# Mirror symmetry, Langlands duality and the Hitchin system

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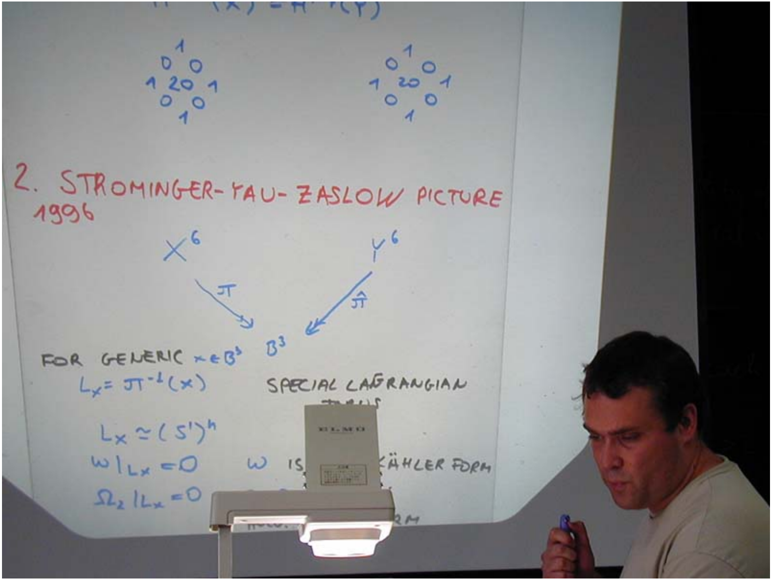

# Mirror Symmetry

- phenomenon first arose in various forms in string theory
- mathematical predictions (Candelas-de la Ossa-Green-Parkes 1991)
- mathematically it relates the symplectic geometry of a Calabi-Yau manifold $X^d$ to the complex geometry of its mirror Calabi-Yau $Y^d$
- first aspect is the *topological mirror test* $h^{p,q}(X) = h^{d-p,q}(Y)$
- compact hyperkähler manifolds satisfy $h^{p,q}(X) = h^{d-p,q}(X)$
- (Kontsevich 1994) suggests *homological mirror symmetry* $\mathcal{D}^b(Fuk(X,\omega)) \cong \mathcal{D}^b(Coh(Y,I))$
- (Strominger-Yau-Zaslow 1996) suggests a geometrical construction how to obtain $Y$ from $X$
- many predictions of mirror symmetry have been confirmed - no general understanding yet

# Hodge diamonds of mirror Calabi-Yaus

Fermat quintic $X$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\hat{X} := X/(\mathbb{Z}_5)^3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

K3 surface $X$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\hat{X}$ mirror K3

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

# Langlands duality

- the Langlands program aims to describe $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via representation theory
- $\mathrm{G}$ reductive group, ${}^L\mathrm{G}$ its Langlands dual
- e.g ${}^L\mathrm{GL}_n = \mathrm{GL}_n$; ${}^L\mathrm{SL}_n = \mathrm{PGL}_n$, ${}^L\mathrm{PGL}_n = \mathrm{SL}_n$
- [Langlands 1967] conjectures that
  $\{\text{homs } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{G}(\mathbb{C})\} \leftrightarrow \{\text{automorphic reps of } {}^L\mathrm{G}(\mathcal{A}_{\mathbb{Q}})\}$
- $\mathrm{G} = \mathrm{GL}_1 \rightsquigarrow$ class field theory
  $\mathrm{G} = \mathrm{GL}_2 \rightsquigarrow$ Shimura-Taniyama-Weil
- function field version: replace $\mathbb{Q}$ with $\mathbb{F}_q(X)$, where $X/\mathbb{F}_q$ is algebraic curve
- [Ngô, 2008] proves fundamental lemma for $\mathbb{F}_q(X) \rightsquigarrow$ FL for $\mathbb{Q}$
- geometric version: replace $\mathbb{F}_q(X)$ with $\mathbb{C}(X)$ for $X/\mathbb{C}$
- [Laumon 1987, Beilinson–Drinfeld 1995]
  Geometric Langlands conjecture
  $\{\mathrm{G}\text{-local systems on } X\} \leftrightarrow \{\text{Hecke eigensheaves on } \mathrm{Bun}_{{}^L\mathrm{G}}(X)\}$
- [Kapustin–Witten 2006] deduces this from reduction of S-duality (electro-magnetic duality) in $N = 4$ SUSY YM in $4d$

