Michigan Math. J. 63 (2014), 571–621

Mirror Symmetry for Stable Quotients Invariants

Yaim Cooper & Aleksey Zinger

Abstract. The moduli space of stable quotients introduced by Marian, Oprea, and Pandharipande provides a natural compactification of the space of morphisms from nonsingular curves to a nonsingular projective variety and carries a natural virtual class. We show that the analogue of Givental's J-function for the resulting twisted projective invariants is described by the same mirror hypergeometric series as the corresponding Gromov–Witten invariants (which arise from the moduli space of stable maps), but without the mirror transform (in the Calabi–Yau case). This implies that the stable quotients and Gromov–Witten twisted invariants agree if there is enough "positivity," but not in all cases. As a corollary of the proof, we show that certain twisted Hurwitz numbers arising in the stable quotients theory are also described by a fundamental object associated with this hypergeometric series. We thus completely answer some of the questions posed by Marian, Oprea, and Pandharipande concerning their invariants. Our results suggest a deep connection between the stable quotients invariants of complete intersections and the geometry of the mirror families. As in Gromov–Witten theory, computing Givental's J-function (essentially a generating function for genus 0 invariants with one marked point) is key to computing stable quotients invariants of higher genus and with more marked points; we exploit this in forthcoming papers.

1. Introduction

Gromov–Witten invariants of a smooth projective variety X are certain counts of curves in X that arise from integrating against the virtual class of the moduli space of stable maps. These are known to possess striking structures, which are often completely unexpected from the classical point of view. For example, the genus 0 Gromov–Witten invariants of a quintic threefold, that is, a degree 5 hypersurface in $\mathbb{P}^4$, are related by a so-called mirror formula to a certain hypergeometric series. This relation was explicitly predicted in [2] and mathematically confirmed in [8] and [13] in the 1990s. In fact, the prediction of [2] has been shown to be a special case of mirror symmetry for certain twisted Gromov–Witten invariants of projective complete intersections of sufficiently small total multidegree [7; 14]; these invariants are associated with direct sums of line bundles (positive and negative) over $\mathbb{P}^n$. This relation is often described by assembling two-point Gromov–Witten invariants (but without constraints on the second marked point) into a generating

Received September 28, 2013. Revision received March 24, 2014.
The second author was partially supported by NSF grants DMS-0635607 and DMS-0846978.

function, known as Givental's J-function. In most cases (in particular, when the anticanonical class of the corresponding complete intersection is at least twice the hyperplane class), the J-function is precisely equal to the appropriate hypergeometric series. In certain borderline cases, they differ by a simple exponential factor. In the remaining Calabi–Yau cases, the correcting factors are more complicated, and the two power series also differ by a change of the power series variable, known as the mirror map.

The gauged linear σ-model of [23] counts rational curves in toric complete intersections by integrating over the natural toric compactifications of the spaces of rational maps into the ambient toric variety. Based on physical considerations, it is shown in [19] that the (three-point) Gromov–Witten and gauged linear σ-model generating functions for the well-studied quintic threefold are related by the mirror map, with a minor additional adjustment; see [19, (4.24), (4.28)], for example. This suggests that the mirror map relating the A (symplectic) side of mirror symmetry to the B (complex geometric) side may be more reflective of the choice of curve counting theory on the A-side than of the mirror symmetry itself. Unfortunately, from the mathematical standpoint, the compactifying spaces in the gauged linear σ-model do not possess many of the nice properties of the spaces of stable maps and require fixing a complex structure on the domain of the maps.

The moduli spaces of stable quotients $\overline{Q}_{g,m}(X,d)$, constructed in [17], provide an alternative to the moduli spaces of stable maps $\overline{\mathfrak{M}}_{g,m}(X,d)$ for compactifying spaces of degree d morphisms from genus g nonsingular curves with m marked points to a projective variety X (with a choice of polarization).[1] In this paper, we show that the genus 0 stable quotients theory, just like the gauged linear σ-model, of Calabi–Yau projective complete intersections is related to their genus 0 Gromov–Witten theory essentially by the mirror map; see (1.9). Based on the approaches of [24] and [26], this relationship between the stable quotients and Gromov–Witten invariants should extend to higher genera; we expect to confirm this in the genus 1 case in the future. In [27], it is shown that the genus 0 three-point stable quotients and Gromov–Witten invariants of Calabi–Yau projective complete intersection threefolds are related precisely by the mirror map. The mirror formula obtained in this paper is central to the computations in [27]. Thus, our paper provides further evidence that the mirror map is an entirely A-side feature and suggests that the stable quotients theory may be the curve counting theory most directly related to the B-side of mirror symmetry. In light of the results in this paper, we also hope that certain properties of the mirror map, such as the integrality of its coefficients [15; 12], can be explained geometrically by comparing the stable quotients and Gromov–Witten invariants.

The moduli space $\overline{Q}_{g,m}(\mathbb{P}^{n-1},d)$ consists of equivalence classes of tuples

$$(\mathcal{C}, y_1, \ldots, y_m; S \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}),$$

[1] These "compactifications" $\overline{Q}_{g,m}(X,d)$ and $\overline{\mathfrak{M}}_{g,m}(X,d)$ are generally just compact spaces containing the spaces of morphisms; the latter need not be dense in $\overline{Q}_{g,m}(X,d)$ or $\overline{\mathfrak{M}}_{g,m}(X,d)$.

where $(\mathcal{C}, y_1, \ldots, y_m)$ is a genus g nodal curve with m marked points, and $S \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}$ is a subsheaf of rank 1 and degree $-d$, that satisfy certain stability and torsion properties; see Section 2. This moduli space is smooth if $g = 0$ or $(g, m) = (1, 0)$ and carries a virtual class in all cases. There is a natural surjective contraction morphism

$$c : \overline{\mathfrak{M}}_{g,m}(\mathbb{P}^{n-1}, d) \longrightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d),$$

which is not injective for $d > 0$ and generally contracts a lot of boundary strata. For example, $\overline{Q}_{1,0}(\mathbb{P}^{n-1}, d)$ is irreducible and has Picard rank just 2; see [5, Theorem 4.1]. Thus, the moduli spaces of stable quotients are much more efficient compactifications than the moduli spaces of stable maps. However, in the case $X = \mathbb{P}^{n-1}$ and $(g, m) = (0, 3)$, this compactification is larger than the gauged linear σ-model compactification; see [19, Section 3.7].

As in the case of stable maps, there are evaluation morphisms

$$\mathrm{ev}_i : \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}, \quad i = 1, 2, \ldots, m,$$

corresponding to each marked point.[2] There is also a universal curve

$$\pi : \mathcal{U} \longrightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$$

with m sections $\sigma_1, \ldots, \sigma_m$ (given by the marked points) and a universal rank 1 subsheaf

$$\mathcal{S} \subset \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{U}}.$$

For each $i = 1, 2, \ldots, m$, let

$$\psi_i = -\pi_*(\sigma_i^2) \in H^2(\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d))$$

be the first Chern class of the universal cotangent line bundle, as usual. By [17, Theorems 2,3], the moduli space $\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$ carries a canonical virtual class, and

$$c_*[\overline{\mathfrak{M}}_{g,m}(\mathbb{P}^{n-1}, d)]^{\mathrm{vir}} = [\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)]^{\mathrm{vir}}. \tag{1.1}$$

Since the evaluation morphisms ev_i and the ψ-classes on the two moduli spaces commute with c and c^*, respectively, (1.1) implies that the (untwisted) Gromov–Witten and stable quotients invariants of $\mathbb{P}^{n-1}$, obtained by integrating pull-backs of cohomology classes on $\mathbb{P}^{n-1}$ by ev_i and powers of ψ-classes against the two virtual classes, are the same; see [17, Theorem 3]. In this paper, we study twisted invariants in genus 0, arising from sums of line bundles over $\mathbb{P}^{n-1}$; they relate invariants of projective complete intersections to the invariants of the ambient space.

For $l \in \mathbb{Z}^{\geq 0}$ and l-tuple $\mathbf{a} = (a_1, \ldots, a_l) \in (\mathbb{Z}^*)^l$ of nonzero integers, let

$$|\mathbf{a}| = \sum_{k=1}^{l} |a_k|, \quad \langle \mathbf{a} \rangle = \prod_{a_k > 0} a_k \Big/ \prod_{a_k < 0} a_k, \quad \mathbf{a}! = \prod_{a_k > 0} a_k!, \quad \mathbf{a}^{\mathbf{a}} = \prod_{k=1}^{l} a_k^{|a_k|},$$

$$\ell^{\pm}(\mathbf{a}) = |\{k : \ (\pm 1)a_k > 0\}|, \qquad \ell(\mathbf{a}) = \ell^+(\mathbf{a}) - \ell^-(\mathbf{a}).$$

[2] The morphism ev_i sends a tuple $(\mathcal{C}, y_1, \ldots, y_m, S)$ to the line $S_{y_i} \subset \mathbb{C}^n$ if S is viewed as a line subbundle of the trivial rank n bundle over $\mathcal{C}$.

If in addition $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$, let

$$\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)} = \bigoplus_{a_k>0} R^0\pi_*(\mathcal{S}^{*a_k}(-\sigma_1)) \oplus \bigoplus_{a_k<0} R^1\pi_*(\mathcal{S}^{*a_k}(-\sigma_1)) \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d), \tag{1.2}$$

where $\pi : \mathcal{U} \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)$ is the universal curve; this sheaf is locally free. The Euler class of the analogue of this sheaf in Gromov–Witten theory describes the genus 0 invariants of the total space of the vector bundle

$$\bigoplus_{a_k<0} \mathcal{O}_{\mathbb{P}^{n-1}}(a_k)|_{X_{(a_k)_{a_k>0}}} \longrightarrow X_{(a_k)_{a_k>0}}, \tag{1.3}$$

where $X_{(a_k)_{a_k>0}} \subset \mathbb{P}^{n-1}$ is a nonsingular complete intersection of multidegree $(a_k)_{a_k>0}$. The situation in the stable quotients theory is similar. If $a_k > 0$ for all k, then the moduli space $\overline{Q}_{0,2}(X_{\mathbf{a}}, d)$ carries a natural virtual fundamental class, and the resulting invariants of $X_{\mathbf{a}}$ are described by the Euler class of (1.2); see [4, Theorem 4.5.2] and [4, Proposition 6.2.3], respectively.

The stable quotients analogue of Givental's J-function is given by

$$Z_{n;\mathbf{a}}(x, \hbar, q) \equiv 1 + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{e(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1}\right] \in H^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]], \tag{1.4}$$

where $\mathrm{ev}_1 : \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$ is as before, and $x \in H^2(\mathbb{P}^{n-1})$ is the hyperplane class. For example, if $|\mathbf{a}| = n$, then this power series is equivalent to the set of numbers

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)} \mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}) \psi_1^p \mathrm{ev}_1^* x^{n-2-p}, \quad d \in \mathbb{Z}^+, 0 \le p \le n-2.$$

By [4, Proposition 6.2.3],

$$\begin{aligned} &\mathrm{SQ}_{n;\mathbf{a}}^{(d)}(\tau_p(x^{n-2-\ell(\mathbf{a})-p}), 1) \\ &\quad \equiv \int_{[\overline{Q}_{0,2}(X_{n;\mathbf{a}},d)]^{\mathrm{vir}}} \psi_1^p \mathrm{ev}_1^* x^{n-2-\ell(\mathbf{a})-p} \\ &\quad = \langle \mathbf{a} \rangle \int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)} \mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}) \psi_1^p \mathrm{ev}_1^* x^{n-2-p} \quad \forall p \le n-2-\ell(\mathbf{a}); \end{aligned} \tag{1.5}$$

in particular, these numbers vanish if $p \le \ell^-(\mathbf{a}) - 2$ (because $x^{n-p} = 0$ on (1.3) if $p \le \ell^+(\mathbf{a})$). The usual Givental's J-function, which we denote by $Z_{n;\mathbf{a}}^{\mathrm{GW}}(x, \hbar, q)$, is defined as in (1.4) with $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)$ replaced by $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d)$.

The hypergeometric series describing Givental's J-function in Gromov–Witten theory is given by

$$Y_{n;\mathbf{a}}(x, \hbar, q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{a_k>0} \prod_{r=1}^{a_k d} (a_k x + r\hbar) \prod_{a_k<0} \prod_{r=0}^{-a_k d-1} (a_k x - r\hbar)}{\prod_{r=1}^{d} (x + r\hbar)^n} \in \mathbb{Q}[x][[\hbar^{-1}, q]]. \tag{1.6}$$

In the pure Calabi–Yau case, that is, $a_k > 0$ for all k and $|\mathbf{a}| = n$, we also need the power series

$$I_{n;\mathbf{a}}(q) = \begin{cases} 1 & \text{if } |\mathbf{a}| - \ell^-(\mathbf{a}) < n, \\ Y_{n;\mathbf{a}}(0,1,q) = \sum_{d=0}^{\infty} q^d \prod_{k=1}^{l} (a_k d)!/(d!)^n \\ \qquad \text{if } |\mathbf{a}| - \ell^-(\mathbf{a}) = n. \end{cases} \tag{1.7}$$

By the following theorem, the stable quotients analogue of Givental's J-function is also described by the hypergeometric series (1.6), but in a more straightforward way.

THEOREM 1. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$, then the stable quotients analogue of Givental's J-function satisfies*

$$Z_{n;\mathbf{a}}(x,\hbar,q) = \frac{Y_{n;\mathbf{a}}(x,\hbar,q)}{I_{n;\mathbf{a}}(q)} \in H^*(\mathbb{P}^{n-1})[[\hbar^{-1},q]]. \tag{1.8}$$

COROLLARY 1. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$ and $|\mathbf{a}| - \ell^-(\mathbf{a}) \leq n-2$, then*

$$Z^{\mathrm{GW}}_{n;\mathbf{a}}(x,\hbar,q) = Z_{n;\mathbf{a}}(x,\hbar,q).$$

COROLLARY 2. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| = n$, then*

$$Z^{\mathrm{GW}}_{n;\mathbf{a}}(x,\hbar,Q) = \mathrm{e}^{-J_{n;\mathbf{a}}(q)x/\hbar} Z_{n;\mathbf{a}}(x,\hbar,q), \quad \textit{where } Q = q \cdot \mathrm{e}^{J_{n;\mathbf{a}}(q)}, \tag{1.9}$$

with the standard change of variables $q \longrightarrow Q$ of Gromov–Witten theory described by

$$\langle \mathbf{a} \rangle J_{n;\mathbf{a}}(q) = \sum_{d=1}^{\infty} q^d \mathrm{SQ}^{(d)}_{n;\mathbf{a}}(\tau_0(x^{n-2-\ell(\mathbf{a})}), 1). \tag{1.10}$$

If $|\mathbf{a}| \leq n$ and $|\mathbf{a}| - \ell^-(\mathbf{a}) \leq n-2$, (1.8) also holds with $Z_{n;\mathbf{a}}(x,\hbar,Q)$ replaced by $Z^{\mathrm{GW}}_{n;\mathbf{a}}(x,\hbar,Q)$; see [7, Theorem 9.1] for the $\ell^-(\mathbf{a}) = 0$ case and [6, Theorem 5.1] for the $\ell^-(\mathbf{a}) \geq 1$ case. Thus, Corollary 1 is an immediate consequence of Theorem 1.

If $|\mathbf{a}| = n$ and $\ell^-(\mathbf{a}) \leq 1$, the Gromov–Witten analogue of (1.8) involves a mirror transform between the power series variable on the left-hand side (now denoted by Q) and the power series variable q on the right-hand side. It takes the form

$$Z^{\mathrm{GW}}_{n;\mathbf{a}}(x,\hbar,Q) = \mathrm{e}^{-J_{n;\mathbf{a}}(q)x/\hbar} \frac{Y_{n;\mathbf{a}}(x,\hbar,q)}{I_{n;\mathbf{a}}(q)}, \quad \text{where } Q = q \cdot \mathrm{e}^{J_{n;\mathbf{a}}(q)}, \tag{1.11}$$

for an explicit power series $J_{n;\mathbf{a}}(q) \in q \cdot \mathbb{Q}[[q]]$; see [7, Theorem 11.8] for the $\ell^-(\mathbf{a}) = 0$ case and [6, Theorem 5.1] for the $\ell^-(\mathbf{a}) = 1$ case. Along with (1.11), Theorem 1 immediately implies the $\ell^-(\mathbf{a}) \leq 1$ case of (1.9); the $\ell^-(\mathbf{a}) \geq 2$ case of (1.9), where $J_{n;\mathbf{a}}(q) = 0$, follows from Corollary 1. By (1.4) and (1.5), the right-hand side of (1.10) is the coefficient of $(h^{-1})^0$ in $Z_{n;\mathbf{a}}(x,\hbar,q)$ times $\langle \mathbf{a} \rangle$. By the string relation of Gromov–Witten theory [11, Section 26.3], the coefficient of $(h^{-1})^0$ in $Z^{\mathrm{GW}}_{n;\mathbf{a}}(x,\hbar,q)$ is zero. Thus, (1.10) follows from (1.9).

Table 1 Some genus 0 GW- and SQ-invariants of the quintic threefold $X_{(5)}$

d	$\frac{\mathrm{GW}^{(d)}_{5;(5)}(\tau_1(x),1)}{d} = -\frac{\mathrm{GW}^{(d)}_{5;(5)}(\tau_2(1),1)}{2}$	$\mathrm{SQ}^{(d)}_{5;(5)}(\tau_0(x^2),1)$	$\frac{\mathrm{SQ}^{(d)}_{5;(5)}(\tau_1(x),1)}{d}$	$-\frac{\mathrm{SQ}^{(d)}_{5;(5)}(\tau_2(1),1)}{2}$
1	2,875	3,850	2,875	2,875
2	$\frac{4{,}876{,}875}{8}$	3,589,125	$\frac{19{,}660{,}875}{8}$	$\frac{13{,}731{,}875}{8}$
3	$\frac{8{,}564{,}575{,}000}{27}$	$\frac{16{,}126{,}540{,}000}{3}$	$\frac{76{,}579{,}948{,}750}{27}$	$\frac{175{,}851{,}761{,}875}{27}$
4	$\frac{15{,}517{,}926{,}796{,}875}{64}$	$\frac{19{,}736{,}572{,}853{,}125}{2}$	$\frac{801{,}135{,}363{,}990{,}625}{192}$	$\frac{1{,}123{,}498{,}525{,}946{,}875}{576}$
5	229,305,888,887,648	20,310,770,587,807,020	$\frac{14{,}274{,}970{,}288{,}322{,}171}{2}$	$\frac{125{,}303{,}832{,}133{,}435{,}229}{48}$

In the remaining case, that is, $|\mathbf{a}|=n-1$ and $\ell^-(\mathbf{a})=0$, the Gromov–Witten analogue of (1.8) is the relation

$$Z^{\mathrm{GW}}_{n;\mathbf{a}}(x,\hbar,q)=\mathrm{e}^{-\mathbf{a}!q/\hbar}\frac{Y_{n;\mathbf{a}}(x,\hbar,q)}{I_{n;\mathbf{a}}(q)}; \tag{1.12}$$

see [7, Theorem 10.7]. Theorem 1 implies that (1.12) holds with $Y_{n;\mathbf{a}}(x,\hbar,q)$ replaced by $Z_{n;\mathbf{a}}(x,\hbar,q)$. The same comparisons apply to the equivariant versions of Givental's J-function for the stable quotients invariants, computed by Theorem 3, and of Givental's J-function for Gromov–Witten invariants computed by [7, Theorems 9.5, 10.7, 11.8] in the $\ell^-(\mathbf{a})=0$ case and [6, Theorem 5.3] in the $\ell^-(\mathbf{a})\ge 1$ case. Thus, the Gromov–Witten and stable quotients invariants are related essentially by the mirror map. By [27, Theorem 1], the primary (without ψ-classes) genus 0 three-point Gromov–Witten and stable quotients invariants of Calabi–Yau complete intersection threefolds are related precisely by the change of variables $Q\longrightarrow q$, and the rescaling $I_{n;\mathbf{a}}(q)$, that is, the exponential factor in (1.9), can be seen as an artifact of the presence of $\hbar$.

Table 1 lists a few Gromov–Witten and stable quotients invariants of the quintic threefold $X_{(5)}\subset\mathbb{P}^4$ obtained from (1.11) and (1.8), respectively. In the first column of this table,

$$\begin{aligned}\mathrm{GW}^{(d)}_{5;(5)}(\tau_p(x^{2-p}),1)&\equiv\int_{[\overline{\mathfrak{M}}_{0,2}(X_{(5)},d)]^{\mathrm{vir}}}\psi_1^p\,\mathrm{ev}_1^*x^{2-p}\\&=5\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^4,d)}\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{5;(5)})\psi_1^p\,\mathrm{ev}_1^*x^{3-p},\end{aligned}$$

where $\dot{\mathcal{V}}^{(d)}_{5;(5)}$ is the usual analogue of (1.2) over $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^4,d)$. By the string, dilaton, and divisor relations [11, Section 26.3],

$$\frac{\mathrm{GW}^{(d)}_{5;(5)}(\tau_1(x),1)}{d}=\deg[\overline{\mathfrak{M}}_{0,0}(X_{(5)},d)]^{\mathrm{vir}}=-\frac{\mathrm{GW}^{(d)}_{5;(5)}(\tau_2(1),1)}{2}. \tag{1.13}$$

These relations are obtained using the forgetful maps

$$\overline{\mathfrak{M}}_{0,2}(X_{(5)},d) \xrightarrow{f_2} \overline{\mathfrak{M}}_{0,1}(X_{(5)},d) \xrightarrow{f_1} \overline{\mathfrak{M}}_{0,0}(X_{(5)},d),$$

which have no analogues in the stable quotients theory. The middle term in (1.13) does not have an analogue in the stable quotients theory either, whereas the analogues of the outer terms in (1.13) are not equal, as Table 1 illustrates. The numbers $\mathrm{GW}_d(\tau_0(x^2),1)$ vanish since the classes $\mathrm{ev}_1^* x^3$ on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^4,d)$ are the pullbacks by the forgetful morphisms f_1 of the classes $\mathrm{ev}_1^* x^3$ and $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^4,d)$. The analogous stable quotients invariants do not vanish; see (1.10).

It is interesting to observe that the numbers $d\mathrm{SQ}_d(\tau_0(x^2),1)$ are integers if $(n,\mathbf{a})=(5,(5))$ and $d\le 1{,}000$; as noted in [27, Section 1], the same is the case for the numbers $d\mathrm{SQ}_d(\tau_0(x),\tau_0(x))$. Two-point GW-invariants of such form are equal to three-point primary GW-invariants, which are integers, when the target is a Calabi–Yau, due to symplectic topology considerations; see [18, Section 7.3] and [21], for example. Since the stable quotients invariants are purely algebro-geometric objects, the apparent integrality of the primary invariants $d\mathrm{SQ}_d(\cdot,\cdot)$ suggests that there should be an algebro-geometric reason behind the integrality of these numbers, as well as of the closely related three-point GW-invariants.

As in the case of mirror symmetry for Gromov–Witten invariants, Theorem 1 follows immediately from its $\mathbb{T}^n$-equivariant version, Theorem 3 in Section 4. The latter is proved using the Atiyah–Bott localization theorem [1] on $\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)$, which reduces the equivariant version of the power series (1.4), the power series $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ defined by (4.1) below, to a sum of rational functions over certain graphs. As in the case of Gromov–Witten invariants, $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ is $\mathfrak{C}$-recursive in the sense of Definition 5.1, with the collection $\mathfrak{C}$ of structure coefficients given by (5.6), and satisfies the self-polynomiality condition of Definition 5.2; the same is the case of the equivariant version of the power series (1.6), the power series $\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ defined by (4.2). Thus, the two power series

$$\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q),\ \mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q) \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[[\hbar^{-1},q]]$$

are determined by their mod $(\hbar^{-1})^2$ part; see Proposition 5.3. It is straightforward to determine the mod $(\hbar^{-1})^2$-part of the power series $\mathcal{Y}_{n;\mathbf{a}}$. The mod $(\hbar^{-1})^2$-part of Givental's J-function in Gromov–Witten theory is 1 in all cases for a simple geometric reason. This approach thus confirms the analogue of Theorem 1 in *Gromov–Witten* theory and thus mirror symmetry for the genus 0 Gromov–Witten invariants of projective complete intersections.

