $\succeq$ on $H_2(X, \mathbb{Q})_{free}$. Explicitly $r \preceq d$ iff $d - r \in \check{K}$. If $\{\check{H}_j\}$ is the dual basis for $\{H_i\}$ in $H_2(X), r \preceq d \Leftrightarrow d - r = d_1\check{H}_1 + \cdots + d_l\check{H}_l$ where $d_i, i = 1, \ldots, l$ are nonnegative integers. Let $X = Fl(n)$ be the complete flag variety. The first Chow ring $A^1(X) \cong H^2(X, \mathbb{Z})$ is generated by $\mathfrak{S}_i = c_1(L_{\lambda_i}), i = 1, \ldots, n-1$ and λ_i is the dominant weight of torus action with $\lambda_i = (1, \ldots, 1, 0, \ldots, 0)$ first i terms are

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1's. Here, L_{λ_i} is the line bundle over X, associated the 1 dimensional representation with respect to weight λ_i. For more on homogenous manifolds, one can consult [1, 2, 3, 14, 15].

2. Basics of Mirror Principle

We will define stable pointed map moduli for a general projective $\mathbb{T}$ space X, where $\mathbb{T}$ is an algebraic torus acting on X. Let $M_{0,n}(d, X)$ be the degree $d \in A_1(X)$ arithmetic genus 0, $n-$pointed stable map moduli stack with target X(see [4], [6]). Following [10], we will not use the bar notation for compactification. A typical element can be represented by $(C, f, x_1, \ldots, x_n)$. This moduli space has a "virtual" fundamental class $[M_{0.n}(d, X)]$ of dimension $dimX + \langle c_1(X), d\rangle + n - 3$ similar to the fundamental class in topology. For more details for constructions, see [7].

Let V be a vector bundle on X. It induces a vector bundle $V_d, d \in A_1(X)$ on $M_{0,n}(d, X)$ whose fiber at $(C, f, x_1, \ldots, x_n)$ is given by $H^0(C, f^*V) \oplus H^1(C, f^*V)$. Another important construction is the graph space $M_d(X)$ for a projective $\mathbb{T}$ manifold X. $M_d(X)$ is the moduli stack of degree $(1, d)$ arithmetic genus 0, 0-pointed stable maps with target $\mathbb{P}^1 \times X$. The standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$, together with the action of $\mathbb{T}$ on X, induces an action of $\mathbb{G} = \mathbb{T} \times \mathbb{C}^*$ on $M_d(X)$. We will denote $\mathbb{G}$-equivariant virtual class by $[M_d(X)] \in A_*^{\mathbb{G}}(M_d(X))$, which has dimension $\langle c_1(X), d\rangle + dimX$.

$\mathbb{C}^*$ fixed points of $M_d(X)$ plays an important role and will be described as

$$F_r := M_{0,1}(r, X) \times_X M_{0,1}(d - r, X).$$

For any $(C_1, f_1, x_1) \times (C_2, f_2, x_2) \in F_r$ we can obtain an element in $M_d(X)$ by gluing C_1 and C_2 to $\mathbb{P}^1$ at 0 and ∞ respectively. New curve C will be mapped to $\mathbb{P}^1 \times X$ as follows: Map $\mathbb{P}^1$ identically $\mathbb{P}^1$ and contract C_1, C_2 to $0, \infty$. Map C_i by f_i and contract $\mathbb{P}^1$ to the point $f_1(x_1) = f_2(x_2)$. This defines an element $(C, f) \in M_d(X)$. Observe that $F_0 = M_{0,1}(d, X) = F_d$ but they will be imbedded in $M_d(X)$ in two different ways. For F_0, we glue the marked point to 0 and glue the marked point to ∞ for F_d in $\mathbb{P}^1$. We will denote inclusion maps $i_r : F_r \hookrightarrow M_d(X)$. Note that each F_r has an evaluation map $e_r^X : F_r \to X$ sending each point to the common image of the marked points in X. Here are some other notations which will be used.

- Let L_r be the universal line bundle on $M_{0,1}(r, X)$ which is the tangent line at the marked point.

- We have natural forgetting and projection maps

$$\rho : M_{0,1}(d, X) \to M_{0,0}(d, X), \qquad \nu : M_d(X) \to M_{0,0}(d, X) \text{ ,(see [8], [9]).}$$

with the commutative diagram

$$\begin{array}{ccc} F_0 = M_{0,1}(d, X) & \xrightarrow{i_0} & M_d(X) \\ & \overset{\rho}{\searrow} & \downarrow \nu \\ & & M_{0,0}(d, X) \end{array} \quad . \tag{1}$$

- Let α be the weight of the standard $\mathbb{C}^*$ action on $\mathbb{P}^1$. Denote by $A_*^{\mathbb{T}}(X)(\alpha)$ the algebra obtained from the polynomial algebra $A_*^{\mathbb{T}}(X)[\alpha]$ by localizing with respect to all invertible elements. For an element $\beta \in A_*^{\mathbb{T}}(X)(\alpha)$ we let $\overline{\beta}$ be the class obtained by $\alpha \mapsto -\alpha$ in β. Introduce formal variables $\zeta = (\zeta_1, \ldots, \zeta_m)$ such that $\overline{\zeta_i} = -\zeta_i, \forall i$. Let $\mathcal{R} = \mathbb{C}[\mathcal{T}^*][\alpha]$, where $\mathcal{T}^*$ is the dual of the Lie algebra of $\mathbb{T}$. When we consider a multiplicative class like the Chern polynomial $c_{\mathbb{T}}(x) = \sum_{i=0}^{r} c_i x^i$, we extend the ground field to $\mathbb{C}(x)$.

- For each d let $\varphi : M_d(X) \to W_d$ be $\mathbb{G}$-equivariant map into smooth manifold (or orbifold) W_d such that $\mathbb{C}^*$ fixed point components in W_d are $\mathbb{G}$- invariant submanifolds Y_r satisfying $\varphi^{-1}(Y_r) = F_r$. Construction of such maps and spaces are given in [8, 9]. In particular, for a smooth manifold X let

$$\tau : X \to \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_l} := Y$$

be an equivariant projective embedding inducing an isomorphism $A^1(X) \simeq A^1(Y)$. Then we have a $\mathbb{G}$- equivariant embedding $M_d(X) \to M_d(Y)$ and we can construct $\mathbb{G}$- equivariant map $M_d(Y) \to W_d := N_{d_1} \times \cdots \times N_{d_l}$ with $N_{d_i} \simeq \mathbb{P}^{(m_i+1)d_i+m_i}$ which are linear sigma models for $\mathbb{P}^{m_i}$(see [8]). Therefore we obtain the map

$$\varphi : M_d(X) \to W_d,$$

satisfying the above condition. Let κ_a be the equivariant hyperplane class in W_d which is pulled back from N_{d_a}; and denote the equivariant hyperplane class on Y by H_a, also pulled back from $\mathbb{P}^{m_a}$ to Y. Let $Y_r, 0 \preceq r \preceq d$ be $\mathbb{C}^*$ fixed point components of W_d which are $\mathbb{G}$-equivariantly isomorphic to $Y = Y_0$ and $j_r : Y_r \hookrightarrow W_d$ be the inclusion map. We have $j_r^* \kappa_a = H_a + \langle r, H_a \rangle \alpha$.

