Mirror symmetry and Langlands duality in the non-Abelian Hodge theory of a curve

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Abstract

This is a survey of results and conjectures on mirror symmetry phenomena in the non-Abelian Hodge theory of a curve. We start with the conjecture of Hausel-Thaddeus which claims that certain Hodge numbers of moduli spaces of flat $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$-connections on a smooth projective algebraic curve agree. We then change our point of view in the non-Abelian Hodge theory of the curve, and concentrate on the $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ character varieties of the curve. Here we discuss a recent conjecture of Hausel-Rodriguez-Villegas which claims, analogously to the above conjecture, that certain Hodge numbers of these character varieties also agree. We explain that for Hodge numbers of character varieties one can use arithmetic methods, and thus we end up explicitly calculating, in terms of Verlinde-type formulas, the number of representations of the fundamental group into the finite groups $SL(n,\mathbb{F}_q)$ and $PGL(n,\mathbb{F}_q)$, by using the character tables of these finite groups of Lie type. Finally we explain a conjecture which enhances the previous result, and gives a simple formula for the mixed Hodge polynomials, and in particular for the Poincaré polynomials of these character varieties, and detail the relationship to results of Hitchin, Gothen, Garsia-Haiman and Earl-Kirwan. One consequence of this conjecture is a curious Poincaré duality type of symmetry, which leads to a conjecture, similar to Faber's conjecture on the moduli space of curves, about a strong Hard Lefschetz theorem for the character variety, which can be considered as a generalization of both the Alvis-Curtis duality in the representation theory of finite groups of Lie type and a recent result of the author on the quaternionic geometry of matroids.

1 Introduction

Non-Abelian Hodge theory ([Hi1], [Si1]) of a genus g smooth complex projective curve C studies three moduli spaces attached to C and a reductive complex algebraic group G, which in this paper will be either $GL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$ or $PGL(n,\mathbb{C})$. They are $\mathcal{M}^d_{Dol}(G)$, the moduli space of semistable G-Higgs bundles on C; $\mathcal{M}^d_{DR}(G)$, the moduli space of flat G-connections on C and $\mathcal{M}^d_B(G)$ the character variety, i.e. the moduli space of representations of $\pi_1(C)$ into G modulo conjugation. With some assumptions these moduli spaces are smooth varieties (or orbifolds when $G = PGL(n, \mathbb{C})$) with the underlying differentiable manifolds canonically identified, which carries a natural hyperkähler metric.

The cohomology of this underlying manifold has mostly been studied from the perspective of $\mathcal{M}^d_{Dol}(G)$. Using a natural circle action on it [Hi1] and [Go] calculated the Poincaré polynomials for $G = SL(2, \mathbb{C})$ and $G = SL(3, \mathbb{C})$ respectively; while [HT1] and [Ma] found a simple set of generators for the cohomology ring for $G = PGL(2, \mathbb{C})$ and $G = PGL(n, \mathbb{C})$ respectively. The paper [HT2] then calculated the cohomology ring explicitly for $G = PGL(2, \mathbb{C})$. The techniques used in these papers do not seem to generalize easily to higher n.

A new perspective for these investigations on the cohomology of $\mathcal{M}^d_{Dol}(G)$ and $\mathcal{M}^d_{DR}(G)$ was introduced in [HT3] and [HT4]. It was shown there that the hyperkähler metrics and the Hitchin systems [Hi2] for $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_{DR}(G^L)$, with $G = SL(n, \mathbb{C})$ and Langlands dual $G^L = PGL(n, \mathbb{C})$ provide the geometrical setup suggested in [SYZ] as a criteria for mirror symmetry. Based on this observation [HT4] conjectured that a version of the topological mirror symmetry also holds, i.e. that certain Hodge numbers for $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_{DR}(G^L)$ agree. Using the above mentioned results of [Hi1] and [Go] this conjecture was checked for $G = SL(2, \mathbb{C}), SL(3, \mathbb{C})$. This mirror symmetry conjecture motivates the study of not just the cohomology but the mixed Hodge structure on the cohomology of the spaces $\mathcal{M}^d_{DR}(G)$, $\mathcal{M}^d_{Dol}(G)$ and $\mathcal{M}^d_B(G)$. While it was shown in [HT4] that the mixed Hodge structure on $\mathcal{M}^d_{Dol}(G)$ and $\mathcal{M}^d_{DR}(G)$ agree, and can be shown to be pure as in Theorem 2.1 or [Me], the mixed Hodge structure on $\mathcal{M}^d_B(G)$ has not been studied until very recently.

An important theme of this survey paper is in fact the mixed Hodge structure on the character variety $\mathcal{M}^d_B(G)$ or alternatively the three variable polynomial $H(x, y, t)$ the so-called mixed Hodge polynomial or shortly H-polynomial which encodes the dimensions of the graded pieces of the mixed Hodge structure on $\mathcal{M}^d_B(G)$. In a recent project [HRV] an arithmetic method was used to calculate the E-polynomial (where the E-polynomial $E(x, y)$ for a smooth variety is defined as $x^n y^n H(1/x, 1/y, -1)$, where n is the complex dimension of the variety) of $\mathcal{M}^d_B(G)$. The idea of [HRV] is to count the rational points of $\mathcal{M}^d_B(G(\mathbb{F}_q))$, the variety $\mathcal{M}^d_B(G)$ over the finite field $\mathbb{F}_q$, where q is a prime power. This count then is possible due to a result of [FQ], and the resulting formula, which resembles the famous Verlinde formula [Ve], is a simple sum over irreducible representations of the finite group of Lie type $G(\mathbb{F}_q)$. Thus the representation theory behind the E-polynomial of the character variety is that of the finite groups of Lie type, which could be considered as an analogue of Nakajima's principle [Na], which states that the representation theory of Kac-Moody algebras are encoded in the cohomology of the (hyperkähler) quiver varieties.

The shape of the E-polynomials of the various character varieties then made us conjecture [HRV] that the mirror symmetry conjecture also holds for the pair $\mathcal{M}^d_B(G)$ and $\mathcal{M}^d_B(G^L)$ in the case of $G = SL(n, \mathbb{C})$ at least. Due to our improved ability to calculate these Hodge numbers via this number theoretical method, we could check this conjecture in the cases when n is 4 or a prime. As the two mirror symmetry conjectures of [HT4] and [HRV] are equivalent on the level of Euler characteristic, we get a proof of the original mirror symmetry conjecture of [HT4] on the level of Euler characteristic in these cases.

Perhaps even more interestingly [HRV] achieves explicit formulas, in terms of a simple generating function, for the E-polynomials of the character variety $\mathcal{M}^d_B(GL(n, \mathbb{C}))$. In particular it can be deduced from this that the Euler characteristic of $\mathcal{M}^d_B(PGL(n, \mathbb{C}))$ is $\mu(n)n^{2g-3}$, where μ is the fundamental number theoretic function: the Mobius function, i.e. the sum of all primitive nth roots of unity. This result, which could not be obtained with other methods, in itself hints at an interesting link between number theory and the topology of our hyperkähler manifolds.

Another consequence of our formula, is that these E-polynomials turn out to be palindromic, i.e. satisfy an unexpected Poincaré duality-type of symmetry. In fact this symmetry can be traced back to the Alvis-Curtis duality [Al, Cu] in the representation theory of finite groups of Lie type.

Finally, a deformation of the formula for the E-polynomial of the character variety $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ is presented, which conjecturally [HRV] should agree with the H-polynomial. Moreover we will later modify this formula to obtain what conjecturally should be the H-polynomial of the Higgs moduli space $P\mathcal{M}_{Dol}^d(GL(n,\mathbb{C}))$. We also explain how, using, as a guide, our mirror symmetry conjectures, one could get conjectures for the corresponding H-polynomials for the varieties associated to $SL(n,\mathbb{C})$.

These conjectures imply a conjecture on the Poincaré polynomials (where the Poincaré polynomial is obtained from the H-polynomial as $H(1,1,t)$) of our manifolds $\mathcal{M}_{Dol}^d(PGL(n,\mathbb{C}))$. This conjecture is similar in flavour to Lusztig's conjecture [Lu] on the Poincaré polynomials of Nakajima's quiver varieties, which is also a hyperkähler manifold, similar to the Higgs moduli space $\mathcal{M}_{Dol}^d(G)$. We should also mention Zagier's [Zag] formula for the Poincaré polynomial of the moduli space $\mathcal{N}^d$ of stable bundles (the "Kähler version" of $\mathcal{M}_{Dol}^d(SL(n,\mathbb{C}))$), where the formula is a similar sum, but is parametrized by ordered partitions of n.