# Hitchin system

- Hamiltonian system: $(X^{2d}, \omega)$ symplectic manifold
  $H : X \to \mathbb{R}$ Hamiltonian function $X_H$ Hamiltonian vector field
  $(dH = \omega(X_H, .))$
- $f : X \to \mathbb{R}$ is a *first integral* if $X_H f = \omega(X_f, X_H) = 0$
- the Hamiltonian system is *completely integrable* if there is
  $f = (H = f_1, \ldots, f_d) : X \to \mathbb{R}^d$ generic such that
  $\omega(X_{f_i}, X_{f_j}) = 0$
- the generic fibre of $f$ has an action of $\mathbb{R}^d = \langle X_{f_1}, \ldots, X_{f_d} \rangle \rightsquigarrow$
  when $f$ is proper generic fibre is a torus $(S^1)^d$
- examples include: Euler and Kovalevskaya tops and the spherical pendulum
- algebraic version when replacing $\mathbb{R}$ by $\mathbb{C} \rightsquigarrow$ many examples can be formulated as a version of the *Hitchin system*
- a Hitchin system is associated to a complex curve $C$ and a complex reductive group $G$
- it arose in the study [Hitchin 1987] of the 2-dimensional reduction of the Yang-Mills equations

# Mirror symmetry for Langlands dual Hitchin systems

- The mirror symmetry proposal of [Hausel–Thaddeus 2003]: "Hitchin systems for Langlands dual groups satisfy Strominger-Yau-Zaslow, so could be considered mirror symmetric; in particular they should satisfy the *topological mirror tests*:"
- the Hitchin systems for $\mathrm{SL}_n$ and $\mathrm{PGL}_n$ become dual special Lagrangian fibrations $\Leftrightarrow$ SYZ

$$\begin{tikzcd}[column sep=small] \mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n) \arrow[dr, "\check{\chi}"'] & & \mathcal{M}^e_{\mathrm{DR}}(\mathrm{PGL}_n) \arrow[dl, "\hat{\chi}"] \\ & \mathcal{A}^0 & \end{tikzcd}$$

**Theorem (Hausel–Thaddeus 2003, "Topological mirror test")**

*$n = 2, 3$; $d, e \in \mathbb{Z}$, s.t. $(d, n) = (e, n) = 1$, we have agreement of certain Hodge numbers of $\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n)$ and $\mathcal{M}^e_{\mathrm{DR}}(\mathrm{PGL}_n)$*

$$E\left(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n); x, y\right) = E^{\hat{B}^d}_{\mathrm{st}}\left(\mathcal{M}^e_{\mathrm{DR}}(\mathrm{PGL}_n); x, y\right).$$

# Diffeomorphic spaces in non-Abelian Hodge theory

- $C$ genus g curve; $\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$

$$\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{G}) := \left\{ \begin{array}{c} \text{moduli space of stable rank } n \\ \text{degree } d \ \mathrm{G}\text{-Higgs bundles } (E,\phi) \\ \text{i.e. } E \text{ rank } n \text{ degree d bundle on } C \\ \phi \in H^0(C, ad(E)\otimes K) \text{ Higgs field} \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{DR}}(\mathrm{G}) := \left\{ \begin{array}{c} \text{moduli space of flat } \mathrm{G}\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\}$$

$$\mathcal{M}^d_{\mathrm{B}}(\mathrm{G}) := \{A_1, B_1, .., A_g, B_g \in \mathrm{G} | \prod_{i=1}^{g} A_i^{-1} B_i^{-1} A_i B_i = e^{\frac{2\pi i d}{n}} Id\} /\!\!/ \mathrm{G}$$

- when $(d,n) = 1$ these are smooth non-compact varieties
- $\Gamma = Jac_C[n] \cong \mathbb{Z}_n^{2g}$ acts on $\mathcal{M}^d(\mathrm{SL}_n)$ by tensoring $\Rightarrow$ $\mathcal{M}^d(\mathrm{PGL}_n) := \mathcal{M}^d(\mathrm{SL}_n)/\Gamma$ is an orbifold