Consider the commutative diagram

$$\begin{array}{ccc} F_r & \xrightarrow{i_r} & M_d(X) \\ \downarrow e & & \downarrow \varphi \\ Y_r & \xrightarrow{j_r} & W_d. \end{array}$$

The following proposition helps us to carry the computations to W_d from $M_d(X)$ which is easier to deal with.

Proposition 2.1 *([10], Lemma 3.2) Given $\omega \in A^*_{\mathbb{G}}(M_d(X))$ we have the following equality on $Y_r \simeq Y$ for $0 \preceq r \preceq d$.*

$$\frac{j_r^* \varphi_*(\omega \cap [M_d(X)])}{e_{\mathbb{G}}(Y_r/W_d)} = e_* \Big(\frac{i_r^* \cap [F_r]^{vir}}{e_{\mathbb{G}}(F_r/M_d(X))} \Big).$$

□

For $d = (d_1, \ldots, d_l), r = (r_1 \ldots, r_l) \preceq d$ we have

$$e_{\mathbb{G}}(Y_r/W_d) = \prod_{a=1}^{l} \prod_{i=0}^{m_a} \prod_{\substack{k=0 \\ k \neq r_a}}^{d_a} (H_a - u_{a,i} - (k - r_a)\alpha),$$

where $u_{i,a}$ are $\mathbb{T}$ weights of $\mathbb{P}^{m_a}$.

Note that a class $\phi \in H^2_{\mathbb{T}}(X)$ has a $\mathbb{G}$-equivariant extension $\hat{\phi} \in H^2_{\mathbb{G}}(W_d)$ determined by $j_r^* \hat{\phi} = \phi + \langle \phi, r \rangle \alpha$ by localization theorem. We denote by $\langle H^2_{\mathbb{T}}(X) \rangle$ the ring generated by $H^2_{\mathbb{T}}(X)$ and R_d the ring generated by their lifts. So we have the following definition from [10].

Definition 2.2 *Let $\Gamma \in H^*_{\mathbb{T}}(Y)$. A list $P : P_d \in H^*_{\mathbb{G}}(W_d), d \succeq 0$ is an Γ- Euler data on if*

$$\Gamma \cdot j_r^* P_d = \overline{j_0^* P_r} \cdot j_0^* P_{d-r}.$$

An immediate observation is when we apply τ^* we get

$$\tau^*\Gamma \cdot \tau^* j_r^* P_d = \overline{\tau^* j_0^* P_r} \cdot \tau^* j_0^* P_{d-r}$$

There is an interesting construction for linear sigma models for a toric variety X(see [9, 10]).

Whenever $t = (t_1, \dots, t_l)$ formal variable we let $d \cdot t = \sum d_i t_i$, $\kappa \cdot t = \sum \kappa_a t_a, H \cdot t = \sum H_a t_a$.

Fix a $\mathbb{T}$ equivariant multiplicative class $b_{\mathbb{T}}$ and an equivariant vector bundle $V = V^+ \oplus V^-$ where $V^\pm$ are convex and concave bundles on X. We will assume $\Omega = \frac{b_{\mathbb{T}}(V^+)}{b_{\mathbb{T}}(V^-)}$ is a well defined class on X. For such a vector bundle we have

$$V_d \to M_{0,0}(d, X), \qquad \mathcal{U}_d \to M_d(X)$$

where $\mathcal{U}_d = \nu^* V_d$. Define the linear maps

$$\begin{aligned} i_r^{vir} : A_{\mathbb{G}}^*(M_d(X)) &\longrightarrow A_*^{\mathbb{T}}(X)(\alpha) \\ i_r^{vir}\omega &:= (e_r^X)_*\left(\frac{i_r^*\omega \cap [F_r]}{e_{\mathbb{G}}(F_r/M_d(X))}\right). \end{aligned}$$

For a given concavex bundle V on X and $b_{\mathbb{T}}$, we put

$$\begin{aligned} A^{V,b_{\mathbb{T}}}(t) = A(t) &:= e^{-H\cdot t/\alpha} \sum_d A_d e^{d\cdot t} \\ A_d &:= i_0^{vir} \nu^* b_T(V_d), \end{aligned}$$

where $A_0 = \Omega$ and the sum is taken over all $d = (d_1, \dots, d_l) \in \mathbb{Z}_+^l$. We call $A(t)$ the A series associated to V and $b_{\mathbb{T}}$. In particular, if we specialize $b_{\mathbb{T}}$ to the unit class, we have

$$\mathbb{I}(t) = e^{-H\cdot t/\alpha} \sum_d \mathbb{I}_d e^{d\cdot t}, \qquad \mathbb{I}_d = i_0^{vir} 1_d. \tag{2}$$

Here 1_d is the unit class in $M_d(X)$.

Definition 2.3 *Let $\Omega \in A_{\mathbb{T}}^*(X)$ be invertible. We call a power series of the form*

$$B(t) := e^{-H\cdot t/\alpha} \sum_d B_d e^{d\cdot t}, \qquad B_d \in A_*^{\mathbb{T}}(X)(\alpha)$$

an Ω - Euler series if

$$\sum_{0 \preceq r \preceq d} \int_X \Omega^{-1} \cap \overline{B_r} \cdot B_{d-r} e^{(H+r\alpha)\cdot\zeta} \in \mathcal{R}[[\zeta]]$$

for all d.

Proposition 2.4 *$A^{V,b_{\mathbb{T}}}(t) = A(t)$ is an Euler series*

Proof. Cf. [10], corr.3.9. □

Theorem 2.5 *([10], Thm 3.11) Let $P : P_d$ be an ΓEuler data. Then*

$$B(t) = e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap \mathbb{I}_d e^{d\cdot t}$$

is an $\tau^*\Gamma$ Euler series. □

Recall that we have a commutative diagram of maps which read $\nu \circ i_0 = \rho$. So we can write

$$A_d = (e_0)_*^X \left(\frac{\rho^* b_{\mathbb{T}}(V_d) \cap [M_{0,1}(d,X)]}{e_{\mathbb{G}}(F_0/M_d(X))}\right).$$

We can also compute $e_{\mathbb{G}}(F_r/M_d(X))$ explicitely for $0 \preceq r \preceq d$. Although the $\mathbb{G}$ equivariant Euler class of the normal bundle of F_0 in $M_d(X)$, that is $N_{F_0/M_d(X)}$, will be used mostly, following lemma gives such a class for every $\mathbb{C}^*$ -fixed point component in $M_d(X)$.