We discuss in detail many checks on these conjectures, by showing how our conjectures imply results obtained by Hitchin [Hi1], Gothen [Go] and Earl-Kirwan [EK]. Already the combinatorics of these formulas are non-trivial, and surprisingly the calculus of Garsia-Haiman [GH] needs to be used to check the conjecture when $g = 0$.

Since a curious Poincaré duality type of symmetry is satisfied for the conjectured Hodge numbers of $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$, we also discuss the conjecture that a certain version of the Hard Lefschetz theorem is satisfied for our non-compact varieties. This is then explained to be a generalization of a result in [Ha3] on the quaternionic geometry of matroids, and an analogue of the Faber conjecture [Fa] on the moduli space of curves.

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2 Abelian and non-Abelian Hodge theory

This section gives some basic definitions on Abelian and non-Abelian Hodge theory which will be used in the paper later. For details on them consult the sources indicated below.

2.1 Hodge-De Rham theory. Fix a smooth complex algebraic variety M. There are various cohomology theories which associate a graded anti-commutative ring to the variety M. First we consider the singular, or Betti, cohomology $H_B^*(M,\mathbb{C})$ of M with complex coefficients. This in fact can be defined for any reasonable topological space. The dimension of $H_B^k(M,\mathbb{C})$ is called the k-th Betti number and denoted $b_k(M)$. The Poincaré polynomial is then formed from these numbers as coefficients:

$$P(t;M) = \sum_k b_k(M)t^k.$$

Next we consider the De Rham cohomology $H^*_{DR}(M, \mathbb{C})$, which is the space of closed differential forms modulo exact ones. This can be defined on any differentiable manifold. The De Rham theorem then shows that these two cohomologies are naturally isomorphic:

$$H^*_B(M, \mathbb{C}) \cong H^*_{DR}(M, \mathbb{C}). \tag{1}$$

Now we assume that our variety is projective. Then we have the Dolbault cohomology $H^*_{Dol}(M, \mathbb{C})$, which is defined as

$$H^k_{Dol}(M, \mathbb{C}) = \bigoplus_{p+q=k} H^q(M, \Omega^p_M).$$

The Hodge theorem then implies that there is a natural isomorphism

$$H^k_{DR}(M, \mathbb{C}) \cong H^k_{Dol}(M, \mathbb{C}). \tag{2}$$

The above two isomorphisms then imply the Hodge decomposition theorem:

$$H^k_B(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M), \tag{3}$$

where $H^{p,q}(M)$ denotes $H^p(M, \Omega^q_M)$. The dimension of $H^{p,q}(M)$ is denoted $h^{p,q}(M)$ and called the Hodge numbers of the variety M. From these numbers we form a two variable polynomial the Hodge polynomial:

$$H(x, y; M) := \sum_{p,q} h^{p,q}(M) x^p y^q.$$

For more details on these cohomology theories see [GrH], while the approach above, which will be suitable later for non-Abelian cohomology, is described in more detail in [KP][Section 3].

2.2 Mixed Hodge structures. Deligne [De2] generalized the Hodge decomposition theorem (3) to any complex variety M, not necessarily smooth or projective, by introducing a so called *mixed Hodge structure* on $H^*_B(M, \mathbb{C})$. This implies a decomposition

$$H^k_B(M, \mathbb{C}) \cong \bigoplus_{p,q} H^{p,q;k}(M),$$

where $p + q$ is called the weight of $H^{p,q;k}(M)$. In the case of a smooth projective variety we have $H^{p,q;k}(M) = H^{p,q}(M)$, i.e. that the weight of $H^{p,q;k}(M)$ is always k, this weight is called *pure weight*. However in general we could have other weights appear in the mixed Hodge structure of a complex algebraic variety; indeed we will see such examples later. The dimensions of $H^{p,q;k}(M)$ are denoted by $h^{p,q;k}(M)$ and are called mixed Hodge numbers. From them we form the three variable polynomial

$$H(x, y, t; M) := \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k. \tag{4}$$

Similarly, Deligne [De2] constructs a mixed Hodge structure on the compactly supported $H^*_{B,cpt}(M, \mathbb{C})$ singular cohomology of our complex algebraic variety M. This yields the decomposition

$$H^k_{B,cpt}(M, \mathbb{C}) \cong \bigoplus_{p,q} H^{p,q;k}_{cpt}(M),$$

and leads to the compactly supported mixed Hodge numbers $h^{p,q;k}_{cpt}(M)$, which is defined as the dimension of $H^{p,q;k}_{cpt}(M)$. Then one can introduce the e-numbers $e^{p,q}(M) = \sum_k (-1)^k h^{p,q;k}_{cpt}(M)$ from which we get the E-polynomial:

$$E(x, y; M) := \sum_{p,q} e^{p,q}(M) x^p y^q. \tag{5}$$

Clearly for a smooth projective variety $E(x, y) = H(x, y)$. Moreover for a smooth variety Poincaré duality implies that

$$E(x, y) = (xy)^n H(1/x, 1/y, -1),$$

where n is the complex dimension of M. The significance of the E-polynomial is that it is additive for decompositions and multiplicative for Zariski locally trivial fibrations.

For more details see the original [De2] or [BD] for more on the E-polynomials.

2.3 Stringy cohomology. Suppose a finite group Γ acts on our variety M. Then by the naturality of the mixed Hodge structure Γ will act on $H^{p,q,k}(M)$ and we have

$$H^{p,q;k}(M/\Gamma) \cong \left(H^{p,q;k}(M)\right)^\Gamma.$$

However for a Calabi-Yau M and Γ preserving the Calabi-Yau structure string theorists [Va, Zas] introduced different Hodge numbers on the Calabi-Yau orbifold M/Γ: the so-called stringy Hodge numbers, which are the right Hodge numbers for mirror symmetry. Their mathematical significance is highlighted by a theorem of Kontsevich [Ko] that the stringy Hodge numbers agree with the ordinary Hodge numbers of any crepant resolution. Following [BD] we can define the stringy E-polynomials:

$$E_{\mathrm{st}}(x, y; M/\Gamma) = \sum_{[\gamma]} E(x, y; M^\gamma)^{C(\gamma)} (xy)^{F(\gamma)}.$$

Here the sum runs over the conjugacy classes of Γ; $C(\gamma)$ is the centralizer of γ; M^γ is the subvariety fixed by γ; and $F(\gamma)$ is an integer called the *fermionic shift*, which is defined as follows. The group element γ has finite order, so it acts on $TM|_{M^\gamma}$ as a linear automorphism with eigenvalues $e^{2\pi i w_1}, \ldots, e^{2\pi i w_n}$, where each $w_j \in [0, 1)$. Let $F(\gamma) = \sum w_j$; this is an integer since, by hypothesis, γ acts trivially on the canonical bundle.

The last cohomology theory we will need is the stringy cohomology of a Calabi-Yau orbifold twisted by a B-field. Following [Hi3] we let $B \in H^2_\Gamma(M, U(1))$ i.e. an isomorphism class of a Γ-equivariant flat unitary gerbe. For any element $\gamma \in \Gamma$ this B-field induces a $C(\gamma)$-invariant local system $[L_{B,\gamma}] \in H^1(M^\gamma/C(\gamma), U(1))$ on the fixed point set M^γ. Using this we can twist the stringy E-polynomial of M/Γ to get:

$$E^B_{\mathrm{st}}(x, y; M/\Gamma) = \sum_{[\gamma]} E(x, y; M^\gamma/\mathbb{C}(\gamma); L_{B,\gamma})(xy)^{F(\gamma)}, \tag{6}$$

where $E(M^\gamma/\mathbb{C}(\gamma); L_{B,\gamma})$ is defined from the mixed Hodge structure on $H^*_{cpt}(M^\gamma/\mathbb{C}(\gamma); L_{B,\gamma})$ analogously to (5).

For more information on the mathematics of stringy cohomology see [BD], for twisting with a B-field see [HT4].

2.4 Non-Abelian Hodge theory. The starting point of non-Abelian Hodge theory is the identification of the space $H^1_B(M,\mathbb{C}^\times)$ with the space of homomorphisms from $\pi_1(M)\to\mathbb{C}^\times$; the space $H^1_{DR}(M,\mathbb{C}^\times)$ with algebraic local systems on M and the space

$$H_{Dol}(M,\mathbb{C}^\times)\cong H^1(M,\mathcal{O}^\times)\oplus H^0(M,\Omega^1)$$

with pairs of a holomorphic line bundle and a holomorphic one form.

