MIXED BOUNDARY VALUE PROBLEMS IN MECHANICS

by

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1. INTRODUCTION

In attempting to formulate a given "equilibrium-type" of continuous system in mechanics one may either use some kind of a variational principle and reduce the problem to a minimization problem subject to certain constraints, or, as is more commonly the case, one may directly apply the equilibrium principles and reduce the problem to a boundary value problem which consists of a (system of) differential equation(s) subject to certain boundary conditions. Even though in most practical applications the minimization problem is further reduced to a boundary value problem, it may also be solved directly by using an approximate technique such as the Ritz's method. To facilitate the definition of certain concepts consider the following boundary value problem in two dimensions:

$$L_{2m}(u) = f(x_1,x_2) \quad , \qquad (x_1, x_2)\varepsilon D \quad , \tag{1.1}$$

$$B_i(u) = g_i(s) \quad , \qquad (i = 1,..,m) \quad , \quad s\varepsilon S \tag{1.2}$$

where L_{2m} is a differential operator of order 2m, x_1,x_2 are the spatial coordinates, D is the domain of definition for the unknown function u, S is the boundary of D, B_i (i = 1,..,m) is a differential operator (containing u and its normal derivatives) of order (at most) $2m-1$, f and g are known functions, and s is any convenient coordinate defining the point on the boundary (say, the arc length). The domain D may contain the point at infinity and may be multiplyconnected. Contours forming the boundary are assumed to consist of piecewise smooth arcs. The points on S at which the tangent has a discontinuous slope will be called the <u>points of geometric singularity</u>.

There is another type of singular point on the boundary which results from the change in the nature of the homogeneous operators $B_i(u)$ specifying the boundary conditions. Such a point on a smooth boundary either side of which at least one of the operators B_i $(i = 1,..,m)$ has different behavior is called a point of flux singularity. Note that if the behavior of an operator B_i changes at a "corner point" of the boundary, this point is then a point of both a geometric as well as a flux singularity. If the homogeneous operators B_i remain unchanged on each closed contour (but not necessarily the same on all contours), then the corresponding problem is called an ordinary boundary value problem. On the other hand if there are points of flux singularity on the boundary, the problem is called a mixed boundary value problem.

In working with mixed boundary value problems it is often advantageous to keep in mind that the physical system has generally two types of quantities, namely, the potential and the flux type quantities. In potential theory the meaning of these concepts is unambigious and clear. They are, for example, identified by temperature, velocity potential, electrostatic potential, mass concentration, or displacement (in anti-plane shear problems) as the potential type quantities, and heat flux, velocity, electrostatic charge, mass rate of diffusion, or stress, as the corresponding flux type quantities. Similarly, in solid mechanics one may classify the displacements and the stresses (or the strains) as respectively the potential and flux type quantities. The physics of the problem requires that the "potential" be bounded and continuous everywhere in D+S, including the points of both geometric and flux singularity.

To fix the ideas, consider the following simple problem in a wedge-shaped domain:

$$\nabla^2 u(r,\theta) = 0 \quad , \qquad 0<r<\infty \quad , \qquad 0<\theta<\theta_o \quad , \tag{1.3}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta} u(r,0) = f_1(r) \quad , \qquad \frac{1}{r}\frac{\partial}{\partial\theta} u(r,\theta_o) = f_2(r) \quad , \quad 0<r<\infty \tag{1.4}$$

where the known functions f_1 and f_2 are such that the global equilibrium is satisfied and $0<\theta_o\leq 2\pi$. The problem is an ordinary boundary value problem and $r=0$ is a point of geometric singularity. If in the neighborhood of $r=0$ f_1 and f_2 are zero, simple application of Mellin transform would indicate that for small values of r the components of the flux vector are of the following form:

$$\frac{1}{r}\frac{\partial u}{\partial\theta} = Kr^{(\pi-\theta_o)/\theta_o} \sin\frac{\pi\theta}{\theta_o} + 0(r^{(2\pi-\theta_o)/\theta_o}) \quad ,$$

$$\frac{\partial u}{\partial r} = -Kr^{(\pi-\theta_o)/\theta_o} \cos\frac{\pi\theta}{\theta_o} + 0(r^{(2\pi-\theta_o)/\theta_o}) \quad , \tag{1.5}$$

where K is a constant. Note that at $r=0$ the flux becomes unbounded for $\pi<\theta_o\leq 2\pi$, the corresponding power of singularity being $0>\frac{\pi-\theta_o}{\theta_o}\geq -0.5$.

If the boundary conditions (1.4) are replaced by

$$u(r,0) = 0 \quad , \qquad \frac{1}{r}\frac{\partial}{\partial\theta} u(r,\theta_o) = f(r) \quad , \qquad 0<r<\infty \tag{1.6}$$

it is seen that the problem is a mixed boundary value problem in which $r=0$ is a point of both geometric and flux singularity. Hence, in this problem one would expect a stronger flux singularity than in the previous problem. The asymptotic solution of the problem for small r may again be expressed as

$$u(r,\theta) = Kr^{\pi/2\theta_o} \sin\frac{\pi\theta}{2\theta_o} + 0(r^{3\pi/2\theta_o}) \quad ,$$

$$\frac{1}{r}\frac{\partial u}{\partial\theta} = K\frac{\pi}{2\theta_o} r^{(\pi-2\theta_o)/2\theta_o} \cos\frac{\pi\theta}{2\theta_o} + 0(r^{(3\pi-2\theta_o)/2\theta_o}) \quad ,$$

$$\frac{\partial u}{\partial r} = K\,\frac{\pi}{2\theta_o}\, r^{(\pi-2\theta_o)/2\theta_o}\,\sin\frac{\pi\theta}{2\theta_o} + O(r^{(3\pi-2\theta_o)/2\theta_o}) \quad . \tag{1.7}$$

From (1.7) it is seen that for $\pi/2 \leq \theta_o \leq 2\pi$ the corresponding power of the flux singularity is $0 \geq \frac{\pi-2\theta_o}{2\theta_o} \geq -3/4$, which is stronger than that found in the previous problem for the same angle θ_o.