Lemma 2.6 *([8],[10]) For $r \neq 0, d$*

$$e_{\mathbb{G}}(F_r/M_d(X)) = \alpha(\alpha + p_0^* c_1(L_r))\alpha(\alpha - p_\infty^* c_1(L_{d-r}))$$

For r=0,d

$$e_{\mathbb{G}}(F_0/M_d(X)) = \alpha(\alpha - c_1(L_d)), \qquad e_{\mathbb{G}}(F_d/M_d(X)) = \alpha(\alpha + c_1(L_d))$$

where $p_0 : F_r \to M_{0,1}(r,X)$ and $p_\infty : F_r \to M_{0,1}(d-r,X)$ are projections. □

Corollary 2.7 *If we denote the degree of α in a class $\omega \in A_*^{\mathbb{T}}(X)(\alpha)$ by $deg_\alpha \omega$ then $deg_\alpha A_d \leq -2$.*

Proof. We have

$$A_d = (e_0)_*^X(\frac{\rho^* b_{\mathbb{T}}(V_d) \cap [M_{0,1}(d,X)]}{e_{\mathbb{G}}(F_0/M_d(X))}) = (e_0)_*^X(\frac{\rho^* b_{\mathbb{T}}(V_d) \cap [M_{0,1}(d,X)]}{\alpha(\alpha - c_1(L_d))})$$

by previous lemma. So $deg_\alpha A_d \leq -2$ □

In particular, when $\mathbb{I}_d$ is concerned, we have a better estimate for α degree.

Proposition 2.8 $\forall d, \quad deg_\alpha \mathbb{I}_d \leq min(-2, -\langle c_1(X), d\rangle)$.

Proof. If $\langle c_1(X), d\rangle \leq 2$, then previous corollary gives the result. So assume $\langle c_1(X), d\rangle > 2$. Recall that the class $[M_{0,1}(d,X)]$ is of dimension $s = exp.dim M_{0,1}(d,X) = \langle c_1(X), d\rangle + dimX - 2$. Set $c = c_1(L_d)$ then $c^k \cap [M_{0,1}(d,X)]$ is of dimension $s-k$ and so $e_*(c^k \cap [M_{0,1}(d,X)]) \in A_{s-k}^{\mathbb{T}}(X)$. But this group is zero unless $s - k \leq dimX$ hence $k \geq s - dimX = \langle c_1(X), d\rangle - 2$. By the lemma , we have

$$\mathbb{I}_d = \sum_{k \geq \langle c_1(X), d\rangle - 2} \frac{1}{\alpha^{k+2}} e_*(c^k \cap [M_{0,1}(d,X)])$$

hence the proposition follows. □

Most of the time, computing $A(t)$ directly from the definition is quite difficult. Nevertheless, provided that some conditions are satisfied it is possible to compute the A-series up to some special operation called "Mirror Transformation"; cf. [8],[10]. The main idea of the process is to consider another special series, which we call B series, and if some analytic conditions are satisfied we can get the A series from this B series by mirror transform. We will now give more explanations.

Definition 2.9 *A projective $\mathbb{T}$ manifold X is called a balloon manifold if the fixed point set $X^{\mathbb{T}}$ is finite and if for $p \in X^{\mathbb{T}}$ the weights of the isotropic representation T_pX are pairwise linearly independent. The second condition is known as the GKM condition.[5]*

We will assume that the balloon manifold has the property that if $p, q \in X^{\mathbb{T}}$ such that $i_p^* c = i_q^* c, \forall c \in A_{\mathbb{T}}^1(X)$ then $p = q$. If two fixed points p, q in X are connected by a $\mathbb{T}$

invariant 2-sphere we call the sphere a balloon and denote it by pq. Balloon manifolds are examined in more detail in [9]

Definition 2.10 *Two Euler series A, B are linked if every balloon pq in X and every $d = \delta[pq] \succ 0$ the function $(A_d - B_d)|_p \in \mathbb{C}(\mathcal{T}^*)(\alpha)$ is regular at $\alpha = \lambda/\delta$ where λ is the weight on the tangent line $T_p(pq) \subset T_pX$.*

Let $B(t) := e^{-H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap \mathbb{I}_d e^{d\cdot t}$ be an $\Omega = \tau^*\Gamma$- Euler series obtained from a Γ-Euler data $P : P_d$. The following theorem is adapted from [10] (thm 4.5 and corrollary 4.6).

Theorem 2.11 *Suppose that at $\alpha = \lambda/\delta$ and $F = (\mathbb{P}^1, f_\delta, 0) \in F_0$ we have $i_p^*\tau^* j_0^* P_d = i_F^* \rho^* b_{\mathbb{T}}(V_d)$ for all $d = \delta[pq]$. Then $B(t)$ is linked to $A^{V,b_{\mathbb{T}}}(t)$.*

Now we state a theorem which relates two Euler series in the previous setting by what we call a mirror transform. Assume $B(t) = e^{H\cdot t/\alpha} \sum_d \tau^* j_0^* P_d \cap \mathbb{I}_d e^{d\cdot t}$ where $\tau^* j_0^* P_d$ satisfies the assertion of the previous theorem. In addition assume for all d we have

$$\tau^* j_0^* P_d = \Omega \alpha^{\langle c_1(X), d\rangle} (a + (a^{'} + a^{''} \cdot H)\alpha^{-1} + \dots .)$$

for some $a, a^{'}, a^{''} \in \mathbb{C}(\mathcal{T}^*)$ depending on d. Note also that $\mathbb{I}_d$ can be expanded as

$$\mathbb{I}_d = \alpha^{-\langle c_1(X), d\rangle} (b + (b^{'} + b^{''} \cdot H)\alpha^{-1} + \dots .)$$

for some $b, b^{'}, b^{''} \in \mathbb{C}(\mathcal{T}^*)$ also depending on d. Then we have the following theorem

Theorem 2.12 *Suppose $A^{V,b_{\mathbb{T}}}(t), B(t)$ are as in the previous theorem and the above assumptions hold. Then there exist power series $f \in \mathcal{R}[[e^{t_1}, \dots , e^{t_m}]], g = (g_1, \dots , g_m), g_j \in \mathcal{R}[[e^{t_1}, \dots , e^{t_m}]]$ without constant terms such that*

$$A^{V,b_{\mathbb{T}}}(t + g) = e^{f/\alpha} B(t).$$

Proof. see [10] . □

There is an explicit method to compute $A(t) = A^{V,b_{\mathbb{T}}}(t)$ in full generality on any balloon manifold X for arbitrary $V, b_{\mathbb{T}}$. Computations are in terms of some $\mathbb{T}$ representations. Observe that by the previous theorems, it is useful to understand the structure of

$i_F^* \rho^* b_{\mathbb{T}}(V_d)$ and obtain an Euler series, satisfying the conditions specified in the previous theorem, so we can compute A series up to mirror transformation. Now we will discuss some part of the method given in [10] to compute $A(t)$.