This then can be generalized to any non-Abelian complex reductive group G. We define $H^1_B(M,G)$ to be conjugacy classes of representations of $\pi_1(M)\to G$. I.e.

$$H^1_B(M,G):=\mathrm{Hom}(\pi_1(M),G)//G,$$

which is the affine GIT quotient of the affine variety $\mathrm{Hom}(\pi_1(M),G)$ by the conjugation action by G. This is sometimes called the *character variety*. The space $H^1_{DR}(M,G)$ can be identified as the moduli space of algebraic G-local systems on M. Finally $H^1_{Dol}(M,G)$ is defined as the moduli space of certain semistable G-Higgs bundles on M. We will give precise definition in the case of a curve below. The identification between $H^1_B(M,G)$ and $H^1_{DR}(M,G)$, which is analogous to the De Rham map (1), is given by the Riemann-Hilbert correspondence [De1, Si4], while the identification between $H^1_{DR}(M,G)$ and $H^1_{Dol}(M,G)$, which is analogous to the Hodge decomposition (2), is given in [Co, Si2] by the theory of harmonic bundles, which is the non-Abelian generalization of Hodge theory.

For an introduction to non-Abelian Hodge theory see [Si1], and [KP][Section 3], for more details on the construction of the spaces appearing in non-Abelian Hodge theory and the maps between them see [Si2, Si3, Si4].

2.5 The case of a curve. We now fix a smooth projective complex curve C of genus g and specify our spaces in the case when $M=C$ and $G=GL(n,\mathbb{C})$. According to the above definitions we have:

$$\begin{aligned}\mathcal{M}_B(GL(n,\mathbb{C})) &:= H^1_B(C,GL(n,\mathbb{C}))=\\ &= \{A_1,B_1,\ldots,A_g,B_g\in GL(n,\mathbb{C})|[A_1,B_1]\cdot\dots\cdot[A_g,B_g]=Id\}//GL(n,\mathbb{C}).\end{aligned}$$

There is a natural way to twist these varieties, and because they will be needed for $PGL(n,\mathbb{C})$ we introduce these twisted varieties here. For an integer d we also consider:

$$\mathcal{M}^d_B(GL(n,\mathbb{C})):=\{A_1,B_1,\ldots,A_g,B_g\in GL(n,\mathbb{C})|[A_1,B_1]\cdot\dots\cdot[A_g,B_g]=e^{\frac{2\pi id}{n}}Id\}//GL(n,\mathbb{C}).$$

The De-Rham space looks like

$$\begin{aligned}\mathcal{M}_{DR}(GL(n,\mathbb{C})) &:= H^1_{DR}(C,GL(n,\mathbb{C}))=\\ &= \{\text{moduli space of flat } GL(n,\mathbb{C})\text{-connections on } C\}\end{aligned}$$

and in the twisted case we need to fix a point $p\in C$, and define

$$\mathcal{M}^d_{DR}(GL(n,\mathbb{C})):=\left\{\begin{array}{c}\text{moduli space of flat } GL(n,\mathbb{C})\text{-connections on } C\setminus\{p\},\\ \text{with holonomy } e^{\frac{2\pi id}{n}}Id \text{ around } p\end{array}\right\}.$$

Finally the Dolbeault spaces are:

$$\begin{aligned}\mathcal{M}_{Dol}(GL(n,\mathbb{C})) &:= H^1_{Dol}(C,GL(n,\mathbb{C}))=\\ &= \{\text{moduli space of semistable rank } n \text{ degree 0 Higgs bundles on } C\},\end{aligned}$$

where a rank n Higgs bundle is a pair (E, ϕ) of a rank n algebraic vector bundle E on C, with degree 0 and Higgs field $\phi \in H^0(C, K\mathrm{End}E)$. A Higgs bundle is called semistable if for any Higgs subbundle (F, ψ) (i.e. a subbundle with compatible Higgs fields) we have $\frac{\deg(F)}{\mathrm{rank}(F)} \leq \frac{\deg(E)}{\mathrm{rank}(E)} = 0$. The twisted version of $\mathcal{M}_{Dol}(GL(n, \mathbb{C}))$ is defined:

$$\mathcal{M}^d_{Dol}(GL(n, \mathbb{C})) := \{ \text{ moduli space of semistable rank } n \text{ degree } d \text{ Higgs bundles on } C\}.$$

The varieties defined above for $GL(n, \mathbb{C})$ are all of dimension $n^2(2g-2)+2$. The Betti space is affine, while the De Rham space is analytically isomorphic, via the Riemann-Hilbert correspondence, to the Betti space but not algebraically, so the De Rham space is a Stein manifold as a complex manifold but not an affine variety as an algebraic variety. Finally the Dolbeault space is a quasi projective variety with large projective subvarieties.

From now on we will consider only the case when $(n, d) = 1$; so we fix such a d. In this case the corresponding twisted spaces are additionally smooth, have a diffeomorphic underlying manifold $\mathcal{M}^d(GL(n, \mathbb{C}))$ which carries a complete hyperkähler metric [Hi1]. The complex structures of $\mathcal{M}^d_{Dol}(GL(n, \mathbb{C}))$ and $\mathcal{M}^d_{DR}(GL(n, \mathbb{C}))$ appear in the hyperkähler structure.

We started this subsection by determining these spaces for $GL(1, \mathbb{C}) \cong \mathbb{C}^\times$. By the identifications explained there we see that

$$\begin{aligned} {\mathcal{M}^d}_B(GL(1, \mathbb{C})) &\cong (\mathbb{C}^\times)^{2g}, \\ \mathcal{M}^d_{Dol}(GL(1, \mathbb{C})) &\cong T^*Jac^d(C) \end{aligned} \tag{7}$$

and $\mathcal{M}^d_{DR}$ is a certain affine bundle over $Jac^d(C)$. Interestingly for $d = 0$ they are all algebraic groups and they act on the corresponding spaces for $GL(n, \mathbb{C})$ and any d by tensorization.

We can consider the map

$$\begin{array}{rccc} \lambda_{Dol}: & \mathcal{M}^d_{Dol}(GL(n, \mathbb{C})) & \to & \mathcal{M}^d_{Dol}(GL(1, \mathbb{C})) \\ & (E, \Phi) & \mapsto & (\det(E), \mathrm{tr}(\phi)). \end{array}$$

The fibres of this map can be shown to be isomorphic using the above tensorization action. So up to isomorphism it is irrelevant which fibre we take, but we usually take a point $(\Lambda, 0) \in \mathcal{M}^d_{Dol}(GL(1, \mathbb{C}))$ and define

$$\mathcal{M}^d_{Dol}(SL(n, \mathbb{C})) := \lambda^{-1}_{Dol}((\Lambda, 0)),$$

for the other two spaces we have:

$$\mathcal{M}^d_{DR}(SL(n, \mathbb{C})) = \left\{ \begin{array}{c} \text{moduli space of flat } SL(n, \mathbb{C})\text{-connections on } C \setminus \{p\} \\ \text{with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\},$$

and

$$\mathcal{M}^d_B(SL(n, \mathbb{C})) = \{A_1, B_1, \dots, A_g, B_g \in SL(n, \mathbb{C}) | [A_1, B_1] \cdot \dots \cdot [A_g, B_g] = e^{\frac{2\pi i d}{n}} Id\} // SL(n, \mathbb{C}).$$

The varieties $\mathcal{M}^d_B(SL(n, \mathbb{C}))$, $\mathcal{M}^d_{DR}(SL(n, \mathbb{C}))$ and $\mathcal{M}^d_{DR}(SL(n, \mathbb{C}))$ are smooth of dimension $(n^2-1)(2g-2)$, with diffeomorphic underlying manifold $\mathcal{M}^d(SL(n, \mathbb{C}))$. The Betti space is affine, and the Betti and De Rham spaces are again analytically, but not algebraically, isomorphic.