Let us now consider a special case of this problem in which $\theta_o = \pi$. Here the boundary is the infinite line; consequently the geometric singularity is removed but $r = 0$ remains to be a point of flux singularity having a power $-1/2$. As will be seen later in this article, $-1/2$ power singularity is quite typical for the points of flux singularity on a smooth boundary. It will also be seen that however, somewhat contrary to the general expectation, this is not always the case in mixed boundary value problems with boundary conditions containing the potential and the flux.

Finally, consider the following (mixed) boundary conditions

$$u(r,0) = 0 \quad , \qquad a<r<b \quad ,$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}\, u(r,0) = g(r) \quad , \qquad 0<r<a \quad , \qquad b<r<\infty \quad ,$$

$$\frac{1}{r}\frac{\partial}{\partial r}\, u(r,\theta_o) = f(r) \quad , \qquad 0<r<\infty \quad . \tag{1.8}$$

where f and g are known functions. It is seen that $r = 0$ is a point of geometric singularity and the points $r = a$ and $r = b$ on the smooth boundary ($\theta = 0$, $0<r<\infty$) are points of flux singularity. The problem is a mixed boundary value problem. Around the geometric singularity $r = 0$ the solution is expected to behave as in (1.5) and around the points of flux singularity it will have a behavior as in (1.7) with $\theta_o = \pi$ and necessary coordinate transformations. It should again be emphasized that the standard $-1/2$ power of the singularity at $r = a > 0$ will be a function of θ_o for $a = 0$ which, depending on the value of θ_o, may be stronger or weaker than $-1/2$. A similar phenomenon will be discussed in connection with a contact problem in elasticity later in this paper.

2. DEFINITIONS: MULTIPLE SERIES EQUATIONS, MULTIPLE INTEGRAL EQUATIONS

In considering the solution of a given mixed boundary value problem perhaps the simplest technique is the direct application of the method of complex potentials provided the problem admits such potentials and the domain and the boundary conditions are suitable for such an application. In this case the problem is reduced to a Riemann-Hilbert problem for a (system of) sectionally holomorphic function(s) which may be solved in a straightforward manner. On the other hand if one applied a more standard technique such as, the separation of variables, integral transforms, or the method of Green's function, the mixed boundary conditions invariably lead to a formulation involving "dual series equations", "dual integral equations" or "singular integral equations". Again, to facilitate the basic understanding of these notions, the definitions will be preceded by the formulation of some simple examples.

2.1 Multiple Series Equations.

As a first example consider the following mixed boundary value problem in potential theory for the unit circle:

$$\nabla^2 u(r,\theta) = 0 \quad , \qquad 0 \le r < 1 \quad , \qquad 0 \le \theta < 2\pi \quad , \tag{2.1}$$

$$\frac{\partial}{\partial r} u(1,\theta) = g(\theta) \quad , \qquad \theta \varepsilon L_1 \quad , \tag{2.2}$$

$$u(1,\theta) = f(\theta) \quad , \qquad \theta \varepsilon L_2 \quad , \tag{2.3}$$

where

$$L_1 = \sum_1^N L_{1i} \quad , \qquad L_{1i} = (r = 1;\ a_i < \theta < b_i) \quad , \tag{2.4}$$

and L_2 is the complement of L_1 on the unit circle. Using the technique of the separation of variables the solution of (2.1) may be expressed as

$$u(r, \ ) = A_o + \sum_1^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \quad . \tag{2.5}$$

Formally, substituting from (2.5) into the boundary conditions one obtains the following system of equations to determine the coefficients A_n and B_n:

$$\sum_1^{\infty} n(A_n \cos n\theta + B_n \sin n\theta) = g(\theta) \quad , \qquad \theta \varepsilon L_1 \quad ,$$

$$A_o + \sum_1^{\infty} (A_n \cos n\theta + B_n \sin n\theta) = f(\theta) \quad , \qquad \theta \varepsilon L_2 \quad . \tag{2.6a,b}$$

In the case of Dirichlet ($L_1 \equiv 0$) or Neumann ($L_2 \equiv 0$) problems (2.6b) or (2.6a) give the unknown coefficients directly by expanding $f(\theta)$ or $g(\theta)$ into Fourier series in $(0,2\pi)$. However, in the present problem the functions $\sin n\theta$ and $\cos n\theta$ $(n = 0,1,2,\ldots)$ are not orthogonal on L_1 and L_2 and hence, (2.6) is at best equivalent to (or can be transformed into) an infinite system of algebraic equations.

The structure of (2.6) is typical of the mixed boundary value problems defined in a bounded domain $a<x<b$ which may be expressed as

$$\sum_1^{\infty} A_n k_1(n,x) = f_1(x) \quad , \qquad x \varepsilon L_1 \quad ,$$

$$\sum_1^{\infty} A_n k_2(n,x) = f_2(x) \quad , \qquad x \varepsilon L_2 \quad , \tag{2.7}$$

or, more generally, for J sets of coefficients A_{jn}, $(j = 1,..,J;\ n = 1,2,\ldots)$ one obtains

$$\sum_{j=1}^{J} \sum_{n=1}^{\infty} k_{1j}^{i}(n,x)A_{jn} = f_1^i(x) \quad , \qquad x\varepsilon L_1 \quad , \qquad (i=1,..,J) \quad ,$$

$$\sum_{j=1}^{J} \sum_{n=1}^{\infty} k_{2j}^{i}(n,x)A_{jn} = f_2^i(x) \quad , \qquad x\varepsilon L_2 \quad , \qquad (i=1,..,J) \quad , \tag{2.8}$$

where again

$$L_1 = \sum_{1}^{N} L_{1k} \quad , \qquad L_{1k} = (a_k,b_k) \quad , \qquad L_1 + L_2 = (a,b) \quad ,$$

$$a_k < b_k < a_{k+1} \quad , \qquad a \leq a_1 \quad , \qquad b_N \leq b \quad , \tag{2.9}$$

and the input functions f_r or f_r^i and the kernel functions k_r or k_{rj}^i are known. The system of equations such as (2.6), (2.7) or (2.8) are defined as dual series equations.