Recall that $V_d \to M_{0,1}(d, X)$ is a vector bundle with fiber at (C, f, x) is given by $H^0(C, f^*V) \oplus H^1(C, f^*V)$. Then for a vector bundle V on X and $F = (\mathbb{P}^1, f_\delta, 0), d = \delta[pq]$ we have a $\mathbb{T}$ representation

$$i_F^* \rho^*(V_d) = H^0(C, f^*V) \oplus H^1(C, f^*V),$$

which is the value of $b_{\mathbb{T}}$ for a trivial bundle over a point. So the method uses the $\mathbb{T}$ representations of related bundles on each balloon $pq \simeq \mathbb{P}^1$

Let V be any $\mathbb{T}$ equivariant vector bundle on X and let

$$0 \longrightarrow V_N \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V \longrightarrow 0$$

be an equivariant resolution. Then by Euler-Poincaré Principal,

$$[H^0(\mathbb{P}^1, f_\delta^* V)] - [H^1(\mathbb{P}^1, f_\delta^* V)] = \sum_a (-1)^{a+1}([H^0(\mathbb{P}^1, f_\delta^* V_a)] - [H^1(\mathbb{P}^1, f_\delta^* V_a)]).$$

Now, suppose each V_a is a direct sum of $\mathbb{T}$ equivariant line bundles. Then each summand L will contribute to $[H^0(\mathbb{P}^1, f_\delta^* V_a)] - [H^1(\mathbb{P}^1, f_\delta^* V_a)]$ the representations

$$\begin{aligned} &c_1(L)|_p - k\lambda/\delta \quad , \quad k = 0, \ldots, l\delta \text{ or} \\ &c_1(L)|_p + k\lambda/\delta \quad , \quad k = 1, \ldots, -l\delta - 1, \end{aligned}$$

depending on the sign of $l = \langle c_1, [pq] \rangle$. For $l \geq 0$, we get the first and for $l < 0$ we have the second kind of contribution.

3. A- series for Fl(n)

Let X=$Fl(n)$ be a complete flag vaiety. $A^1(X)$ is generated by $\mathfrak{S}_i = c_1(L_{\lambda_i}), i = 1, \ldots, n-1$ and λ_i is the dominant weight $\lambda_i = (\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0)$. Note that these are Schubert polynomials. Let $d = (d_1, \ldots, d_{n-1})$ be a class of a curve in the Kähler cone. Since Kähler cone of X is generated by $d = \sum_{i=1}^{n-1} d_i \check{\mathfrak{S}}_i$ where $\{\check{\mathfrak{S}}_i\}$ forms a dual basis for $\{\mathfrak{S}_i\}$. These are the Poincaré duals of the Schubert polynomials. Now consider

$$Fl(n) \xrightarrow{\quad \tau \quad} \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_{n-1}} \xrightarrow{\quad j_0 \quad} W_d := N_{d_1} \times \cdots \times N_{d_{n-1}},$$

where j_0 is the imbedding of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_{n-1}}$ as a $\mathbb{C}^*$ fixed point component of W_d; and for $0 \preceq r \preceq d$, all fixed point components are $\mathbb{T}$ equivariantly isomorphic to $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_{n-1}}$. τ is the Plucker embedding. Here $N_{d_i} \simeq \mathbb{P}^{(m_i+1)d_i+m_i}$ and W_d is the linear sigma model. Finally, $m_i = \binom{n}{i} - 1$. Let H_i be the equivariant hyperplane classes in $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_{n-1}}$. Pull back of each H_i gives the corresponding $\mathfrak{S}_i$. There exists $\mathbb{G}=\mathbb{C}^* \times \mathbb{T}$ -equivariant hyperplane classes κ_i in W_d with the property that $j_r^* \kappa_i = H_i + \langle H_i, r \rangle \alpha$ for $0 \preceq r \preceq d$. By the pull back of $\tau \circ j_0$, these κ_i are taken to $\mathfrak{S}_i$. Again, we are using the same notation for equivalent and ordinary cohomology. We will compute the A series of $Fl(n)$ for $\mathbb{T}$ equivariant tangent bundle and Chern polynomial.

Lemma 3.1 *Let $[pq]$ be a class of balloon joining p, q. Then*

$$\langle y_a, [pq] \rangle = \int_{[pq] \simeq \mathbb{P}^1} y_a = \begin{cases} 1 & \text{if } i \geq a \\ 0 & \text{if } a \neq i, j \\ -1 & \text{if } j = a, \end{cases}$$

where $p = \omega, q = \omega(ij) \in S_n$ are permutations representing the fixed points and (ij) is a transposition.

Proof. We know that $(pq) \simeq X_{\omega}^{\omega(ij)} \simeq \mathbb{P}^1$, Richardson variety and $y_a = c_1(L_{\gamma_a}), \gamma_a = (\underbrace{0, \ldots, 1}_{a}, 0, \ldots, 0)$ is a weight of $\mathbb{T}$. Then

$$\langle y_a, [pq] \rangle = \langle c_1(L^*_{\gamma_i}), [pq] \rangle = \langle c_1(\mathcal{O}(\gamma_{a,i} - \gamma_{a,j})), [pq] \rangle = \gamma_{a,i} - \gamma_{a,j},$$

where $\gamma_{a,i}$ means the i-th entry of γ_a. So considering possibilities we obtain the lemma. □

Recall that in the equivariant Grothendieck group, we have

$$[TFl(n)] = \sum_{i=1}^{n} [U^*_{n-1} \otimes S_{\chi_i}] - \sum_{i=1}^{n-1} [U^*_i \otimes U_i] + \sum_{i=1}^{n-2} [U^*_i \otimes U_{i+1}]. \tag{3}$$