Moreover we see that a finite subgroup namely $Jac[n] \cong \mathbb{Z}_n^{2g} \subset \mathcal{M}_{Dol}(GL(1,\mathbb{C}))$ preserves the fibration λ_{Dol} and thus acts on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. The quotient then is:

$$\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C})) := \mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))/\mathrm{Jac}[n]$$

and similarly

$$\mathcal{M}^d_{DR}(PGL(n,\mathbb{C})) := \mathcal{M}^d_{DR}(SL(n,\mathbb{C}))/\mathrm{Jac}[n],$$

and

$$\mathcal{M}^d_{B}(PGL(n,\mathbb{C})) = \mathcal{M}^d_{B}(SL(n,\mathbb{C}))/\mathbb{Z}_n^{2g}.$$

This shows that the spaces $\mathcal{M}^d_B(PGL(n,\mathbb{C}))$, $\mathcal{M}^d_{DR}(PGL(n,\mathbb{C}))$ and $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$ are hyperkähler orbifolds of dimension $(n^2-1)(2g-2)$. As they are orbifolds we can talk about their stringy mixed Hodge numbers as defined above in 2.2. Moreover there are natural orbifold B-fields on them, which we construct now. First we consider a universal Higgs-pair $(\mathbf{E},\mathbf{\Phi})$ on $\mathcal{M}^d_{\mathrm{Dol}}(SL(n,\mathbb{C}))\times C$, it exists because $(d,n)=1$. Restrict $\mathbf{E}$ to $\mathcal{M}^d_{\mathrm{Dol}}\times\{p\}$ to get the vector bundle $\mathbf{E}_p$ on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. Now we can consider the projective bundle $\mathbb{P}\mathbf{E}_p$ of $\mathbf{E}_p$ which is a $PGL(n,\mathbb{C})$-bundle. The bundle $\mathbf{E}_p$ is a $GL(n,\mathbb{C})$ bundle but not a $SL(n,\mathbb{C})$-bundle, because it has non-trivial determinant. The obstruction class to lift the $PGL(n,\mathbb{C})$ bundle $\mathbb{P}\mathbf{E}$ to an $SL(n,\mathbb{C})$ bundle is a class $B\in H^2(\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}),\mathbb{Z}_n)\subset H^2(\mathcal{M}^d_{Dol}(SL(n,\mathbb{C})),U(1))$, which gives us our B-field on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. Moreover B has [HT4][Section 3] a natural equivariant extension $\hat{B}\in H^2_\Gamma(\mathcal{M}^d_{Dol}(SL(n,\mathbb{C})),\mathbb{C})$, giving us our B-field on $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$. This B-field will come handy for our mirror symmetry discussions below.

For non-Abelian Hodge theory on a curve, see [Hi1], which gives a gauge theoretical approach, and yields the natural hyperkähler metrics on our spaces. On the geometry and cohomology of $\mathcal{M}^1_{Dol}(SL(2,\mathbb{C}))$ see [Ha2].

2.6 The mixed Hodge structure on non-Abelian Hodge cohomologies. The main subject of this survey paper is the mixed Hodge polynomial of the (sometimes stringy, sometimes with a B-field) cohomology of the spaces $\mathcal{M}^d_{Dol}(G)$, $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_B(G)$, for our three choices for $G=GL(n,\mathbb{C}),PGL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$. As a notational convenience we may omit G and simply write $\mathcal{M}^d_B$, $\mathcal{M}^d_{DR}$ and $\mathcal{M}^d_{Dol}$, when it is clear what G should be, or if G could be any of our groups.

First take $G=GL(1,\mathbb{C})$. From (7) we can easily calculate the mixed Hodge polynomials as follows:

$$H(x,y,t;\mathcal{M}^d_B(GL(1,\mathbb{C}))) = (1+xyt)^{2g}$$
$$H(x,y,t;\mathcal{M}^d_{Dol}(GL(1,\mathbb{C}))) = H(x,y,t;\mathcal{M}^d_{DR}(GL(1,\mathbb{C}))) = (1+xt)^g(1+yt)^g.$$

It is remarkable that $H(x,y,t;\mathcal{M}^d_B(GL(1,\mathbb{C})))\neq H(x,y,t;\mathcal{M}^d_{DR}(GL(1,\mathbb{C})))$ even though the spaces are analytically isomorphic. Moreover we can explicitly see that the mixed Hodge structure on $H^k(\mathcal{M}^d_{Dol}(GL(1,\mathbb{C})),\mathbb{C})$ and $H^k(\mathcal{M}^d_{DR}(GL(1,\mathbb{C})),\mathbb{C})$ are pure, while on $H^k(\mathcal{M}^d_B(GL(1,\mathbb{C})),\mathbb{C})$ it is not.

From a Künneth argument we also see that:

$$H(x,y,t;\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))) = H(x,y,t;\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C})))H(x,y,t;\mathcal{M}^d_{Dol}(GL(1,\mathbb{C}))),$$

and similarly for the other two spaces. Thus the calculation for $GL(n,\mathbb{C})$ is equivalent with the calculation for $PGL(n,\mathbb{C})$.

Now we list what we know about the cohomologies $H^*(\mathcal{M}^d,\mathbb{C})$. The Poincaré polynomials $P(t;\mathcal{M}^1(SL(2,\mathbb{C})))$ and $P(t;\mathcal{M}^1(PGL(2,\mathbb{C})))$ were calculated in [Hi1], while the polynomials $P(t;\mathcal{M}^1(SL(3,\mathbb{C})))$ and $P(t;\mathcal{M}^1(PGL(3,\mathbb{C})))$ have been calculated in [Go]. In both papers Morse theory for a natural $\mathbb{C}^\times$ action on $\mathcal{M}^d_{Dol}$ was used (acting by multiplication of the Higgs field). The idea is to calculate the Poincaré polynomial of the various fixed point components of this action, and then sum them up with a certain shift. The largest of the fixed point components, when $\phi=0$, is an important and well-studied space itself so we define it here as

$$\mathcal{N}^d(SL(n,\mathbb{C})) := \\ \{\text{the moduli space of stable vector bundles of fixed determinant bundle of degree d}\}. \quad (8)$$

The Poincaré polynomial of this space was calculated in [HN] by arithmetic and in [AB] by gauge theoretical methods with explicit formulas in [Zag]. Thus its contribution to $P(t;\mathcal{M}^d(SL(n,\mathbb{C})))$ is easy to handle. However the other components of the fixed point set of the natural circle action is more cumbersome to determine already when $n=4$, and consequently this Morse theory approach has not been completed for $n\geq 4$.

As a running example here we calculate from [Hi1] the Poincaré polynomial of $\mathcal{M}^1(PGL(2,\mathbb{C}))$ when $g=3$:

$$P(t;\mathcal{M}^1(PGL(2,\mathbb{C}))) = \\ = 3\,t^{12}+12\,t^{11}+18\,t^{10}+32\,t^9+18\,t^8+12\,t^7+17\,t^6+6\,t^5+2\,t^4+6\,t^3+t^2+1 \quad (9)$$

The cohomology ring of $\mathcal{M}^1_{Dol}(PGL(2,\mathbb{C}))$ has been described explicitly by generators [HT1] and relations [HT2]. A result which proved to be essential to produce our main Conjecture 5.1. Finally Markman [Ma] showed that for $PGL(n,\mathbb{C})$ the universal cohomology classes do generate the cohomology ring.

Considering the mixed Hodge structure on the cohomology of our spaces the following result first appeared in [Me] using a construction of [HT1]. Here we present a simple proof.

Theorem 2.1 *The mixed Hodge structure on $H^k(\mathcal{M}^d_{Dol},\mathbb{C})$ is pure of weight k.*

Proof: Recall the compactification $\overline{\mathcal{M}}^d_{Dol}$ of $\mathcal{M}^d_{Dol}$ constructed in [Ha1]. From that paper it follows that $\overline{\mathcal{M}}^d_{Dol}$ is a projective orbifold, so its mixed Hodge structure on $H^k(\overline{\mathcal{M}}^d_{Dol},\mathbb{C})$ is pure of weight k. Now [Ha1] also implies that the natural map $H^*(\overline{\mathcal{M}}^d_{Dol},\mathbb{C})\to H^*(\mathcal{M}^d_{Dol},\mathbb{C})$ is surjective. The functoriality of mixed Hodge structures [De2] completes the proof. □

One can similarly prove the same result for $\mathcal{M}^d_{DR}$.