**Theorem (Non-Abelian Hodge Theorem)**

$$\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{G}) \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{DR}}(\mathrm{G}) \overset{diff}{\cong} \mathcal{M}^d_{\mathrm{B}}(\mathrm{G})$$

# Hitchin map

- the characteristic polynomial of $\phi \in H^0(C, End(E) \otimes K)$
  $\chi(\phi) \in H^0(C, K) \oplus H^0(C, K^2) \oplus \cdots \oplus H^0(C, K^n)$
  defines *Hitchin map*

$$\chi_{\mathrm{GL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{GL}_n) \to \mathcal{A}_{\mathrm{GL}_n} = \oplus_{i=1}^n H^0(C, K^i)$$

$$\chi_{\mathrm{SL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n) \to \mathcal{A}_{\mathrm{SL}_n} = \oplus_{i=2}^n H^0(C, K^i)$$

$$\chi_{\mathrm{PGL}_n} : \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n) \to \mathcal{A}_{\mathrm{PGL}_n} = \oplus_{i=2}^n H^0(C, K^i)$$

**Theorem (Hitchin 1987, Nitsure 1991, Faltings 1993)**

*$\chi$ is* proper *and a completely integrable Hamiltonian system.*
*($\omega(X_{\chi_i}, X_{\chi_j}) = 0$)*
*Over a generic point $a \in \mathcal{A}$ the fibre $\chi^{-1}(a)$ is a torsor for an Abelian variety.*

## Strominger-Yau-Zaslow

**Theorem (Hausel, Thaddeus 2003)**

*For a generic* $a \in \mathcal{A}_{\mathrm{SL}_n} \cong \mathcal{A}_{\mathrm{PGL}_n}$ *the fibres* $\chi^{-1}_{\mathrm{SL}_n}(a)$ *and* $\chi^{-1}_{\mathrm{PGL}_n}(a)$ *are torsors for* dual *Abelian varieties.*

$$\begin{array}{ccc} \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n) & \leftarrow & \mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n) \\ \downarrow \chi_{\mathrm{PGL}_n} & & \downarrow \chi_{\mathrm{SL}_n} \\ \mathcal{A}_{\mathrm{PGL}_n} & \cong & \mathcal{A}_{\mathrm{SL}_n}. \end{array}$$

$\Rightarrow \mathcal{M}^d_{\mathrm{DR}}(\mathrm{PGL}_n)$ *and* $\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n)$ *satisfy the SYZ construction for a pair of mirror symmetric Calabi-Yau manifolds.*

- (Kontsevich 1994)'s homological mirror symmetry proposal $\Rightarrow$ $\mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{SL}_n))) \sim \mathcal{D}^b(Fuk(\mathcal{M}^d_{\mathrm{DR}}(\mathrm{PGL}_n)))$
- $\rightsquigarrow$ Geometric Langlands program of (Beilinson-Drinfeld 1995)
- (Kapustin-Witten 2007) $\Rightarrow$ above from reduction of S-duality (electro-magnetic duality) in $N = 4$ SUSY YM in $4d$
- $\overset{semi-classical}{\rightsquigarrow}$ $\mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{SL}_n))) \sim \mathcal{D}^b(Coh(\mathcal{M}^d_{\mathrm{Dol}}(\mathrm{PGL}_n)))$ $\rightsquigarrow$ fibrewise Fourier-Mukai transform?

# Topological mirror tests

- (Deligne 1971) $\rightsquigarrow$ weight filtration for any complex algebraic variety $X$: $W_0 \subset \cdots \subset W_k \subset \cdots \subset W_{2d} = H_c^d(X;\mathbb{Q})$ , plus a pure Hodge structure on $W_k/W_{k-1}$ of weight $k$
- define $E(X;x,y) = \sum (-1)^d x^i y^j h^{i,j}\left(W_k/W_{k-1}(H_c^d(X,\mathbb{C}))\right)$
  $E_{st}^B(M/\Gamma) = \sum_{[\gamma]\in[\Gamma]} E(M^\gamma; L_\gamma^B)^{C(\gamma)}(uv)^{F(\gamma)}$
- if $Y \to X/\Gamma$ is crepant then (Kontsevich 1996) $\rightsquigarrow$
  $E_{st}(X/\Gamma;x,y) = E(Y;x,y)$