We know that $L_{\chi_i} = U_i/U_{i-1}, i = 1, \dots, n-1$. Of course we are using induced bundles for $\mathbb{T}$ -action without changing the notation. Then we have

$$0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow U_2/U_1 \longrightarrow 0$$

short exact sequence. So in Grothendieck group $[U_2] = [L_{\chi_1}] + [L_{\chi_2}]$. We can proceed for $i = 1, \dots, n-1$ and obtain $[U_i] = \sum_{j=1}^{i}[L_{\chi_j}]$. Since the duality of vector bundles yields an involution $[V] \mapsto [V^*]$ in Grothendieck group. We have $[U_i^*] = \sum_{j=1}^{i}[L_{\chi_i}^*]$. So equation (3) can be decomposed further to be

$$[TFl(n)] = \sum_{i=1}^{n}\sum_{a=1}^{n-1}[L_{\chi_a}^* \otimes S_{\chi_i}] - \sum_{i=1}^{n-1}\sum_{1\le a,b\le i}[L_{\chi_a}^* \otimes L_{\chi_b}] + \sum_{i=1}^{n-2}\sum_{\substack{1\le a\le i\\ 1\le b\le i+1}}[L_{\chi_a}^* \otimes L_{\chi_b}].$$

So we obtained a decomposition of $\mathbb{T}$ equivariant tangent bundle into line bundles in Grothendieck group. Therefore given a balloon $pq \in Fl(n)$ and $d = \delta[pq]$ together with $F = (\mathbb{P}^1, f_\delta, 0) \in M_{0,1}(Fl(n), d)$ we have for $V = TFl(n)$ the representation $R = [H^0(\mathbb{P}^1, f_\delta^* V)] - [H^1(\mathbb{P}^1, f_\delta^* V)]$ is equal to

$$\begin{aligned} R \;=\; & \sum_{i=1}^{n}\sum_{a=1}^{n-1}[H^0(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes S_{\chi_i}))] - [H^1(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes S_{\chi_i}))] \qquad (4)\\ & - \sum_{i=1}^{n-1}\sum_{1\le a,b\le i}[H^0(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes L_{\chi_b}))] - [H^1(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes L_{\chi_b}))]\\ & + \sum_{i=1}^{n-2}\sum_{\substack{1\le a\le i\\ 1\le b\le i+1}}[H^0(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes L_{\chi_b}))] - [H^1(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes L_{\chi_b}))]. \end{aligned}$$

Considering (4), and using the method of [10], we can compute $i_{\rho(F)}^* b_{\mathbb{T}}(V_d)$ for equivariant Chern polynomial. We will consider three cases.

Case 1) $[H^0(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes L_{\chi_b}))] - [H^1(\mathbb{P}^1, f_\delta^*(L_{\chi_a}^* \otimes L_{\chi_b}))], 1 \le a \le i, 1 \le b \le i+1$.

Note $c_1(L_{\chi_a}^* \otimes L_{\chi_b})|_p = (y_a - y_b)|_p = u_{\omega(a)} - u_{\omega(b)}, \omega \in S_n$ corresponds to p and $i_p^* y_a = u_{\omega(a)}$([15]). We also have

$$l_{ab} = \langle L_{\chi_a}^* \otimes L_{\chi_b}, [pq]\rangle = \langle y_a - y_b, [pq]\rangle = \langle \mathcal{O}(\lambda_s - \lambda_t), [pq]\rangle = \lambda_s - \lambda_t,$$

where $\lambda = \chi_b - \chi_a$ and $pq \simeq X_\omega^{\omega(st)}, q = \omega(st)$. So as in lemma (3.1) we can compute l_{ab}. Namely assuming $a < b$ we obtain

$$l_{ab} = \begin{cases} 0 & \text{if } s, t \neq a, b \\ -1 & \text{if } t = a \text{ or } s = b \\ 1 & \text{if } s = a, t \neq b \text{ or } s = b, t \neq a \\ 2 & \text{if } s = a, t = b, \end{cases}$$

and for $b < a$ we have

$$l_{ab} = \begin{cases} 0 & \text{if } s, t \neq a, b \\ -1 & \text{if } s = b \text{ and } t \neq a \text{ or } s = a, t \neq b \\ 1 & \text{if } s = a \text{ or } t = b \\ -2 & \text{if } s = b, t = a \end{cases}$$

for $1 \leq a \leq i, 1 \leq b \leq i+1, 1 \leq i \leq n-2$. This contributes as $(x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p - k\lambda/\delta)$ for $l_{ab} \geq 0, k = 0, \ldots, l_{ab}\delta$ and $(x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p + k\lambda/\delta)$ for $l_{ab} < 0, k = 1, \ldots, -l_{ab}\delta - 1$ and eventually we get

$$\frac{\prod_{l_{ab} \geq 0} \prod_{k=0}^{l_{ab}\delta} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p - k\lambda/\delta)}{\prod_{l_{ab} < 0} \prod_{k=1}^{-l_{ab}\delta - 1} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p + k\lambda/\delta)}..$$

Case 2) $-[H^0(\mathbb{P}^1, f^*_\delta(L^*_{\chi_a} \otimes L_{\chi_b}))] - [H^1(\mathbb{P}^1, f^*_\delta(L^*_{\chi_a} \otimes L_{\chi_b}))], 1 \leq a, b \leq n-1$.
Similarly, $i^*_p(y_a - y_b) = u_{\omega(a) - \omega(b)}$ and set

$$l_{ab} = \langle c_1(L^*_{\chi_a} \otimes L_{\chi_b}), [pq] \rangle,$$

which can be computed as before and we obtain

$$\frac{\displaystyle\prod_{l_{ab} < 0} \prod_{k=1}^{-l_{ab}\delta - 1} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p + k\lambda/\delta)}{\displaystyle\prod_{l_{ab} \geq 0} \prod_{k=0}^{l_{ab}\delta} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p - k\lambda/\delta)}$$

because of the negative sign in front.

Case 3) $[H^0(\mathbb{P}^1, f^*_\delta(L^*_{\chi_a} \otimes S_{\chi_i}))] - [H^1(\mathbb{P}^1, f^*_\delta(L^*_{\chi_a} \otimes S_{\chi_i}))], 1 \leq i \leq n, 1 \leq a \leq n-1$.

This time we have $c_1(L^*_{\chi_a} \otimes S_{\chi_i})|_p = u_{\omega(a)} - u_i$ and

$$l_a = \langle c_1(L^*_{\chi_a} \otimes S_{\chi_i}), [pq] \rangle.$$

The contribution will be

$$\frac{\prod_{l_a \geq 0} \prod_{k=0}^{l_a \delta} (x + c_1(L^*_{\chi_a} \otimes S_{\chi_i})|_p - k\lambda/\delta)}{\prod_{l_a < 0} \prod_{i=1}^{-l_a - 1} (x + c_1(L^*_{\chi_a} \otimes S_{\chi_i})|_p + k\lambda/\alpha)}.$$

Combining all of the above we obtain the next theorem.