In the *stable quotients* theory, the situation with the mod $(\hbar^{-1})^2$-part of $\mathcal{Z}_{n;\mathbf{a}}$ is different. It is still 1, for dimensional reasons, if $|\mathbf{a}|\le n-2$. If $|\mathbf{a}|=n-1$, the mod $(\hbar^{-1})^2$-part of $\mathcal{Z}_{n;\mathbf{a}}$ vanishes in the q-degrees 2 and higher; it is straightforward to see that the coefficient of q^1 mod $(\hbar^{-1})^2$ is $\mathbf{a}!/\hbar$ if $\ell^-(\mathbf{a})=0$ and 0 otherwise.[3] So, in these cases, the proof of mirror symmetry for Gromov–Witten invariants carries over to the stable quotients invariants. However, in the Calabi–

[3] Even this is not necessary due to our approach to the Calabi–Yau case.

Yau case, $|\mathbf{a}|=n$, the mod $(\hbar^{-1})^2$-part of $\mathcal{Z}_{n;\mathbf{a}}$ is not zero in all q-degrees if $\ell^-(\mathbf{a})\le 1$, and we see no a priori reason for the coefficients of positive q-degrees to vanish even if $\ell^-(\mathbf{a})\ge 2$. Thus, the proof of mirror symmetry for Gromov–Witten invariants can*not* directly carry over to the stable quotients invariants in the Calabi–Yau cases.

Since the coefficients of q^0 on the two sides of the identity in Theorem 3 are the same (both are 1), it is equivalent to the equality of the auxiliary coefficients $\mathcal{Y}_i^r(d)$ and $\mathcal{Z}_i^r(d)$ in the recursions (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$, respectively. By a direct algebraic computation, the coefficients $\mathcal{Y}_i^r(d)$ are expressible in terms of certain residues of $\mathcal{Y}$; see Lemma 5.4. Analyzing the relevant graphs, one can show that the coefficients $\mathcal{Z}_i^r(d)$ are likewise expressible in terms of certain residues of $\mathcal{Z}$, but in a different way; see Proposition 6.1. Thus, for each pair $(n,\mathbf{a})$ with $|\mathbf{a}|\le n$, the identity in Theorem 3 is equivalent to certain identities for the residues of $\mathcal{Y}_{n;\mathbf{a}}$; see Lemma 8.2. Since $\mathcal{Y}_{n;\mathbf{a}}=\mathcal{Z}_{n;\mathbf{a}}$ whenever $|\mathbf{a}|\le n-2$, these identities hold whenever $|\mathbf{a}|\le n-2$. By the proof of Proposition 8.3, the validity of these identities is independent of n, and thus they hold for all pairs $(n,\mathbf{a})$. This yields Theorem 3 and thus Theorem 1.

The relations of Lemma 8.2 involve twisted Hurwitz numbers arising from certain moduli spaces of weighted stable curves $\overline{\mathcal{M}}_{0,2|d}$; see Section 2. These relations in turn uniquely determine the twisted Hurwitz numbers, even equivariantly, in terms of a key power series associated with $\mathcal{Y}_{n;\mathbf{a}}$; see Theorems 2 and 4 in Sections 2 and 4, respectively. Based on developments in Gromov–Witten theory, one would expect these closed formulas to be a key ingredient in computing twisted genus 1 stable quotients invariants and thus answering yet another question raised in [17].

The proof that the equivariant version of Givental's J-function in Gromov–Witten theory satisfies the self-polynomiality condition of Definition 5.2 uses the localization theorem [1] to compute integrals over the moduli space $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^1\times\mathbb{P}^{n-1},(1,d))$. Our proof that the equivariant stable quotients analogue of Givental's J-function satisfies the self-polynomiality condition uses the moduli space of stable pairs of quotients $\overline{Q}_{0,2}(\mathbb{P}^1\times\mathbb{P}^{n-1},(1,d))$ in a similar way; see Section 7. This moduli space is a special case of the moduli space

$$\overline{Q}_{g,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\ldots,d_p))$$

of stable p-tuples of quotients, which we describe in Section 2 by extending the notion of stable quotients introduced in [17].

The Gromov–Witten analogues of Theorem 1 and its equivariant version, Theorem 3 in Section 4, extend to the so-called concavex bundles over products of projective spaces, that is, vector bundles of the form

$$\bigoplus_{k=1}^{l}\mathcal{O}_{\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1}}(a_{k;1},\ldots,a_{k;p})\longrightarrow\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},$$

where for each given $k = 1, 2, \ldots, l$, either $a_{k;1}, \ldots, a_{k;p} \in \mathbb{Z}^{\geq 0}$ or $a_{k;1}, \ldots, a_{k;p} \in \mathbb{Z}^{-}$. The stable quotients analogue of these bundles are the sheaves

$$\bigoplus_{k=1}^{l} \mathcal{S}_1^{*a_{k;1}} \otimes \cdots \otimes \mathcal{S}_p^{*a_{k;p}} \longrightarrow \mathcal{U} \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}, (d_1, \ldots, d_p)) \qquad (1.14)$$

with the same condition on $a_{k;i}$, where $\mathcal{S}_i \longrightarrow \mathcal{U}$ is the universal subsheaf corresponding to the ith factor; see Section 2. In this case, we compare two power series

$$Y_{n_1,\ldots,n_p;\mathbf{a}}(x_1, \ldots, x_p, \hbar, q_1, \ldots, q_p) \in \mathbb{Q}[x_1, \ldots, x_p][[\hbar^{-1}, q_1, \ldots, q_p]], \qquad (1.15)$$

$$Z_{n_1,\ldots,n_p;\mathbf{a}}(x_1, \ldots, x_p, \hbar, q_1, \ldots, q_p) \in H^*(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1})[[\hbar^{-1}, q_1, \ldots, q_p]], \qquad (1.16)$$

where $x_1, \ldots, x_p \in H^*(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1})$ are the pullbacks of the hyperplane classes by the projection maps. The coefficient of $q_1^{d_1} \cdots q_p^{d_p}$ in (1.16) is defined by the same pushforward as in (1.4), with the degree d of the stable quotients replaced by $(d_1, \ldots, d_p)$. The coefficient of $q_1^{d_1} \cdots q_p^{d_p}$ in (1.15) is given by

$$\prod_{a_{k;1} \geq 0} \prod_{r=1}^{\sum_{s=1}^{p} a_{k;s} d_s} \left(\sum_{s=1}^{p} a_{k;s} x_s + r\hbar \right) \prod_{a_{k;1} < 0} \prod_{r=0}^{-\sum_{s=1}^{p} a_{k;s} d_s - 1} \left(\sum_{s=1}^{p} a_{k;s} x_s - r\hbar \right) \Big/ \prod_{s=1}^{p} \prod_{r=1}^{d_s} (x_s + r\hbar)^{n_s}.$$

The condition $|\mathbf{a}| \leq n$ should be replaced by the conditions

$$|a_{1;s}| + \cdots + |a_{l;s}| \leq n_s \quad \forall s = 1, \ldots, p.$$

Our proof of Theorem 3 (and thus of Theorem 1) extends directly to this situation; we will comment on the necessary modifications in each step of the proof.

Mirror formulas for the two-point versions of (1.4) and (4.1), that is, with ev_1 and $(\hbar - \psi_1)$ replaced by $\mathrm{ev}_1 \times \mathrm{ev}_2$ and $(\hbar_1 - \psi_1)(\hbar_2 - \psi_2)$, as well as their generalizations to products of projective spaces, can now be readily obtained using the approaches of [25; 20] in Gromov–Witten theory; see [27]. They are related to the corresponding formulas in Gromov–Witten theory in the same ways as the one-point formulas; see the paragraph following Theorem 1. Similarly to developments in Gromov–Witten theory, these two-point genus 0 formulas are one of the key steps in computing twisted genus 1 stable quotients invariants.

A notable feature of the mirror formula of Theorem 1 and its two-point analogue is that they are invariant under replacing $(n, (a_1, \ldots, a_k))$ by $(n + 1,$

$(a_1, \dots, a_k, 1)$); their extensions to products of projective spaces have a similar feature.[4] This is consistent with [4, Proposition 6.4.1].

2. Moduli Spaces of Stable Quotients

We begin this section by reviewing the notion of stable quotients for products of projective spaces. Propositions 2.1 and 2.2 describing moduli spaces of such objects are a special case of [3, Theorems 3.2.1, 4.0.1] and precisely the statement of [3, Example 7.2.6], respectively. We include proofs of these statements, extending [17] from the case of projective spaces, for the sake of completeness, since [3] treats the general toric case and is thus more involved. We then introduce related moduli spaces of weighted curves. We conclude this section with a closed formula for twisted Hurwitz numbers arising from integrals over these moduli spaces of curves; see Theorem 2.

By a nodal genus g curve we will mean a reduced connected scheme $\mathcal{C}$ over $\mathbb{C}$ of pure dimension 1 with at worst nodal singularities and $h^1(\mathcal{C}, \mathcal{O}_\mathcal{C}) = g$. Let $\mathcal{C}^* \subset \mathcal{C}$ denote the nonsingular locus of such a curve. A quasi-stable genus g m-marked curve is a tuple $(\mathcal{C}, y_1, \dots, y_m)$ consisting of a nodal genus g curve and distinct points $y_i \in \mathcal{C}^*$. A (corank 1) quasi-stable quotient of the trivial rank n sheaf on such a curve is a rank 1 subsheaf $S \subset \mathbb{C}^n \otimes \mathcal{O}_\mathcal{C}$ such that the corresponding quotient sheaf Q, given by

$$0 \longrightarrow S \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_\mathcal{C} \longrightarrow Q \longrightarrow 0,$$

is locally free on $(\mathcal{C} - \mathcal{C}^*) \cup \{y_1, \dots, y_m\}$, that is, at the nodes and markings of C. A tuple $(S_1, \dots, S_p)$ of quasi-stable quotients on $(\mathcal{C}, y_1, \dots, y_m)$ is stable if the $\mathbb{Q}$-line bundle

$$\omega_\mathcal{C}(y_1 + \dots + y_m) \otimes (S_1^* \otimes \dots \otimes S_p^*)^\varepsilon \longrightarrow \mathcal{C}$$

is ample for all $\varepsilon \in \mathbb{Q}^+$; this implies that $2g - 2 + m \geq 0$. An isomorphism

$$\phi : (\mathcal{C}, y_1, \dots, y_m, S_1, \dots, S_p) \longrightarrow (\mathcal{C}', y_1', \dots, y_m', S_1', \dots, S_p')$$

between tuples of quasi-stable quotients is an isomorphism $\phi : \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\phi(y_i) = y_i' \quad \forall i = 1, \dots, m, \qquad \phi^* S_j' = S_j \subset \mathbb{C}^{n_j} \otimes \mathcal{O}_\mathcal{C} \quad \forall j = 1, \dots, p.$$

The automorphism group of any stable tuple of quotients is finite.

PROPOSITION 2.1. *If $g, m, d_1, \dots, d_p \in \mathbb{Z}^{\geq 0}$ and $n_1, \dots, n_p \in \mathbb{Z}^+$, the moduli space*

$$\overline{Q}_{g,m}(\mathbb{P}^{n_1-1} \times \dots \times \mathbb{P}^{n_p-1}, (d_1, \dots, d_p)) \tag{2.1}$$

parameterizing the stable p-tuples of quotients

$$(\mathcal{C}, y_1, \dots, y_m, S_1, \dots, S_p), \tag{2.2}$$

with $h^1(\mathcal{C}, \mathcal{O}_\mathcal{C}) = g$, $S_i \subset \mathbb{C}^{n_i} \otimes \mathcal{O}_\mathcal{C}$, and $\deg(S_i) = -d_i$, is a separated and proper Deligne–Mumford stack of finite type over $\mathbb{C}$ and carries a canonical two-term obstruction theory.

[4] This replacement does not change the total space of the vector bundle (1.3).

Proof. The construction of $\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$ in [17] carries through with minor changes. We sketch the modification here.

I. *Construction of the moduli space.* Let $g, m, d_1, \ldots, d_p$ satisfy

$$2g - 2 + m + \varepsilon(d_1 + \cdots + d_p) > 0 \quad \forall \varepsilon > 0.$$

Let $d = d_1 + \cdots + d_p$. Fix a stable p-tuple of quotients $(\mathcal{C}, y_1, \ldots, y_m, S_1, \ldots, S_p)$, where

$$0 \longrightarrow S_i \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow Q_i \longrightarrow 0. \tag{2.3}$$

By assumption, the line bundle

$$\mathcal{L}_\varepsilon = \omega_{\mathcal{C}}(y_1 + \cdots + y_m) \otimes (S_1^* \otimes \cdots \otimes S_p^*)^\varepsilon$$

is ample for all $\varepsilon > 0$. Fix $\varepsilon = 1/(d+1)$ and let $f = 5(d+1)$. By [17, Lemma 5], the line bundle $\mathcal{L}_\varepsilon^f$ is very ample and has no higher cohomology. Therefore,

$$h^0(\mathcal{C}, \mathcal{L}_\varepsilon^f) = 1 - g + 5(d+1)(2g - 2 + m) + 5d$$

is independent of the choice of the stable p-tuple of quotients. Let

$$V = H^0(\mathcal{C}, \mathcal{L}_\varepsilon^f)^*.$$

The line bundle $\mathcal{L}_\varepsilon^f$ induces an embedding $\iota : \mathcal{C} \hookrightarrow \mathbb{P}(V)$. Let Hilb denote the Hilbert scheme of curves in $\mathbb{P}(V)$ of genus g and degree

$$5(d+1)(2g - 2 + m) + 5d = \deg \mathcal{L}_\varepsilon^f.$$

Each stable quotient gives rise to a point in

$$\mathcal{H} = \mathsf{Hilb} \times \mathbb{P}(V)^m,$$

where the last factors record the locations of the markings $y_1, \ldots, y_m$.

Points in $\mathcal{H}$ correspond to tuples $(\mathcal{C}, y_1, \ldots, y_m)$. Denote by $\mathcal{H}' \subset \mathcal{H}$ the quasi-projective subscheme consisting of the tuples such that

(i) the points $y_1, \ldots, y_m$ are contained in $\mathcal{C}$,
(ii) the curve $(\mathcal{C}, y_1, \ldots, y_m)$ is quasi-stable.

Let $\pi : \mathcal{U}' \longrightarrow \mathcal{H}'$ be the universal curve over $\mathcal{H}'$. For $i = 1, \ldots, p$, let

$$\mathsf{Quot}(n_i, d_i) \longrightarrow \mathcal{H}'$$

be the π-relative Quot scheme parameterizing rank $n_i - 1$ degree d_i quotients

$$0 \longrightarrow S_i \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow Q_i \longrightarrow 0$$

on the fibers of π. Denote by $\mathcal{Q}$ the fiber product

$$\mathcal{Q} = \mathsf{Quot}(n_1, d_1) \times_{\mathcal{H}'} \cdots \times_{\mathcal{H}'} \mathsf{Quot}(n_p, d_p) \longrightarrow \mathcal{H}'$$

and by $\mathcal{Q}' \subset \mathcal{Q}$ the locally closed subscheme consisting of the tuples such that

(iii) Q_i is locally free at the nodes and at the marked points of $\mathcal{C}$,
(iv) the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$ to $\mathcal{C}$ agrees with the line bundle

$$(\omega_{\mathcal{C}}(y_1 + \cdots + y_m))^{5(d+1)} \otimes (S_1^* \otimes \cdots \otimes S_p^*)^5.$$

The action of $\mathrm{PGL}(V)$ on $\mathcal{H}$ induces actions on $\mathcal{H}'$ and $\mathcal{Q}'$. A $\mathrm{PGL}(V)$-orbit in $\mathcal{Q}'$ corresponds to a stable quotient up to isomorphism. By stability, each orbit has finite stabilizers. The moduli space (2.1) is the stack quotient $[\mathcal{Q}'/\mathrm{PGL}(V)]$.

II. *Separateness.* We prove that the moduli stack (2.1) is separated by the valuative criterion. Let $(\Delta, 0)$ be a nonsingular pointed curve and $\Delta^0 = \Delta - \{0\}$. Take two flat families of quasi-stable pointed curves

$$\mathcal{X}^j \longrightarrow \Delta, \qquad y_1^j, \ldots, y_m^j : \Delta \longrightarrow \mathcal{X}^j,$$

and two flat families of stable quotients

$$0 \longrightarrow S_i^j \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{X}_i} \longrightarrow Q_i^j \longrightarrow 0,$$

with $j = 1, 2$ and $i = 1, \ldots, p$. Assume that the two families are isomorphic away from the central fiber. By [17, Section 6.2], an isomorphism between these two families over $\Delta - 0$ extends to the families of curves $\mathcal{X}^j \longrightarrow \Delta$ in a manner preserving the sections and hence extends to each pair of families of stable quotients.

III. *Properness.* We prove that the moduli stack (2.1) is proper, again by the valuative criterion. Let

$$\pi^0 : \mathcal{X}^0 \longrightarrow \Delta^0, \qquad y_1, \ldots, y_m : \Delta^0 \longrightarrow \mathcal{X}^0$$

carry a flat family of stable p-tuples of quotients

$$0 \longrightarrow S_i \longrightarrow \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{X}^0} \longrightarrow Q_i \longrightarrow 0.$$

By [17, Section 6.3], each stable quotient individually extends, possibly after base-change, and hence the p-tuple extends. In particular, the blowup procedure in [17, Section 6.3] yielding the sheaf $\widetilde{S}$ in [17, (18)] can be applied to each sheaf S_i separately to yield sheaves $\widetilde{S}_i$ over a flat family $\widetilde{\mathcal{X}} \longrightarrow \Delta$ so that the corresponding quotients $\tilde{Q}_i$ are locally free at the nodes and at the marked points of the central fiber. After a base change and altering each quotient sheaf at finitely points, we obtain a flat family of quasi-stable quotients Q_i'' over a flat family as in [17, (19)]. The final blowdown step of [17, Section 6.3] is applied with the unstable genus 0 curves P such that $S_i''|_P = \mathcal{O}_P$ for all $i = 1, \ldots, p$ and the line bundle $\mathfrak{L}$ obtained from the one in [17] by replacing $\Lambda^r(S'')$ with $S_1'' \otimes \cdots \otimes S_p''$. The resulting p-tuple of push-forward sheaves over the central fiber is then stable.

IV. *Obstruction theory.* We follow the argument in [17, Section 3.2]. Let $\phi : \mathcal{C} \longrightarrow \mathcal{M}_{g,m}$ be the universal curve over the Artin stack of pointed curves, and $\mathbf{Q}(n, d) \longrightarrow \mathcal{M}_{g,m}$ be the relative Quot scheme of rank $n-1$ degree d quotients of $\mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}$ along the fibers of ϕ. Denote by

$$\mathbf{Q}'(n, d) \subset \mathbf{Q}(n, d)$$

the locus consisting of locally free subsheaves and by

$$\nu\colon\ \mathbf{Q}' \equiv \mathbf{Q}'(n_1, d_1) \times_{\mathcal{M}_{g,m}} \cdots \times_{\mathcal{M}_{g,m}} \mathbf{Q}'(n_p, d_p) \times_{\mathcal{M}_{g,m}} \mathcal{C} \longrightarrow \mathcal{M}_{g,m}$$

the fiber product. The universal sequence of sheaves

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

over $\mathbf{Q}'(n,d)\times_{\mathcal{M}_{g,m}}\mathcal{C}$ gives rise to a universal sequence

$$0\longrightarrow\bigoplus_{i=1}^{p}\mathcal{S}_i\longrightarrow\bigoplus_{i=1}^{p}(\mathbb{C}^{n_i}\otimes\mathcal{O}_{\mathcal{C}})\longrightarrow\bigoplus_{i=1}^{p}\mathcal{Q}_i\longrightarrow 0$$

over $\mathbf{Q}'\times_{\mathcal{M}_{g,m}}\mathcal{C}$. Let $\pi:\mathbf{Q}'\times_{\mathcal{M}_{g,m}}\mathcal{C}\longrightarrow\mathbf{Q}'$ be the projection map. By [22, Proposition 4.4.4] with

$$\mathcal{K}=\bigoplus_{i=1}^{p}\mathcal{S}_i,\qquad \mathcal{H}=\bigoplus_{i=1}^{p}\mathbb{C}^{n_i}\otimes\mathcal{O}_{\mathcal{C}},\quad\text{and}\quad \mathcal{F}=\bigoplus_{i=1}^{p}\mathcal{Q}_i$$

the relative deformation–obstruction theory of $\nu:\mathbf{Q}'\longrightarrow\mathcal{M}_{g,m}$ is given by

$$RHom_{\pi}(\mathcal{S}_1,\mathcal{Q}_1)\oplus\cdots\oplus RHom_{\pi}(\mathcal{S}_p,\mathcal{Q}_p)=\bigoplus_{i=1}^{p}R\pi_*Hom(\mathcal{S}_i,\mathcal{Q}_i);$$

the equality above holds because each $\mathcal{S}_i$ is a locally free sheaf. By [16, Section 2], $R\pi_*Hom(\mathcal{S}_i,\mathcal{Q}_i)$ can be resolved by a two-step complex of vector bundles. Thus,

$$\nu^A:\overline{Q}_{g,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\ldots,d_p))\longrightarrow\mathcal{M}_{g,m}$$

admits a two-term relative deformation–obstruction theory. Along with the smoothness of $\mathcal{M}_{g,m}$, this induces an absolute two-term deformation–obstruction theory of the moduli space (2.1); see [9, Appendix B]. □

PROPOSITION 2.2 ([3, Example 7.2.6]). *If* $g=0$ *or* $(g,m)=(1,0)$ *and* $d_1,\ldots,d_p$, $n_1,\ldots,n_p\geq 1$, *the moduli space*

$$\overline{Q}_{g,m}(\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(d_1,\ldots,d_p))\tag{2.4}$$

is a nonsingular irreducible Deligne–Mumford stack of the expected dimension.

Proof. By part IV in the proof of Proposition 2.1, the moduli space (2.4) is smooth at a point $(\mathcal{C},y_1,\ldots,y_m,S_1,\ldots,S_p)$ if

$$\bigoplus_{i=1}^{p}\mathrm{Ext}^1(S_i,Q_i)=0.\tag{2.5}$$

Since each S_i is locally free, this is the case if

$$H^1(S_i^*\otimes Q_i)=0\tag{2.6}$$

for each $i=1,\ldots,p$. From the cohomology long exact sequence for the short exact sequence

$$0\longrightarrow\mathcal{O}_{\mathcal{C}}\longrightarrow\mathbb{C}^{n_i}\otimes S_i^*\longrightarrow Q_i\otimes S_i^*\longrightarrow 0$$

we see that (2.6) holds if $H^1(S_i^*)=0$.

If $g=0$, $\mathcal{C}$ is a rational curve, and thus there are no special line bundles on $\mathcal{C}$ that have a nonnegative degree on every component of $\mathcal{C}$. If $(g,m)=(1,0)$, then $\mathcal{C}$ is either a nonsingular curve of genus 1 or a cycle of rational curves; thus, there are no special line bundles of positive degree on $\mathcal{C}$ that have nonnegative degree on each component of $\mathcal{C}$. In either case, we conclude that $H^1(S_i^*)=0$ for each $i=$

$1, \dots, p$, and so (2.5) holds. Thus, the moduli space (2.4) is smooth at every point and hence is a nonsingular Deligne–Mumford stack of the expected dimension.

It remains to show that it is also irreducible. Let U denote the open locus in the moduli space where the domain curve is smooth. In the $g = 0$ case, U is dominated by the product of projective spaces $(\mathbb{P}^1)^m \times \prod_i \mathrm{Proj}(H^0(\mathcal{O}(d_i))^{n_i})$. In the $(g, m) = (1, 0)$ case, U is dominated by the bundle $\prod_i \mathrm{Proj}(H^0(\mathcal{O}(dp))^{n_i})$ over $\overline{\mathcal{M}}_{1,1}$, where p is the marked point. Thus, U is irreducible in both cases. Since the moduli space (2.4) is unobstructed, U is dense in (2.4), and thus the latter is also irreducible. □

A stable tuple as in (2.2) such that each quotient sheaf $Q_i = \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{C}}/S_i$ is locally free corresponds to a stable morphism

$$\mathcal{C} \longrightarrow \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}$$

with marked points $y_1, \dots, y_m$. As in the $p = 1$ case considered in [17, Section 3.1], there are evaluation morphisms

$$\mathrm{ev}_i : \overline{Q}_{g,m}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}, (d_1, \dots, d_p)) \longrightarrow \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}$$

with $i = 1, 2, \dots, m$. There is also a universal curve

$$\pi : \mathcal{U} \longrightarrow \overline{Q}_{g,m}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}, (d_1, \dots, d_p))$$

with m sections $\sigma_1, \dots, \sigma_m$ and universal rank 1 subsheaves $\mathcal{S}_i \subset \mathbb{C}^{n_i} \otimes \mathcal{O}_{\mathcal{U}}$.