Going back to the problem for the unit circle (2.1), if one assumes that the boundary $L = (0,2\pi)$ is divided into three parts with the following boundary conditions[(1)]

$$\frac{\partial}{\partial r} u(1,\theta) = f_1(\theta) \quad , \qquad \theta\varepsilon L_1 \quad ,$$

$$u(1,\theta) = f_2(\theta) \quad , \qquad \theta\varepsilon L_2 \quad ,$$

$$h_1 u(1,\theta) + h_2 \frac{\partial}{\partial r} u(1,\theta) = f_3(\theta) \quad , \qquad \theta\varepsilon L_3 \quad , \tag{2.10a-c}$$

where L_r, $(r=1,2,3)$ is the union of nonintersecting arcs L_{rj}, $(r=1,2,3;$ $j=1,2,..,J_r)$ on the unit circle with $L_1 + L_2 + L_3 = L = (0,2\pi)$. h_1 and h_2 may be functions of θ. Again, formally from (2.1) and (2.10) it follows that

$$\sum_{1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta) = f_1(\theta) \quad , \qquad \theta\varepsilon L_1 \quad ,$$

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(1) This is a problem, for example, in heat conduction in which, in addition to specifying the heat flux and temperature on parts of the boundary L_1 and L_2, there is free convection taking place along L_3 where h_1 is the coefficient of heat convection, h_2 is the coefficient of heat conduction, and f_3 is related to the environmental temperature u_∞ through $h_1 u_\infty = f_3$.

$$A_o + \sum_1^\infty (A_n \cos n\theta + B_n \sin n\theta) = f_2(\theta) \quad , \qquad \theta \epsilon L_2 \quad ,$$

$$h_1 A_o + h_1 \sum_1^\infty (A_n \cos n\theta + B_n \sin n\theta)$$
$$+ h_2 \sum_1^\infty n(A_n \cos n\theta + B_n \sin n\theta) = f_3(\theta) \quad , \qquad \theta \epsilon L_3 \quad . \qquad (2.11a\text{-}c)$$

In this problem as seen from (2.10) the boundary conditions are defined in terms of three distinct operators on three separate parts of the boundary giving rise to a set of three series equations described by (2.11). Thus, these equations will be called <u>triple series equations</u>. In general, then a set of series equations for a system of unknown coefficients A_1, A_2,... of the form

$$\sum_1^\infty A_n k_i(n,x) = f_i(x) \quad , \qquad x \epsilon L_i \quad , \qquad \sum_1^M L_i = L \quad ,$$
$$(i = 1,..,M) \quad , \qquad (2.12)$$

will be called <u>multiple series equations</u> (of multiplicity M)[(1)].

In the mixed boundary value problems described by (2.6) and (2.11) the intersections of the boundary segments L_{ij} are points of "flux singularity". The discussion given in the previous section for the wedge would indicate that at least at some of these points the flux will be unbounded, at others the behavior of the solution is not known beforehand. Hence, the technique developed to solve the multiple series equations will not only have to be sufficiently general to apply to a great diversity of problems but will also have to lend itself to the correct treatment of the singular nature

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[(1)] Note that this definition is different than that found in current literature (e.g., [1]) on "dual series" and "dual integral" equations where the multiplicity of the equations is taken to be the number of independent segments on the boundary rather than the number of independent operators defining the boundary conditions.

of the solution. Aside from the method of complex potentials whenever applicable, it appears that the method of singular integral equations is the only approach which fulfills these requirements. Because of the singularities, since the infinite series giving the components of the flux vector will be divergent at certain points on the boundary, it is clear that any direct method reducing the multiple series equations to an infinite system of algebraic equations in the unknown coefficients (which has no closed form solution) will not be acceptable.

2.2 Multiple Integral Equations

Consider now the equivalent problem in potential theory for the half plane $y>0$. The problem is stated as follows:

$$\nabla^2 u(x,y) = 0 \quad , \qquad (-\infty<x<\infty \quad , \quad 0<y<\infty) \quad , \tag{2.13}$$

$$\frac{\partial}{\partial y} u(x,0) = f_1(x) \quad , \qquad x\varepsilon L_1 \quad , \tag{2.14}$$

$$u(x,0) = f_2(x) \quad , \qquad x\varepsilon L_2 \quad , \tag{2.15}$$

$$L_1 = \sum_1^N L_{1i} \quad , \qquad L_{1i} = (a_i,b_i) \quad , \qquad a_i<b_i<a_{i+1} \quad ,$$

$$-\infty\le a_1 \quad , \qquad b_N\le\infty \quad , \tag{2.16}$$

where L_2 is the complement of L_1 on $(-\infty<x<\infty)$, f_1 and f_2 are known functions and are such that $u\to 0$ as $x^2 + y^2\to\infty$ (i.e., any homogeneous solution behaving differently at infinity has been separated). Using Fourier transforms, the solution of (2.13) may be expressed as

$$u(x,y) = \int_{-\infty}^{\infty} A(\alpha)e^{-y|\alpha|-i\alpha x}\,d\alpha \quad , \tag{2.17}$$

where $A(\alpha)$ is an unknown function. Substituting from (2.17) into (2.14) and (2.15) formally we find

$$-\int_{-\infty}^{\infty} |\alpha| A(\alpha) e^{-i\alpha x}\, d\alpha = f_1(x) \quad , \quad x \epsilon L_1 \quad ,$$

$$\int_{-\infty}^{\infty} A(\alpha) e^{-i\alpha x}\, d\alpha = f_2(x) \quad , \quad x \epsilon L_2 \quad . \tag{2.18a,b}$$