Theorem 3.2 *Let $X = Fl(n), F = (\mathbb{P}^1, f_\delta, 0)$ with $d = \delta[pq]$ for a balloon $pq \subset X$ where $p = \omega, q = \omega(jn)$ and $pq \simeq X_\omega^{\omega(jn)}$. Then at $\alpha = \lambda/\delta$*

$$\begin{aligned} i^*_{\rho(F)} b_{\mathbb{T}}(V_d) \quad &= \quad \prod_{i=1}^{n} \prod_{a=1}^{n-1} \frac{\prod_{l_a \geq 0} \prod_{k=0}^{l_a \delta} (x + c_1(L^*_{\chi_a} \otimes S_{\chi_i})|_p - k\lambda/\delta)}{\prod_{l_a < 0} \prod_{i=1}^{-l_a - 1} (x + c_1(L^*_{\chi_a} \otimes S_{\chi_i})|_p + k\lambda/\alpha)} \\ &\cdot \prod_{i=1}^{n-1} \prod_{1 \leq a, b \leq i} \frac{\prod_{l_{ab} < 0} \prod_{k=1}^{-l_{ab}\delta - 1} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p + k\lambda/\delta)}{\prod_{l_{ab} \geq 0} \prod_{k=0}^{l_{ab}\delta} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p - k\lambda/\delta)} \\ &\cdot \prod_{i=1}^{n-2} \prod_{\substack{1 \leq a \leq i \\ 1 \leq b \leq i+1}} \frac{\prod_{l_{ab} \geq 0} \prod_{k=0}^{l_{ab}\delta} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p - k\lambda/\delta)}{\prod_{l_{ab} < 0} \prod_{k=1}^{-l_{ab}\delta - 1} (x + c_1(L^*_{\chi_a} \otimes L_{\chi_b})|_p + k\lambda/\delta)}. \end{aligned}$$

□

For $d = \sum d_i \check{\mathfrak{S}}_i$, in $A^*_{\mathbb{G}}(W_d)$ define

$$
\begin{aligned}
Q_d &= \prod_{i=1}^{n}\prod_{a=1}^{n-1} \frac{\prod\limits_{d_a-d_{a-1}\geq 0}\prod\limits_{k=0}^{d_a-d_{a-1}}(x+\kappa_a-\kappa_{a-1}-u_i-k\alpha)}{\prod\limits_{d_a-d_{a-1}<0}\prod\limits_{k=1}^{d_{a-1}-d_a-1}(x+\kappa_a-\kappa_{a-1}-u_i+k\alpha)} \\
&\cdot \prod_{i=1}^{n-1}\prod_{1\leq a,b\leq i} \frac{\prod\limits_{d_{ab}<0}\prod\limits_{k=1}^{-d_{ab}-1}(x+\kappa_{ab}+k\alpha)}{\prod\limits_{d_{ab}\geq 0}\prod\limits_{k=0}^{d_{ab}}(x+\kappa_{ab}-k\alpha)} \\
&\cdot \prod_{i=1}^{n-2}\prod_{\substack{1\leq a\leq i\\ 1\leq b\leq i+1}} \frac{\prod\limits_{d_{ab}\geq 0}\prod\limits_{k=0}^{d_{ab}}(x+\kappa_{ab}-k\alpha)}{\prod\limits_{d_{ab}<0}\prod\limits_{k=1}^{-d_{ab}-1}(x+\kappa_{ab}+k\alpha)},
\end{aligned}
$$

where

$$
\begin{aligned}
d_{ab} &= \langle y_a - y_b, d\rangle = \langle \mathfrak{S}_a - \mathfrak{S}_{a-1} - \mathfrak{S}_b + \mathfrak{S}_{b-1}, d\rangle = d_a - d_{a-1} - d_b + d_{b-1}, \\
\kappa_{ab} &= \kappa_a - \kappa_{a-1} - (\kappa_b - \kappa_{b-1}).
\end{aligned}
$$

Proposition 3.3 *With the notations of the previous theorem,* $i_p^*\tau^*j_0^*Q_d = i^*_{\rho(F)}b_{\mathbb{T}}(V_d)$.

Proof. We have $d = \delta[pq], \alpha = \lambda/\delta$ and note that $d_i = \langle \mathfrak{S}_i, \delta[pq]\rangle$ and $\tau^* j_0^*\kappa_a = \tau^* H_i = \mathfrak{S}_i$. So $i_p\tau^*j_0^*Q_d$ will give the same expression as theorem (3.2). □

Proposition 3.4 $B(t) = e^{-\mathfrak{S}\cdot t/\alpha}\sum_d \tau^* j_0^* Q_d \cap \mathbb{I}_d e^{d\cdot t}$ *is an* $\Omega-$*Euler series. Here* $\mathfrak{S} = (\mathfrak{S}_1, \ldots, \mathfrak{S}_{n-1})$.

Proof. Equivariant Chern polynomial of $Fl(n)$ is given by

$$\Omega = \tau^*\Gamma = b_{\mathbb{T}}(V) = \frac{\prod_{i=1}^{n}\prod_{a=1}^{n-1}(x+y_a-u_i)\prod_{i=1}^{n-2}\prod_{\substack{1\le a\le i\\ 1\le b\le i+1}}(x+y_a-y_b)}{\prod_{i=1}^{n-1}\prod_{1\le a,b\le i}(x+y_a-y_b)}$$

where Γ is a $\mathbb{T}$ equivariant class in $H^*_{\mathbb{T}}(Y)$ given by

$$\Gamma = \frac{\prod_{i=1}^{n}\prod_{a=1}^{n-1}(x+H_a-H_{a-1}-u_i)\prod_{i=1}^{n-2}\prod_{\substack{1\le a\le i\\ 1\le b\le i+1}}(x+H_{ab})}{\prod_{i=1}^{n-1}\prod_{1\le a,b\le i}(x+H_{ab})},$$

where $H_{ab} = H_a - H_{a-1} - (H_b - H_{b-1})$.