We will also need a certain moduli space of weighted curves; this is the stable quotients counterpart of the Deligne–Mumford moduli space of stable genus g marked curves in Gromov–Witten theory. A d-tuple of flecks on a quasi-stable m-marked curve $(\mathcal{C}, y_1, \dots, y_m)$ is a d-tuple $(\hat{y}_1, \dots, \hat{y}_d)$ of points of $\mathcal{C}^* - \{y_1, \dots, y_m\}$. Such a tuple is stable if the $\mathbb{Q}$-line bundle

$$\omega_{\mathcal{C}}(y_1 + \cdots + y_m + \varepsilon(\hat{y}_1 + \cdots + \hat{y}_d)) \longrightarrow \mathcal{C}$$

is ample for all $\varepsilon \in \mathbb{Q}^+$; this again implies that $2g - 2 + m \geq 0$. An isomorphism

$$\phi : (\mathcal{C}, y_1, \dots, y_m, \hat{y}_1, \dots, \hat{y}_d) \longrightarrow (\mathcal{C}', y'_1, \dots, y'_m, \hat{y}'_1, \dots, \hat{y}'_d)$$

between curves with m marked points and d flecks is an isomorphism $\phi : \mathcal{C} \longrightarrow \mathcal{C}'$ such that

$$\phi(y_i) = y'_i \quad \forall i = 1, \dots, m, \qquad \phi(\hat{y}_j) = \hat{y}'_j \quad \forall j = 1, \dots, d.$$

The automorphism group of any stable curve with m marked points and d flecks is finite.

PROPOSITION 2.3. *If $g, m, d \in \mathbb{Z}^{\geq 0}$, then the moduli space $\overline{\mathcal{M}}_{g,m|d}$ parameterizing the stable genus g curves with m marked points and d flecks,*

$$(\mathcal{C}, y_1, \dots, y_m, \hat{y}_1, \dots, \hat{y}_d), \tag{2.7}$$

is a nonsingular, irreducible, proper Deligne–Mumford stack.

Proof. The moduli space $\overline{\mathcal{M}}_{g,m|d}$ is the moduli space of weighted pointed stable curves, defined in [10, Section 2], with m points of weight 1 and d points of weight $1/d$ (if $d > 0$). Thus, this proposition is a special case of [10, Theorem 2.1]. □

Any tuple as in (2.7) induces a quasi-stable quotient

$$\mathcal{O}_{\mathcal{C}}(-\hat{y}_1 - \cdots - \hat{y}_d) \subset \mathcal{O}_{\mathcal{C}} \equiv \mathbb{C}^1 \otimes \mathcal{O}_{\mathcal{C}}.$$

For any ordered partition $d = d_1 + \cdots + d_p$ with $d_1, \ldots, d_p \in \mathbb{Z}^{\geq 0}$, this correspondence gives rise to a morphism

$$\overline{\mathcal{M}}_{g,m|d} \longrightarrow \overline{Q}_{g,m}(\mathbb{P}^0 \times \cdots \times \mathbb{P}^0, (d_1, \ldots, d_p)).$$

In turn, this morphism induces an isomorphism

$$\phi : \overline{\mathcal{M}}_{g,m|d}/\mathbb{S}_{d_1} \times \cdots \times \mathbb{S}_{d_p} \xrightarrow{\sim} \overline{Q}_{g,m}(\mathbb{P}^0 \times \cdots \times \mathbb{P}^0, (d_1, \ldots, d_p)), \tag{2.8}$$

with the symmetric group $\mathbb{S}_{d_1}$ acting on $\overline{\mathcal{M}}_{g,m|d}$ by permuting the points $\hat{y}_1, \ldots, \hat{y}_{d_1}$, $\mathbb{S}_{d_2}$ acting on $\overline{\mathcal{M}}_{g,m|d}$ by permuting the points $\hat{y}_{d_1+1}, \ldots, \hat{y}_{d_1+d_2}$, and so on.

There is again a universal curve

$$\pi : \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{g,m|d}$$

with sections $\sigma_1, \ldots, \sigma_m$ and $\hat{\sigma}_1, \ldots, \hat{\sigma}_d$. Let

$$\psi_i = -\pi_*(\sigma_i^2), \; \hat{\psi}_j = -\pi_*(\hat{\sigma}_j^2) \in H^2(\overline{\mathcal{M}}_{g,m|d}) \tag{2.9}$$

be the first Chern classes of the universal cotangent line bundles.

Lemma 2.4 ([17, Section 4.5]). *If $d \in \mathbb{Z}^+$ and $a_1, a_2, b_1, \ldots, b_d \in \mathbb{Z}^{\geq 0}$, then*

$$\int_{\overline{\mathcal{M}}_{0,2|d}} \psi_1^{a_1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \cdots \hat{\psi}_d^{b_d} = \binom{d-1}{a_1, a_2} \cdot \begin{cases} 1 & \text{if } b_1, \ldots, b_d = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Proof. If $d > 1$, then there is a forgetful morphism

$$f : \overline{\mathcal{M}}_{0,2|d} \longrightarrow \overline{\mathcal{M}}_{0,2|d-1},$$

dropping the fleck $\hat{y}_d$. For $i = 1, 2$, let $D_i \subset \overline{\mathcal{M}}_{0,2|d}$ denote the divisor whose generic element consists of two components, with one of them containing y_i and $\hat{y}_d$ (and no other marked points). By (2.9),

$$\psi_i = f^*\psi_i + D_i \quad \forall i = 1, 2, \qquad \hat{\psi}_j = f^*\psi_j \quad \forall j = 1, \ldots, d-1. \tag{2.11}$$

Under the canonical identification of $D_i \approx \overline{\mathcal{M}}_{0,2|d-1} \times \overline{\mathcal{M}}_{0,2|1}$ with $\overline{\mathcal{M}}_{0,2|d-1}$,

$$\begin{aligned} D_i|_{D_i} &= -\psi_i, \quad D_1 \cdot D_2 = 0, \quad \psi_i|_{D_i}, \hat{\psi}_d|_{D_i} = 0, \\ \psi_{3-i}|_{D_i} &= \psi_{3-i}, \qquad \hat{\psi}_j|_{D_i} = \hat{\psi}_j \quad \forall j = 1, \ldots, d-1. \end{aligned} \tag{2.12}$$

If the left-hand side of (2.10) is not zero, then the sum of the exponents is $d-1$. Thus, by symmetry, we can assume that $b_d = 0$. By (2.11) and (2.12),

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,2|d}} \psi_1^{a_1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \cdots \hat{\psi}_d^{b_d} = &\int_{\overline{\mathcal{M}}_{0,2|d-1}} \psi_1^{a_1-1} \psi_2^{a_2} \hat{\psi}_1^{b_1} \cdots \hat{\psi}_d^{b_d} \\ &+ \int_{\overline{\mathcal{M}}_{0,2|d-1}} \psi_1^{a_1} \psi_2^{a_2-1} \hat{\psi}_1^{b_1} \cdots \hat{\psi}_d^{b_d}. \end{aligned}$$

This implies (2.10) by induction on d (if $d = 1$, $\overline{\mathcal{M}}_{0,2|d}$ is a single point). □

Our proof of Theorems 1 and 3 immediately leads to a closed formula for certain twisted equivariant Hurwitz numbers; see Theorem 4 in Section 4. We conclude this section with a nonequivariant version of this formula.

Let $x \in H^2(\mathbb{P}^\infty)$ denote the hyperplane class. For any $d \in \mathbb{Z}^+$, let

$$\mathcal{S}^*(x) \equiv \pi_{\mathbb{P}^\infty}^* \mathcal{O}_{\mathbb{P}^\infty}(1) \otimes \pi_{\mathcal{U}}^* \mathcal{S}^* \longrightarrow \mathbb{P}^\infty \times \mathcal{U} \longrightarrow \mathbb{P}^\infty \times \overline{\mathcal{M}}_{0,2|d},$$

where $\pi_{\mathbb{P}^\infty}, \pi_{\mathcal{U}} : \mathbb{P}^\infty \times \mathcal{U} \longrightarrow \mathbb{P}^\infty, \mathcal{U}$ are the two projections. In particular,

$$\mathbf{e}(\mathcal{S}^*(x)) = x \times 1 + 1 \times e(\mathcal{S}^*) \in H^*(\mathbb{P}^\infty \times \mathcal{U}) = \mathbb{Q}[x] \otimes H^*(\mathcal{U}).$$

Similarly to (1.2), let

$$\begin{aligned}\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(x) = \bigoplus_{a_k>0} R^0\pi_*(\mathcal{S}^*(x)^{a_k}(-\sigma_1)) \oplus \bigoplus_{a_k<0} R^1\pi_*(\mathcal{S}^*(x)^{a_k}(-\sigma_1))\\ \longrightarrow \overline{\mathcal{M}}_{0,2|d},\end{aligned} \tag{2.13}$$

where $\pi : \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2|d}$ is the projection as before; this sheaf is locally free. We define power series $L_{\mathbf{a}}, \xi_{\mathbf{a}} \in \mathbb{Q}[x][[q]]$ by

$$L_{\mathbf{a}} \in x + q\mathbb{Q}[x][[q]], \quad L_{\mathbf{a}}(x,q) - q\mathbf{a}^{\mathbf{a}} L_{\mathbf{a}}(x,q)^{|\mathbf{a}|} = x^n,$$
$$\xi_{\mathbf{a}} \in q\mathbb{Q}[x][[q]], \quad x + q\frac{\mathrm{d}}{\mathrm{d}q}\xi_{\mathbf{a}}(x,q) = L_{\mathbf{a}}(x,q).$$

THEOREM 2. *If $l \in \mathbb{Z}^{\ge 0}$ and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$\begin{aligned}1 + (\hbar_1+\hbar_2)\sum_{d=1}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{e(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(x))}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\\ = \mathrm{e}^{\xi_{\mathbf{a}}(x,q)/\hbar_1 + \xi_{\mathbf{a}}(x,q)/\hbar_2} \in \mathbb{Q}[x][[\hbar_1^{-1},\hbar_2^{-1},q]].\end{aligned}$$

Proof. This is obtained from Theorem 4 by setting $n=1$, $i=1$, and $\alpha_1 = x$. □

In the case $l=0$, the left-hand side of the expression in Theorem 2 reduces to

$$\begin{aligned}1 + \sum_{a_1,a_2\ge 0} (\hbar_1^{-a_1}\hbar_1^{-(a_2+1)} + \hbar_1^{-(a_1+1)}\hbar_1^{-a_2}) \frac{q^{a_1+a_2+1}}{(a_1+a_2+1)!}\int_{\overline{\mathcal{M}}_{0,2|a_1+a_2+1}} \psi_1^{a_1}\psi_2^{a_2}\\ = 1 + \sum_{a_1,a_2\ge 0} (\hbar_1^{-a_1}\hbar_1^{-(a_2+1)} + \hbar_1^{-(a_1+1)}\hbar_1^{-a_2}) \frac{q^{a_1+a_2+1}}{(a_1+a_2+1)!}\binom{a_1+a_2}{a_1}\\ = \mathrm{e}^{q/\hbar_1 + q/\hbar_2};\end{aligned}$$

the first equality above holds by Lemma 2.4. Since $\xi_{(\cdot)}(x,q) = q$, this agrees with Theorem 2.

3. Equivariant Cohomology

In this section, we review the notion of equivariant cohomology and set up related notation that will be used throughout the rest of the paper. For the most part, our notation agrees with [11, Chapters 29, 30]; the main difference is that we work with $\mathbb{P}^{n-1}$ instead of $\mathbb{P}^n$.

For any $n \in \mathbb{Z}^+$, let

$$[n] = \{1, \dots, n\}.$$

We denote by $\mathbb{T}$ the n-torus $(\mathbb{C}^*)^n$. It acts freely on $E\mathbb{T} = (\mathbb{C}^\infty)^n - 0$:

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n).$$

Thus, the classifying space for $\mathbb{T}$ and its group cohomology are given by

$$B\mathbb{T} \equiv E\mathbb{T}/\mathbb{T} = (\mathbb{P}^\infty)^n \quad \text{and} \quad H^*_{\mathbb{T}} \equiv H^*(B\mathbb{T}; \mathbb{Q}) = \mathbb{Q}[\alpha_1, \dots, \alpha_n],$$

where $\alpha_i = \pi_i^* c_1(\gamma^*)$ if

$$\pi_i : (\mathbb{P}^\infty)^n \longrightarrow \mathbb{P}^\infty \quad \text{and} \quad \gamma \longrightarrow \mathbb{P}^\infty$$

are the projection onto the ith component and the tautological line bundle, respectively. Let

$$\mathcal{H}^*_{\mathbb{T}} = \mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha_1, \dots, \alpha_n)$$

be the field of fractions of $H^*_{\mathbb{T}}$.

A representation ρ of $\mathbb{T}$, that is, a linear action of $\mathbb{T}$ on $\mathbb{C}^k$, induces a vector bundle over $B\mathbb{T}$:

$$V_\rho \equiv E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}^k.$$

If ρ is one-dimensional, we will call

$$c_1(V_\rho^*) = -c_1(V_\rho) \in H^*_{\mathbb{T}} \subset \mathbb{Q}_\alpha$$

the weight of ρ. For example, α_i is the weight of the representation

$$\pi_i : \mathbb{T} \longrightarrow \mathbb{C}^*, \qquad (t_1, \dots, t_n) \cdot z = t_i z. \tag{3.1}$$

More generally, if a representation ρ of $\mathbb{T}$ on $\mathbb{C}^k$ splits into one-dimensional representations with weights $\beta_1, \dots, \beta_k$, we will call $\beta_1, \dots, \beta_k$ the weights of ρ. In such a case,

$$e(V_\rho^*) = \beta_1 \cdot \dots \cdot \beta_k. \tag{3.2}$$

We will call the representation ρ of $\mathbb{T}$ on $\mathbb{C}^n$ with weights $\alpha_1, \dots, \alpha_n$ the standard representation of $\mathbb{T}$.

If $\mathbb{T}$ acts on a topological space M, let

$$H^*_{\mathbb{T}}(M) \equiv H^*(BM; \mathbb{Q}), \quad \text{where } BM = E\mathbb{T} \times_{\mathbb{T}} M,$$

denote the corresponding equivariant cohomology of M. The projection map $BM \longrightarrow B\mathbb{T}$ induces an action of $H^*_{\mathbb{T}}$ on $H^*_{\mathbb{T}}(M)$. Let

$$\mathcal{H}^*_{\mathbb{T}}(M) = H^*_{\mathbb{T}}(M) \otimes_{H^*_{\mathbb{T}}} \mathcal{H}^*_{\mathbb{T}}.$$

If the $\mathbb{T}$-action on M lifts to an action on a (complex) vector bundle $V \longrightarrow M$, then

$$BV \equiv E\mathbb{T} \times_{\mathbb{T}} V$$

is a vector bundle over BM. Let

$$\mathbf{e}(V) \equiv e(BV) \in H^*_{\mathbb{T}}(M) \subset \mathcal{H}^*_{\mathbb{T}}(M)$$

denote the equivariant Euler class of V.

Throughout the paper, we work with the standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$, that is, the action induced by the standard action ρ of $\mathbb{T}$ on $\mathbb{C}^n$:

$$(t_1,\ldots,t_n)\cdot[z_1,\ldots,z_n]=[t_1z_1,\ldots,t_nz_n].$$

Since $B\mathbb{P}^{n-1}=\mathbb{P}V_\rho$,

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\equiv H^*(\mathbb{P}V_\rho;\mathbb{Q})=\mathbb{Q}[\mathbf{x},\alpha_1,\ldots,\alpha_n]/(\mathbf{x}^n+c_1(V_\rho)\mathbf{x}^{n-1}+\cdots+c_n(V_\rho)),$$

where $\mathbf{x}=c_1(\tilde{\gamma}^*)$ and $\tilde{\gamma}\longrightarrow\mathbb{P}V_\rho$ is the tautological line bundle. Since

$$c(V_\rho)=(1-\alpha_1)\cdots(1-\alpha_n),$$

it follows that

$$\begin{aligned}H^*_{\mathbb{T}}(\mathbb{P}^{n-1})&=\mathbb{Q}[\mathbf{x},\alpha_1,\ldots,\alpha_n]/(\mathbf{x}-\alpha_1)\cdots(\mathbf{x}-\alpha_n),\\ \mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1})&=\mathbb{Q}_\alpha[\mathbf{x}]/(\mathbf{x}-\alpha_1)\cdots(\mathbf{x}-\alpha_n).\end{aligned}\tag{3.3}$$

The standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ has n fixed points:

$$P_1=[1,0,\ldots,0],\quad P_2=[0,1,0,\ldots,0],\quad\ldots,\quad P_n=[0,\ldots,0,1].$$

For each $i=1,2,\ldots,n$, let

$$\phi_i=\prod_{k\neq i}(\mathbf{x}-\alpha_k)\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}).\tag{3.4}$$

By equation (3.9) below, ϕ_i is the equivariant Poincaré dual of P_i. We also note that $\tilde{\gamma}|_{BP_i}=V_{\pi_i}$, where π_i is as in (3.1). Thus, the restriction map on the equivariant cohomology induced by the inclusion $P_i\longrightarrow\mathbb{P}^{n-1}$ is given by

$$H^*_{\mathbb{T}}(\mathbb{P}^{n-1})=\mathbb{Q}[\mathbf{x},\alpha_1,\ldots,\alpha_n]\Big/\prod_{k=1}^{k=n}(\mathbf{x}-\alpha_k)\longrightarrow H^*_{\mathbb{T}}(P_i)=\mathbb{Q}[\alpha_1,\ldots,\alpha_n],$$
$$\mathbf{x}\longrightarrow\alpha_i.$$

In particular, if $\mathcal{F}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})$, then

$$\mathcal{F}=0\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\quad\Longleftrightarrow\quad\mathcal{F}(\mathbf{x}=\alpha_i)\equiv\mathcal{F}|_{P_i}=0\in\mathbb{Q}[\alpha_1,\ldots,\alpha_n]\subset\mathbb{Q}_\alpha\quad\forall i\in[n].\tag{3.5}$$

The tautological line bundle $\gamma_{n-1}\longrightarrow\mathbb{P}^{n-1}$ is a subbundle of $\mathbb{P}^{n-1}\times\mathbb{C}^n$ preserved by the diagonal action of $\mathbb{T}$. Thus, the action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ naturally lifts to an action on γ_{n-1}, and

$$\mathbf{e}(\gamma^*_{n-1})|_{P_i}=\alpha_i\quad\forall i=1,2,\ldots,n.\tag{3.6}$$

The $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ also has a natural lift to the vector bundle $T\mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}$ so that there is a short exact sequence

$$0\longrightarrow\gamma^*_{n-1}\otimes\gamma_{n-1}\longrightarrow\gamma^*_{n-1}\otimes\mathbb{C}^n\longrightarrow T\mathbb{P}^{n-1}\longrightarrow0$$

of $\mathbb{T}$-equivariant vector bundles on $\mathbb{P}^{n-1}$. By (3.2), (3.6), and (3.4),

$$\mathbf{e}(T\mathbb{P}^{n-1})|_{P_i}=\prod_{k\neq i}(\alpha_i-\alpha_k)=\phi_i|_{P_i}\quad\forall i=1,2,\ldots,n.\tag{3.7}$$

If $\mathbb{T}$ acts smoothly on a smooth compact oriented manifold M, then there is a well-defined integration-along-the-fiber homomorphism

$$\int_M : H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}$$

for the fiber bundle $BM \longrightarrow B\mathbb{T}$. The classical localization theorem of [1] relates it to integration along the fixed locus of the $\mathbb{T}$-action. The latter is a union of smooth compact orientable manifolds F; $\mathbb{T}$ acts on the normal bundle $\mathcal{N}F$ of each F. Once an orientation of F is chosen, there is a well-defined integration-along-the-fiber homomorphism

$$\int_F : H^*_{\mathbb{T}}(F) \longrightarrow H^*_{\mathbb{T}}.$$

The localization theorem states that

$$\int_M \eta = \sum_F \int_F \frac{\eta|_F}{\mathbf{e}(\mathcal{N}F)} \in \mathbb{Q}_\alpha \quad \forall \eta \in H^*_{\mathbb{T}}(M), \tag{3.8}$$

where the sum is taken over all components F of the fixed locus of $\mathbb{T}$. Part of the statement of (3.8) is that $\mathbf{e}(\mathcal{N}F)$ is invertible in $\mathcal{H}^*_{\mathbb{T}}(F)$. In the case of the standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$, (3.8) implies that

$$\eta|_{P_i} = \int_{\mathbb{P}^{n-1}} \eta\phi_i \in \mathbb{Q}_\alpha \quad \forall \eta \in \mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1}), i = 1, 2, \ldots, n; \tag{3.9}$$

see also (3.7).

Finally, if $f : M \longrightarrow M'$ is a $\mathbb{T}$-equivariant map between two compact oriented manifolds, there is a well-defined pushforward homomorphism

$$f_* : H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}(M').$$

It is characterized by the property that

$$\int_{M'} (f_*\eta)\eta' = \int_M \eta(f^*\eta') \quad \forall \eta \in H^*_{\mathbb{T}}(M), \eta' \in H^*_{\mathbb{T}}(M'). \tag{3.10}$$

The homomorphism $\int_M$ of the previous paragraph corresponds to M' being a point. It is immediate from (3.10) that

$$f_*(\eta(f^*\eta')) = (f_*\eta)\eta' \quad \forall \eta \in H^*_{\mathbb{T}}(M), \eta' \in H^*_{\mathbb{T}}(M'). \tag{3.11}$$

4. Equivariant Mirror Theorem

In this section, we state an equivariant version of Theorem 1, Theorem 3, which immediately implies Theorem 1. It is proved in the rest of this paper, as outlined in Section 1 after the statement of Theorem 1. We then formulate an equivariant version of Theorem 2, Theorem 4, providing a closed formula for equivariant Hurwitz numbers. This theorem immediately implies Theorem 2 and is obtained in Section 8 by combining Proposition 8.3 in this paper with some results from [26]. Throughout the paper, we use calligraphic letters, for example, $\mathcal{Y}$ and $\mathcal{Z}$, for equivariant generating functions.

The standard $\mathbb{T}$-representation on $\mathbb{C}^n$ (as well as any other representation) induces a $\mathbb{T}$-action on the trivial rank n sheaf over any quasi-stable curve $(\mathcal{C}, y_1, \ldots, y_m)$,

$$\mathbb{T} \cdot \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}} \longrightarrow \mathbb{C}^n \otimes \mathcal{O}_{\mathcal{C}}, \quad (t_1, \ldots, t_n) \cdot (f_1, \ldots, f_n) = (t_1 f_1, \ldots, t_n f_n),$$

and thus on the rank 1 subsheaves of this sheaf. This action preserves the degree of the subsheaf and the torsion and stability properties of Section 2 and thus induces a $\mathbb{T}$-action on the moduli space $\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)$, with respect to which the evaluation maps

$$\mathrm{ev}_i : \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}, \quad i = 1, 2, \ldots, m,$$

are $\mathbb{T}$-equivariant. This action lifts to a $\mathbb{T}$-action on the universal subsheaf $\mathcal{S} \longrightarrow \mathcal{U}$ and thus to $\mathbb{T}$-actions on the locally free sheaves

$$\pi_*(\sigma_i^2) \longrightarrow \overline{Q}_{g,m}(\mathbb{P}^{n-1}, d), \qquad \dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)} \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d).$$

This gives rise to $\mathbb{T}$-equivariant cohomology classes

$$\psi_i \in H_{\mathbb{T}}^*(\overline{Q}_{g,m}(\mathbb{P}^{n-1}, d)), \qquad \mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)}) \in H_{\mathbb{T}}^*(\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)).$$

The stable quotients analogue of the equivariant version of Givental's J-function is given by

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) \equiv 1 + \sum_{d=1}^{\infty} q^d \mathrm{ev}_{1*}\left[\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})}{\hbar - \psi_1}\right] \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]], \tag{4.1}$$

where $\mathrm{ev}_1 : \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d) \longrightarrow \mathbb{P}^{n-1}$ is as before. The equivariant analogue of the power series (1.6) is given by

$$\begin{aligned}\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) &\equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{a_k>0} \prod_{r=1}^{a_k d}(a_k \mathbf{x} + r\hbar) \prod_{a_k<0} \prod_{r=0}^{-a_k d-1}(a_k \mathbf{x} - r\hbar)}{\prod_{r=1}^{d} \prod_{k=1}^{n}(\mathbf{x} - \alpha_k + r\hbar)} \\ &\in \mathbb{Q}[\alpha_1, \ldots, \alpha_n, \mathbf{x}][[\hbar^{-1}, q]].\end{aligned} \tag{4.2}$$

We view (4.1) and (4.2) as power series in $\hbar^{-1}$ and q, by expanding around $\hbar = \infty$ and $q = 0$. The coefficients of powers of $\hbar^{-1}$ and q in (4.2) are polynomials in $\alpha_1, \ldots, \alpha_n$ and $\mathbf{x}$; the coefficients in (4.2) are $\mathbb{T}$-equivariant cohomology classes on $\mathbb{P}^{n-1}$, which can also be represented by polynomials.