If L_1 or L_2 is zero, (2.18) may be solved in closed form in terms of inversion integrals. Integral equations of the form (2.18) or more generally the pair of integral equations

$$\int_L k_1(x,\alpha) A(\alpha) d\alpha = f_1(x) \quad , \quad x \epsilon L_1 \quad ,$$

$$\int_L k_2(x,\alpha) A(\alpha) d\alpha = f_2(x) \quad , \quad x \epsilon L_2 \quad , \quad L_1 + L_2 = L \quad , \tag{2.19}$$

in which the kernels k_1 and k_2 are different, are called a set of dual integral equations for the unknown function $A(\alpha)$. If the problem involves more than one unknown function, the boundary conditions would lead to a system of dual integral equations for the unknown functions $A_j(\alpha)$, $(j=1,..,J)$ of the following form:

$$\int_L \sum_{j=1}^{J} k_{1j}^{i} \ (x,\alpha) A_j(\alpha) d\alpha = f_1^i(x) \quad , \quad x \epsilon L_1 \quad , \quad i = 1,..,J,$$

$$\int_L \sum_{1}^{J} k_{2j}^{i} \ (x,\alpha) A_j(\alpha)\ d\alpha = f_2^i(x) \quad , \quad x \epsilon L_2 \quad , \quad i = 1,..,J,$$

$$L_1 + L_2 = L \quad . \tag{2.20}$$

In this problem too one may consider the following more general boundary conditions:

$$\frac{\partial}{\partial y} u(x,0) = f_1(x) \quad , \quad x \epsilon L_1 \quad ,$$

$$u(x,0) = f_2(x) \quad , \quad x \epsilon L_2 \quad ,$$

$$h_1 u(x,0) + h_2 \frac{\partial}{\partial y} u(x,0) = f_3(x) \quad , \quad x \epsilon L_3 \quad , \tag{2.21a-c}$$

where $L_1 + L_2 + L_3 = (-\infty,\infty)$, and L_i, $(i = 1,2,3)$ consists of nonintersecting line segments L_{ik} $(k = 1,..,K_i)$ on the real line. From (2.17) and (2.21) it follows that

$$-\int_{-\infty}^{\infty} |\alpha| A(\alpha) e^{-i\alpha x}\, d\alpha = f_1(x) \quad , \quad x\varepsilon L_1 \quad ,$$

$$\int_{-\infty}^{\infty} A(\alpha) e^{-i\alpha x}\, d\alpha = f_2(x) \quad , \quad x\varepsilon L_2 \quad ,$$

$$\int_{-\infty}^{\infty} (h_1 - h_2|\alpha|) A(\alpha) e^{-i\alpha x}\, d\alpha = f_3(x) \quad , \quad x\varepsilon L_3 \quad , \qquad (2.22a\text{-}c)$$

Equations (2.22a-c) form a set of <u>triple integral equations</u> for the unknown function $A(\alpha)$. More generally

$$\int_L \sum_1^J k_{mj}^n(x,\alpha) A_j(\alpha) d\alpha = f_m^n(x) \quad , \quad x\varepsilon L_m \quad ,$$

$$\sum_{m=1}^{M} L_m = L \quad , \quad n = 1,..,J \quad , \quad m = 1,..,M \qquad (2.23)$$

is called a <u>system of multiple integral equations</u> (of multiplicity M) for the unknown functions $A_1,..,A_J$.

As in multiple series equations, in problems formulated in terms of multiple integral equations the singular nature of the solution is generally not known beforehand. Therefore, in these problems too it is important that the method of solution developed to solve the integral equations be not only sufficiently general and effective but also give the correct behavior of existing singularities. In this respect, particularly in dealing with somewhat unusual mixed boundary value problems, the methods of complex potentials and singular integral equations appear to be far superior to the standard operational techniques. An extensive treatment of the operational techniques for the solution of dual series and dual integral equations may be found in a recent book by Sneddon [1]. [2-13] are some of the outstanding references on the theory and applications of the complex potentials and the singular integral equations. In this article the primary emphasis will be on the recent developments concerning the methods of solution of the singular integral equations and particularly the applications to some mixed boundary value problems with uncommon singularities.

3. APPLICATION OF COMPLEX POTENTIALS

In this section the direct application of complex potentials will be described by considering some relatively simple examples.

3.1 A Problem in Potential Theory

Consider the mixed boundary value problem in potential theory for the half plane ($-\infty<x<\infty$, $y>0$) which is formulated by (2.13)-(2.16)(Figure 1). Let the harmonic function $u(x,y)$ be the real part of a complex potential $F(z)$, $z = x + iy$. $F(z)$ is holomorphic in the upper half plane S^+ where the derivatives of u may be expressed as

$$2\frac{\partial u}{\partial x} = F'(z) + \bar{F}'(\bar{z}) \;, \qquad -2i\frac{\partial u}{\partial y} = F'(z) - \bar{F}'(\bar{z}) \;. \tag{3.1}$$

Noting that if $z\varepsilon S^+$ and $z \to t + i0$ then $\bar{z}\varepsilon S^-$ and $\bar{z}\to t-i0$, taking the boundary values of (3.1), and using the conditions (2.14) and (2.15) we obtain

$$F'^+(t) - \bar{F}'^-(t) = -2if_1(t) \;, \qquad t\varepsilon L_1$$

$$F'^+(t) + \bar{F}'^-(t) = 2f_2'(t) \;, \qquad t\varepsilon L_2 \tag{3.2a,b}$$

where (2.15) is used in differentiated form which means that for single-valuedness the solution must satisfy the following conditions:

$$\int_{a_k}^{b_k} \frac{\partial}{\partial x} u(x,0)\, dx = f_2(b_k) - f_2(a_k) \tag{3.3}$$

where $k = 1,\ldots,N$ if $a_1 > -\infty$, $b_N < \infty$, and $k = 2,\ldots,N-1$ if $a_1 = -\infty$ and $b_N = \infty$.