Theorem 2.2 *The mixed Hodge structure on $H^k(\mathcal{M}^d_{DR},\mathbb{C})$ is pure of weight k.*

Proof: As explained in [HT4][Theorem 6.2] one can deform the complex structure of $\overline{\mathcal{M}}^d_{Dol}$ to the projective orbifold $\overline{\mathcal{M}}^d_{DR}$, which is the compactification of $\mathcal{M}^d_{DR}$ given by Simpson in [Si5]. Now this way we see that the natural map $H^*(\overline{\mathcal{M}}^d_{DR},\mathbb{C})\to H^*(\mathcal{M}^d_{DR},\mathbb{C})$ is a surjection, getting our result as in the previous proof. □

In fact the argument in [HT4][Theorem 6.2] shows that

Theorem 2.3 (HT4) *The mixed Hodge structure on $H^*(\mathcal{M}^d_{Dol}, \mathbb{C})$ is isomorphic with the mixed Hodge structure on $H^*(\mathcal{M}^d_{DR}, \mathbb{C})$.*

The mixed Hodge structure on $\mathcal{M}^d_B$ however has not been studied in the literature. We will start the study of it later in this paper, where we will see that this Hodge structure will in fact be very much not pure. But for now we explain our reason to be interested in these mixed Hodge structures on the spaces $\mathcal{M}^d_{Dol}$, $\mathcal{M}^d_{DR}$ and $\mathcal{M}^d_B$. The reason is mirror symmetry:

3 Mirror symmetry conjectures

Our starting point in [HT4] was the observation that the pair $\mathcal{M}^d_{DR}(SL(n,\mathbb{C}))$ together with the B-field B^e and $\mathcal{M}^d_{DR}(PGL(n,\mathbb{C}))$ with the B-field $\hat{B}^d$ satisfy the geometric picture for mirror symmetry conjectured by Strominger-Yau-Zaslow [SYZ] and modified for B-fields by Hitchin in [Hi3]. This geometric picture requires the existence of a special Lagrangian fibration on both spaces, so that the fibres are dual. In fact in [HT4] it is shown that the so-called Hitchin map [Hi2] provides the required special Lagrangian fibration on our spaces, with dual Abelian varieties as fibers. For details on this see [HT4][Section 3].

Our focus in this survey is the topological implications of this mirror symmetry. The following conjecture is what we call the topological mirror test for our SYZ-mirror partners.

Conjecture 3.1 ([HT4]) *For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$, we have*

$$E^{B^e}_{\mathrm{st}}\big(x, y; \mathcal{M}^d_{\mathrm{DR}}(SL(n,\mathbb{C}))\big) = E^{\hat{B}^d}_{\mathrm{st}}\big(x, y; \mathcal{M}^e_{\mathrm{DR}}(PGL(n,\mathbb{C}))\big).$$

Remark 3.1.1 Since $M^d_{\mathrm{DR}}(SL(n,\mathbb{C}))$ is smooth, the left-hand side actually equals $E\big(M^d_{\mathrm{DR}}(SL(n,\mathbb{C}))\big)$, which is independent of e. This motivates the following:

Conjecture 3.2 (HT4) *For any two d_1 and d_2 as long as $(d_1, n) = (d_2, n) = 1$ we have:*

$$E\big(x, y; \mathcal{M}^{d_1}_{Dol}(SL(n,\mathbb{C}))\big) = E\big(x, y; \mathcal{M}^{d_2}_{Dol}(SL(n,\mathbb{C}))\big). \tag{10}$$

This, if so, is quite surprising as the Betti numbers of $\mathcal{N}^d(SL(n,\mathbb{C}))$, the moduli space of stable vector bundles, with fixed determinant of degree d, which can be considered as the "Kähler version" of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$, is known to depend on d. Already when $n = 5$, Zagier's explicit formula [Zag] for $P(t; \mathcal{N}^1(SL(5,\mathbb{C})))$ and $P(t; \mathcal{N}^3(SL(5,\mathbb{C})))$ are different. We will see strong support for this Conjecture 3.2 later in Corollary 3.4.

Remark 3.2.2 Conjecture 3.1 was proved for $n = 2$ and $n = 3$ in [HT4]. The proof proceeds by first transforming the calculation to $\mathcal{M}^d_{Dol}$ via Theorem 2.3 and then uses the Morse theoretic method of [Hi1] and [Go]. It is unclear, however, how this method can be extended for $n \geq 4$.

Remark 3.2.3 An important ingredient of the proofs was a modification of a result of Narasimhan-Ramanan [NR] to Higgs bundles. It describes the fixed points of the action by the elements of $Jac[n]$ on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. The fixed point sets will be some lower rank $m|n$ Higgs moduli spaces $\mathcal{M}^d_{Dol}(SL(m,\mathbb{C}); \tilde{C})$ for a certain covering $\tilde{C}$ of C. Their cohomology enters in the stringy contribution to the right hand side of Conjecture 3.1 (recall (6)).

3.1 Number theory to the rescue. Although our mirror symmetry Conjecture 3.1 is still open for $n \geq 4$, recently some support for the validity of it has been achieved in form of progress on another related conjecture.

Conjecture 3.3 (HRV) *For all $d, e \in \mathbb{Z}$, such that $(d, n) = (e, n) = 1$, we have*

$$E_{\mathrm{st}}^{B^e}\big(x, y, \mathcal{M}_{\mathrm{B}}^d(SL(n, \mathbb{C}))\big) = E_{\mathrm{st}}^{\hat{B}^d}\big(x, y, \mathcal{M}_{\mathrm{B}}^e(PGL(n, \mathbb{C}))\big).$$

This conjecture has been proved [HRV] when n is a prime and when $n = 4$; which implies Conjecture 3.1 on the level of Euler characteristic in these cases. The method of proof is arithmetic. We count rational points of the variety $\mathcal{M}_B^d$ over a finite field $\mathbb{F}_q$, when n divides $q - 1$, where $q = p^r$ is a prime power. Because this count will turn out to be a polynomial in q we have the following result, which is basically the Weil conjecture for our special smooth affine varieties:

Theorem 3.1 ([HRV]) *The E-polynomial of $\mathcal{M}_B^d$ has only $x^k y^k$ type terms, and the polynomial $E(q) := E(\sqrt{q}, \sqrt{q})$ agrees with the number of rational points of $\mathcal{M}_B^d(G)$ over $\mathbb{F}_q$:*

$$E(q) = \#(\mathcal{M}_B^d(G)(\mathbb{F}_q))$$

The count is then possible because we only need to count the solutions of the equation:

$$[A_1, B_1] \cdot \dots \cdot [A_g, B_g] = \xi_n,$$

in the finite group of Lie type $G(F_q)$, i.e. so that $A_i, B_i \in G(\mathbb{F}_q)$, where $\xi_n \in G$ is a central element of order n.

A simple modification of a theorem of Freed-Quinn [FQ][(5.19)] then implies:

Theorem 3.2 *Let $G = SL(n, \mathbb{C})$ or $G = GL(n, \mathbb{C})$. Then the number of rational points on $\mathcal{M}_B^d(G)$ over a finite field $\mathbb{F}_q$, where $q = p^r$ is a prime power, with $n|(q-1)$ is given by the character formula:*

$$\#(\mathcal{M}_B^d(G)(\mathbb{F}_q)) = \sum_{\chi \in \mathrm{Irr}(\mathrm{G}(\mathbb{F}_\mathrm{q}))} \frac{|G|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n),$$

where the sum is over all irreducible characters of the finite group $G(\mathbb{F}_q)$ of Lie type.

The two above theorems then imply

Corollary 3.3 ([HRV]) *The E-polynomial of the character variety $\mathcal{M}_B^d(G)$ is given by the character formula:*

$$E(q) = \sum_{\chi \in \mathrm{Irr}(\mathrm{G}(\mathbb{F}_\mathrm{q}))} \frac{|G|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n). \tag{11}$$

Remark 3.3.1 An immediate consequence of this formula is the Betti analogue of Conjecture 3.2. This follows from Corollary 3.3 as that character formula transforms by a Galois automorphism when one changes from d_1 to d_2. Moreover because our varieties $\mathcal{M}_B^{d_1}(G)$ and $\mathcal{M}_B^{d_2}(G)$ are Galois conjugate themselves, we can deduce that their Betti numbers agree, and presumably their mixed Hodge structure should also agree. In summary we have

Corollary 3.4 ([HRV]) *For all $d_1, d_2 \in \mathbb{Z}$ as long as $(d_1, n) = (d_2, n) = 1$ we have*

$$E\big(x, y; \mathcal{M}_B^{d_1}(G)\big) = E\big(x, y; \mathcal{M}_B^{d_2}(G)\big) \tag{12}$$

and

$$P\big(t; \mathcal{M}_B^{d_1}(G)\big) = P\big(t; \mathcal{M}_B^{d_2}(G)\big). \tag{13}$$

Thus we get an affirmative answer for Conjecture 3.2 on the level of the Poincaré polynomials. In general Galois conjugate varieties tend to be (although need not be see e.g. [Se]) homeomorphic over $\mathbb{C}$.

Problem 3.5 *Are $\mathcal{M}_B^{d_1}(G)$ and $\mathcal{M}_B^{d_2}(G)$ homeomorphic for $(n, d_1) = (n, d_2) = 1$?*

Remark 3.5.1 In order to calculate the character formula in Corollary 3.3, we will need to know the values of irreducible characters of G on central elements. Fortunately for $GL(n, \mathbb{F}_q)$ this has been calculated by Green [Gr] and for $SL(n, \mathbb{F}_q)$ the required information, i.e. the value of the characters on central elements, was obtained by Lehrer in [Le]. In the next section we will show an explicit result for the character formula for $GL(n, \mathbb{F}_q)$.