THEOREM 3. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$ are such that $|\mathbf{a}| \leq n$, then the equivariant stable quotients analogue of Givental's J-function satisfies*

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q) = \frac{\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)}{I_{n;\mathbf{a}}(q)} \in H_{\mathbb{T}}^*(\mathbb{P}^{n-1})[[\hbar^{-1}, q]]. \tag{4.3}$$

Restricting to a fiber of the projection

$$B\mathbb{P}^{n-1} \equiv E\mathbb{T} \times_{\mathbb{T}} \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1},$$

we send $\mathbf{x}$ to x and α_i to 0; this gives Theorem 1. The relation of Theorem 3 to its Gromov–Witten analogue is the same as the relation of Theorem 1 to its Gromov–Witten analogue; see the paragraph following the statement of Theorem 1 in Section 1. In particular, the twisted equivariant stable quotients invariants of $\mathbb{P}^{n-1}$ determined by a tuple $\mathbf{a}$ are the same as the corresponding Gromov–Witten invariants if $|\mathbf{a}|-\ell^-(\mathbf{a})\le n-2$, but not if $|\mathbf{a}|-\ell^-(\mathbf{a})=n-1,n$.

We prove Theorem 3 through a two-pronged approach. We show that the power series $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ are $\mathfrak{C}$-recursive in the sense of Definition 5.1 with the collection $\mathfrak{C}$ given by (5.6) and satisfy the self-polynomiality condition of Definition 5.2; see Lemma 5.4 and Propositions 6.1 and 7.1. Proposition 5.3 then implies that these power series are determined by their mod $\hbar^{-2}$-parts, that is, the coefficients of $\hbar^0$ and $\hbar^{-1}$ in this case. It is straightforward to determine the mod $\hbar^{-2}$-part of $\mathcal{Y}_{n;\mathbf{a}}$ in all cases ($\mathcal{Y}_{n;\mathbf{a}}$ is given by an explicit algebraic expression) and the mod $\hbar^{-2}$-part of $\mathcal{Z}_{n;\mathbf{a}}$ if $|\mathbf{a}|\le n-2$, thus establishing Theorem 3 whenever $|\mathbf{a}|\le n-2$; see Corollary 8.1.

In order to establish Theorem 3 in all cases, we show that the secondary coefficients $\mathcal{Y}_i^r(d)$ and $\mathcal{Z}_i^r(d)$, instead of $\mathcal{F}_i^r(d)$, in the recursions (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ are the same. By induction on d, this implies that the coefficients of q^d on the two sides of (4.3) are the same because this is the case for $d=0$ (when both coefficients are 1). As part of the proof of $\mathfrak{C}$-recursivity for $\mathcal{Y}_{n;\mathbf{a}}$, we show that $\mathcal{Y}_i^r(d)$ is determined by the expansion of $\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)$ around $h=0$; see Lemma 5.4. As part of the proof of $\mathfrak{C}$-recursivity for $\mathcal{Z}_{n;\mathbf{a}}$, we show that $\mathcal{Z}_i^r(d)$ is also determined by the expansion of $\mathcal{Z}_{n;\mathbf{a}}(\alpha_i,\hbar,q)$ around $h=0$; see Proposition 6.1. In contrast to $\mathcal{Y}_i^r(d)$, $\mathcal{Z}_i^r(d)$ is determined by lower-degree coefficients of $\mathcal{Z}_{n;\mathbf{a}}$ or equivalently by $\mathcal{Z}_j^s(d')$ with $d'<d$; this relation thus completely determines $\mathcal{Z}_{n;\mathbf{a}}$ (assuming $\mathfrak{C}$-recursivity). It follows that (4.3) holds if and only if the coefficients $\mathcal{Y}_i^r(d)$ for $\mathcal{Y}_{n;\mathbf{a}}$ satisfy the same relation; see Lemma 8.2. The coefficients in this relation involve twisted Hurwitz numbers over the moduli spaces $\overline{\mathcal{M}}_{0,2|d}$. These are not easy to compute, but they can be described qualitatively in a way independent of n. This implies that the validity of the desired recursion for the secondary coefficients $\mathcal{Y}_i^r(d)$ for $\mathcal{Y}_{n;\mathbf{a}}$ is *independent* of n. Since this recursion is equivalent to (4.3) whenever $|\mathbf{a}|\le n$ and (4.3) holds whenever $|\mathbf{a}|\le n-2$ (by Corollary 8.1), it follows that the recursion holds in all cases (see Proposition 8.3) and (4.3) holds whenever $|\mathbf{a}|\le n$, as claimed.

As stated in Section 1, Theorem 3 extends to products of projective spaces and concavex sheaves (1.14). The relevant torus action is then the product of the actions on the components described in Section 3. If its weights are denoted by $\alpha_{s;k}$, with $s=1,\dots,p$ and $k=1,\dots,n_s$, then

$$\begin{aligned}
&\mathcal{Y}_{n_1,\dots,n_p;\mathbf{a}}(\mathbf{x}_1,\dots,\mathbf{x}_p,\hbar,q_1,\dots,q_p)\\
&\qquad\in\mathbb{Q}[\alpha_{1;1},\dots,\alpha_{p;n_p},\mathbf{x}_1,\dots,\mathbf{x}_p][[\hbar^{-1},q_1,\dots,q_p]], \qquad (4.4)\\
&\mathcal{Z}_{n_1,\dots,n_p;\mathbf{a}}(\mathbf{x}_1,\dots,\mathbf{x}_p,\hbar,q_1,\dots,q_p)\\
&\qquad\in H^*_{\mathbb{T}}(\mathbb{P}^{n_1-1}\times\dots\times\mathbb{P}^{n_p-1})[[\hbar^{-1},q_1,\dots,q_p]], \qquad (4.5)
\end{aligned}$$

and $\mathbf{x}_1, \ldots, \mathbf{x}_p \in H^*(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1})$ correspond to the pullbacks of the equivariant hyperplane classes by the projection maps. The coefficient of $q_1^{d_1} \cdots q_p^{d_p}$ in (4.5) is defined by the same pushforward as in (4.1), with the degree d of the stable quotients replaced by $(d_1, \ldots, d_p)$. The coefficient of $q_1^{d_1} \cdots q_p^{d_p}$ in (4.4) is given by

$$\prod_{a_{k;1}\geq 0} \prod_{r=1}^{\sum_{s=1}^{p} a_{k;s}d_s} \Big(\sum_{s=1}^{p} a_{k;s}\mathbf{x}_s + r\hbar\Big) \prod_{a_{k;1}<0} \prod_{r=0}^{-\sum_{s=1}^{p} a_{k;s}d_s-1} \Big(\sum_{s=1}^{p} a_{k;s}\mathbf{x}_s - r\hbar\Big)$$
$$\Big/ \prod_{s=1}^{p}\prod_{r=1}^{d_s}\prod_{k=1}^{n_s} (\mathbf{x}_s - \alpha_{s;k} + r\hbar).$$

Our proof of Theorem 3 extends directly to this situation.

We conclude this section with an equivariant version of Theorem 2. For any $d \in \mathbb{Z}^+$ and $\beta \in H^2_{\mathbb{T}}$, denote by

$$\mathcal{S}^*(\beta) \longrightarrow \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2|d} \tag{4.6}$$

the universal sheaf with the $\mathbb{T}$-action so that

$$\mathbf{e}(\mathcal{S}^*(\beta)) = \beta \times 1 + 1 \times e(\mathcal{S}^*) \in H^*_{\mathbb{T}}(\mathcal{U}) = H^*_{\mathbb{T}} \otimes H^*(\mathcal{U}).$$

Similarly to (1.2), let

$$\begin{aligned}\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\beta) = \bigoplus_{a_k>0} R^0\pi_*(\mathcal{S}^*(\beta)^{a_k}(-\sigma_1)) \oplus \bigoplus_{a_k<0} R^1\pi_*(\mathcal{S}^*(\beta)^{a_k}(-\sigma_1))&\\ \longrightarrow \overline{\mathcal{M}}_{0,2|d},&\end{aligned} \tag{4.7}$$

where $\pi : \mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2|d}$ is the projection as before; this sheaf is locally free. The bundle

$$\dot{\mathcal{V}}_1^{(d)}(\beta) \equiv \dot{\mathcal{V}}_{(1)}^{(d)}(\beta) = R^0\pi_*(\mathcal{S}^*(\beta)(-\sigma_1)) \longrightarrow \overline{\mathcal{M}}_{0,2|d} \tag{4.8}$$

plays a central role in the deformation theory of stable quotients as explained in Section 6. We define the power series $L_{n;\mathbf{a}}, \xi_{n;\mathbf{a}} \in \mathbb{Q}_\alpha[\mathbf{x}][[q]]$ by

$$\begin{aligned}&L_{n;\mathbf{a}} \in \mathbf{x} + q\mathbb{Q}_\alpha[\mathbf{x}][[q]],\\ &\qquad \prod_{k=1}^{n}(L_{n;\mathbf{a}}(\mathbf{x},q) - \alpha_k) - q\mathbf{a}^{\mathbf{a}} L_{n;\mathbf{a}}(\mathbf{x},q)^{|\mathbf{a}|} = \prod_{k=1}^{n}(\mathbf{x} - \alpha_k),\\ &\xi_{n;\mathbf{a}} \in q\mathbb{Q}_\alpha[\mathbf{x}][[q]], \quad \mathbf{x} + q\,\frac{\mathrm{d}}{\mathrm{d}q}\xi_{n;\mathbf{a}}(\mathbf{x},q) = L_{n;\mathbf{a}}(\mathbf{x},q).\end{aligned}$$

THEOREM 4. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$\begin{aligned}1 + (\hbar_1+\hbar_2)\sum_{d=1}^{\infty}\frac{q^d}{d!}\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))(\hbar_1-\psi_1)(\hbar_2-\psi_2)}&\\ = \mathrm{e}^{\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_1 + \xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_2} \in \mathbb{Q}_\alpha[[\hbar_1^{-1}, \hbar_2^{-1}, q]]&\end{aligned}$$

for every $i = 1, \ldots, n$.

5. Algebraic Observations

In this section, we describe a number of properties of power series, such as $\mathcal{Y}_{n;\mathbf{a}}$ in (4.2) and $\mathcal{Z}_{n;\mathbf{a}}$ in (4.1), that determine them completely. We also show that $\mathcal{Y}_{n;\mathbf{a}}$ indeed satisfies these properties.

If R is a ring, then denote by

$$R\lceil\lceil\hbar\rceil\rceil \equiv R[[\hbar^{-1}]] + R[\hbar]$$

the R-algebra of Laurent series in $\hbar^{-1}$ (with finite principal part). If $f \in R[[q]]$ and $d \in \mathbb{Z}^{\geq 0}$, then let $[[f]]_{q;d} \in R$ denote the coefficient of q^d in f. If $p \in \mathbb{Z}^{\geq 0}$ and

$$\mathcal{F}(\hbar, q) = \sum_{d=0}^{\infty}\left(\sum_{r=-N_d}^{\infty} \mathcal{F}_d^{(r)}\hbar^{-r}\right)q^d \in R\lceil\lceil\hbar\rceil\rceil[[q]]$$

for some $\mathcal{F}_d^{(r)} \in R$, we define

$$\mathcal{F}(\hbar, q) \cong \sum_{d=0}^{\infty}\left(\sum_{r=-N_d}^{p-1} \mathcal{F}_d^{(r)}\hbar^{-r}\right)q^d \quad (\text{mod } \hbar^{-p}),$$

that is, we drop $\hbar^{-p}$ and higher powers of $\hbar^{-1}$ instead of higher powers of $\hbar$. If R is a field, let

$$R(\hbar) \hookrightarrow R\lceil\lceil\hbar\rceil\rceil$$

be the embedding given by taking the Laurent series of rational functions at $\hbar^{-1} = 0$.

If $f = f(z)$ is a rational function in z and possibly some other variables, for any $z_0 \in \mathbb{P}^1 \supset \mathbb{C}$, let

$$\mathfrak{R}_{z=z_0} f(z) \equiv \frac{1}{2\pi\mathrm{i}}\oint f(z)\,\mathrm{d}z, \tag{5.1}$$

where the integral is taken over a positively oriented loop around $z = z_0$ with no other singular points of $f\,\mathrm{d}z$, denote the residue of the 1-form $f\,\mathrm{d}z$. If $z_1, \ldots, z_k \in \mathbb{P}^1$ is any collection of points, let

$$\mathfrak{R}_{z=z_1,\ldots,z_k} f(z) \equiv \sum_{i=1}^{i=k} \mathfrak{R}_{z=z_i} f(z). \tag{5.2}$$

By the residue theorem on S^2,

$$\sum_{\mathbf{x}_0 \in S^2} \mathfrak{R}_{\mathbf{x}=\mathbf{x}_0}\{f(\mathbf{x})\} = 0$$

for every rational function $f = f(\mathbf{x})$ on $S^2 \supset \mathbb{C}$. If f is regular at $z = 0$, let $[[f]]_{z;p}$ denote the coefficient of z^p in the power series expansion of f around $z = 0$.

DEFINITION 5.1. Let $C \equiv (C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ be any collection of elements of $\mathbb{Q}_\alpha$. A power series $\mathcal{F} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\lceil\lceil\hbar\rceil\rceil[[q]]$ is C-recursive if the following holds: if $d^* \in \mathbb{Z}^{\ge 0}$ is such that

$$[[\mathcal{F}(\mathbf{x}=\alpha_i,\hbar,q)]]_{q;d^*-d} \in \mathbb{Q}_\alpha(\hbar) \subset \mathbb{Q}_\alpha\lceil\lceil\hbar\rceil\rceil \quad \forall d \in [d^*], i \in [n],$$

and $[[\mathcal{F}(\alpha_i,\hbar,q)]]_{q;d}$ is regular at $\hbar = (\alpha_i - \alpha_j)/d$ for all $d < d^*$ and $i \neq j$, then

$$[[\mathcal{F}(\alpha_i,\hbar,q)]]_{q;d^*} - \sum_{d=1}^{d^*}\sum_{j\neq i}\frac{C_i^j(d)}{\hbar-(\alpha_j-\alpha_i)/d}[[\mathcal{F}(\alpha_j,z,q)]]_{q;d^*-d}|_{z=(\alpha_j-\alpha_i)/d} \in \mathbb{Q}_\alpha[\hbar,\hbar^{-1}] \subset \mathbb{Q}_\alpha\lceil\lceil\hbar\rceil\rceil. \tag{5.3}$$

Thus, if $\mathcal{F} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\lceil\lceil\hbar\rceil\rceil[[q]]$ is C-recursive for any collection C, then

$$\mathcal{F}(\mathbf{x}=\alpha_i,\hbar,q) \in \mathbb{Q}_\alpha(\hbar)[[q]] \subset \mathbb{Q}_\alpha\lceil\lceil\hbar\rceil\rceil[[q]] \quad \forall i \in [n],$$

as can be seen by induction on d, and

$$\mathcal{F}(\alpha_i,\hbar,q) = \sum_{d=0}^{\infty}\sum_{r=-N_d}^{N_d}\mathcal{F}_i^r(d)\hbar^r q^d + \sum_{d=1}^{\infty}\sum_{j\neq i}\frac{C_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d} \times \mathcal{F}(\alpha_j,(\alpha_j-\alpha_i)/d,q) \quad \forall i \in [n], \tag{5.4}$$

for some $\mathcal{F}_i^r(d) \in \mathbb{Q}_\alpha$. The nominal issue with defining C-recursivity by (5.4), as is normally done, is that a priori the evaluation of $\mathcal{F}(\alpha_j,\hbar,q)$ at $\hbar = (\alpha_j - \alpha_i)/d$ need not be well defined since $\mathcal{F}(\alpha_j,\hbar,q)$ is a power series in q with coefficients in the Laurent series in $\hbar^{-1}$; a priori they may not converge anywhere. However, taking the coefficient of each power of q in (5.4) shows by induction on the degree d that this evaluation does make sense; this is the substance of Definition 5.1.

DEFINITION 5.2. For any $\mathcal{F} \equiv \mathcal{F}(\mathbf{x},\hbar,q) \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\lceil\lceil\hbar\rceil\rceil[[q]]$, let

$$\Phi_{\mathcal{F}}(\hbar,z,q) \equiv \sum_{i=1}^{n}\frac{\langle\mathbf{a}\rangle\alpha_i^{\ell(\mathbf{a})}\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{F}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z})\mathcal{F}(\alpha_i,-\hbar,q) \in \mathbb{Q}_\alpha\lceil\lceil\hbar\rceil\rceil[[z,q]]. \tag{5.5}$$

A power series $\mathcal{F} \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\lceil\lceil\hbar\rceil\rceil[[z,q]]$ satisfies the self-polynomiality condition if $\Phi_{\mathcal{F}} \in \mathbb{Q}_\alpha[\hbar][[z,q]]$.

PROPOSITION 5.3 ([11, Lemma 30.3.2]). *Let $\mathcal{F},\mathcal{F}' \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})\lceil\lceil\hbar\rceil\rceil[[q]]$. If $\mathcal{F}$ and $\mathcal{F}'$ are C-recursive for some collection $C \equiv (C_i^j(d))_{d,i,j\in\mathbb{Z}^+}$ of elements of $\mathbb{Q}_\alpha$, satisfy the self-polynomiality condition, and*

$$\mathcal{F}(\mathbf{x}=\alpha_i,\hbar,q), \mathcal{F}'(\mathbf{x}=\alpha_i,\hbar,q) \in \mathbb{Q}^*_\alpha + q\cdot\mathbb{Q}_\alpha\lceil\lceil\hbar\rceil\rceil[[q]] \subset \mathbb{Q}_\alpha\lceil\lceil\hbar\rceil\rceil[[q]] \quad \forall i \in [n],$$

then $\mathcal{F} \cong \mathcal{F}' \pmod{\hbar^{-2}}$ if and only if $\mathcal{F} = \mathcal{F}'$.

Let

$$\mathfrak{C}_i^j(d) \equiv \prod_{a_k>0}\prod_{r=1}^{a_kd}\Big(a_k\alpha_i + r\frac{\alpha_j-\alpha_i}{d}\Big)\prod_{a_k<0}\prod_{r=0}^{-a_kd-1}\Big(a_k\alpha_i - r\frac{\alpha_j-\alpha_i}{d}\Big)$$
$$\Big/\Big(d\prod_{\substack{r=1\\(r,k)\neq(d,j)}}^{d}\prod_{k=1}^{n}\Big(\alpha_i-\alpha_k+r\frac{\alpha_j-\alpha_i}{d}\Big)\Big) \in \mathbb{Q}_\alpha. \tag{5.6}$$

LEMMA 5.4. *If $l\in\mathbb{Z}^{\geq 0}$, $n\in\mathbb{Z}^+$, and $\mathbf{a}\in(\mathbb{Z}^*)^l$ are such that $|\mathbf{a}|\leq n$, the power series $\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ given by (4.2) is $\mathfrak{C}$-recursive, with the auxiliary coefficients in the recursion (5.4) for $\mathcal{Y}_{n;\mathbf{a}}$ given by*

$$\sum_{d=0}^{\infty}\mathcal{Y}_i^r(d)q^d = \begin{cases}\mathfrak{R}_{\hbar=0}\{\hbar^{-r-1}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\} & \text{if } r<0,\\ I_{n;\mathbf{a}}(q) & \text{if } r=0;\\ 0 & \text{if } r>0.\end{cases} \tag{5.7}$$

Furthermore, $\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ satisfies the self-polynomiality condition.

Proof. This is well known from the various proofs of mirror symmetry for Gromov–Witten invariants (e.g., [7, Section 11], [11, Chapter 30], [6, Section 4]); we include a proof for the sake of completeness.

We first view $\mathcal{Y}_{n;\mathbf{a}}$ as an element of $\mathbb{Q}_\alpha(\mathbf{x},\hbar)[[q]]$. Splitting the coefficient of $q^{d+d'}$ in (4.2) into the factors with $r\leq d$ and $r>d$, plugging in $(\alpha_j-\alpha_i)/d$ into all factors other than the $(r,k)=(d,j)$ factor in the denominator, and simplifying, we obtain

$$\mathfrak{R}_{z=(\alpha_j-\alpha_i)/d}\Big\{\frac{1}{\hbar-z}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,z,q)\Big\} = \frac{\mathfrak{C}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d}\mathcal{Y}_{n;\mathbf{a}}(\alpha_j,(\alpha_j-\alpha_i)/d,q).$$

By the residue theorem on S^2,

$$\begin{aligned}\sum_{d=1}^{\infty}\sum_{j\neq i}&\frac{\mathfrak{C}_i^j(d)q^d}{\hbar-(\alpha_j-\alpha_i)/d}\mathcal{Y}_{n;\mathbf{a}}(\alpha_j,(\alpha_j-\alpha_i)/d,q)\\ &= -\mathfrak{R}_{z=\hbar,0,\infty}\Big\{\frac{1}{\hbar-z}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,z,q)\Big\}\\ &= \mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q) - \mathfrak{R}_{z=0,\infty}\Big\{\frac{1}{\hbar-z}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,z,q)\Big\}.\end{aligned} \tag{5.8}$$

Since the coefficients of $(\hbar^{-1})^0$ in $\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)$ and $Y_{n;\mathbf{a}}(\alpha_i,\hbar,q)$ are the same,

$$\mathfrak{R}_{z=\infty}\Big\{\frac{1}{\hbar-z}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,z,q)\Big\} = I_{n;\mathbf{a}}(q)$$

by (1.7). Since the coefficient of q^d in $\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)$ has a pole of order d at $\hbar=0$,

$$\underset{z=0}{\mathfrak{R}}\left\{\frac{1}{\hbar-z}[[\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,z,q)]]_{q;d}\right\} = \left[\left[\frac{1}{\hbar-z}\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd}(a_k\alpha_i+rz)\prod_{a_k<0}\prod_{r=0}^{-a_kd-1}(a_k\alpha_i-rz)}{d!\prod_{r=1}^{d}\prod_{k\neq i}(\alpha_i-\alpha_k+rz)}\right]\right]_{z;d-1};$$

the last expression is a polynomial in $\hbar^{-1}$ with coefficients in $\mathbb{Q}_\alpha$ of degree at most d. This establishes that $\mathcal{Y}_{n;\mathbf{a}}$ is $\mathfrak{C}$-recursive and (5.7) holds; the $r<0$ case of (5.7) follows from (5.4) with $\mathcal{F}$ replaced by $\mathcal{Y}_{n;\mathbf{a}}$.