Since $F'(z)$ is holomorphic in S^+, $\bar{F}'(z)$ will be holomorphic in S^-. If L_1 is finite we define a new sectionally holomorphic function by

$$G(z) = \begin{cases} F'(z) , & z\varepsilon S^+ \\ -\bar{F}'(z) , & z\varepsilon S^- \end{cases} . \tag{3.4}$$

From (3.2) and (3.4) it then follows that

$$G^+(t) + G^-(t) = -2if_1(t) , \quad t\varepsilon L_1 ,$$

$$G^+(t) - G^-(t) = 2f_2'(t) , \quad t\varepsilon L_2 \tag{3.5a,b}$$

From (3.4) and (3.5) it is seen that $G(z)$ is holomorphic everywhere in the complex plane except on L_1 and on that part of L_2 on which $f_2'(t)$ is not zero. Depending on the behavior of u, G may also have a pole of finite degree at infinity. For example, if there is a uniform "flux" at infinity given by

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = p_1 + ip_2 , \quad (x^2+y^2\to\infty) , \tag{3.6}$$

we have

$$\lim_{|z|\to\infty} F'(z) = p_1 - ip_2 . \tag{3.7}$$

Consider now the related homogeneous Riemann-Hilbert problem given by

$$X^+(t) + X^-(t) = 0 , \quad t\varepsilon L_1 , \quad L_1 = \sum_1^N L_{1k}$$

$$X^+(t) - X^-(t) = 0 , \quad t\varepsilon L_2 \tag{3.8}$$

where $X(z)$ is the fundamental solution of the original Riemann-Hilbert problem (3.5) which is clearly determinate within an arbitrary multiplicative analytic function. Referring to, for example, [3], the general solution of (3.8) may be expressed as

$$X(z) = P(z) \prod_{1}^{N} (z-a_k)^{-\frac{1}{2} + \alpha_k} (z-b_k)^{\frac{1}{2} + \beta_k} \tag{3.9}$$

where α_k and $\beta_k (k = 1,..,N)$ are arbitrary integers (positive, negative, or zero) and $P(z)$ is an arbitrary polynomial. At this point it should be strongly emphasized that in (3.9) as well as in the applications of the function-theoretic methods to the singular integral equations elsewhere in this article the arbitrary integers α_k, β_k cannot be determined from purely mathematical considerations. To do this the physics of the problem has to be properly taken into account. In the present problem the points a_k and b_k are the type of singular points at which flux vector has an integrable singularity. Therefore, $\alpha_k = 0$, $\beta_k = -1$, $(k = 1,..,N)$, and ignoring the arbitrary polynomial, the fundamental solution of the problem becomes

$$X(z) = \prod_{1}^{N} (z-a_k)^{-\frac{1}{2}} (z-b_k)^{-\frac{1}{2}} \tag{3.10}$$

where for the particular branch considered $\lim_{|z|\to\infty} z^n X(z) = 1$.

Dividing both sides of (3.5) by $X^+(t)$ and using (3.8) we find

$$\left(\frac{G(t)}{X(t)}\right)^+ - \left(\frac{G(t)}{X(t)}\right)^- = \begin{cases} -2if_1(t)/X^+(t) , & t\varepsilon L_1 , \\ 2f'_2(t)/X^+(t) , & t\varepsilon L_2 \end{cases} \tag{3.11}$$

Equation (3.11) is now a simple boundary value problem the general solution of which having a finite degree at infinity may be written as [3],

$$\frac{G(z)}{X(z)} = -\frac{1}{2\pi i}\int_{L_1} \frac{2if_1(t)}{(t-z)X^+(t)} dt + \frac{1}{2\pi i}\int_{L_2} \frac{2f'_2(t)}{(t-z)X^+(t)} dt + P_k(z) \tag{3.12}$$

where $P_k(z)$ is an arbitrary polynomial of degree k. From (3.4) and (3.7) it is seen that $G(z)$ has a pole of order zero at infinity. Thus, from (3.10) and (3.12) it follows that the degree of P_k is N, and (3.12) becomes

$$G(z) = -\frac{X(z)}{\pi} \int_{L_1} \frac{f_1(t)dt}{(t-z)X^+(t)} + \frac{X(z)}{\pi i} \int_{L_2} \frac{f_2'(t)dt}{(t-z)X^+(t)} + X(z)\sum_o^N c_k z^k. \quad (3.13)$$

where $c_o,..,c_N$ are arbitrary constants. From (3.4), (3.7) and (3.13) it is easily seen that

$$c_N = p_1 - ip_2 \quad (3.14)$$

Noting that L_1 is finite, the remaining N constants $c_o,..,c_{N-1}$ are determined from the single-valuedness conditions (3.3). From

$$2\frac{\partial}{\partial x} u(x,+0) = F'^+(x) + \bar{F}'^-(x) = G^+(x) - G^-(x) \quad (3.15)$$

these conditions may be expressed as

$$\frac{1}{2}\int_{a_k}^{b_k} [G^+(t) - G^-(t)]dt = f_2(b_k) - f_2(a_k) \quad , \quad (k = 1,..,N) \quad (3.16)$$

giving, with (3.13), a system of N linear algebraic equations in $c_o,..,c_{N-1}$. This completes the solution for finite L_1.