We must show that $\Gamma \cdot j_r^* Q_d = \overline{j_0^* Q_r} \cdot j_0^* Q_{d-r}, 0 \preceq r \preceq d$. We will consider several cases. Let's fix $1 \le i \le n$ and $1 \le a \le n-1$. For $0 \preceq r \preceq d$:

- If $d_a - d_{a-1} \ge 0, r_a - r_{a-1} \ge 0$ and $d_a - d_{a-1} - (r_a - r_{a-1}) \ge 0$, then we will have a term $\prod_{k=0}^{d_a-d_{a-1}}(x+\kappa_a-\kappa_{a-1}-u_i-k\alpha)$ in Q_d. Isolate $(x+H_a-H_{a-1}-u_i)$, a part of Ω , to compute

$$(x+H_a-H_{a-1}-u_i)\cdot j_r^*\prod_{k=0}^{d_a-d_{a-1}}(x+\kappa_a-\kappa_{a-1}-u_i-k\alpha) =$$

$$(x+y_a-u_i)\prod_{k=0}^{d_a-d_{a-1}}(x+H_a-H_{a-1}-u_i+((r_a-r_{a-1}-k)\alpha) \tag{5}$$

  On the other hand we have $r_a - r_{a-1} \ge 0$ and $d_a - d_{a-1} - (r_a - r_{a-1}) \ge 0$. Consider

$$\overline{j_0^*\prod_{k=0}^{r_a-r_{a-1}}(x+\kappa_a-\kappa_{a-1}-u_i-k\alpha)}\cdot j_0^*\prod_{k=0}^{(d-r)_a}(x+\kappa_a-\kappa_{a-1}-u_i-k\alpha)$$

where $(d-a)_a = d_a - d_{a-1} - (r_a - r_{a-1})$. This becomes

$$\prod_{k=0}^{r_a - r_{a-1}} (x + H_a - H_{a-1} - u_i + k\alpha) \prod_{k=0}^{(d-a)_a} (x + H_a - H_{a-1} - u_i - k\alpha), \tag{6}$$

Expanding (5) and comparing to (6) we clearly see that they are equal. This is also contained as an example of Euler data in [8].

- $d_a - d_{a-1} \geq 0$, $r_a - r_{a-1} \geq 0$ but $(d-a)_a < 0$. In this case we still have (5) but this time we must consider the division

$$\frac{\prod_{k=0}^{r_a - r_{a-1}} (x + H_a - H_{a-1} - u_i + k\alpha)}{\prod_{k=1}^{-(d-a)_a - 1} (x + H_a - H_{a-1} - u_i + k\alpha).} \tag{7}$$

Recall that $d_a - d_{a-1} - (r_a - r_{a-1}) < 0 \Rightarrow -(d-a)_a - 1 < r_a - r_{a-1}$. Moreover, expanding (7) we see that the only remaining term is

$$(x + H_a - H_{a-1} - u_i)(x + H_a - H_{a-1} - u_i - (d-r)_a\alpha) \cdots (x + H_a - H_{a-1} - u_i$$
$$+(r_a - r_{a-1})\alpha),$$

which is equal to (5).

- $d_a - d_{a-1} \geq 0$ and $r_a - r_{a-1} < 0$. In this case, we have $(d-a)_a > 0$ and obtain the equality

$$\begin{aligned}(x + H_a - H_{a-1} - u_i) \quad &\cdot \quad \prod_{k=0}^{d_a - d_{a-1}} (x + H_a - H_{a-1} - u_i + (r_a - r_{a-1} - k)\alpha) \\ &= \frac{\prod_{k=0}^{(d-a)_a} (x + H_a - H_{a-1} - u_i - k\alpha)}{\prod_{k=1}^{r_{a-1} - r_a - 1} (x + H_a - H_{a-1} - u_i - k\alpha),}\end{aligned}$$

which is in fact

$$(x + H_a - H_{a-1} - u_i) \cdot j_r^* \prod_{k=0}^{d_a - d_{a-1}} (x + \kappa_a - \kappa_{a-1} - u_i - k\alpha) \quad = $$

$$\frac{j_0^* \displaystyle\prod_{k=0}^{(d-a)_a} (x + \kappa_a - \kappa_{a-1} - u_i - k\alpha)}{\overline{\Big(j_0^* \displaystyle\prod_{k=1}^{r_{a-1} - r_a - 1} (x + \kappa_a - \kappa_{a-1} - u_i + k\alpha)\Big)}}$$

- $d_a - d_{a-1} < 0, r_a - r_{a-1} \geq 0$. Obviously this implies $(d-r)_a < 0$. We compare

$$\frac{(x + H_a - H_{a-1} - u_i)}{j_r^* \displaystyle\prod_{k=1}^{d_{a-1} - d_a - 1} (x + \kappa_a - \kappa_{a-1} - u_i + k\alpha)} \qquad \text{and} \tag{8}$$

$$\frac{\overline{\Big(j_0^* \displaystyle\prod_{k=0}^{r_a - r_{a-1}} (x + \kappa_a - \kappa_{a-1} - u_i - k\alpha)\Big)}}{j_0^* \displaystyle\prod_{k=1}^{-(d-a)_a - 1} (x + \kappa_a - \kappa_{a-1} - u_i + k\alpha)} \tag{9}$$

If $r_a - r_{a-1} = -(d-r)_a - 1 = r_a - r_{a-1} - d_a + d_{a-1} + 1$ then we have $d_a - d_{a-1} = -1$ and no term on (8) except $(x + H_a - H_{a-1} - u_i)$ appears. Clearly only the same term survives on (9) after cancellation. Otherwise, observing $-(d-r)_a - 1 > r_a - r_{a-1}$, and expanding (9) accordingly, we obtain the equality of (8) and (9).

- $d_a - d_{a-1} < 0, r_a - r_{a-1} < 0, (d-r) \geq 0$. This time we will compare

$$\frac{(x + H_a - H_{a-1} - u_i)}{\displaystyle\prod_{k=1}^{d_{a-1} - d_a - 1} (x + H_a - H_{a-1} - u_i + (r_a - r_{a-1} + k)\alpha)} \qquad \text{and} \tag{10}$$

$$\frac{\displaystyle\prod_{k=0}^{(d-r)_a}(x+H_a-H_{a-1}-u_i-k\alpha)}{\displaystyle\prod_{k=1}^{r_{a-1}-r_a-1}(x+H_a-H_{a-1}-u_i-k\alpha).} \tag{11}$$

Observe that

$$r_{a-1}-r_a-1-(d-r)_a=-1-(d_a-d_{a-1})=\begin{cases} >0 & d_a-d_{a-1}<-1 \\ 0 & d_a-d_{a-1}=-1 \end{cases}$$

If $d_a - d_{a-1} = -1$, (10) is just $(x + H_a - H_{a-1} - u_i)$ and same for (11). Note if $d_a - d_{a-1} < -1$ then after cancellations on (11) we obtain the equality again. Finally:

- $d_a - d_{a-1} < 0, r_a - r_{a-1} < 0, (d-r)_a < 0$. Then we will have the equality of

$$\frac{(x+H_a-H_{a-1}-u_i)}{\displaystyle\prod_{k=1}^{d_{a-1}-d_a-1}(x+H_a-H_{a-1}-u_i+(r_a-r_{a-1}+k)\alpha)}$$

and

$$\frac{1}{\displaystyle\prod_{k=1}^{-(d-r)_a-1}(x+H_a-H_{a-1}-u_i+k\alpha)\cdot\prod_{k=1}^{r_{a-1}-r_a-1}(x+H_a-H_{a-1}-u_i-k\alpha)}$$

since $(d-r)_a < 0 \Rightarrow d_{a-1} - d_a > r_{a-1} - r_a$ and we will obtain the term $(x + H_a - H_{a-1} - u_i)$ in the first expression when $k = r_{a-1} - r_a$.