We now expand $\mathcal{Y}_{n;\mathbf{a}}$ as a power series in $\hbar^{-1}$ and q with coefficients in $\mathbb{Q}_\alpha[\mathbf{x}]$. Thus,

$$\frac{\langle\mathbf{a}\rangle\alpha_i^{\ell(\mathbf{a})}\mathrm{e}^{\mathbf{x}z}}{\prod_{k=1}^{n}(\mathbf{x}-\alpha_k)}\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q\mathrm{e}^{\hbar z})\mathcal{Y}_{n;\mathbf{a}}(-\mathbf{x},\hbar,q)\in\mathbb{Q}_\alpha(\mathbf{x})[[\hbar^{-1},z,q]]$$

viewed as a function of $\mathbf{x}$ has residues only at $\mathbf{x}=\alpha_i$ with $i\in[n]$ and $\mathbf{x}=\infty$. By (4.2),

$$\frac{\langle\mathbf{a}\rangle\alpha_i^{\ell(\mathbf{a})}\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z})\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,-\hbar,q) = \underset{\mathbf{x}=\alpha_i}{\mathfrak{R}}\left\{\frac{\langle\mathbf{a}\rangle\mathbf{x}^{\ell(\mathbf{a})}\mathrm{e}^{\mathbf{x}z}}{\prod_{k=1}^{n}(\mathbf{x}-\alpha_k)}\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q\mathrm{e}^{\hbar z})\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},-\hbar,q)\right\}.$$

Thus, by the residue theorem on S^2,

$$\Phi_{\mathcal{Y}_{n;\mathbf{a}}}(\hbar,z,q) = -\underset{\mathbf{x}=0,\infty}{\mathfrak{R}}\left\{\frac{\langle\mathbf{a}\rangle\mathbf{x}^{\ell(\mathbf{a})}\mathrm{e}^{\mathbf{x}z}}{\prod_{k=1}^{n}(\mathbf{x}-\alpha_k)}\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q\mathrm{e}^{\hbar z})\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},-\hbar,q)\right\} \equiv -\mathfrak{R}_0-\mathfrak{R}_\infty.$$

Since the coefficients of positive powers of q in $\mathcal{Y}_{n;\mathbf{a}}$ are divisible by $\mathbf{x}^{\ell^-(\mathbf{a})}$,

$$\mathfrak{R}_0=\langle\mathbf{a}\rangle\left[\left[\frac{\mathrm{e}^{\mathbf{x}z}}{\prod_{k=1}^{n}(\mathbf{x}-\alpha_k)}\right]\right]_{\mathbf{x};-\ell(\mathbf{a})-1}\in\mathbb{Q}_\alpha[z]\subset\mathbb{Q}_\alpha[\hbar][[z,q]].$$

The residue $\mathfrak{R}_\infty$ is computed by replacing $\mathbf{x}$ with $1/w$ and simplifying. Since the coefficient of q^d in $\mathcal{Y}_{n;\mathbf{a}}(1/w,\hbar,q^d)$ vanishes to order $(n-|\mathbf{a}|)d$ at $w=0$, a direct computation gives

$$-\mathfrak{R}_\infty=\langle\mathbf{a}\rangle\sum_{d_1,d_2=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^{n-1-\ell(\mathbf{a})+p+(n-|\mathbf{a}|)(d_1+d_2)}}{(n-1-\ell(\mathbf{a})+p+(n-|\mathbf{a}|)(d_1+d_2))!}q^{d_1+d_2}\mathrm{e}^{\hbar d_1 z}$$
$$\times\left[\left[\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd_1}(a_k+r\hbar w)\prod_{a_k<0}\prod_{r=0}^{-a_kd_1-1}(a_k-r\hbar w)}{\prod_{k=1}^{n}(1-\alpha_k w)}\cdot\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd_2}(a_k-r\hbar w)\prod_{a_k<0}\prod_{r=0}^{-a_kd_2-1}(a_k+r\hbar w)}{\prod_{r=1}^{d_1}\prod_{k=1}^{n}(1-(\alpha_k-r\hbar)w)\prod_{r=1}^{d_2}\prod_{k=1}^{n}(1-(\alpha_k+r\hbar)w)}\right]\right]_{w;p}.$$

The (d_1, d_2, p)-summand above is $q^{d_1+d_2}$ times an element of $\mathbb{Q}_\alpha[\hbar][[z]]$. $\square$

In the case of products of projective spaces and concavex sheaves (1.14), Definition 5.1 becomes inductive on the total degree $d_1 + \cdots + d_p$ of $q_1^{d_1} \cdots q_p^{d_p}$. The power series $\mathcal{F}$ is evaluated at $(\mathbf{x}_1, \ldots, \mathbf{x}_p) = (\alpha_{1;i_1}, \ldots, \alpha_{p;i_p})$ for the purposes of the C-recursivity condition (5.3) and (5.4). The relevant primary structure coefficients are of the form

$$\mathfrak{C}^j_{i_1\cdots i_p}(s;d) \equiv \prod_{a_{k;1}\geq 0} \prod_{r=1}^{a_{k;s}d} \left(\sum_{t=1}^{p} a_{k;t}\alpha_{t;i_t} + r\frac{\alpha_{s;j}-\alpha_{s;i_s}}{d} \right) \times \frac{\prod_{a_{k;1}<0} \prod_{r=0}^{-a_{k;s}d-1} \big(\sum_{t=1}^{p} a_{k;t}\alpha_{t;i_t} - r(\alpha_{s;j}-\alpha_{s;i_s})/d\big)}{d \prod_{\substack{r=1\\(r,k)\neq(d,j)}}^{d} \prod_{k=1}^{n_s} \big(\alpha_{s;i_s} - \alpha_{s;k} + r(\alpha_{s;j}-\alpha_{s;i_s})/d\big)}$$

with $s \in [p]$ and $j \neq i_s$. The double sums in these equations are then replaced by triple sums over $s \in [p]$, $j \in [n_s] - i_s$, and $d \in \mathbb{Z}^+$, and with $\mathcal{F}$ evaluated at

$$\mathbf{x}_t = \begin{cases} \alpha_{s;j} & \text{if } t = s, \\ \alpha_{t;i_t} & \text{if } t \neq s, \end{cases} \qquad z = \frac{\alpha_{s;j}-\alpha_{s;i_s}}{d}.$$

The secondary coefficients $\mathcal{F}^r_i(d)$ in (5.4) now become $\mathcal{F}^r_{i_1\cdots i_p}(d_1, \ldots, d_p)$, with $i_s \in [n_s]$ and $d_s \in \mathbb{Z}^{\geq 0}$. In the analogue of Definition 5.2, $\Phi_{\mathcal{F}}$ is a power series in $z_1, \ldots, z_p$ and $q_1, \ldots, q_p$, the sum taken is over all elements $(i_1, \ldots, i_p)$ of $[n_1] \times \cdots \times [n_p]$, the leading fraction is replaced by

$$\frac{\prod_{a_{k;1}\geq 0} \sum_{s=1}^{p} a_{k;s}\alpha_{s;i_s}}{\prod_{a_{k;1}<0} \sum_{s=1}^{p} a_{k;s}\alpha_{s;i_s}} \cdot \frac{\mathrm{e}^{\alpha_{1;i_1}z_1+\cdots+\alpha_{p;i_p}z_p}}{\prod_{s=1}^{p} \prod_{k\neq i_s} (\alpha_{s;i_s}-\alpha_{s;k})},$$

and the $q\mathrm{e}^{\hbar z}$-insertion in the first power series is replaced by the insertions $q_1\mathrm{e}^{\hbar z_1}, \ldots, q_p\mathrm{e}^{\hbar z_p}$. The conclusion of Lemma 5.4 holds with i, d, and q^d replaced by $(i_1, \ldots, i_p)$, $(d_1, \ldots, d_p)$, and $q_1^{d_1} \cdots q_p^{d_p}$, respectively. The proof is nearly identical, except that the last claim involves p applications of the residue theorem on S^2. Instead of the residue at $\mathbf{x} = 0$ of the coefficient of q^0, there may be a residue at a value of $\mathbf{x}_s$ dependent on the values of the other variables $\mathbf{x}_t$, but it again would not involve $\hbar$.

6. Recursivity for Stable Quotients

In this section, we use the classical localization theorem [1] to show that the equivariant stable quotients analogue of Givental's J-function, the power series $\mathcal{Z}_{n;\mathbf{a}}$ given by (4.1), is $\mathfrak{C}$-recursive with the collection $\mathfrak{C}^j_i(d)$ given by (5.6). We also describe the secondary terms $\mathcal{Z}^r_i(d)$ in the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$, establishing the following statement.

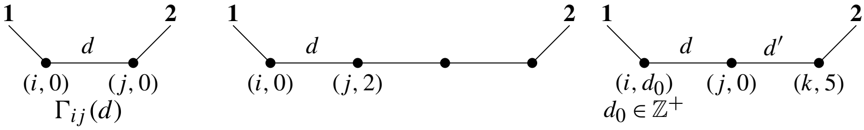

Figure 1 Two strands with $\mathfrak{d}(v_{\min})=0$ and a strand with $\mathfrak{d}(v_{\min})>0$

PROPOSITION 6.1. *If $l\in\mathbb{Z}^{\ge 0}$, $n\in\mathbb{Z}^+$, and $\mathbf{a}\in(\mathbb{Z}^*)^l$, then the power series $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)$ is $\mathfrak{C}$-recursive, with the auxiliary coefficients in the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$ given by*

$$\mathcal{Z}_i^r(d)=0 \quad \forall r\in\mathbb{Z}^+, \qquad \mathcal{Z}_i^0(d)=\delta_{0d},$$

and for all $r\in\mathbb{Z}^-$,

$$\sum_{d=1}^{\infty}\mathcal{Z}_i^r(d)q^d=\sum_{d=1}^{\infty}\frac{q^d}{d!}\sum_{b=0}^{d+r}\left(\left(\int_{\overline{\mathcal{M}}_{0,2|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^{-r-1}\psi_2^b}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))}\right)\times\underset{\hbar=0}{\mathfrak{R}}\left\{\frac{(-1)^b}{\hbar^{b+1}}\mathcal{Z}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\right\}\right).$$

The proof involves a localization computation on $\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)$. Thus, we need to describe the fixed loci of the $\mathbb{T}$-action on $\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)$, their normal bundles, and the restrictions of the relevant cohomology classes to these fixed loci.

As in the case of stable maps described in [11, Section 27.3], the fixed loci of the $\mathbb{T}$-action on $\overline{Q}_{0,m}(\mathbb{P}^{n-1},d)$ are indexed by connected decorated graphs that have no loops. However, in the case $m=2$, the relevant graphs consist of a single strand (possibly consisting of a single vertex) with the two marked points attached at the opposite ends of the strand. Such a graph can be described by an ordered set (Ver, $<$) of vertices, where $<$ is a strict order on the finite set Ver. Given such a strand, denote by $v_{\min}$ and $v_{\max}$ its minimal and maximal elements and by Edg its set of edges, that is, of pairs of consecutive elements. A decorated strand is a tuple

$$\Gamma=(\mathrm{Ver},<;\mu,\mathfrak{d}), \tag{6.1}$$

where (Ver, $<$) is a strand as above, and

$$\mu:\mathrm{Ver}\longrightarrow[n] \quad\text{and}\quad \mathfrak{d}:\mathrm{Ver}\sqcup\mathrm{Edg}\longrightarrow\mathbb{Z}^{\ge 0}$$

are maps such that

$$\mu(v_1)\neq\mu(v_2)\quad\text{if }\{v_1,v_2\}\in\mathrm{Edg},\qquad \mathfrak{d}(e)\neq 0\quad\forall e\in\mathrm{Edg}. \tag{6.2}$$

In Figure 1, the vertices of a decorated strand Γ are indicated by dots in the increasing order, with respect to $<$, from left to right. The values of the map $(\mu,\mathfrak{d})$ on some of the vertices are indicated next to those vertices. Similarly, the values of the map $\mathfrak{d}$ on some of the edges are indicated next to them. By (6.2) no two consecutive vertices have the same first label, and thus $j\neq i$.

Figure 2 The substrands corresponding to the edges of the last graph in Figure 1

With Γ as in (6.1), let

$$|\Gamma| \equiv \sum_{v\in\mathrm{Ver}} \mathfrak{d}(v) + \sum_{e\in\mathrm{Edg}} \mathfrak{d}(e)$$

be the degree of Γ. If $e=\{v_1,v_2\}\in\mathrm{Edg}$ is any edge in Γ, let Γ_e denote the single-edge graph with vertices v_1 and v_2, which are ordered in the same way as in Γ and assigned values $(\mu(v_1),0)$ and $(\mu(v_2),0)$, and with the edge assigned the value $\mathfrak{d}(e)$ as in the original graph; see Figure 2.

As described in [17, Section 7.3], the fixed locus $\mathcal{Q}_\Gamma$ of $\overline{\mathcal{Q}}_{0,2}(\mathbb{P}^{n-1},|\Gamma|)$ corresponding to a decorated strand Γ consists of the stable quotients

$$(\mathcal{C}, y_1, y_2, S\subset\mathbb{C}^n\otimes\mathcal{O}_\mathcal{C})$$

over quasi-stable rational 2-marked curves that satisfy the following conditions. The components of $\mathcal{C}$ on which the corresponding quotient is torsion-free are rational and correspond to the edges of Γ; the restriction of S to any such component corresponds to a morphism to $\mathbb{P}^{n-1}$ of the opposite degree to that of the subsheaf. Furthermore, if $e=\{v_1,v_2\}$ is an edge, the corresponding morphism f_e is a degree-$\mathfrak{d}(e)$ cover of the line

$$\mathbb{P}^1_{\mu(v_1),\mu(v_2)}\subset\mathbb{P}^{n-1}$$

passing through the fixed points $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$; it is ramified only over $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. In particular, f_e is unique up to isomorphism. The remaining components of $\mathcal{C}$ are indexed by the vertices $v\in\mathrm{Ver}$ with $\mathfrak{d}(v)\in\mathbb{Z}^+$. The restriction of S to such a component $\mathcal{C}_v$ of $\mathcal{C}$ (or possibly a connected union of irreducible components) is a subsheaf of the trivial subsheaf $P_{\mu(v)}\subset\mathbb{C}^n\otimes\mathcal{O}_{\mathcal{C}_v}$ of degree $-\mathfrak{d}(v)$; thus, the induced morphism takes $\mathcal{C}_v$ to the fixed point $P_{\mu(v)}\in\mathbb{P}^{n-1}$. Each such component $\mathcal{C}_v$ also carries two distinguished marked points corresponding to the nodes and/or the marked points of $\mathcal{C}$; if neither of the marked points of $\mathcal{C}$ lies on $\mathcal{C}_v$, we denote the marked point corresponding to the node of $\mathcal{C}_v$ separating $\mathcal{C}_v$ from the first marked point by 1 and the other marked point by 2. Thus, as stacks,

$$\begin{aligned}\mathcal{Q}_\Gamma &\approx \prod_{\substack{v\in\mathrm{Ver}\\ \mathfrak{d}(v)>0}} \overline{\mathcal{Q}}_{0,2}(\mathbb{P}^0,\mathfrak{d}(v)) \times \prod_{e\in\mathrm{Edg}} \mathcal{Q}_{\Gamma_e} \approx \prod_{\substack{v\in\mathrm{Ver}\\ \mathfrak{d}(v)>0}} \overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)} \times \prod_{e\in\mathrm{Edg}} \mathcal{Q}_{\Gamma_e}\\ &\approx \Bigg(\prod_{\substack{v\in\mathrm{Ver}\\ \mathfrak{d}(v)>0}} \overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)}\Bigg)\Big/\prod_{e\in\mathrm{Edg}}\mathbb{Z}_{\mathfrak{d}(e)},\end{aligned} \tag{6.3}$$

with each cyclic group $\mathbb{Z}_{\mathfrak{d}(e)}$ acting trivially. For example, in the case of the last diagram in Figure 1,

$$Q_\Gamma \approx (\overline{\mathcal{M}}_{0,2|d_0}/\mathbb{S}_{d_0} \times \overline{\mathcal{M}}_{0,2|5}/\mathbb{S}_5)/\mathbb{Z}_d \times \mathbb{Z}_{d'}$$

is a fixed locus in $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d_0+5+d+d')$.

If Γ is a decorated strand as above and $e \in \mathrm{Edg}$, let

$$\pi_e : Q_\Gamma \longrightarrow Q_{\Gamma_e} \subset \overline{Q}_{0,2}(\mathbb{P}^{n-1}, \mathfrak{d}(e))$$

be the projection in the decomposition (6.3). Similarly, for each $v \in \mathrm{Ver}$ such that $\mathfrak{d}(v) > 0$, let

$$\pi_v : Q_\Gamma \longrightarrow \overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)}$$

be the corresponding projection. If $e = \{v_1, v_2\} \in \mathrm{Edg}$ with $v_1 < v_2$, let

$$\omega_{e;v_1} = -\pi_e^*\psi_1, \omega_{e;v_2} = -\pi_e^*\psi_2, \psi_{v_1;e} = \pi_{v_1}^*\psi_2, \psi_{v_2;e} = \pi_{v_2}^*\psi_1 \in H^2(Q_\Gamma). \tag{6.4}$$

By [11, Section 27.2],

$$\omega_{e;v_i} = \frac{\alpha_{\mu(v_i)} - \alpha_{\mu(v_{3-i})}}{\mathfrak{d}(e)}, \quad i = 1, 2. \tag{6.5}$$

For each $v \in \mathrm{Ver} - \{v_{\min}\}$, let $e_-(v) = \{v_-, v\} \in \mathrm{Edg}$ denote the edge with $v_- < v$; for each $v \in \mathrm{Ver} - \{v_{\max}\}$, let $e_+(v) = \{v, v_+\} \in \mathrm{Edg}$ denote the edge with $v < v_+$.

By [17, Section 7.4] the Euler class of the normal bundle of Q_Γ in $\overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma|)$ is given by

$$\begin{aligned}
\frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{\mu(v_{\min})}\mathbb{P}^{n-1})} = & \prod_{\substack{v\in\mathrm{Ver}\\ \mathfrak{d}(v)>0}} \prod_{k\neq\mu(v)} \pi_v^*\mathbf{e}(\dot{\mathcal{V}}_1^{(\mathfrak{d}(v))}(\alpha_{\mu(v)}-\alpha_k)) \\
& \times \prod_{e\in\mathrm{Edg}} \pi_e^*\mathbf{e}(H^0(f_e^*T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C}) \\
& \times \prod_{\substack{v\in\mathrm{Ver}-v_{\min}-v_{\max}\\ \mathfrak{d}(v)=0}} (\omega_{e_-(v);v} + \omega_{e_+(v);v}) \\
& \times \prod_{\substack{v\in\mathrm{Ver}-v_{\min}\\ \mathfrak{d}(v)>0}} (\omega_{e_-(v);v} - \psi_{v;e_-(v)}) \\
& \times \prod_{\substack{v\in\mathrm{Ver}-v_{\max}\\ \mathfrak{d}(v)>0}} (\omega_{e_+(v);v} - \psi_{v;e_+(v)}),
\end{aligned} \tag{6.6}$$

where $\dot{\mathcal{V}}_1^{(\mathfrak{d}(v))}(\alpha_{\mu(v)} - \alpha_k)$ is as in (4.8), and $\mathbb{C} \subset H^0(f_e^*T\mathbb{P}^n \otimes \mathcal{O}(-y_1))$ is the trivial $\mathbb{T}$-representation. The terms on the second line in (6.6) describe the standard deformations of the domain; they are given by the direct sum of the tensor products of the tangent line bundles at the two branches of each node. The terms on the first line in (6.6) correspond to the deformations of the sheaf without changing the domain $\mathcal{C}$; they are obtained by relating these deformations to the deformations on each component of $\mathcal{C}$ and applying (3.7) to the deformations over the components $\mathcal{C}_v$ corresponding to the vertices. The first term on the right-hand side of (6.6) and the first two terms on the second line of (6.6) are the contributions of nondegenerate vertices described in [17, Section 7.4.2]. The second term on the right-hand side of (6.6) is the edge contributions, which are the same as in Gromov–Witten theory. Finally, by (1.2) and (4.7),

$$\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)})|_{\mathcal{Q}_\Gamma} = \prod_{\substack{v\in\mathrm{Ver}\\ \mathfrak{d}(v)>0}} \pi_v^*\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(\mathfrak{d}(v))}(\alpha_{\mu(v)})) \cdot \prod_{e\in\mathrm{Edg}} \pi_e^*\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)}). \tag{6.7}$$

LEMMA 6.2. *For every edge $e=\{v_1,v_2\}$ with $v_1<v_2$ in Γ as above,*

$$\int_{\mathcal{Q}_{\Gamma_e}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{\mathfrak{d}(e)})}{\mathbf{e}(H^0(f_e^*T\mathbb{P}^n\otimes\mathcal{O}(-y_1))/\mathbb{C})} = \mathfrak{C}_{\mu(v_1)}^{\mu(v_2)}(\mathfrak{d}(e)) \tag{6.8}$$

with $\mathfrak{C}_{\mu(v_1)}^{\mu(v_2)}(\mathfrak{d}(e))$ given by (5.6).

Proof. Since the edge contributions are the same as in Gromov–Witten theory, (6.8) is standard; we recall its derivation for the sake of completeness. Let $i=\mu(v_1)$, $j=\mu(v_2)$, and $d=\mathfrak{d}(e)$.

By [11, Exercise 27.2.3],

$$\mathbf{e}(H^0(f_e^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k))) = \prod_{r=0}^{a_k d} \frac{(a_k d-r)\alpha_i + r\alpha_j}{d} \quad \forall a_k\in\mathbb{Z}^{\ge 0}. \tag{6.9}$$

Since $\mathbf{e}(\mathcal{O}_{\mathbb{P}^{n-1}}(a_k))|_{P_i}=a_k\alpha_i$ and the sequence

$$0\longrightarrow H^0(f_e^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k)\otimes\mathcal{O}(-y_1))\longrightarrow H^0(f_e^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k))$$
$$\longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(a_k)|_{P_i}\longrightarrow 0$$

is exact, the product of (6.9) without the $r=0$ factor over k with $a_k>0$, that is, the first product in the numerator of (5.6), is the equivariant Euler class of the first summand in (1.2) restricted to f_e. By Serre duality and [11, Exercises 27.2.2, 27.2.3],

$$\mathbf{e}(H^1(f_e^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k))) = \prod_{r=1}^{-a_k d-1} \frac{(a_k d+r)\alpha_i - r\alpha_j}{d} \quad \forall a_k\in\mathbb{Z}^{-}. \tag{6.10}$$

Since the sequence

$$0\longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(a_k)|_{P_i}\longrightarrow H^1(f_e^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k)\otimes\mathcal{O}(-y_1))$$
$$\longrightarrow H^1(f_e^*\mathcal{O}_{\mathbb{P}^{n-1}}(a_k))\longrightarrow 0$$

is exact, the product of (6.10) with the extra $r=0$ factor over k with $a_k<0$, that is, the second product in the numerator of (5.6), is the equivariant Euler class of the second summand in (1.2) restricted to f_e. Thus, the numerators in (6.8) and (5.6) are the same.