Remark 3.5.2 Our mirror symmetry Conjecture 3.3 then can be translated to a complicated formula which is valid for the character tables of $PGL(n, \mathbb{F}_q)$ and $SL(n, \mathbb{F}_q)$. In particular we believe that by introducing punctures for our Riemann surfaces a similar mirror symmetry conjecture would in fact capture the exact difference between the full character tables of $PGL(n, \mathbb{F}_q)$ and $SL(n, \mathbb{F}_q)$ (not just on central elements as above). This way our mirror symmetry proposal could be phrased as follows: *the differences between the character tables of $PGL(n, \mathbb{F}_q)$ and its Langlands dual $SL(n, \mathbb{F}_q)$ are governed by mirror symmetry.*

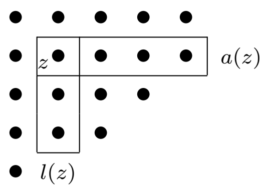
4 Explicit formulas for the E-polynomials

Here we calculate the E-polynomials of $\mathcal{M}_B^d(PGL(n, \mathbb{C}))$, which we denote by $E_n(q)$. We need to start with partitions.

We write a partition of n as $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$, so that $\sum \lambda_i = n$. The *Ferrers diagram* $d(\lambda)$ of λ is the set of lattice points

$$\{(i, j) \in \mathbb{Z}_{\leq 0} \times \mathbb{N} : j < \lambda_{-i+1}\}. \tag{14}$$

The *arm length* $a(z)$ and *leg length* $l(z)$ of a point $z \in d(\lambda)$ denote the number of points strictly to the right of z and below z, respectively, as indicated in this example:



where $\lambda = (5, 5, 4, 3, 1)$, $z = (-1, 1)$, $a(z) = 3$ and $l(z) = 2$. The *hook length* then is defined as

$$h(z) = l(z) + a(z) + 1.$$

Let

$$V_n(q) = E_n(q)q^{(1-g)n(n-1)}(q-1)^{2g-2},$$

and

$$Z_n(q,T) = \exp\left(\sum_{r\geq 1} V_n(q^r)\frac{T^r}{r}\right).$$

We define the Hook polynomials for a partition λ as follows :

$$\mathcal{H}^\lambda(q) = \prod_{z\in d(\lambda)} q^{-l(z)}(1-q^{h(z)}).$$

We can now formulate

Theorem 4.1 ([HRV]) *The E-polynomials of the character varieties $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ for $n = 1,2,3,\ldots$ are given by the following generating function :*

$$\prod_{n=1}^{\infty} Z_n(q,T^n) = \sum_{\lambda\in\mathcal{P}} (\mathcal{H}^\lambda(q))^{2g-2}T^{|\lambda|}, \tag{15}$$

where $\mathcal{P}$ is the set of all partitions.

One simple corollary of this gives a new topological result:

Corollary 4.2 ([HRV]) *The Euler characteristic of $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ (and thus of $\mathcal{M}_{Dol}^d(PGL(n,\mathbb{C}))$ and $\mathcal{M}_{DR}^d(PGL(n,\mathbb{C}))$) is $\mu(n)n^{2g-3}$, where μ is the Möbius function, i.e. $\mu(n)$ is the sum of primitive nth root of unities.*

Another interesting application of the theorem is the following:

Corollary 4.3 *The E-polynomial $E(q) := E(q;\mathcal{M}_B^d(PGL(n,\mathbb{C})))$ is palindromic, i.e. it satisfies, what we call, the* curious *Poincaré duality:*

$$q^{2N}E(1/q) = E(q),$$

where $2N = (n^2-1)(2g-2)$ is the complex dimension of $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$.

Remark 4.3.1 In fact this result originates in the so-called Alvis-Curtis duality [Al, Cu] in the character theory of $GL(n,\mathbb{F}_q)$, which is a duality between irreducible representations of $GL(n,\mathbb{F}_q)$. In particular, if $\chi,\chi' \in \mathrm{Irr}(GL(n,\mathbb{F}_q))$ are dual representations then the dimension $\chi(1)$ is a polynomial in q which satisfies

$$q^{\frac{n(n-1)}{2}}\chi(1)(1/q) = \chi'(1)(q).$$

For example when $n=2$ Theorem 4.1 gives:

$$E_2(q) = (q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}, \tag{16}$$

when $g=3$ this gives

$$E(x,y;\mathcal{M}_B^1(PGL(2,\mathbb{C}))) = q^{12} - 4\,q^{10} + 6\,q^8 - 14\,q^6 + 6\,q^4 - 4\,q^2 + 1, \tag{17}$$

which is a palindromic polynomial indeed. Note also that there does not seem to be much in common with the Poincaré polynomial (9).

5 A conjectured formula for mixed Hodge polynomials

Here we present the conjecture of [HRV] on the H-polynomials of the spaces $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$. As usual we fix the curve C and its genus g and the group $PGL(n,\mathbb{C})$ and write $\mathcal{M}_B^d$ for $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$. Let $H_n(x,y,t)$ denote the mixed Hodge polynomial (4) of the $\mathcal{M}_B^d$ character variety. Because for the character variety we have that $h^{i,j;k}(\mathcal{M}_B^d)=0$ provided that $i\neq j$ we introduce $h_i^k(\mathcal{M}_B^d)=h^{i,i;k}(\mathcal{M}_B^d)$ and a new form for the H-polynomial

$$H_n(q,t)=H_n(\sqrt{q},\sqrt{q},t)=\sum h_i^k(\mathcal{M}_B^d)q^it^k,$$

which contains the same amount of information.

Now let

$$V_n(q,t)=H_n(q,t)\frac{(qt^2)^{(1-g)n(n-1)}(qt+1)^{2g}}{(qt^2-1)(q-1)},$$

and

$$Z_n(q,t,T)=\exp\left(\sum_{r\geq 1}V_n(q^r,-(-t)^r)\frac{T^r}{r}\right).$$

We define the t-deformed Hook polynomials for genus g and partition λ as follows:

$$\mathcal{H}_g^\lambda(q,t)=\prod_{x\in d(\lambda)}\frac{(qt^2)^{(2-2g)l(x)}(1+q^{h(x)}t^{2l(x)+1})^{2g}}{(1-q^{h(x)}t^{2l(x)+2})(1-q^{h(x)}t^{2l(x)})}.$$

We can now formulate our next

Conjecture 5.1 ([HRV]) *The mixed Hodge polynomials of the character varieties $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ for $n=1,2,3,\ldots$ are given by the following generating function:*

$$\prod_{n=1}^{\infty}Z_n(q,t,T^n)=\sum_{\lambda\in\mathcal{P}}\mathcal{H}_g^\lambda(q,t)T^{|\lambda|}. \tag{18}$$

Because the above formula (18) could be taken as a combinatorial definition of the expressions of $H_n(q,t)$ (which are a priori only certain rational functions in q and t), we can formalize in this combinatorial form our expectations from $H_n(q,t)$, with the addition of a curious, Poincaré duality-type of symmetry, which was in fact our most important guide to come up with these formulas:

Conjecture 5.2 *The rational functions $H_n(q,t)$ defined in the generating function of (18) satisfy the following properties:*

- *$H_n(q,t)$ is a polynomial in q and t.*
- *Both the q degree and the t degree of the polynomial $H_n(q,t)$ agree with $2N=2(n^2-1)(g-1)$. In fact the largest degree monomial in both variables is $(qt)^{2(n^2-1)(g-1)}$.*
- *All coefficients of $H_n(q,t)$ are non-negative integers.*
- *The coefficients of $H_n(q,t)$ satisfy, what we call the* curious Poincaré duality*:*

$$h_{N-j}^{i-j}=h_{N+j}^{i+j} \tag{19}$$

In the following we list some checks and implications of the above conjectures:

Remark 5.2.2 Computer calculations with Maple gives $H_n(q,t)$ from the above generating function when $n = 2, 3, 4$. In all these cases for small g we do get a polynomial in q and t with the expected degree and positive coefficients, satisfying the curious symmetry (19).