If L_2 instead of L_1 is finite (i.e., $a_1 = -\infty, b_N = \infty$) the procedure to solve the problem is quite similar to that given above with the following main differences: the branch cut should be introduced along L_2 by defining G(z) as

$$G(z) = \begin{cases} F'(z) \quad , & z\varepsilon S^+ \\ \bar{F}'(z) \quad , & z\varepsilon S^- \end{cases} \quad (3.17)$$

which would give the fundamental solution as follows:

$$X(z) = \prod_1^{N-1} (z-b_k)^{-1/2}(z-a_{k+1})^{-1/2} \quad . \quad (3.18)$$

Considering only the solution for which

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial y} u(x,0)dx = Q = \text{finite} \quad (3.19)$$

and noting that $X(z) \to z^{1-N}$ for $|z| \to \infty$, it may be seen that in the expression of G(z) similar to (3.13), the degree of the arbitrary polynomial will be N-2, i.e., the solution will contain N-1 arbitrary constants $c_o, \ldots, c_{N-2}$. These constants are then determined from (3.19) and (3.3), completing the solution.

3.2 The Case of Periodic Cuts

In the problem considered in the previous section let the cuts (a_k, b_k) be equal in length and be equally spaced. Thus, one may define the end points of 2m+1 cuts by

$$a_k = 2kb - a \quad , \quad b_k = 2kb + a \quad , \quad (k = 0, \mp 1, \ldots, \mp m)$$

$$L_{1k} = (a_k, b_k) \quad , \quad L_1 = \sum_{-m}^{m} L_{1k} \quad . \tag{3.20}$$

The fundamental solution (3.10) of the problem then becomes

$$X(z) = \prod_{-m}^{m} (z-a_k)^{-1/2}(z-b_k)^{-1/2}$$
$$= A[(z+a) \prod_{1}^{m} (1-(\frac{z+a}{2kb})^2)(z-a) \prod_{1}^{m} (1-(\frac{z-a}{2kb})^2)]^{-1/2} \tag{3.21}$$

where A is a constant given by

$$A = (-1)^m (2b)^{-2m} (m!)^{-2}$$

Because of the homogeneous nature of (3.8), since X(z) is determinate within a multiplicative analytic function, the constant A may be (and will be) ignored. It is seen that if we now let $m \to \infty$, geometrically the problem becomes that of a plane with periodic cuts. In addition to this if the functions f_1 and f_2 are assumed to be periodic then we have a problem for a half plane with periodic mixed boundary conditions. For simplicity, here it will be assumed that any homogeneous "loading" condition at infinity has been separated through a proper superposition and consequently in the problem under consideration $F(z)\to 0$ as $|z|\to\infty$. First ignoring the constant A and then using the relation [14]

$$\sin\theta = \theta \prod_{1}^{\infty} \left(1 - \frac{\theta^2}{k^2\pi^2}\right) , \tag{3.22}$$

in limit from (3.21) X(z) may be obtained as

$$X(z) = \left(\sin^2 \frac{\pi z}{2b} - \sin^2 \frac{\pi a}{2b}\right)^{-1/2} , \tag{3.23}$$

where the particular branch which is positive for $a<Re(z) = t<2b-a$ will be considered.

With X(z) as given by (3.23) the solution (3.12) is still valid. Since the geometry and the boundary conditions in the problem are periodic in x with a period of 2b, the potential G(z) must also be periodic in z with a real period 2b. On the other hand from (3.23) it is seen that

$$X^+(t) = - X^-(t) = - X^+(t+2b) = X^+(t+4b) , \quad t\varepsilon L_1 ,$$

$$X^+(t) = X^-(t) = - X^+(t+2b) = X^+(t+4b) , \quad t\varepsilon L_2 \tag{3.24}$$

namely, X(z) is periodic with a period of 4b. Therefore, in order to have a periodic potential with (real) period 2b, the arbitrary polynomial in (3.12) must be of the following form

$$P_k(z) = \left(B_1 \sin \frac{\pi z}{2b} + B_2 \cos \frac{\pi z}{2b}\right)Q(z) \tag{3.25}$$

where Q(z) is an arbitrary periodic analytic function with period 2b and B_1 and B_2 are arbitrary constants. From

$$\lim_{|z|\to\infty} G(z) = 0 = \lim_{|z|\to\infty} X(z)P_k(z) = (B_1+iB_2)Q(\infty) \tag{3.26}$$

it follows that $Q(\infty) = 0$, or since Q is analytic in the entire plane, $Q(z) = 0$.

The solution of the problem may then be written as

$$G(z) = -\frac{X(z)}{\pi} \sum_{-\infty}^{\infty} \int_{a_k}^{b_k} \frac{f_1(t)dt}{(t-z)X^+(t)} + \frac{X(z)}{\pi i} \sum_{-\infty}^{\infty} \int_{b_k}^{a_{k+1}} \frac{f_2'(t)dt}{(t-z)X^+(t)} , \quad (3.27)$$

where X(z) is given by (3.23). Letting

$$t = \tau+2kb , \quad f_1(t) = f_1(\tau+2kb) = g_1(\tau) ,$$

$$f_2'(t) = g_2(\tau) , \quad X^+(t) = X(\tau) , \quad -b<\tau<b , \quad k = 0,\mp 1,.. \quad (3.28)$$

and using (3.24) the integrals in (3.27) may be modified as follows:

$$S_1 = \sum_{-\infty}^{\infty} \int_{a_k}^{b_k} \frac{f_1(t)dt}{(t-z)X^+(t)} = \int_{-a}^{a} \frac{g_1(\tau)d\tau}{X(\tau)} \sum_{-\infty}^{\infty} \frac{(-1)^k}{(\tau-z)+2kb}$$

$$= \int_{-a}^{a} \frac{g_1(\tau)d\tau}{X(\tau)} \left[\frac{1}{\tau-z} + \sum_{1}^{\infty} \frac{(-1)^k 2(\tau-z)}{(\tau-z)^2-(2kb)^2}\right] . \quad (3.29)$$