The denominator in (6.8) is computed using the exact sequence

$$0 \longrightarrow H^0(f_e^* T\mathbb{P}^1_{i,j} \otimes \mathcal{O}(-y_1))/\mathbb{C} \longrightarrow H^0(f_e^* T\mathbb{P}^n \otimes \mathcal{O}(-y_1))/\mathbb{C} \\ \longrightarrow \bigoplus_{k\neq i,j} H^0(f_e^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) \otimes \mathbb{C}_{\alpha_i-\alpha_k} \otimes \mathcal{O}(-y_1)) \longrightarrow 0, \qquad (6.11)$$

where $\mathbb{C}_{\alpha_i-\alpha_k}$ is the topologically trivial line bundle with equivariant Euler class $\alpha_i - \alpha_k$; this sequence is obtained from the equivariant Euler sequence for $\mathbb{P}^{n-1}$. The equivariant Euler class of each summand on the second line of (6.11) is given by (6.9) with $a_k = 1$, each factor increased by $\alpha_i - \alpha_k$ (because of the tensor product with the line bundle $\mathbb{C}_{\alpha_i-\alpha_k}$), and the $r = 0$ factor again dropped. Thus, the equivariant Euler class of the vector space on the second line of (6.11) is the product of the factors in the denominator of (5.6) with $k \neq i, j$. By [11, Exercise 27.2.3],

$$\mathbf{e}(H^0(f_e^* T\mathbb{P}^1_{i,j})) = \prod_{r=0}^{2d} \frac{(d-r)(\alpha_i-\alpha_j) + r(\alpha_j-\alpha_i)}{d}. \qquad (6.12)$$

Since $\mathbf{e}(T\mathbb{P}^1_{i,j})|_{P_i} = \alpha_i - \alpha_j$ and the sequence

$$0 \longrightarrow H^0(f_e^* T\mathbb{P}^1_{i,j} \otimes \mathcal{O}(-y_1)) \longrightarrow H^0(f_e^* T\mathbb{P}^1_{i,j}) \longrightarrow T\mathbb{P}^1_{i,j}|_{P_i} \longrightarrow 0$$

is exact, (6.11) and (6.12) give

$$\mathbf{e}(H^0(f_e^* T\mathbb{P}^1_{i,j} \otimes \mathcal{O}(-y_1))/\mathbb{C}) = \prod_{r=1}^{d} \frac{r(\alpha_j-\alpha_i)}{d} \cdot \prod_{r=1}^{d-1} \frac{r(\alpha_i-\alpha_j)}{d}.$$

Thus, the denominator in (6.8) equals to the product in the denominator of (5.6). The remaining factor d in the denominator of (5.6) accounts for the automorphism group of Q_{Γ_e}. □

Proposition 6.1 is proved by applying the localization theorem to

$$\mathcal{Z}_{n;\mathbf{a}}(x=\alpha_i, \hbar, q) = 1 + \sum_{d=1}^{\infty} q^d \int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)} \frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}})\mathrm{ev}_1^*\phi_i}{\hbar - \psi_1} \\ \in \mathbb{Q}_\alpha[[\hbar^{-1}, q]], \qquad (6.13)$$

where ϕ_i is the equivariant Poincaré dual of the fixed point $P_i \in \mathbb{P}^{n-1}$; see (3.4), (3.9), and (3.10). Since $\phi_i|_{P_j} = 0$ unless $j = i$, a decorated strand as in (6.1) contributes to (6.13) only if the first marked point is attached to a vertex labeled i, that is, $\mu(v_{\min}) = i$ for the smallest element $v_{\min} \in \mathrm{Ver}$. We show that, just as with Givental's J-function, the (d, j)-summand in (5.4) with $C = \mathfrak{C}$ and $\mathcal{F} = \mathcal{Z}_{n;\mathbf{a}}$, that is,

$$\frac{\mathfrak{C}_i^j(d)q^d}{\hbar - (\alpha_j - \alpha_i)/d} \mathcal{Z}_{n;\mathbf{a}}(\alpha_j, (\alpha_j - \alpha_i)/d, q),$$

is the sum over all strands such that $\mu(v_{\min}) = i$, that is, the first marked point is mapped to the fixed point $P_i \in \mathbb{P}^{n-1}$, $v_{\min}$ is a bivalent vertex, that is, $\mathfrak{d}(v_{\min}) = 0$, the only edge leaving this vertex is labeled d, and the other vertex of this edge is labeled j. We also show that the first sum on the right-hand side of (5.4) is 1

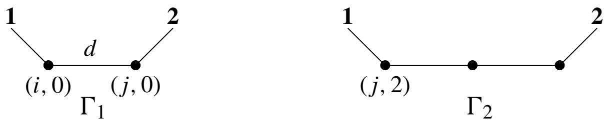

Figure 3 The two substrands of the second strand in Figure 1

(for the degree 0 term) plus the sum over all strands such that $\mu(v_{\min})=i$ and $\mathfrak{d}(v_{\min})>0$.

If Γ is a decorated strand with $\mu(v_{\min})=i$ as above,

$$\mathrm{ev}_1^*\phi_i|_{Q_\Gamma}=\prod_{k\neq i}(\alpha_i-\alpha_k)=\mathbf{e}(T_{\mu(v_{\min})}\mathbb{P}^{n-1}). \tag{6.14}$$

Suppose in addition that $\mathfrak{d}(v_{\min})=0$. Let $v_1\equiv(v_{\min})_+$ be the immediate successor of $v_{\min}$ in Γ, and $e_1=\{v_{\min},v_1\}$ be the edge leaving $v_{\min}$. If $|\mathrm{Edg}|>1$ or $\mathfrak{d}(v_1)>0$ (i.e., Γ is not as in the first diagram in Figure 1), we break Γ at v_1 into two "substrands":

(i) $\Gamma_1=\Gamma_{e_1}$ consisting of the vertices $v_{\min}<v_1$, the edge $\{v_{\min},v_1\}$, and the $\mathfrak{d}$-value of 0 at both vertices;

(ii) Γ_2 consisting all vertices and edges of Γ, other than the vertex $v_{\min}$ and the edge $\{v_{\min},v_1\}$;

see Figure 3. By (6.3),

$$Q_\Gamma\approx Q_{\Gamma_1}\times Q_{\Gamma_2}.$$

Let $\pi_1,\pi_2: Q_\Gamma\longrightarrow Q_{\Gamma_1},Q_{\Gamma_2}$ be the two component projection maps. By (6.7) and (6.6),

$$\begin{aligned}\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)})|_{Q_\Gamma}&=\pi_1^*\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_1|)})\cdot\pi_2^*\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_2|)}),\\ \frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}&=\pi_1^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right)\\ &\qquad\cdot\pi_2^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}{\mathbf{e}(T_{P_{\mu(v_1)}}\mathbb{P}^{n-1})}\right)\cdot(\omega_{e_1;v_1}-\pi_2^*\psi_1).\end{aligned}$$

Combining this with (6.5), (6.8), and (6.14), we find that

$$\begin{aligned}&q^{|\Gamma|}\int_{Q_\Gamma}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)})\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_1)\mathbf{e}(\mathcal{N}Q_\Gamma)}\\ &\quad=\frac{\mathfrak{C}_i^{\mu(v_1)}(\mathfrak{d}(e_1))q^{\mathfrak{d}(e_1)}}{\hbar-(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1)}\\ &\qquad\cdot\left(q^{|\Gamma_2|}\left\{\int_{Q_{\Gamma_2}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_2|)})\mathrm{ev}_1^*\phi_{\mu(v_1)}}{(\hbar-\psi_1)\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}\right\}\bigg|_{\hbar=(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1)}\right).\end{aligned} \tag{6.15}$$

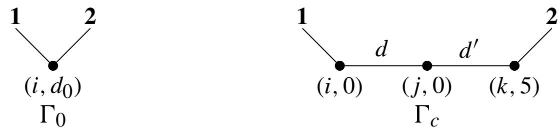

Figure 4 The two substrands of the last strand in Figure 1

By (6.13) with i replaced by $\mu(v_1)$ and the localization formula (3.8), the sum of the last factors over all possibilities for Γ_2, with Γ_1 held fixed, is

$$\mathcal{Z}_{n;\mathbf{a}}(\alpha_{\mu(v_1)}, (\alpha_{\mu(v_1)} - \alpha_i)/\mathfrak{d}(e_1), q) - 1.$$

On the other hand, the contribution of the graph $\Gamma_{i\mu(v_1)}(\mathfrak{d}(e_1))$ as in the first diagram in Figure 1 is precisely the first factor on the right-hand side of (6.15). Thus, the contribution to (6.13) from all strands Γ such that $\mu(v_1) = j$ and $\mathfrak{d}(e_1) = d$ is

$$\frac{\mathfrak{C}_i^j(d)q^d}{\hbar - (\alpha_j - \alpha_i)/d}\mathcal{Z}_{n;\mathbf{a}}(\alpha_j, (\alpha_j - \alpha_i)/d, q),$$

that is, the (d, j)-summand in the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$.

Suppose next that Γ is a strand such that $\mu(v_{\min}) = i$ and $\mathfrak{d}(v_{\min}) > 0$. If $|\mathrm{Ver}| > 1$, that is, Γ is not as in the first diagram in Figure 4, we break Γ at $v_{\min}$ into two “substrands”:

(i) Γ_0 consisting of the vertex $\{v_{\min}\}$ only, with the same μ and $\mathfrak{d}$-values as in Γ;
(ii) Γ_c consisting all vertices and edges of Γ, but with the $\mathfrak{d}$-value of $v_{\min}$ replaced by 0;

see Figure 4. By (6.3),

$$Q_\Gamma \approx Q_{\Gamma_0} \times Q_{\Gamma_c} = (\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}/\mathbb{S}_{\mathfrak{d}(v_{\min})}) \times Q_{\Gamma_c}; \tag{6.16}$$

if $|\mathrm{Ver}| = 1$, then this decomposition holds with $Q_{\Gamma_c} \equiv \{pt\}$ and $\mathfrak{d}(v_{\min}) = |\Gamma|$. Let π_0, π_c be the two component projection maps in (6.16). Since

$$\psi_1|_{Q_\Gamma} = \pi_0^*\psi_1,$$

$\mathbb{T}$ acts trivially on $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}$,

$$\psi_1 = 1 \times \psi_1 \in H_{\mathbb{T}}^*(\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}) = H_{\mathbb{T}}^* \otimes H^*(\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}),$$

that is, $\mathbb{T}$ acts trivially on the universal cotangent line bundle for the first marked point on $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}$, and the dimension of $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v_{\min})}$ is $\mathfrak{d}(v_{\min}) - 1$,

$$\frac{1}{\hbar - \psi_1}\bigg|_{Q_\Gamma} = \sum_{r=0}^{\mathfrak{d}(v_{\min})-1} \hbar^{-(r+1)}\pi_0^*\psi_1^r. \tag{6.17}$$

Since $|\mathfrak{d}(v_{\min})| \le |\Gamma|$ and Γ contributes to the coefficient of $q^{|\Gamma|}$ in (6.13), it follows that $\mathcal{Z}_{n;\mathbf{a}}$ satisfies (5.4) with $\mathcal{F} = \mathcal{Z}_{n;\mathbf{a}}$, $C_i^j(d) = \mathfrak{C}_i^j(d)$, $N_d = d$, $\mathcal{Z}_i^r(d) = 0$ for $r \in \mathbb{Z}^+$, and $\mathcal{Z}_i^0(d) = \delta_{0d}$. In particular, $\mathcal{Z}_{n;\mathbf{a}}$ is $\mathfrak{C}$-recursive.

It remains to verify the last identity in Proposition 6.1. We continue with the notation as in the previous paragraph. If $|\mathrm{Ver}|=1$, then the second factor in (6.16) is trivial; in this case, (6.7) and (6.6) immediately give

$$q^{|\Gamma|}\int_{Q_\Gamma}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)})\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_1)\mathbf{e}(\mathcal{N}Q_\Gamma)} = \sum_{r=0}^{|\Gamma|-1}\hbar^{-(r+1)}\frac{q^{|\Gamma|}}{(|\Gamma|)!}\int_{\overline{\mathcal{M}}_{0,2||\Gamma|}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(|\Gamma|)}(\alpha_i))\psi_1^r}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(|\Gamma|)}(\alpha_i-\alpha_k))}. \tag{6.18}$$

Suppose next that $|\mathrm{Ver}|>1$. By (6.7) and (6.6),

$$\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)})|_{Q_\Gamma} = \pi_0^*\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(|\Gamma_0|)}(\alpha_i))\cdot\pi_c^*\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_c|)}),$$
$$\frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} = \pi_0^*\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(|\Gamma_0|)}(\alpha_i-\alpha_k)) \cdot\pi_c^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_c})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right)\cdot(\omega_{e_1;v_{\min}}-\pi_0^*\psi_2),$$

where e_1 is the edge leaving $v_{\min}$. By (6.4),

$$\frac{1}{\omega_{e_1;v_{\min}}-\pi_0^*\psi_2} = \sum_{b=0}^{\infty}\pi_0^*\psi_2^b(-\pi_c^*\psi_1)^{-(b+1)}.$$

Combining the last four identities, we find that

$$q^{|\Gamma|}\int_{Q_\Gamma}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)})\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_1)\mathbf{e}(\mathcal{N}Q_\Gamma)} = \sum_{r=0}^{d_0-1}\sum_{b=0}^{d_0-1-r}\hbar^{-(r+1)}\Bigg(\frac{q^{d_0}}{d_0!}\int_{\overline{\mathcal{M}}_{0,2|d_0}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d_0)}(\alpha_i))\psi_1^r\psi_2^b}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d_0)}(\alpha_i-\alpha_k))} \times(-1)^{b+1}q^{|\Gamma_c|}\int_{Q_{\Gamma_c}}\psi_1^{-(b+1)}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_c|)})\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_c})}\Bigg), \tag{6.19}$$

where $d_0=\mathfrak{d}(v_{\min})=|\Gamma_0|$.

We now sum up the last factors in (6.19) over all possibilities for Γ_c with $|\Gamma_c|>0$ by decomposing Γ_c into substrands $\Gamma_1=\Gamma_{ij}(d)$, for some $j\in[n]-i$ and $d\in\mathbb{Z}^+$, and Γ_2, as in the case $\mathfrak{d}(v_{\min})=0$ above. If $\Gamma_c\neq\Gamma_1$, (6.15) with Γ replaced by Γ_c gives

$$q^{|\Gamma_c|}\int_{Q_{\Gamma_c}}\psi_1^{-(b+1)}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_c|)})\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}Q_{\Gamma_c})} = \mathfrak{C}_i^{\mu(v_1)}(\mathfrak{d}(e_1))q^{\mathfrak{d}(e_1)}\left(\frac{\alpha_{\mu(v_1)}-\alpha_i}{\mathfrak{d}(e_1)}\right)^{-(b+1)} \times\left(q^{|\Gamma_2|}\left\{\int_{Q_{\Gamma_2}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_2|)})\mathrm{ev}_1^*\phi_{\mu(v_1)}}{\hbar-\psi_1}\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}\right\}\bigg|_{\hbar=(\alpha_{\mu(v_1)}-\alpha_i)/\mathfrak{d}(e_1)}\right).$$

The sum of the last factors above over all possibilities for Γ_2, with Γ_1 held fixed, including the case Γ_2 is empty (when this factor is taken to be 1 for the equality to hold), is

$$\mathcal{Z}_{n;\mathbf{a}}(\alpha_{\mu(v_1)}, (\alpha_{\mu(v_1)} - \alpha_i)/\mathfrak{d}(e_1), q),$$

as before. Comparing with the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$, we conclude

$$\begin{aligned}\sum_{\substack{\Gamma_c, |\Gamma_c|>0 \\ \mu(v_1)=j, \mathfrak{d}(e_1)=d}} & q^{|\Gamma_c|} \int_{Q_{\Gamma_c}} \psi_1^{-(b+1)} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_c|)}) \mathrm{ev}_1^* \phi_i}{\mathbf{e}(\mathcal{N} Q_{\Gamma_c})} \\ &= \mathfrak{C}_i^j(d) q^d \left(\frac{\alpha_j - \alpha_i}{d}\right)^{-(b+1)} \mathcal{Z}_{n;\mathbf{a}}(\alpha_j, (\alpha_j - \alpha_i)/d, q) \\ &= \underset{\hbar=(\alpha_j-\alpha_i)/d}{\mathfrak{R}} \{\hbar^{-(b+1)} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q)\}.\end{aligned}$$

Thus, by the recursion (5.4) for $\mathcal{Z}_{n;\mathbf{a}}$ and the residue theorem on S^2,

$$\begin{aligned}\sum_{\Gamma_c, |\Gamma_c|>0} q^{|\Gamma_c|} & \int_{Q_{\Gamma_c}} \psi_1^{-(b+1)} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_c|)}) \mathrm{ev}_1^* \phi_i}{\mathbf{e}(\mathcal{N} Q_{\Gamma_c})} \\ &= - \underset{\hbar=0,\infty}{\mathfrak{R}} \{\hbar^{-(b+1)} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q)\} \\ &= - \underset{\hbar=0}{\mathfrak{R}} \{\hbar^{-(b+1)} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q)\} + \delta_{0b}.\end{aligned}$$

Combining this with (6.19) and (6.18), we obtain

$$\begin{aligned}\sum_{\Gamma, \mathfrak{d}(v_{\min})>0} & q^{|\Gamma|} \int_{Q_\Gamma} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma|)}) \mathrm{ev}_1^* \phi_i}{\hbar - \psi_1} \bigg|_{Q_\Gamma} \frac{1}{\mathbf{e}(\mathcal{N} Q_\Gamma)} \\ &= \sum_{d=1}^{\infty} \frac{q^d}{d!} \sum_{r=0}^{d-1} \hbar^{-(r+1)} \sum_{b=0}^{d-1-r} \left(\left(\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i)) \psi_1^r \psi_2^b}{\prod_{k \neq i} \mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i - \alpha_k))} \right) \right. \\ &\quad \left. \times \underset{\hbar=0}{\mathfrak{R}} \left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \right).\end{aligned}$$

This concludes the proof of Proposition 6.1.

In the case of products of projective spaces and concavex sheaves (1.14), we need analogues of (4.6) and (4.7) for every pair of tuples

$$\mathbf{d} \equiv (d_1, \dots, d_p) \in (\mathbb{Z}^{\geq 0})^p - 0, \qquad \beta = (\beta_1, \dots, \beta_p) \in H_{\mathbb{T}}^2.$$

Thus, we define the sheaves $\mathcal{S}_1^*, \dots, \mathcal{S}_p^*$ over the universal curve $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,2||\mathbf{d}|}$ by

$$\mathcal{S}_1^* \equiv \mathcal{O}_{\mathcal{U}}(\sigma_1 + \cdots + \sigma_{d_1}), \mathcal{S}_2^* \equiv \mathcal{O}_{\mathcal{U}}(\sigma_{d_1+1} + \cdots + \sigma_{d_1+d_2}), \ldots \longrightarrow \mathcal{U}$$

and denote by $\mathcal{S}_i^*(\beta_i)$, with $i = 1, \dots, p$, the sheaves such that

$$\mathbf{e}(\mathcal{S}_i^*(\beta_i)) = \beta_i \times 1 + 1 \times e(\mathcal{S}_i^*) \in H_{\mathbb{T}}^*(\mathcal{U}) = H_{\mathbb{T}}^* \otimes H^*(\mathcal{U}).$$

Similarly to (4.7), let

$$\dot{\mathcal{V}}_{\mathbf{a}}^{(\mathbf{d})}(\beta) = \bigoplus_{a_{k;1}\ge 0} R^0\pi_*(\mathcal{S}_1^*(\beta_1)^{a_{k;1}} \otimes \cdots \otimes \mathcal{S}_1^*(\beta_p)^{a_{k;p}}(-\sigma_1)) \oplus \bigoplus_{a_{k;1}<0} R^1\pi_*(\mathcal{S}_1^*(\beta_1)^{a_{k;1}} \otimes \cdots \otimes \mathcal{S}_1^*(\beta_p)^{a_{k;p}}(-\sigma_1)) \longrightarrow \overline{\mathcal{M}}_{0,2||\mathbf{d}|}.$$

The fixed points of the $\mathbb{T}$-action on $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_p-1}$ are

$$P_{i_1\cdots i_p} \equiv P_{i_1} \times \cdots \times P_{i_p}, \quad i_s \in [n_s];$$

thus, the function μ on vertices now takes values in the tuples $(i_1, \ldots, i_p)$. The function $\mathfrak{d}$ on vertices now takes values in $(\mathbb{Z}^{\ge 0})^p$, with the space $\overline{\mathcal{M}}_{0,2|\mathfrak{d}(v)}/\mathbb{S}_{\mathfrak{d}(v)}$ above replaced by

$$\overline{\mathcal{M}}_{0,2|\mathfrak{d}_1(v)+\cdots+\mathfrak{d}_p(v)}/\mathbb{S}_{\mathfrak{d}_1(v)} \times \cdots \times \mathbb{S}_{\mathfrak{d}_p(v)},$$

in light of (2.8). The $\mathbb{T}$-fixed curves are the lines between the points $P_{i_1\cdots i_p}$ and $P_{j_1\cdots j_p}$ such that

$$|\{s \in [p]:\ i_s \neq j_s\}| = 1;$$

thus, the vertices of any edge now differ by precisely one of the indices $(i_1, \ldots, i_p)$, with the ω-classes in (6.5) described by the difference in the weights of this index. The strands with $\mathfrak{d}(v_{\min}) = 0$ now give rise to a triple sum, with the summation index $s \in [p]$ on the outer sum indicating which of the indices $(i_1, \ldots, i_p)$ changes. The computation of the contribution from the strands with $\mathfrak{d}(v_{\min}) > 0$ proceeds exactly as above, but the denominator in the integrand for $\overline{\mathcal{M}}_{0,2|d_0}$ above is replaced by the product of factors corresponding to each of the p factors. This results in a similar formula for the secondary coefficients $\mathcal{Z}^r_{i_1\cdots i_p}$ in (5.4):

$$\begin{aligned}\sum_{(d_1,\ldots,d_p)\in(\mathbb{Z}^{\ge 0})-0}^{\infty} & \mathcal{Z}^r_{i_1\cdots i_p}(d_1,\ldots,d_p) q_1^{d_1}\cdots q_p^{d_p} \\ &= \sum_{\mathbf{d}\in(\mathbb{Z}^{\ge 0})-0} \frac{q_1^{d_1}\cdots q_p^{d_p}}{d_1!\cdots d_p!} \\ &\quad \times \sum_{b=0}^{|\mathbf{d}|+r} \Bigg(\Bigg(\int_{\overline{\mathcal{M}}_{0,2||\mathbf{d}|}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(\mathbf{d})}(\alpha_{i_1},\ldots,\alpha_{i_p}))\psi_1^{-r-1}\psi_2^b}{\prod_{s=1}^p \prod_{k\neq i_s} \mathbf{e}(\dot{\mathcal{V}}_{e_s}^{(d_s)}(\alpha_{s;i_s}-\alpha_{s;k}))}\Bigg) \\ &\quad \times \mathfrak{R}_{\hbar=0}\Bigg\{\frac{(-1)^b}{\hbar^{b+1}} \mathcal{Z}_{n;\mathbf{a}}(\alpha_{i_1},\ldots,\alpha_{i_p},\hbar,q_1,\ldots,q_p)\Bigg\}\Bigg), \end{aligned} \tag{6.20}$$

whenever $r \in \mathbb{Z}^-$ and $i_s \in [n_s]$, if $e_s \in (\mathbb{Z}^+)^p$ is the sth coordinated vector.

7. Polynomiality for Stable Quotients

In this section, we adopt the argument in [11, Section 30.2], showing that the equivariant version of Givental's J-function satisfies the self-polynomiality condition of Definition 5.2, to show that the equivariant stable quotients analogue of Givental's J-function, the power series $\mathcal{Z}_{n;\mathbf{a}}$ defined by (4.1), also satisfies the self-polynomiality condition. Proposition 7.1 is an immediate consequence of Lemma 7.2 below, which provides a geometric description of the power series $\Phi_{\mathcal{Z}_{n;\mathbf{a}}}$.

PROPOSITION 7.1. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^{+}$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then the power series $\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x}, \hbar, q)$ satisfies the self-polynomiality condition.*

The proof involves applying the classical localization theorem [1] with $(n+1)$-torus

$$\widetilde{\mathbb{T}} \equiv \mathbb{C}^* \times \mathbb{T},$$

where $\mathbb{T} = (\mathbb{C}^*)^n$ as before. We denote the weight of the standard action of the one-torus $\mathbb{C}^*$ on $\mathbb{C}$ by $\hbar$. Thus, by Section 3,

$$H^*_{\mathbb{C}^*} \approx \mathbb{Q}[\hbar], \qquad H^*_{\widetilde{\mathbb{T}}} \approx \mathbb{Q}[\hbar, \alpha_1, \dots, \alpha_n] \quad \Longrightarrow \quad \mathcal{H}^*_{\widetilde{\mathbb{T}}} \approx \mathbb{Q}_\alpha(\hbar).$$

Throughout this section, $V = \mathbb{C} \oplus \mathbb{C}$ denotes the representation of $\mathbb{C}^*$ with the weights 0 and $-\hbar$. The induced action on $\mathbb{P}V$ has two fixed points:

$$q_1 \equiv [1, 0], \qquad q_2 \equiv [0, 1].$$

With $\gamma_1 \longrightarrow \mathbb{P}V$ denoting the tautological line bundle,

$$\mathbf{e}(\gamma_1^*)|_{q_1} = 0, \quad \mathbf{e}(\gamma_1^*)|_{q_2} = -\hbar, \quad \mathbf{e}(T_{q_1}\mathbb{P}V) = \hbar, \quad \mathbf{e}(T_{q_2}\mathbb{P}V) = -\hbar; \tag{7.1}$$

this follows from our definition of the weights in Section 3.