**Conjecture (Hausel–Thaddeus 2003, "DR-TMS", "Dol-TMS")**

*For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$, we have*

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{DR}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{DR}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right)$$

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{Dol}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{Dol}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right)$$

**Conjecture ( Hausel–Villegas 2004,"B-TMS")**

$$E_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{B}}^d(\mathrm{SL}_n(\mathbb{C}));x,y\right) = E_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{B}}^e(\mathrm{PGL}_n(\mathbb{C}));x,y\right)$$

# Results-Problems

- (Hausel–Thaddeus 2003) Dol-TMS ($\Leftrightarrow$ DR-TMS) for $n = 2, 3$ and $(d, n) = 1$ using description of $H^*(\mathcal{M}^1_{\mathrm{Dol}}(\mathrm{SL}_n))$ of (Hitchin 1987) for $n = 2$ and of (Gothen 1994) for $n = 3$
- (Hausel–Villegas $\geq$ 2004, Mereb $\geq$ 2009) B-TMS for $n$ is prime and $n = 4$ using arithmetic techniques and character tables of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{SL}_n(\mathbb{F}_q)$
- Three main problems with this picture
  1. Why two different topolocial mirror symmetry conjectures (Dol-TMS & DR-TMS vs. B-TMS)?
  2. Why the same Hodge numbers, why not mirrored ones?
  3. Why geometric Langlands and not classical Langlands?

# Hard Lefschetz for Weight and Perverse Filtrations

- Weight filtration: $W_0 \subset \cdots \subset W_i \subset \cdots \subset W_{2k} = H^k(X)$
- Alvis-Curtis duality in $R(\mathrm{GL}_n(\mathbb{F}_q))$
  $\rightsquigarrow$ Curious Hard Lefschetz Conjecture (theorem for $\mathrm{PGL}_2$):
  $$L^l : \begin{array}{ccc} Gr^W_{d-2l}(H^{i-l}(\mathcal{M}_{\mathrm{B}})) & \xrightarrow{\cong} & Gr^W_{d+2l}H^{i+l}(\mathcal{M}_{\mathrm{B}}) \\ x & \mapsto & x \cup \alpha^l \end{array},$$
  where $\alpha \in W_4H^2(\mathcal{M}_{\mathrm{B}})$
- Perverse filtration: $P_0 \subset \cdots \subset P_i \subset \dots P_k(X) \cong H^k(X)$
  for $f : X \to Y$ proper $X$ smooth $Y$ affine
  (de Cataldo-Migliorini, 2008):
  take $Y_0 \subset \cdots \subset Y_i \subset \dots Y_d = Y$
  s.t. $Y_i$ generic with $\dim(Y_i) = i$ then
  $$P_{k-i-1}H^k(X) = \mathrm{ker}(\mathrm{H^k(X)} \to \mathrm{H^k(f^{-1}(Y_i))})$$
- the Relative Hard Lefschetz Theorem holds:
  $$L^l : \begin{array}{ccc} Gr^P_{d-l}(H^*(X)) & \xrightarrow{\cong} & Gr^P_{d+l}H^{*+2l}(X) \\ x & \mapsto & x \cup \alpha^l \end{array}$$
  where $\alpha \in H^2(X)$ is a relative ample class

# $P = W$ conjecture

- recall Hitchin map $\begin{array}{cccc} \chi : & \mathcal{M}_{\mathrm{Dol}} & \to & \mathcal{A} \\ & (E, \phi) & \mapsto & \mathrm{charpol}(\phi) \end{array}$ is proper, thus induces perverse filtration on $H^*(\mathcal{M}_{\mathrm{Dol}})$

**Conjecture ("P=W", de Cataldo-Hausel-Migliorini 2008)**

$P_k(\mathcal{M}_{\mathrm{Dol}}) \cong W_{2k}(\mathcal{M}_{\mathrm{B}})$ *under the isomorphism* $H^*(\mathcal{M}_{\mathrm{Dol}}) \cong H^*(\mathcal{M}_{\mathrm{B}})$ *from non-Abelian Hodge theory.*

**Theorem (de Cataldo-Hausel-Migliorini 2009)**

$P = W$ *when* $G = \mathrm{GL}_2, \mathrm{PGL}_2$ *or* $\mathrm{SL}_2$.