In (3.29) the series may be summed by using [14]

$$\frac{1}{\theta} + \sum_{1}^{\infty} \frac{2\theta}{\theta^2-n^2} = \pi\cot\pi\theta , \quad \sum_{1}^{\infty} \frac{2\theta}{\pi^2(\frac{2n-1}{2})^2-\theta^2} = \tan\theta \quad (3.30)$$

which gives

$$S_1 = \frac{\pi}{4b} \int_{-a}^{a} \frac{g_1(\tau)d\tau}{X(\tau)} \left[\cot\frac{\pi}{4b}(\tau-z) + \tan\frac{\pi}{4b}(\tau-z)\right]$$

$$= \frac{\pi}{2b} \int_{-a}^{a} \frac{g_1(\tau)d\tau}{X(\tau)\sin\frac{\pi}{2b}(\tau-z)} . \quad (3.31)$$

Similarly

$$\sum_{-\infty}^{\infty} \int_{b_k}^{a_{k+1}} \frac{f_2'(t)dt}{(t-z)X^+(t)} = \frac{\pi}{2b} \int_{a}^{2b-a} \frac{g_2(\tau)d\tau}{X(\tau)\sin\frac{\pi}{2b}(\tau-z)} . \quad (3.32)$$

Thus, the solution becomes

$$G(z) = -\frac{X(z)}{2b}\int_{-a}^{a}\frac{g_1(\tau)d\tau}{X(\tau)\sin\frac{\pi}{2b}(\tau-z)} + \frac{X(z)}{2bi}\int_{a}^{2b-a}\frac{g_2(\tau)d\tau}{X(\tau)\sin\frac{\pi}{2b}(\tau-z)} \quad (3.33)$$

where $X(z)$ is given by (3.23) and

$$X(\tau) = X^+(t) = (\sin^2\frac{\pi t}{2b} - \sin^2\frac{\pi a}{2b})^{-1/2} \quad . \quad (3.34)$$

Consider, for example, the simple case of an infinite plane containing uniformly "loaded" periodic cuts for which $f_1(t) = -p_o$ and $f_2(t) = 0$. From (3.33) G may be obtained as

$$G(z) = ip_o[1-X(z)\sin\frac{\pi z}{2b}] \quad . \quad (3.35)$$

Using now the relations

$$\frac{\partial}{\partial x}u(x,+0) = \frac{1}{2}[G^+(x) - G^-(x)] \quad ,$$

$$\frac{\partial}{\partial y}u(x,+0) = -\frac{1}{2i}[G^+(x) + G^-(x)] \quad , \quad (3.36a,b)$$

the components of the flux vector on the boundary may be expressed as follows:

$$\frac{\partial}{\partial x}u(x,+0) = \begin{cases} 0 \quad , & a<|x|<b \quad , \\ \dfrac{-p_o\sin(\pi x/2b)}{[\sin^2(\pi a/2b)-\sin^2(\pi x/2b)]^{1/2}} \quad , & 0\leq|x|<a \quad , \end{cases}$$

$$\frac{\partial}{\partial y}u(x,+0) = \begin{cases} -p_o \quad , & 0\leq|x|<a \\ \dfrac{p_o\sin(\pi x/2b)}{[\sin^2(\pi x/2b)-\sin^2(\pi a/2b)]^{1/2}} - p_o \quad , & a<|x|<b \end{cases} \quad (3.37)$$

In limit, for $b\to\infty$, a = finite, (3.35)-(3.37) reduce to the following results for a plane with a single cut, (-a,a) for which $\frac{\partial}{\partial y}u(x,0) = -p_o$,

(-a<x<a) is the only external disturbance:

$$G(z) = ip_o[1 - z(z^2-a^2)^{-1/2}] \quad ,$$

$$\frac{\partial}{\partial x} u(x,0) = -\frac{p_o x}{(a^2-x^2)^{1/2}} \quad , \qquad 0 \le |x| < a \quad ,$$

$$\frac{\partial}{\partial y} u(x,+0) = \frac{p_o x}{(x^2-a^2)^{1/2}} - p_o \quad , \qquad a < |x| < \infty \quad . \tag{3.38}$$

In the general case the real and imaginary parts of G(z) give the components of the flux vector in the upper half plane and (3.36) that on the boundary. $G^+(x)$ and $G^-(x)$ may in turn be obtained from (3.33) by using (3.24) and the following Plemelj formulas [3]:

$$\phi(z) = \frac{1}{2\pi i} \int_L \frac{f(t)dt}{\sin \frac{\pi}{2b}(t-z)} \quad ,$$

$$\phi^+(x) - \phi^-(x) = \begin{cases} 0 \quad , & x \varepsilon L' \\ f(x) \quad , & x \varepsilon L \quad , \end{cases}$$

$$\phi^+(x) + \phi^-(x) = \begin{cases} 2\phi(x) \quad , & x \varepsilon L' \\ \frac{1}{\pi i} \int_L \frac{f(t)dt}{\sin \frac{\pi}{2b}(t-x)} \quad , & x \varepsilon L \quad , \end{cases} \tag{3.39a-c}$$

where f(t) is any Hölder-continuous function defined on the open interval L, L + L' = (-b,b).