For each $d \in \mathbb{Z}^{\geq 0}$, the action of $\widetilde{\mathbb{T}}$ on $\mathbb{C}^n \otimes \mathrm{Sym}^d V^*$ induces an action on

$$\overline{\mathfrak{X}}_d \equiv \mathbb{P}(\mathbb{C}^n \otimes \mathrm{Sym}^d V^*).$$

It has $(d+1)n$ fixed points:

$$P_i(r) \equiv [\tilde{P}_i \otimes u^{d-r} v^r], \quad i \in [n], r \in \{0\} \cup [d],$$

where (u, v) are the standard coordinates on V and $\tilde{P}_i \in \mathbb{C}^n$ is the ith coordinate vector (so that $[\tilde{P}_i] = P_i \in \mathbb{P}^{n-1}$). Let

$$\Omega \equiv \mathbf{e}(\gamma^*) \in H^*_{\widetilde{\mathbb{T}}}(\overline{\mathfrak{X}}_d)$$

denote the equivariant hyperplane class.

For all $i \in [n]$ and $r \in \{0\} \cup [d]$,

$$\Omega|_{P_i(r)} = \alpha_i + r\hbar,$$

$$\mathbf{e}(T_{P_i(r)}\overline{\mathfrak{X}}_d) = \left\{ \prod_{\substack{s=0 \\ (s,k)\neq(r,i)}}^{d} \prod_{k=1}^{n} (\Omega - \alpha_k - s\hbar) \right\}\Big|_{\Omega=\alpha_i+r\hbar}. \quad (7.2)$$[5]

Since

$$B\overline{\mathfrak{X}}_d = \mathbb{P}(B(\mathbb{C}^n \otimes \mathrm{Sym}^d V^*)) \longrightarrow B\widetilde{\mathbb{T}} \quad \text{and}$$

$$c(B(\mathbb{C}^n \otimes \mathrm{Sym}^d V^*)) = \prod_{s=0}^{d}\prod_{k=1}^{n}(1-(\alpha_k + s\hbar)) \in H^*(B\widetilde{\mathbb{T}}),$$[6]

the $\widetilde{\mathbb{T}}$-equivariant cohomology of $\overline{\mathfrak{X}}_d$ is given by

$$\begin{aligned} H^*_{\widetilde{\mathbb{T}}}(\overline{\mathfrak{X}}_d) \equiv H^*(B\overline{\mathfrak{X}}_d) &= H^*(B\widetilde{\mathbb{T}})[\Omega] \Big/ \prod_{s=0}^{d}\prod_{k=1}^{n}(\Omega - (\alpha_k + s\hbar)) \\ &\approx \mathbb{Q}[\Omega, \hbar, \alpha_1, \ldots, \alpha_n] \Big/ \prod_{s=0}^{d}\prod_{k=1}^{n}(\Omega - \alpha_k - s\hbar) \\ &\subset \mathbb{Q}_\alpha[\hbar, \Omega] \Big/ \prod_{s=0}^{d}\prod_{k=1}^{n}(\Omega - \alpha_k - s\hbar). \end{aligned}$$

In particular, every element of $H^*_{\widetilde{\mathbb{T}}}(\overline{\mathfrak{X}}_d)$ is a polynomial in Ω with coefficients in $\mathbb{Q}_\alpha[\hbar]$ of degree at most $(d+1)n-1$.

By [13, Lemma 2.6], there is a natural $\widetilde{\mathbb{T}}$-equivariant morphism

$$\Theta : \overline{\mathfrak{M}}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1,d)) \longrightarrow \overline{\mathfrak{X}}_d.$$

A general element of b of $\overline{\mathfrak{M}}_{0,m}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1,d))$ determines a morphism

$$(f, g) : \mathbb{P}^1 \longrightarrow (\mathbb{P}V, \mathbb{P}^{n-1}),$$

up to an automorphism of the domain $\mathbb{P}^1$. Thus, the morphism

$$g \circ f^{-1} : \mathbb{P}V \longrightarrow \mathbb{P}^{n-1}$$

is well defined and determines an element $\Theta(b) \in \overline{\mathfrak{X}}_d$. Let

$$\begin{aligned} \mathfrak{X}_d &= \{b \in \overline{\mathfrak{M}}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1,d)) : \\ &\qquad \mathrm{ev}_1(b) \in q_1 \times \mathbb{P}^{n-1}, \mathrm{ev}_2(b) \in q_2 \times \mathbb{P}^{n-1}\}, \\ \mathfrak{X}'_d &= \{b' \in \overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1,d)) : \\ &\qquad \mathrm{ev}_1(b') \in q_1 \times \mathbb{P}^{n-1}, \mathrm{ev}_2(b') \in q_2 \times \mathbb{P}^{n-1}\}. \end{aligned} \quad (7.3)$$

---

[5] The weight (i.e., negative first Chern class) of the $\widetilde{\mathbb{T}}$-action on the line $P_i(r) \subset \mathbb{C}^n \otimes \mathrm{Sym}^d V^*$ is $\alpha_i + r\hbar$. The tangent bundle of $\overline{\mathfrak{X}}_d$ at $P_i(r)$ is the direct sum of the lines $P_i(r)^* \otimes P_k(s)$ with $(k,s) \neq (i,r)$.

[6] The vector space $\mathbb{C}^n \otimes \mathrm{Sym}^d V^*$ is the direct sum of the one-dimensional representations $P_k(s)$ of $\widetilde{\mathbb{T}}$.

Since the morphism to $\mathbb{P}^1$ corresponding to any element of $b' \in \mathfrak{X}'_d$ takes the two marked points to q_1 and q_2, it is not constant. Thus, the restriction of the morphism Θ to $\mathfrak{X}_d$ is constant along the fibers of the natural surjective morphism $c : \mathfrak{X}_d \longrightarrow \mathfrak{X}'_d$.[7] It follows that the restriction of Θ to $\mathfrak{X}_d$ descends via c to a morphism

$$\theta = \theta_d : \mathfrak{X}'_d \longrightarrow \overline{\mathfrak{X}}_d.$$

For $d > 0$, there is also a natural forgetful morphism

$$F : \overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d)) \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d),$$

which drops the first sheaf in the pair and contracts one component of the domain if necessary. Similarly to (1.2), for each $d \in \mathbb{Z}^+$, let

$$\mathcal{V}^{(d)}_{n;\mathbf{a}} = \bigoplus_{a_k>0} R^0\pi_*(\mathcal{S}^{*a_k}) \oplus \bigoplus_{a_k<0} R^1\pi_*(\mathcal{S}^{*a_k}) \longrightarrow \overline{Q}_{0,2}(\mathbb{P}^{n-1}, d).$$

From the usual short exact sequence for the restriction along σ_1, we find that

$$\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}}) = \langle \mathbf{a} \rangle \mathrm{ev}_1^* \mathbf{x}^{\ell(\mathbf{a})} \mathbf{e}(\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}}) \in H^*_{\mathbb{T}}(\overline{Q}_{0,2}(\mathbb{P}^{n-1}, d)). \tag{7.4}$$

In the case $d = 0$, we set

$$F^*\mathbf{e}(\mathcal{V}^{(0)}_{n;\mathbf{a}}) = \langle \mathbf{a} \rangle \mathrm{ev}_1^*(1 \times \mathbf{x}^{\ell(\mathbf{a})}) \in H^*(\overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, 0)));$$

this is used in Lemma 7.2 below.

LEMMA 7.2. *If $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, and $\mathbf{a} \in (\mathbb{Z}^*)^l$, then*

$$\Phi_{\mathcal{Z}_{n;\mathbf{a}}}(\hbar, z, q) = \sum_{d=0}^{\infty} q^d \int_{\mathfrak{X}'_d} \mathrm{e}^{(\theta^*\Omega)z} F^*\mathbf{e}(\mathcal{V}^{(d)}_{n;\mathbf{a}}) \in H^*_{\mathbb{T}}[[z, q]] \subset \mathbb{Q}_\alpha[\hbar][[z, q]]. \tag{7.5}$$

We prove Lemma 7.2 in the remainder of this section by applying the localization theorem of [1] to the $\widetilde{\mathbb{T}}$-action on $\mathfrak{X}'_d$. We show that each fixed locus of the $\widetilde{\mathbb{T}}$-action on $\mathfrak{X}'_d$ contributing to the right-hand side of (7.5) corresponds to a pair (Γ_1, Γ_2) of decorated strands as in (6.1), with Γ_1 and Γ_2 contributing to $\mathcal{Z}_{n;\mathbf{a}}(\alpha_i, \hbar, q\mathrm{e}^{\hbar z})$ and $\mathcal{Z}_{n;\mathbf{a}}(\alpha_i, -\hbar, q)$, respectively, for some $i \in [n]$.

Similarly to Section 6, the fixed loci of the $\widetilde{\mathbb{T}}$-action on $\overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (d', d))$ correspond to decorated strands Γ with two marked points at the opposite ends. The map $\mathfrak{d}$ should now take values in pairs of nonnegative integers, indicating the degrees of the two subsheaves. The map μ should similarly take values in the pairs (i, j) with $i \in [2]$ and $j \in [n]$, indicating the fixed point (q_i, P_j) to

[7] For a stable map b, $\Theta(b)$ depends only on the restriction of b to the irreducible component $\mathcal{C}_{b;1}$ of its domain $\mathcal{C}_b$ on which the degree of the map to $\mathbb{P}^1$ is not zero, the nodes of $\mathcal{C}_{b;1}$, and the degrees of the restrictions of b to the connected components of $\mathcal{C}_b - \mathcal{C}_{b;1}$. In contrast, $c(b)$ depends on the restriction of b to the minimal connected union (chain) of irreducible components $\mathcal{C}'_b$ of its domain that contains the two marked points, the nodes of $\mathcal{C}'_b$, and the degrees of the restrictions of b to the connected components of $\mathcal{C}_b - \mathcal{C}'_b$. Whenever $b \in \mathfrak{X}_d$, $\mathcal{C}_{b;1} \subset \mathcal{C}'_b$. Thus, the restriction of Θ to $\mathfrak{X}_d$ contracts everything that the restriction of c contracts.

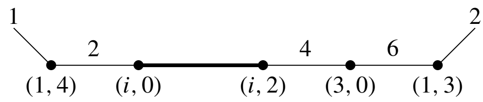

Figure 5 A strand representing a fixed locus in $\mathfrak{X}'_d$; $i \neq 1, 3$

which the vertex is mapped. The μ-values on consecutive vertices must differ by precisely one of the two components.

The situation for the $\widetilde{\mathbb{T}}$-action on

$$\mathfrak{X}'_d \subset \overline{Q}_{0,2}(\mathbb{P}V \times \mathbb{P}^{n-1}, (1, d))$$

is simpler, however. There is a unique edge of positive $\mathbb{P}V$-degree; we draw it as a thick line in Figure 5. The first component of the value of $\mathfrak{d}$ on all other edges and on all vertices must be 0; so we drop it. The first component of the value of μ on the vertices changes only when the thick edge is crossed. Thus, we drop the first components of the vertex labels as well, with the convention that these components are 1 on the left side of the thick edge and 2 on the right. In particular, the vertices to the left of the thick edge (including the left endpoint) lie in $q_1 \times \mathbb{P}^{n-1}$, and the vertices to its right lie in $q_2 \times \mathbb{P}^{n-1}$. Thus, by (7.3), the marked point 1 is attached to a vertex to the left of the thick edge, and the marked point 2 is attached to a vertex to the right. Finally, the remaining second component of μ takes the same value $i \in [n]$ on the two vertices of the thick edge.

Let $\mathcal{A}_i$ denote the set of strands as above so that the μ-value on the two endpoints of the thick edge is labeled i; see Figure 5. We break each strand $\Gamma \in \mathcal{A}_i$ into three substrands:

(i) Γ_1 consisting of all vertices of Γ to the left of the thick edge, including its left vertex v_1 with its $\mathfrak{d}$-value, but in the opposite order, and a new marked point attached to v_1;
(ii) Γ_0 consisting of the thick edge e_0, its two vertices v_1 and v_2, with $\mathfrak{d}$-values set to 0, and new marked points 1 and 2 attached to v_1 and v_2, respectively;
(iii) Γ_2 consisting of all vertices to the right of the thick edge, including its right vertex v_2 with its $\mathfrak{d}$-value, and a new marked point attached to v_2;

see Figure 6. From (6.3) we then obtain a splitting of the fixed locus in $\mathfrak{X}'_d$ corresponding to Γ:

$$\begin{aligned} Q_\Gamma &\approx Q_{\Gamma_1} \times Q_{\Gamma_0} \times Q_{\Gamma_2} \\ &\subset \overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma_1|) \times \overline{Q}_{0,2}(\mathbb{P}V, 1) \times \overline{Q}_{0,2}(\mathbb{P}^{n-1}, |\Gamma_2|). \end{aligned} \tag{7.6}$$

The exceptional cases are $|\Gamma_1| = 0$ and $|\Gamma_2| = 0$; the above isomorphism then holds with the corresponding component replaced by a point.

Let π_1, π_0, and π_2 denote the three component projection maps in (7.6). By (7.4), (6.7), and (6.6),

$$F^*\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma|)})|_{Q_\Gamma} = \langle \mathbf{a} \rangle \alpha_i^{\ell(\mathbf{a})} \cdot \pi_1^*\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_1|)}) \cdot \pi_2^*\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_2|)}),$$

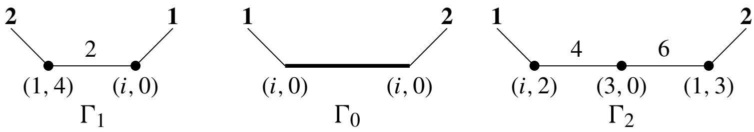

Figure 6 The three substrands of the strand in Figure 5

$$\frac{\mathbf{e}(\mathcal{N}Q_\Gamma)}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})} = \pi_1^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right)\cdot\pi_2^*\left(\frac{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}{\mathbf{e}(T_{P_i}\mathbb{P}^{n-1})}\right) \cdot(\omega_{e_0;v_1}-\pi_1^*\psi_1)(\omega_{e_0;v_2}-\pi_2^*\psi_1). \tag{7.7}$$

Since Q_{Γ_0} consists of a degree 1 map, by the last two identities in (7.1)

$$\omega_{e_0;v_1}=\hbar, \qquad \omega_{e_0;v_2}=-\hbar. \tag{7.8}$$

The morphism θ takes the locus Q_Γ to a fixed point $P_k(r)\in\overline{\mathfrak{X}}_d$. It is immediate that $k=i$. By continuity considerations, $r=|\Gamma_1|$. Thus, by the first identity in (7.2),

$$\theta^*\Omega|_{Q_\Gamma}=\alpha_i+|\Gamma_1|\hbar. \tag{7.9}$$

Combining (7.7)–(7.9), we obtain

$$\begin{aligned} q^{|\Gamma|}\int_{Q_\Gamma}&\frac{\mathrm{e}^{(\theta^*\Omega)z}F^*\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(|\Gamma|)})|_{Q_\Gamma}}{\mathbf{e}(\mathcal{N}Q_\Gamma)}\\ &=\frac{\langle\mathbf{a}\rangle\alpha_i^{\ell(\mathbf{a})}\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)}\left\{\mathrm{e}^{|\Gamma_1|\hbar z}q^{|\Gamma_1|}\int_{Q_{\Gamma_1}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_1|)})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\bigg|_{Q_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}\right\}\\ &\quad\times\left\{q^{|\Gamma_2|}\int_{Q_{\Gamma_2}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_2|)})\mathrm{ev}_1^*\phi_i}{(-\hbar)-\psi_1}\bigg|_{Q_{\Gamma_2}}\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}\right\}. \end{aligned} \tag{7.10}$$

This identity remains valid with $|\Gamma_1|=0$ and/or $|\Gamma_2|=0$ if we set the corresponding integral to 1.

We now sum up (7.10) over all $\Gamma\in\mathcal{A}_i$. This is the same as summing over all pairs (Γ_1,Γ_2) of decorated strands such that:

(1) Γ_1 is a 2-point strand of degree $d_1\geq 0$ such that the marked point 1 is attached to a vertex labeled i;

(2) Γ_2 is a 2-point strand of degree $d_2\geq 0$ such that the marked point 1 is attached to a vertex labeled i.

By the localization formula (3.8),

$$\begin{aligned} 1+\sum_{\Gamma_1}(q\mathrm{e}^{\hbar z})^{|\Gamma_1|}&\left\{\int_{Q_{\Gamma_1}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_1|)})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}\bigg|_{Q_{\Gamma_1}}\frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_1})}\right\}\\ &=1+\sum_{d=1}^{\infty}(q\mathrm{e}^{\hbar z})^d\int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})\mathrm{ev}_1^*\phi_i}{\hbar-\psi_1}=\mathcal{Z}_{n;\mathbf{a}}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z}); \end{aligned}$$

$$1+\sum_{\Gamma_2} q^{|\Gamma_2|}\left\{\int_{Q_{\Gamma_2}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(|\Gamma_2|)})\mathrm{ev}_1^*\phi_i}{(-\hbar)-\psi_1}\bigg|_{Q_{\Gamma_2}} \frac{1}{\mathbf{e}(\mathcal{N}Q_{\Gamma_2})}\right\}$$

$$=1+\sum_{d=0}^{\infty} q^d \int_{\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)} \frac{\mathbf{e}(\dot{\mathcal{V}}_{n;\mathbf{a}}^{(d)})\mathrm{ev}_1^*\phi_i}{(-\hbar)-\psi_1} = \mathcal{Z}_{n;\mathbf{a}}(\alpha_i,-\hbar,q). \tag{7.11}$$

Finally, by (3.8), (7.10), and (7.11),

$$\begin{aligned}\sum_{d=0}^{\infty} q^d \int_{\mathfrak{X}'_d} &\mathrm{e}^{(\theta^*\Omega)z} F^*\mathbf{e}(\mathcal{V}_{n;\mathbf{a}}^{(d)}) \\ &= \sum_{i=1}^{n} \frac{\langle \mathbf{a}\rangle \alpha_i^{\ell(\mathbf{a})}\mathrm{e}^{\alpha_i z}}{\prod_{k\neq i}(\alpha_i-\alpha_k)} \mathcal{Z}_{n;\mathbf{a}}(\alpha_i,\hbar,q\mathrm{e}^{\hbar z})\mathcal{Z}_{n;\mathbf{a}}(\alpha_i,-\hbar,q) \\ &= \Phi_{\mathcal{Z}_{n;\mathbf{a}}}(\hbar,z,q),\end{aligned}$$

as claimed in (7.5).

In the case of products of projective spaces and concavex sheaves (1.14), the spaces

$$\overline{Q}_{0,2}(\mathbb{P}V\times\mathbb{P}^{n-1},(1,d)) \quad \text{and} \quad \overline{\mathfrak{X}}_d=\mathbb{P}(\mathbb{C}^n\otimes \mathrm{Sym}^d V^*)$$

are replaced by

$$\overline{Q}_{0,2}(\mathbb{P}V\times\mathbb{P}^{n_1-1}\times\cdots\times\mathbb{P}^{n_p-1},(1,d_1,\ldots,d_p)) \quad \text{and}$$
$$\mathbb{P}(\mathbb{C}^{n_1}\otimes\mathrm{Sym}^{d_1}V^*)\times\cdots\times\mathbb{P}(\mathbb{C}^{n_p}\otimes\mathrm{Sym}^{d_p}V^*),$$

respectively. Lemma 7.2 then becomes

$$\begin{aligned}&\Phi_{\mathcal{Z}_{n_1,\ldots,n_p;\mathbf{a}}}(\hbar,z_1,\ldots,z_p,q_1,\ldots,q_p) \\ &\quad= \sum_{d_1,\ldots,d_p\geq 0} q_1^{d_1}\cdots q_p^{d_p}\int_{\mathfrak{X}'_{d_1,\ldots,d_p}} \mathrm{e}^{(\theta^*\Omega_1)z_1+\cdots+(\theta^*\Omega_p)z_p}\pi_1^*\mathbf{e}(\mathcal{V}_{n_1,\ldots,n_p;\mathbf{a}}^{(d_1,\ldots,d_p)}).\end{aligned}$$

The vertices of the thick edge in Figure 5 are now labeled by a tuple $(i_1,\ldots,i_p)$ with $i_s\in[n_s]$, as needed for the extension of (5.5) described at the end of Section 5. Relation (7.9) becomes

$$\theta^*\Omega_s|_{Q_\Gamma}=\alpha_{s;i_s}+|\Gamma_1|_s\hbar,$$

where $|\Gamma_1|_s$ is the sum of the sth components of the values of $\mathfrak{d}$ on the vertices and edges of Γ_1 (corresponding to the degree of the maps to $\mathbb{P}^{n_s-1}$). Otherwise, the proof is identical.

8. Proof of Theorems 3 and 4

This section concludes the proof of Theorem 3 stated in Section 4. Sections 5–7 reduce this theorem to conditions on the power series $\mathcal{Y}_{n;\mathbf{a}}$ defined in (4.2); see Lemma 8.2. Based on qualitative, primarily algebraic, considerations, we show in the proof of Proposition 8.3 that this power series does indeed satisfy these conditions and thus establish Theorem 3. The only geometric considerations entering the proof of Proposition 8.3 concern moduli spaces of stable curves $\overline{\mathcal{M}}_{0,2|d}$, not moduli spaces of stable quotients $\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)$. We conclude this section by showing that these conditions on $\mathcal{Y}_{n;\mathbf{a}}$ determine certain integrals on $\overline{\mathcal{M}}_{0,2|d}$ and finish the proof of Theorem 4 stated in Section 4.

COROLLARY 8.1. *Let* $l\in\mathbb{Z}^{\ge 0}$, $n\in\mathbb{Z}^+$ *and* $\mathbf{a}\in(\mathbb{Z}^*)^l$. *If* $|\mathbf{a}|\le n-2$, *then*

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[[\hbar^{-1},q]].$$

Proof. Both sides of this identity are $\mathfrak{C}$-recursive and satisfy the self-polynomiality condition (no matter what n and $\mathbf{a}$ are); see Lemma 5.4 and Propositions 6.1 and 7.1. By (4.2),

$$\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\cong 1 \mod \hbar^{-2}$$

whenever $|\mathbf{a}|\le n-2$. If in addition $d\in\mathbb{Z}^+$, then

$$\dim\overline{Q}_{0,2}(\mathbb{P}^{n-1},d)-\operatorname{rk}\dot{\mathcal{V}}^{(d)}_{n;\mathbf{a}}=(n-|\mathbf{a}|)d+(n-2)>n-1=\dim\mathbb{P}^{n-1}.$$

Thus,

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)\cong 1 \mod \hbar^{-2}$$

whenever $|\mathbf{a}|\le n-2$. The claim now follows from Proposition 5.3. □

LEMMA 8.2. *If* $l\in\mathbb{Z}^{\ge 0}$, $n\in\mathbb{Z}^+$, *and* $\mathbf{a}\in(\mathbb{Z}^*)^l$ *are such that* $|\mathbf{a}|\le n$, *then*

$$\mathcal{Z}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)=\frac{\mathcal{Y}_{n;\mathbf{a}}(\mathbf{x},\hbar,q)}{I_{n;\mathbf{a}}(q)}\in H^*_{\mathbb{T}}(\mathbb{P}^{n-1})[[\hbar^{-1},q]] \tag{8.1}$$

if and only if

$$\begin{aligned}\mathop{\mathfrak{R}}_{\hbar=0}\{\hbar^r\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\}&\\ =\sum_{d=1}^{\infty}\frac{q^d}{d!}\sum_{b=0}^{d-1-r}&\Bigg(\bigg(\int_{\overline{\mathcal{M}}_{0,2|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}^{(d)}_{\mathbf{a}}(\alpha_i))\psi_1^r\psi_2^b}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}^{(d)}_1(\alpha_i-\alpha_k))}\bigg)\\ &\times\mathop{\mathfrak{R}}_{\hbar=0}\bigg\{\frac{(-1)^b}{\hbar^{b+1}}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\bigg\}\Bigg)\end{aligned} \tag{8.2}$$

for all $i\in[n]$ *and* $r\in\mathbb{Z}^{\ge 0}$.