To summarize, we obtain for $0 \preceq r \preceq d$

$$\underbrace{\prod_{i=1}^{n}\prod_{a=1}^{n-1}(x+H_a-H_{a-1}-u_i)}_{\Gamma^1}\cdot j_r^* Q_d^1 = \overline{j_0^* Q_r^1}\cdot j_0^* Q_{d-r}^1$$

where

$$Q_d^1 = \prod_{i=1}^{n}\prod_{a=1}^{n-1} \frac{\prod_{d_a-d_{a-1}\geq 0}\prod_{k=0}^{d_a-d_{a-1}}(x+\kappa_a-\kappa_{a-1}-u_i-k\alpha)}{\prod_{d_a-d_{a-1}<0}\prod_{k=1}^{d_{a-1}-d_a-1}(x+\kappa_a-\kappa_{a-1}-u_i+k\alpha)}.$$

In fact preceding argument can easily be seen to be true for the other two parts composing Q_d. Namely, once we set

$$\Gamma_2 = \frac{1}{\prod_{i=1}^{n-1}\prod_{1\leq a,b\leq i}(x+H_{ab})}, Q_d^2 = \prod_{i=1}^{n-1}\prod_{1\leq a,b\leq i}\frac{\prod_{d_{ab}<0}\prod_{k=1}^{-d_{ab}-1}(x+\kappa_{ab}+k\alpha)}{\prod_{d_{ab}\geq 0}\prod_{k=0}^{d_{ab}}(x+\kappa_{ab}-k\alpha)}$$

and

$$\Gamma_3 = \prod_{i=1}^{n-2}\prod_{\substack{1\leq a\leq i\\ 1\leq b\leq i+1}}(x+H_{ab}), Q_d^3 = \prod_{i=1}^{n-2}\prod_{\substack{1\leq a\leq i\\ 1\leq b\leq i+1}}\frac{\prod_{d_{ab}\geq 0}\prod_{k=0}^{d_{ab}}(x+\kappa_{ab}-k\alpha)}{\prod_{d_{ab}<0}\prod_{k=1}^{-d_{ab}-1}(x+\kappa_{ab}+k\alpha)},$$

where d_{ab} and κ_{ab} are as before. Note $Q_0 = \Omega, Q_d = Q_d^1 \cdot Q_d^2 \cdot Q_d^3, \Omega = \Omega_1 \cdot \Omega_2 \cdot \Omega_3$ and combining all of above, we obtain

$$\Gamma \cdot j_r^* Q_d = \overline{j_0^* Q_r} \cdot j_0^* Q_{d-r}, 0 \preceq r \preceq d.$$

This shows that the list $Q : Q_d$ gives an Γ-Euler data and then by theorem (2.5) we obtain the desired result. □

Now we want to compute the α degree of Q_d. Observing closely we find that after possible cancellations are done $Q_d^2 \cdot Q_d^3$ can be written as

$$Q_d^2 \cdot Q_d^3 = \prod_{i=1}^{n-1} \prod_{a=1}^{i} \Big(\frac{\prod_{d_{ia}<0} \prod_{k=1}^{d_{ia}-1} (x + \kappa_{ia} + k\alpha)}{\prod_{d_{ia}\geq 0} \prod_{k=0}^{d_{ia}} (x + \kappa_{ia} - k\alpha)} \Big)$$

The α degree of this expression is less than $\sum_{i=1}^{n-1} \sum_{a=1}^{i} (-d_{ia} - 1)$. In addition,

$$deg_\alpha \Big(\prod_{i=1}^{n} \prod_{a=1}^{n-1} \frac{\prod_{d_a - d_{a-1} \geq 0} \prod_{k=0}^{d_a - d_{a-1}} (x + \kappa_a - \kappa_{a-1} - u_i - k\alpha)}{\prod_{d_a - d_{a-1} < 0} \prod_{k=1}^{d_{a-1} - d_a - 1} (x + \kappa_a - \kappa_{a-1} + k\alpha)} \Big) \leq n d_{n-1}.$$

So we obtain $deg_\alpha Q_d \leq n d_{n-1} - \sum_{i=1}^{n-1} \sum_{a=1}^{i} (d_{ia} + 1)$. Recall that $c_1(X) = 2(\mathfrak{S}_1 + \cdots + \mathfrak{S}_{n-1})$ for $X = Fl(n)$. Then

$$\langle c_1(X), d \rangle - deg_\alpha Q_d \geq 2 \sum_{i=1}^{n-1} d_i - n d_{n-1} + \sum_{i=1}^{n-1} \sum_{a=1}^{i} (d_{ia} + 1). \tag{12}$$

We know $d_{ia} = d_i - d_{i-1} - (d_a - d_{a-1})$. Then

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{a=1}^{i} (d_{ia} + 1) &= \sum_{i=1}^{n-1} i(d_i - d_{i-1}) - \sum_{i=1}^{n-1} \sum_{a=1}^{i} (d_a - d_{a-1}) \\ &= -d_1 - \cdots - d_{n-1} + n d_{n-1} - (d_1 + \cdots + d_{n-1}) \\ &= -2(d_1 + \cdots + d_{n-1}) + n d_{n-1}. \end{aligned}$$

Therefore (12) becomes

$$\langle c_1(X), d \rangle - deg_\alpha Q_d \geq \sum_{i=1}^{n-1} i \geq 0.$$

As a result we conclude that $\tau^* j_0^* Q_d$ satisfies the conditions of theorem (2.12) and we have the following theorem

Theorem 3.5 *Let $X = Fl(n)$ and $V = TX$ be the equivariant tangent bundle. The A-series $A^{V,b_{\mathbb{T}}(X)}$ with equivariant Chern polynomial $b_{\mathbb{T}}$ can be computed as*

$$A(t+g) = e^{f/\alpha} B(t)$$

where $B(t) = e^{-y\cdot t/\alpha} \sum_d \tau^* j_0^* Q_d \cap \mathbb{I}_d$ and f, g are formal power series given as in theorem (2.12). □

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