Remark 5.2.3 Using the explicit description of the cohomology ring of $\mathcal{M}_B^d$ in [HT2], one can write down a monomial basis for $H^*(\mathcal{M}_B^d, \mathbb{C})$ in the tautological generators. Then in turn one can figure out the action of the Frobenius on these generators, which in turn provides a formula for the mixed Hodge polynomial of $\mathcal{M}_B^d$. This formula can be brought to the form

$$\begin{aligned} H_2(q,t) &= \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \\ &- \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)}, \end{aligned} \tag{20}$$

which agrees with the conjectured one through (18), and clearly reduces to (4.1), when $t = -1$. For example when $g = 3$, this gives

$$\begin{aligned} &H(q,t;\mathcal{M}^d(PGL(2,\mathbb{C}))) = \\ &= t^{12}q^{12} + t^{12}q^{10} + 6\,t^{11}q^{10} + t^{12}q^8 + t^{10}q^{10} + 6\,t^{11}q^8 + 16\,t^{10}q^8 + 6\,t^9q^8 + t^{10}q^6 + t^8q^8 + 26\,t^9q^6 + \\ &+ 16\,t^8q^6 + 6\,t^7q^6 + t^8q^4 + t^6q^6 + 6\,t^7q^4 + 16\,t^6q^4 + 6\,t^5q^4 + t^4q^4 + t^4q^2 + 6\,t^3q^2 + t^2q^2 + 1, \end{aligned} \tag{21}$$

which is a common refinement of (9) when $q = 1$ and of (17) when $t = -1$. Note also how the curious Poincaré duality appears when one refines the Poincaré polynomial (9), which does not possess any kind of symmetry, to the mixed Hodge polynomial (21).

Remark 5.2.4 Note that $P_n(t) = H_n(1,t)$ should be the Poincaré polynomial of the character variety, which is the same as the Poincaré polynomial of the diffeomorphic Higgs moduli space $\mathcal{M}_{Dol}^d$. For $n = 2$ Hitchin in [Hi1] calculated the Poincaré polynomial of this latter space, and an easy calculation shows that if one substitutes $q = 1$ into (20) we get $P_2(t) = H_2(1,t)$, the Poincaré polynomial of Hitchin. For $n = 3$ Gothen in [Go] calculated $P_3(t)$ by some non-trivial geometric argument and a fairly complicated computation. He ended up with a rather lengthy formula for $P_3(t)$. Because it is so pleasant to work with a formula like (20), we also give what our Conjecture 18 gives in the $n = 3$ case:

$$\begin{aligned} H_3(q,t) &= \frac{(q^3t^5+1)^{2g}(q^2t^3+1)^{2g}}{(q^3t^6-1)(q^3t^4-1)(q^2t^4-1)(q^2t^2-1)} + \frac{q^{6g-6}t^{12g-12}(q^3t+1)^{2g}(q^2t+1)^{2g}}{(q^3t^2-1)(q^3-1)(q^2t^2-1)(q^2-1)} + \\ &+ \frac{q^{4g-4}t^{8g-8}(q^3t^3+1)^{2g}(qt+1)^{2g}}{(q^3t^4-1)(q^3t^2-1)(qt^2-1)(q-1)} + \frac{1}{3}\frac{q^{6g-6}t^{12g-12}\left((qt+1)^{2g}\right)^2}{(qt^2-1)^2(q-1)^2} - \\ &- \frac{1}{3}\frac{q^{6g-6}t^{12g-12}(q^2t^2-qt+1)^{2g}}{(q^2t^4+qt^2+1)(q^2+q+1)} - \frac{q^{4g-4}t^{8g-8}(q^2t^3+1)^{2g}(qt+1)^{2g}}{(q^2t^4-1)(q^2t^2-1)(qt^2-1)(q-1)} - \\ &- \frac{q^{6g-6}t^{12g-12}(q^2t+1)^{2g}(qt+1)^{2g}}{(q^2t^2-1)(q^2-1)(qt^2-1)(q-1)}. \end{aligned}$$

Now it is a nice exercise to show that $H_3(1,t)$ does indeed produce Got hen's complicated looking formula in [Go].

It is also worth noting that many of the individual terms in $H_n(q,t)$ have poles at $q=1$, however according to our conjecture these poles somehow cancel each other.

Remark 5.2.5 When $g=0$, we know from the definitions that $H_1(q,t)=1$ and $H_n(q,t)=0$ otherwise. One can deduce the same from Conjecture 5.1 by applying part f of Theorem 2.10 in [GH] to calculate the right hand side of (18). Moreover Conjecture 5.2 has the same flavour as the main conjecture in [GH] about q,t Cat Alan numbers, which was then proved by Haiman in [Hai] using some subtle intersection theory on the Hilbert scheme of n points on $\mathbb{C}^2$. Apart from the fact that this Hilbert scheme is also a hyperkähler manifold, the similarities between the two conjectures are rather surprising.

Remark 5.2.6 When $g=1$ we have $H_n(q,t)=1$ for every n, but this we could not prove from (18).

Remark 5.2.7 Let us look at the conjecture (19). Recall that H^2 of our varieties are exactly one dimensional, generated by a class, call it $[\omega]$, which is the Kähler class in the complex structure of $\mathcal{M}^d_{Dol}$. This carries the weight q^2t^2 in the mixed Hodge structure. We have the following hard Lefschetz type of conjecture which enhances the curious Poincaré duality of the conjecture of (19):

Conjecture 5.3 *If L denotes the map by multiplication with $[\omega]$, then we conjecture that the map*

$$L^k : H^{N-k,N-k;i-k}(\mathcal{M}^d_B(PGL(n,\mathbb{C}))) \to H^{N+k,N+k;i+k}(\mathcal{M}^d_B(PGL(n,\mathbb{C})))$$

is an isomorphism.

Interestingly this conjecture implies a theorem of [Ha3] that the Lefschetz map $L^k : H^{N-k} \to H^{N+k}$ is injective for $\mathcal{M}^d_{Dol}$, and it is explained there how this weak version of Hard Lefschetz, when applied to boric hyperkähler varieties, yields new inequalities for the h-numbers of matroids. See also [HS] for the original argument on toric hyperkähler varieties. Furthermore this conjecture can be proved when $n=2$ using the explicit description of the cohomology ring in [HT2]. The general case can also be thought of as an analogue of the Faber conjecture [Fa] on the cohomology of the moduli space of curves, which is another non-compact variety whose cohomology ring is conjectured to satisfy a certain form of the Hard Lefschetz theorem.

Remark 5.3.8 There are two subspaces of the cohomology $H^*(\mathcal{M}^d_B,\mathbb{C})$ which are particularly interesting. One is the middle dimensional cohomology $H^{2N}(\mathcal{M}^d_B,\mathbb{C})$, which is the top non-vanishing cohomology. The mixed Hodge structure will break it into parts with respect to the q-degree. The curious-Poincaré dual (19) of these spaces are also interesting: it is easy to see that they are exactly the pure part of the mixed Hodge structure i.e. spaces of the form $H^{i,i;2i}$. (Another significance of the pure part is that if there is a smooth projective compactification of the variety which surjects in cohomology then its image is exactly this pure part.) Thus it would already be interesting to get the pure part of $H_n(q,t)$. In fact it is easy to identify the pure part in our case with what we call the Pure ring, which is the subring of $H^*(\mathcal{M}^d,\mathbb{C})$ generated by the tautological classes $a_i \in H^{2i}(\mathcal{M}^d,\mathbb{C})$ for $i=2,..,n$ (the other tautological classes, which generate the cohomology ring, are not pure classes).

For example when $n=2$, it was known [Hi1] that the middle degree cohomology of the Higgs moduli space $\mathcal{M}^d_{Dol}(PGL(2,\mathbb{C}))$ is g dimensional. The Pure ring was determined in [HT2], and it was found to be g dimensional due to the relation $\beta^g=0$ (where $\beta=a_2$). Thus these two seemingly unrelated observations are dual to each other via our curious Poincaré duality (19). To see this curious duality in action let us recall the formula (21). The terms which contain the top degree 12 in t are $t^{12}q^{12}$, $t^{12}q^{10}$ and $t^{12}q^{8}$, which are curious Poincaré dual via (19) to the terms 1, t^4q^2 and t^8q^4, which is exactly the ring generated by the degree four class β, which have additive basis 1, β and β^2.