- Define $PE(\mathcal{M}_{\mathrm{Dol}}; x, y, q) := \sum q^k E(Gr_k^P(H^*(\mathcal{M}_{\mathrm{Dol}})); x, y)$
- $PE(\mathcal{M}_{\mathrm{Dol}}; x, y, 1) = E(\mathcal{M}_{\mathrm{Dol}}; x, y) = E(\mathcal{M}_{\mathrm{DR}}; x, y)$
- Conjecture $P = W \Rightarrow PE(\mathcal{M}_{\mathrm{Dol}}; 1, 1, q) = E(\mathcal{M}_{\mathrm{B}}; q)$
- RHL $\rightsquigarrow PE(\mathcal{M}_{\mathrm{Dol}}; x, y, q) = (xyq)^d PE(\mathcal{M}_{\mathrm{Dol}}; x, y; \frac{1}{qxy}) \rightsquigarrow$

**Conjecture (Topological Mirror test, TMS)**

$$PE_{\mathrm{st}}^{B^e}\left(\mathcal{M}_{\mathrm{Dol}}^d(\mathrm{SL}_n); x, y, q\right) = (xyq)^d PE_{\mathrm{st}}^{\hat{B}^d}\left(\mathcal{M}_{\mathrm{Dol}}^e(\mathrm{PGL}_n); x, y, \frac{1}{qxy}\right)$$

# Conclusion

- The TMS above unifies the previous Dol,DR,B-TMS conjectures (Theorem when $n = 2$)
- Fibrewise Fourier-Mukai transform aka S-duality should identify
  $S : H_p^{r,s}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{SL}_n)) \cong H_{st,d-p}^{r+d/2-p,s+d/2-p}(\mathcal{M}_{\mathrm{Dol}}(\mathrm{PGL}_n))$
  this solves the mirror problem
  (Theorem over regular locus of $\chi$)
- (Ngô 2008) proves the fundamental lemma in the Langlands program by proving "geometric stabilisation of the trace formula" which for $\mathrm{SL}_n$ and $\mathrm{PGL}_n$ can be reformulated to prove TMS over integral spectral curves, which when $n$ is a prime, can be extended to a proof of TMS everywhere.

# Example

- let $\check{\mathcal{M}}$ be moduli space of $\mathrm{SL}_2$ parabolic Higgs bundles on elliptic curve $E$ with one parabolic point
- $\mathbb{Z}_2$ acts on $E$ and $\mathbb{C}$ as additive inverse $x \mapsto -x$
- $\check{\mathcal{M}} \to E \times \mathbb{C}/\mathbb{Z}_2$ blowing up; $\chi : \check{\mathcal{M}} \to \mathbb{C}/\mathbb{Z}_2 \cong \mathbb{C}$ is elliptic fibration with $\hat{D}_4$ singular fiber over 0
- $\Gamma = E[2] \cong \mathbb{Z}_2^2$ acts on $\check{\mathcal{M}}$ by multiplying on $E$
- $\hat{\mathcal{M}}$ the $\mathrm{PGL}_2$ moduli space is $\check{\mathcal{M}}/\Gamma$ an orbifold, elliptic fibration over $\mathbb{C}$ with $A_1$ singular fiber with three $\mathbb{C}^2/\mathbb{Z}_2$-orbifold points on one of the components
- blowing up the three orbifold singularities is crepant gives $\check{\mathcal{M}}$
- the topological mirror test: $E_{st}(\hat{\mathcal{M}}; x, y) \overset{Kontsevich}{=} E(\check{\mathcal{M}}; x, y)$
- $P_1(H^2(\check{\mathcal{M}})) = \ker(H^2(\check{\mathcal{M}}) \to H^2(\chi^{-1}(pt)) \cong \mathrm{im}(H^2_{cpt}(\check{\mathcal{M}}) \to H^2(\check{\mathcal{M}}))$ has dimension 4
- $E(\check{\mathcal{M}}; x, y) = 1 + 5xy$ non-symmetric but $PE(\check{\mathcal{M}}; q, x, y) = 1 + 4xyq + xyq^2$ symmetric by RHL $PE(\check{\mathcal{M}}; x, y, q) = (xyq)^2 PE(\check{\mathcal{M}}; x, y; \frac{1}{qxy})$