Proof. Since both sides of (8.1) are $\mathfrak{C}$-recursive with the *same* collection (5.6) of the primary coefficients (see Lemma 5.4 and Proposition 6.1) and have the same q^0-coefficients, (8.1) holds if and only if the secondary coefficients

$$\frac{1}{I_{n;\mathbf{a}}(q)}\sum_{d=0}^{\infty}\mathcal{Y}_i^r(d)q^d \quad\text{and}\quad \sum_{d=0}^{\infty}\mathcal{Z}_i^r(d)q^d,$$

instead of $\mathcal{F}_i^r(d)$, in the recursions (5.4) for $\mathcal{Y}_{n;\mathbf{a}}/I_{n;\mathbf{a}}$ and $\mathcal{Z}_{n;\mathbf{a}}$ are the same (this would make the two recursions the same). Since Proposition 6.1 describes the coefficients $\mathcal{Z}_i^r(d)$ recursively on d, (8.1) holds if and only if the coefficients $\mathcal{Y}_i^r(d)$ satisfy the same description. By Lemma 5.4 and Proposition 6.1, this is the case if and only if (8.2) holds (r in Lemma 5.4 and Proposition 6.1 corresponds to $-r-1$ in the notation of (8.2)). □

PROPOSITION 8.3. *If* $l \in \mathbb{Z}^{\geq 0}$, $n \in \mathbb{Z}^+$, *and* $\mathbf{a} \in (\mathbb{Z}^*)^l$, *then*

$$\begin{aligned}
&\mathfrak{R}_{\hbar=0}\{\hbar^r \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q)\} \\
&\quad = \sum_{d=1}^{\infty} \frac{q^d}{d!} \sum_{b=0}^{d-1-r} \left(\left(\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^r \psi_2^b}{\prod_{k\neq i} \mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i - \alpha_k))} \right) \right. \\
&\qquad \left. \times \mathfrak{R}_{\hbar=0}\left\{ \frac{(-1)^b}{\hbar^{b+1}} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i, \hbar, q) \right\} \right)
\end{aligned} \tag{8.3}$$

for all $i \in [n]$ *and* $r \in \mathbb{Z}^{\geq 0}$.

Proof. Let $i \in [n]$ be fixed throughout the proof.

(1) Whenever $d \in \mathbb{Z}^+$ and $s, t \in [d]$, where $[d] = \{1, \ldots, d\}$ as before, let

$$\Delta_{st} = \{[\mathcal{C}, y_1, y_2, \hat{y}_1, \ldots, \hat{y}_d] \in \overline{\mathcal{M}}_{0,2|d} \colon \ \hat{y}_s = \hat{y}_t\} \in H^2(\overline{\mathcal{M}}_{0,2|d})$$

denote the class of the corresponding "diagonal" and define

$$\Delta_s = \sum_{t=s+1}^{d} \Delta_{st} \in H^2(\overline{\mathcal{M}}_{0,2|d}).$$

For any $a_k > 0$, $s \in [d]$, and $r \in [a_k]$, there is a short exact sequence

$$\begin{aligned}
0 &\longrightarrow R^0\pi_*\mathcal{O}\left((r-1)\hat{\sigma}_s + \sum_{t=s+1}^{d} a_k \hat{\sigma}_t - \sigma_1 \right) \\
&\longrightarrow R^0\pi_*\mathcal{O}\left(r\hat{\sigma}_s + \sum_{t=s+1}^{d} a_k \hat{\sigma}_t - \sigma_1 \right) \\
&\longrightarrow R^0\pi_*\mathcal{O}\left(\left(r\hat{\sigma}_s + \sum_{t=s+1}^{d} a_k \hat{\sigma}_t - \sigma_1 \right)\Big|_{\hat{\sigma}_s} \right) \\
&\longrightarrow 0.
\end{aligned}$$

This gives

$$\begin{aligned}
a_k > 0 \quad \Longrightarrow \quad \mathbf{e}(\dot{\mathcal{V}}_{a_k}^{(d)}(\alpha_i)) &= \prod_{s=1}^{d}\prod_{r=1}^{a_k} (a_k\alpha_i - r\hat{\psi}_s + a_k\Delta_s) \\
&= a_k^{a_k d}\alpha_i^{a_k d} \\
&\quad \times \prod_{s=1}^{d}\prod_{r=1}^{a_k} \left(1 - \frac{r}{a_k}\alpha_i^{-1}\hat{\psi}_s + \alpha_i^{-1}\Delta_s \right).
\end{aligned} \tag{8.4}$$

For any $a_k < 0$, $s \in [d]$, and $r = 0, 1, \ldots, -a_k - 1$, there is a short exact sequence

$$0 \longrightarrow R^0\pi_*\mathcal{O}\left(\left(-r\hat{\sigma}_s + \sum_{t=s+1}^{d} a_k \hat{\sigma}_t - \sigma_1 \right)\Big|_{\hat{\sigma}_s} \right)$$

$$\longrightarrow R^1\pi_*\mathcal{O}\Big((-r-1)\hat{\sigma}_s+\sum_{t=s+1}^{d}a_k\hat{\sigma}_t-\sigma_1\Big)$$

$$\longrightarrow R^1\pi_*\mathcal{O}\Big(-r\hat{\sigma}_s+\sum_{t=s+1}^{d}a_k\hat{\sigma}_t-\sigma_1\Big)\longrightarrow 0.$$

This gives

$$a_k<0 \quad\Longrightarrow\quad \mathbf{e}(\dot{\mathcal{V}}_{a_k}^{(d)}(\alpha_i))=\prod_{s=1}^{d}\prod_{r=0}^{-a_k-1}(a_k\alpha_i+r\hat{\psi}_s+a_k\Delta_s)$$
$$=a_k^{-a_kd}\alpha_i^{-a_kd}$$
$$\times\prod_{s=1}^{d}\prod_{r=0}^{a_k-1}\Big(1+\frac{r}{a_k}\alpha_i^{-1}\hat{\psi}_s+\alpha_i^{-1}\Delta_s\Big). \tag{8.5}$$

Similarly to (8.4),

$$\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))=(\alpha_i-\alpha_k)^d\prod_{s=1}^{d}(1-(\alpha_i-\alpha_k)^{-1}\hat{\psi}_s+(\alpha_i-\alpha_k)^{-1}\Delta_s). \tag{8.6}$$

(2) For $d\in\mathbb{Z}^{\geq 0}$, let

$$C_i(\alpha)=\frac{\prod_{a_k>0}(a_k^{a_k}\alpha_i^{a_k})\prod_{a_k<0}(a_k^{-a_k}\alpha_i^{-a_k})}{\prod_{k\neq i}(\alpha_i-\alpha_k)}.$$

We denote by $\mathfrak{s}_1,\mathfrak{s}_2,\dots$ the elementary symmetric polynomials in

$$\{\beta_k\}=\{(\alpha_i-\alpha_k)^{-1}:\ k\neq i\}$$

for any given number of formal variables β_k. Note that

$$\frac{(-1)^b}{d!}\int_{\overline{\mathcal{M}}_{0,2|d}}\prod_{a_k>0}\prod_{s=1}^{d}\prod_{r=1}^{a_k}\Big(1-\frac{r}{a_k}y\hat{\psi}_s+y\Delta_s\Big)$$
$$\times\prod_{a_k<0}\prod_{s=1}^{d}\prod_{r=0}^{a_k-1}\Big(1+\frac{r}{a_k}y\hat{\psi}_s+y\Delta_s\Big)\psi_1^r\psi_2^b$$
$$\Big/\prod_{k=1}^{n-1}\prod_{s=1}^{d}(1-\beta_k\hat{\psi}_s+\beta_k\Delta_s)$$
$$=\mathcal{H}_{\mathbf{a};d}^{r,b}(y,\mathfrak{s}_1,\dots,\mathfrak{s}_{d-1})\in\mathbb{Q}[y,\beta_1,\dots,\beta_{n-1}] \tag{8.7}$$

for some $\mathcal{H}_{\mathbf{a};d}^{r,b}\in\mathbb{Q}[y,\mathfrak{s}_1,\dots,\mathfrak{s}_{d-1}]$ independent of n. Such $\mathcal{H}_{\mathbf{a};d}^{r,b}$ exists because the integrand on the left-hand side of (8.7) is symmetric in $\{\beta_k\}$ and whatever $\mathcal{H}_{\mathbf{a};d}^{r,b}$ works for $n\geq d-r-b$ works for all n (this can be seen by setting the extra

β_k to 0). By (8.4)–(8.7),

$$\frac{(-1)^b}{d!}\int_{\overline{\mathcal{M}}_{0,2|d}}\frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^r\psi_2^b}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))} = C_i(\alpha)^d\mathcal{H}_{\mathbf{a};d}^{r,b}(\alpha_i^{-1},\mathfrak{s}_1,\ldots,\mathfrak{s}_{d-1}) \quad \forall d\in\mathbb{Z}^{\ge 0}. \tag{8.8}$$

Similarly, for any $d,d'\in\mathbb{Z}^{\ge 0}$, there exists $\mathcal{Y}_{\mathbf{a};d,d'}\in\mathbb{Q}[y,\mathfrak{s}_1,\ldots,\mathfrak{s}_{d'}]$, independent of n, such that

$$\left[\!\left[\frac{\prod_{a_k>0}\prod_{r=1}^{a_kd}(1+(r/a_k)y\hbar)\prod_{a_k<0}\prod_{r=0}^{-a_kd-1}(1-(r/a_k)ry\hbar)}{d!\prod_{r=1}^{d}\prod_{k=1}^{n-1}(1+r\beta_k\hbar)}\right]\!\right]_{\hbar;d'} = \mathcal{Y}_{\mathbf{a};d,d'}(y,\mathfrak{s}_1,\ldots,\mathfrak{s}_{d'}). \tag{8.9}$$

By (4.2) and (8.9),

$$[[\hbar^d[[\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)]]_{q;d}]]_{\hbar;d'} = C_i(\alpha)^d\mathcal{Y}_{\mathbf{a};d,d'}(\alpha_i^{-1},\mathfrak{s}_1,\ldots,\mathfrak{s}_{d'}) \quad \forall d,d'\in\mathbb{Z}^{\ge 0}. \tag{8.10}$$

(3) By (8.8) and (8.10), (8.3) is equivalent to

$$\mathcal{Y}_{\mathbf{a};d,d-1-r}(y,\mathfrak{s}_1,\mathfrak{s}_2,\ldots) = \sum_{\substack{d_1+d_2=d\\ d_1\ge 1}}\sum_{b=0}^{d_1-1-r}\mathcal{H}_{\mathbf{a};d_1}^{r,b}(y,\mathfrak{s}_1,\mathfrak{s}_2,\ldots) \times\mathcal{Y}_{\mathbf{a};d_2,d_2+b}(y,\mathfrak{s}_1,\mathfrak{s}_2,\ldots) \quad \forall d\in\mathbb{Z}^+. \tag{8.11}$$

This equivalence is obtained by taking the coefficients of q^d of the two sides of (8.3), factoring out $C_i(\alpha)^d$, and replacing α_i^{-1} by y and $\{(\alpha_i-\alpha_k)^{-1}:\ k\neq i\}$ by $\{\beta_1,\ldots,\beta_{n-1}\}$. By Lemma 8.2 and Corollary 8.1, (8.11) holds whenever $|\mathbf{a}|\le n-2$. Since (8.11) does not involve n, it holds for all $\mathbf{a}$. Thus, (8.3) holds for all pairs $(n,\mathbf{a})$. □

Proof of Theorem 4. For each $d\in\mathbb{Z}^+$, denote by $D_{1\hat{1};2}\subset\overline{\mathcal{M}}_{0,2|d}$ the divisor whose general element is a two-component rational curve, with one of the components carrying the marked point 1 and the fleck $\hat{1}$ and the other component carrying the marked point 2. The second component must then carry at least one of the remaining flecks. The irreducible components $D_{1\hat{1};2I}$ of $D_{1\hat{1};2}$ thus correspond to the nonempty subsets I of $\{2,\ldots,d\}$ indexing the flecks on the second component. There is a natural isomorphism

$$D_{1\hat{1};2I}\approx\overline{\mathcal{M}}_{0,2|(d-|I|)}\times\overline{\mathcal{M}}_{0,2||I|}. \tag{8.12}$$

If π_1,π_2 are the two component projection maps, then

$$\psi_i|_{D_{1\hat{1};2I}}=\pi_i^*\psi_i,\quad i=1,2,$$
$$\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\beta))|_{D_{1\hat{1};2I}}=\pi_1^*\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d-|I|)}(\beta))\cdot\pi_2^*\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(|I|)}(\beta)). \tag{8.13}$$

On the other hand, by the first identity in (2.11) and induction on d,

$$\psi_2 = D_{1\hat{1};2} \in H^2(\overline{\mathcal{M}}_{0,2|d}). \tag{8.14}$$

By (8.12)–(8.14),

$$\begin{aligned}
&\int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^{b_1}\psi_2^{b_2}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))} \\
&\quad = \int_{D_{1\hat{1};2}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^{b_1}\psi_2^{b_2-1}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))} \\
&\quad = \sum_{\substack{d_1,d_2\geq 1\\ d_1+d_2=d}} \binom{d-1}{d_1-1}\bigg(\int_{\overline{\mathcal{M}}_{0,2|d_1}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d_1)}(\alpha_i))\psi_1^{b_1}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d_1)}(\alpha_i-\alpha_k))}\bigg) \\
&\qquad \times \bigg(\int_{\overline{\mathcal{M}}_{0,2|d_2}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d_2)}(\alpha_i))\psi_2^{b_2-1}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d_2)}(\alpha_i-\alpha_k))}\bigg)
\end{aligned} \tag{8.15}$$

whenever $b_2 \in \mathbb{Z}^+$.

For any $b_1, b_2 \in \mathbb{Z}^{\geq 0}$, let

$$\mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2)}(\alpha_i, q) = \sum_{d=1}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,2|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^{b_1}\psi_2^{b_2}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))} \in q\mathbb{Q}_\alpha[[q]].$$

By (8.15),

$$D\mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2)}(\alpha_i, q) = D\mathcal{F}_{n;\mathbf{a}}^{(b_1,0)}(\alpha_i, q) \cdot \mathcal{F}_{n;\mathbf{a}}^{(0,b_2-1)}(\alpha_i, q) \quad \forall b_2 \in \mathbb{Z}^+, \tag{8.16}$$

where $D\mathcal{F} \equiv q\,\frac{\mathrm{d}}{\mathrm{d}q}\mathcal{F}$. By induction on b_2 this gives

$$\mathcal{F}_{n;\mathbf{a}}^{(0,b_2)}(\alpha_i, q) = \frac{1}{(b_2+1)!}\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q)^{b_2+1}.$$

Combining this with (8.16) and using symmetry, we obtain

$$\begin{aligned}
&D\mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2)}(\alpha_i, q) = \frac{1}{b_1!}\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q)^{b_1} D\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q) \cdot \frac{1}{b_2!}\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q)^{b_2} \\
&\quad \Longrightarrow \quad \mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2)}(\alpha_i, q) = \frac{1}{(b_1+b_2+1)!}\binom{b_1+b_2}{b_1}\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q)^{b_1+b_2+1}.
\end{aligned} \tag{8.17}$$

Thus, the $r=0$ case of (8.3) is equivalent to

$$\mathop{\mathfrak{R}}_{\hbar=0}\{\mathrm{e}^{-\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i,q)/\hbar}\mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\} = 0. \tag{8.18}$$

By [24, Section 2.1], this relation determines $\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q) \in q\mathbb{Q}_\alpha[[q]]$ uniquely. Thus, by [26, Remark 4.5], $\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i, q) = \xi_{n;\mathbf{a}}(\alpha_i; q)$.[8] It follows that (8.17) is equivalent to the identity in Theorem 4. □

[8] Only the case $\ell^-(\mathbf{a}) = 0$ is considered in [26], but the same reasoning applies in all cases.

REMARK 8.4. By (8.17), for any $r^* \in \mathbb{Z}^{\geq 0}$, the set of equations (8.3) with $r = 0, 1, \dots, r^*$ is an invertible linear combination of the set of relations

$$\underset{\hbar=0}{\mathfrak{R}} \{\hbar^r e^{-\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i,q)/\hbar} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q)\} = 0, \quad r = 0, 1, \dots, r^*.$$

Thus, by (8.17), the statement of Proposition 8.3 is equivalent to the condition that the coefficients of the power series

$$e^{-\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i,q)/\hbar} \mathcal{Y}_{n;\mathbf{a}}(\alpha_i,\hbar,q) \in \mathbb{Q}_\alpha(\hbar)[[q]]$$

are regular at $\hbar = 0$. This is indeed the case for $\mathcal{F}_{n;\mathbf{a}}^{(0,0)}(\alpha_i,q) = \xi_{n;\mathbf{a}}(\alpha_i;q)$ by [26, Remark 4.5].

REMARK 8.5. The above approach can be used to eliminate ψ-classes from twisted integrals over $\overline{\mathcal{M}}_{0,m|d}$ with $m \geq 3$. For example, let

$$\mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2,b_3)}(\alpha_i,q) = \sum_{d=0}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,3|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))\psi_1^{b_1}\psi_2^{b_2}\psi_3^{b_3}}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))}.$$

Using $\psi_3 = D_{12;3}$ on $\overline{\mathcal{M}}_{0,3|d}$, we find that

$$\begin{aligned} &\mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2,b_3)}(\alpha_i,q) = \mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2,0)}(\alpha_i,q)\cdot\mathcal{F}_{n;\mathbf{a}}^{(0,b_3-1)}(\alpha_i,q) \quad \forall b_3 \in \mathbb{Z}^+ \\ &\implies \quad \mathcal{F}_{n;\mathbf{a}}^{(b_1,b_2,b_3)}(\alpha_i,q) = \frac{\xi_{n;\mathbf{a}}(\alpha_i,q)^{b_1+b_2+b_3}}{b_1!b_2!b_3!}\mathcal{F}_{n;\mathbf{a}}^{(0,0,0)}(\alpha_i,q). \end{aligned}$$

Multiplying the last equation by $\hbar_1^{-b_1-1}\hbar_2^{-b_2-1}\hbar_3^{-b_3-1}$ and summing over $b_1, b_2, b_3 \geq 0$, we obtain

$$\begin{aligned} &\sum_{d=0}^{\infty} \frac{q^d}{d!} \int_{\overline{\mathcal{M}}_{0,3|d}} \frac{\mathbf{e}(\dot{\mathcal{V}}_{\mathbf{a}}^{(d)}(\alpha_i))}{\prod_{k\neq i}\mathbf{e}(\dot{\mathcal{V}}_1^{(d)}(\alpha_i-\alpha_k))(\hbar_1-\psi_1)(\hbar_2-\psi_2)(\hbar_3-\psi_3)} \\ &\quad = \frac{1}{\hbar_1\hbar_2\hbar_3} e^{\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_1+\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_2+\xi_{n;\mathbf{a}}(\alpha_i,q)/\hbar_3}\mathcal{F}_{n;\mathbf{a}}^{(0,0,0)}(\alpha_i,q) \\ &\quad \in \mathbb{Q}_\alpha[[\hbar_1^{-1},\hbar_2^{-1},\hbar_3^{-1},q]]. \end{aligned}$$

The power series $\mathcal{F}_{n;\mathbf{a}}^{(0,0,0)}$ is described in [27, Section 3].

In the case of products of projective spaces and concavex sheaves (1.14), α_i and q in (8.2) and (8.3) are replaced by $(\alpha_{i_1},\dots,\alpha_{i_p})$ with $i_s \in [n_s]$ and $(q_1,\dots,q_p)$ with the right-hand sides modified as in (6.20). In the proof of Proposition 8.3, we then obtain relations between elementary symmetric polynomials in

$$\{\alpha_{1;1},\dots,\alpha_{1;n_1}\}, \quad \dots, \quad \{\alpha_{p;1},\dots,\alpha_{p;n_p}\}$$

that depend on $\mathbf{a}$, but not on $n_1,\dots,n_p$. They again hold if $|a_{1;s}|+\dots+|a_{l;s}| \leq n_s - 2$ for all $s \in [p]$ and thus in all cases.

ACKNOWLEDGMENTS. We would like to thank the referee for the detailed comments and suggestions that helped improve the exposition, R. Pandharipande for bringing moduli spaces of stable quotients and the problems addressed in this paper to our attention, as well as for helpful comments, A. Popa for suggesting precise references for mirror theorems in Gromov–Witten theory and pointing out typos in the original version of this paper, and M. Alim, I. Ciocan-Fontanine, M. Gross, J. Kollár, R. Plesser, and E. Witten for useful discussions. The second author would also like to thank the School of Mathematics at IAS for its hospitality during the period when the results in this paper were obtained and the paper itself was completed.

References

[1] M. Atiyah and R. Bott, *The moment map and equivariant cohomology,* Topology 23 (1984), 1–28.

[2] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory,* Nucl. Phys. B 359 (1991), 21–74.

[3] I. Ciocan-Fontanine and B. Kim, *Moduli stacks of stable toric quasimaps,* Adv. Math. 225 (2010), no. 6, 3022–3051.

[4] I. Ciocan-Fontanine, B. Kim, and D. Maulik, *Stable quasimaps to GIT quotients,* arXiv:1106.3724.

[5] Y. Cooper, *The geometry of stable quotients in genus one,* Math. Ann. (to appear), arXiv:1109.0331.

[6] A. Elezi, *Mirror symmetry for concavex vector bundles on projective spaces,* Int. J. Math. Math. Sci. 3 (2003), 159–197.

[7] A. Givental, *Equivariant Gromov–Witten invariants,* Int. Math. Res. Not. 13 (1996), 613–663.

[8] ———, *The mirror formula for quintic threefolds,* Amer. Math. Soc. Transl. Ser. 2 (1999), no. 196, 49–62.

[9] T. Graber and R. Pandharipande, *Localization of virtual classes,* Invent. Math. 135 (1999), no. 2, 487–518.

[10] B. Hassett, *Moduli spaces of weighted pointed stable curves,* Adv. Math. 173 (2003), 316–352.

[11] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry,* Clay Math. Inst., Cambridge, MA, AMS, Providence, RI, 2003.

[12] C. Krattenthaler and T. Rivoal, *On the integrality of the Taylor coefficients of mirror maps,* Duke Math. J. 151 (2010), 175–218.

[13] B. Lian, K. Liu, and S. T. Yau, *Mirror principle I,* Asian J. Math. 1 (1997), no. 4, 729–763.

[14] ———, *Mirror principle III,* Asian J. Math. 3 (1999), no. 4, 771–800.

[15] B. Lian and S.-T. Yau, *Integrality of certain exponential series,* Algebra and geometry, Taipei, 1995, Lect. Algebra Geom., 2, pp. 215–227, Int. Press, Cambridge, MA, 1998.

[16] A. Marian and D. Oprea, *Virtual intersections on the Quot scheme and Vafa–Intriligator formulas,* Duke Math. J. 136 (2007), no. 1, 81–113.

[17] A. Marian, D. Oprea, and R. Pandharipande, *The moduli space of stable quotients,* Geom. Topol. 15 (2011), no. 3, 1651–1706.

[18] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology,* AMS, Providence, RI, 2004.
[19] D. Morrison and R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties,* Nucl. Phys. B 440 (1995), no. 1–2, 279–354.
[20] A. Popa, *Two-point Gromov–Witten formulas for symplectic toric manifolds,* arXiv:1206.2703.
[21] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology,* JDG 42 (1995), no. 2, 259–367.
[22] E. Sernesi, *Deformations of algebraic schemes,* Springer, Berlin, 2006.
[23] E. Witten, *Phases of $N = 2$ theories in two dimensions,* Nucl. Phys. B 403 (1993), no. 1–2, 159–222.
[24] A. Zinger, *The reduced genus 1 Gromov–Witten invariants of Calabi–Yau hypersurfaces,* J. Am. Math. Soc. 22 (2009), no. 3, 691–737.
[25] ———, *Genus zero two-point hyperplane integrals in Gromov–Witten theory,* Commun. Anal. Geom. 17 (2010), no. 5, 1–45.
[26] ———, *The genus 0 Gromov–Witten invariants of projective complete intersections,* Geom. Top (to appear), arXiv:1106.1633.
[27] ———, *Double and triple Givental's J-function for stable quotients invariants,* arXiv:1305.2142.

Y. Cooper
Department of Mathematics
Princeton University
Princeton, NJ 08540
USA

yaim@math.princeton.edu

A. Zinger
Department of Mathematics
SUNY Stony Brook
NY 11794-3651
USA

azinger@math.sunysb.edu