The analogous ring, generated by the corresponding classes $a_2, \ldots, a_n \in H^*(\mathcal{N}^d, \mathbb{C})$, which a priori is a quotient of our Pure ring (as $\mathcal{N}^d \subset \mathcal{M}^d_{Dol}$ naturally), was studied for the moduli space $\mathcal{N}^d$ of rank n, degree d stable bundles (with $(n,d)=1$) in [EK], where they in particular found the top non-vanishing degree of this ring to be $2n(n-1)(g-1)$. Computer calculations for our conjecture for $n = 2, 3, 4$ also show that our conjectured Pure ring has the same 1-dimensional top degree. This and the known situation for $n = 2$ (see [HT2]), yields the following

Conjecture 5.4 *The Pure rings of $\mathcal{M}^d_{Dol}$ and $\mathcal{N}^d$, i.e. the subrings of the cohomology rings generated by the classes $a_2, \ldots, a_n$ are isomorphic. In particular, unlike the whole cohomology ring of $\mathcal{N}^d$, it does not depend on d.*

Now we explain a combinatorial consequence of this conjecture. First we extract a conjectured formula for $PP_n(t)$ the Poincaré polynomial of the Pure ring. Indeed we only have to deal with monomials in Conjecture 5.1 whose t-degree is double of their q-degree.

Thus let

$$PV_n(t) = PP_n(t)\frac{t^{2(1-g)n(n-1)}}{(t^2-1)},$$

and

$$PZ_n(t,T) = \exp\left(\sum_{r\geq 1} PV_n(t^r)\frac{T^r}{r}\right).$$

We now define the pure part of the t-deformed Hook polynomials for genus g and partition λ as follows:

$$\mathcal{PH}^\lambda_g(t) = t^{4(1-g)n(\lambda')} \prod_{x\in d(\lambda); a(x)=0} \frac{1}{(1-t^{2h(x)})},$$

where

$$n(\lambda') := \sum_{z\in d(\lambda)} l(z).$$

Thus we get the conjecture that $PP_n(t)$ is given by

$$\prod_{n=1}^{\infty} PZ_n(t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{PH}^\lambda_g(t)T^{|\lambda|}. \tag{22}$$

Now combining the two conjectures above we can formulate a new conjecture:

Conjecture 5.5 *The rational functions $PP_n(t)$ defined in (22) satisfy*

- *$PP_n(t)$ is a polynomial in t*
- *all coefficients of $PP_n(t)$ are non-negative integers*
- *The degree of $PP_n(t)$ is $2n(n-1)(g-1)$, and the coefficients of the leading term is* 1

So for example when $n=3$ our conjecture gives for the Poincaré polynomial of the Pure ring:

$$PP_3(t)=\frac{1}{(t^6-1)(t^4-1)}+t^{12g-12}-\frac{t^{8g-8}}{t^2-1}+\frac{1}{3}\frac{t^{12g-12}}{(t^2-1)^2}-\frac{1}{3}\frac{t^{12g-12}}{t^4+t^2+1}-\frac{t^{8g-8}}{(t^4-1)(t^2-1)}+\frac{t^{12g-12}}{t^2-1}$$

Remark 5.5.9 Interestingly we can modify the formula of Conjecture 5.1 to get a conjectured formula for the mixed Hodge polynomial of $\mathcal{M}^d_{Dol}$. Recall from Theorem 2.1 that the mixed Hodge structure on $H^k(\mathcal{M}^d_{Dol},\mathbb{C})$ is pure of weight k, thus this mixed Hodge polynomial is equivalent with the E-polynomial.

We now introduce polynomials $H_n(q,x,y)$ of three variables. Let

$$V_n(q,x,y)=H_n(q,x,y)\frac{(qxy)^{(1-g)n(n-1)}(qx+1)^g(qy+1)^g}{(qxy-1)(q-1)},$$

and

$$Z_n(q,x,y,T)=\exp\left(\sum_{r\geq 1}V_n(q^r,-(-x)^r,-(-y)^r)\frac{T^r}{r}\right).$$

We define the (x,y)-deformed Hook polynomials for genus g and partition λ as follows:

$$\mathcal{H}^\lambda_g(q,x,y)=\prod_{z\in d(\lambda)}\frac{(qxy)^{(2-2g)l(z)}(1+q^{h(z)}y^{l(z)}x^{l(z)+1})^g(1+q^{h(z)}x^{l(z)}y^{l(z)+1})^g}{(1-q^{h(z)}(xy)^{l(z)+1})(1-q^{h(z)}(xy)^{l(z)})}. \tag{23}$$

The following generating function defines $H_n(q,x,y)$:

$$\prod_{n=1}^{\infty}Z_n(q,x,y,T^n)=\sum_{\lambda\in\mathcal{P}}\mathcal{H}^\lambda_g(q,x,y)T^{|\lambda|}. \tag{24}$$

Clearly we have $H_n(q,t,t)=H_n(q,t)$ which says that a specialization of $H_n(q,x,y)$ gives the mixed Hodge polynomial $H_n(q,t)$ of $\mathcal{M}^d_B$. The following conjecture says that another specialization gives the mixed Hodge polynomial of $\mathcal{M}^d_{Dol}$ and $\mathcal{M}^d_{DR}$.

Conjecture 5.6 *$H_n(q,x,y)$ is a polynomial with non-negative integer coefficients with specialization $H_n(1,x,y)$ equal to the E-polynomial of the Higgs moduli space $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$.*

Thus we have a mysterious formula $H_n(q,x,y)$ which specializes, on one hand to the H-polynomial of the character variety, and on the other hand to the mixed Hodge polynomial of the Higgs (or equivalently flat connection) moduli space. It would be very interesting to find a geometrical meaning for $H_n(q,x,y)$.

Checks on this Conjecture 5.6 include a proof for $n=2$ and $n=3$, (one can easily modify Hitchin's and Gothen's argument to get the Hodge polynomial instead of the Poincaré polynomial of the Higgs moduli space) and also computer checks that the shape of the polynomial $H_n(1,x,y)$ is the expected one when $n=4$.

Consider now the specification $H_n(q,-1,y)$. Interestingly, the corresponding specification of the (x,y)-deformed Hook polynomials (23) becomes a polynomial, showing that at least $H_n(q,-1,y)$ is a polynomial. We get an even nicer formula if we make the further specification $H_n(1,-1,y)$ which by Conjecture 5.6 should be the Hirzebruch y-genus of the moduli space of Higgs bundles $\mathcal{M}^d_{Dol}$. Namely, for $g>1$, most of the (x,y) deformed Hook polynomials vanish, when one substitutes first $x=-1$ and then $q=1$. Indeed, the only partitions which will have a non-zero contribution to the y-genus are the partitions of the form $n=1+1+\cdots+1$; when $l(z)=0$ only for once. This in turn gives the following closed formula for the conjectured y-genus of $\mathcal{M}^d_{Dol}$:

Conjecture 5.7 *The Hirzebruch y-genus of $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$, for $g>1$, equals*

$$(1-y+\cdots+(-y)^{n-1})^{g-1}\sum_{m|n}\frac{\mu(m)}{m}\left((-y)^{n(n-n/m)}m\prod_{i=1}^{n/m-1}(1-(-y)^{mi})^2\right)^{g-1}$$

In particular note that the term corresponding to $m=1$, is exactly the known y-genus of $\mathcal{N}^d$ (see [NR]). The rest thus should be thought as contribution of the other fixed point components of the circle action on $\mathcal{M}^d_{Dol}$. Of course this conjectured formula gives the known specialization of Corollary 4.2 at $y=-1$, while the $y=1$ specialization gives $\mu(n)n^{g-2}$ when n is odd, and 0 when n is even. The specialization at $y=1$ can be thought of as the signature of the pairing on the rationalized circle equivariant cohomology of $\mathcal{M}^d_{Dol}$ as defined in [HP].

Remark 5.7.10 Finally we discuss how to obtain a conjecture for the mixed Hodge polynomial of $\mathcal{M}^d_B(SL(n,\mathbb{C}))$. For the mixed Hodge polynomial of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$ the mirror symmetry conjecture 3.1, together with Conjecture 5.6 imply a conjecture. For $\mathcal{M}^d_B(SL(n,\mathbb{C}))$ the mixed Hodge polynomial contains more information than the E-polynomial. In order thus to have a conjecture on $H_n(x,y,t;\mathcal{M}^d_B(SL(n,\mathbb{C})))$ a mirror symmetry conjecture is needed on the level of the H-polynomial. We finish by formulating such a conjecture, generalizing Conjecture 3.3 for H-polynomials:

Conjecture 5.8 *For all $d,e\in\mathbb{Z}$, with $(d,n)=(e,n)=1$ we have*

$$H^{B^e}_{\rm st}\big(x,y,t;\mathcal{M}^d_{\rm B}(SL(n,\mathbb{C}))\big)=H^{\hat{B}^d}_{\rm st}\big(x,y,t;\mathcal{M}^e_{\rm B}(PGL(n,\mathbb{C}))\big),$$

where $H^B_{\rm st}$ is the stringy mixed Hodge polynomial twisted with a B-field, which can be defined identically as $E^B_{\rm st}$ is defined in (6).

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