From (3.13) and (3.33), or more specifically, from (3.37) it is seen that the components of the flux vector $\partial u/\partial x$ and $\partial u/\partial y$ have integrable singularities at the points of intersection a_k, b_k of L_1 and L_2 with a power of - 1/2 (see also (1.7) for $\theta_o = \pi$). A close examination of (3.13) and (3.33) would indicate that

$$G(z) = F_1(z) + X(z)F_2(z) \tag{3.40}$$

where F_1 and F_2 are holomorphic in the entire plane. By letting $z-a = re^{i\theta}$, for small values of r G(z) may be expressed as

$$G(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{1}{\sqrt{r}} \frac{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}{[\frac{\pi}{2b}\sin\frac{\pi a}{b}]^{1/2}} F_2(a) + O(1)$$

$$0 \leq \theta < \pi \tag{3.41}$$

For example, in the case of uniformly "loaded" cuts, from (3.35) it follows that [(1)]

$$G(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = -(\frac{b}{\pi}\tan\frac{\pi a}{2b})^{1/2}\frac{p_o}{\sqrt{r}}(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}) + O(1)$$

$$0 \leq \theta \leq \pi - 0 \tag{3.42}$$

Note that in this simple example $u(x,0) = 0$ on L_1 $\frac{\partial}{\partial y}u(x,0) = -p_o$ on L_2. Hence, the boundary conditions around the flux singularity x = a are identical to (1.6). It would then be expected that the resulting asymptotic solutions be the same which may indeed be seen from (1.7) and (3.42).

3.3 An Elasticity Problem for a Nonhomogeneous Plane

As another example for the direct application of complex potentials to mixed boundary value problems consider the following elasticity problem: Let two linearly elastic isotropic half planes with material constants [(2)] μ_i, κ_i ($i = 1$ for $y>0$, $-\infty<x<\infty$, i.e., S^+ and $i = 2$ for $y<0$, $-\infty<x<\infty$, i.e., S^-) be bonded along the (nonintersecting) line segments $L_k = (a_k, b_k)$, $(k = 1,..N)$ on the real axis. Let $L = \sum_1^N L_k$ be finite and the x and y- components of

[(1)] In (3.41) and (3.42) the terms O(1) stands for bounded terms and come from $F_1(z)$ in (3.40). Note that the second term in (3.40) is of the asymptotic form $O(r^{-1/2}) + O(r^{1/2})$.

[(2)] Where μ_i is the shear modulus, $\kappa_i = 3-4\nu_i$ for plane strain, and $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for plane stress, ν_i being the Poisson's ratio.

the resulting force acting on the half planes at infinity be Q and P, respectively. Let L' be the complement of L on $(-\infty<x<\infty,\ y=0)$ and $-p_1(x)$ and $p_2(x)$ be the normal tractions acting on the half planes along L'. The problem will have to be solved under the following conditions:(1)

$$\sigma^+_{1yy}(t) - i\,\sigma^+_{1xy}(t) = \sigma^-_{2yy}(t) - i\,\sigma^-_{2xy}(t) \quad , \qquad t\epsilon L+L'$$

$$\sigma^+_{1yy}(t) - i\,\sigma^+_{1xy}(t) = -\,[p_1(t) + ip_2(t)] = -\,p(t) \quad , \qquad t\epsilon L'$$

$$[u^+_1(t) + iv^+_1(t)] - [u^-_2(t) + iv^-_2(t)] = f_1(t) + if_2(t) = f(t),\ t\epsilon L \ ,$$

$$\int_{L+L'} (\sigma^-_{2yy} - i\,\sigma^-_{2xy})dt = P - iQ \ . \qquad (3.43a\text{-}d)$$

where the superscripts + and − refer to limits as $y\to+0$ and $y\to-0$, u_k, v_k, are the x,y- components of the displacement vector, and $\sigma_{kj\ell}$, $(k = 1,2;\ (j,\ell) = (x,y)$ is the stress.

The simplest method to solve this problem would be the use of complex potentials known as Kolosov-Muskhelishvili or Goursat functions. In terms of these potentials the stresses and displacements may be expressed as follows [2,8-10]:

$$\sigma_{kxx} + \sigma_{kyy} = 2[\Phi_k(z) + \bar{\Phi}_k(\bar{z})] \quad ,$$

$$\sigma_{kyy} - \sigma_{kxx} + 2i\sigma_{kxy} = 2[\bar{z}\Phi'_k(z) - \Psi_k(z)] \quad ,$$

$$2\mu_k(u_k + iv_k) = \kappa_k\phi_k(z) - z\bar{\Phi}_k(\bar{z}) - \bar{\psi}_k(\bar{z}) \ ,$$

$$\Phi_k(z) = \phi'_k(z) \quad , \quad \Psi_k(z) = \psi'_k(z) \quad , \quad (k = 1,2) \qquad (3.44a\text{-}e)$$

(1) The input function f(t) may be non-zero in, for example, thermal stress and residual stress problems [15].

where $z\varepsilon S^+$ for k =1 and $z\varepsilon S^-$ for k = 2. Noting that Φ_1 and Ψ_1 are defined in S^+ and Φ_2 and Ψ_2 are defined in S^- only, by extending the definition of Φ_1 into S^- and Φ_2 into S^+ in such a way that they are holomorphic on the unloaded parts of the boundary (i.e., the real axis), one could make the following substitution [2]:

$$\Psi_k(z) = -\Phi_k(z) - \bar{\Phi}_k(z) - z\Phi_k'(z) \tag{3.45}$$

where $z\varepsilon S^+$ for k = 1 and $z\varepsilon S^-$ for k = 2. From (3.44) and (3.45) it then follows that

$$\sigma_{kyy} - i\sigma_{kxy} = \Phi_k(z) - \Phi_k(\bar{z}) + (z-\bar{z})\bar{\Phi}_k'(\bar{z}) \ , \quad (k = 1,2) \tag{3.46}$$

$$2\mu_k \frac{\partial}{\partial x}(u_k+iv_k) = \kappa_k\Phi_k(z) + \Phi_k(\bar{z}) - (z-\bar{z})\Phi_k'(\bar{z}) \ . \quad (k = 1,2) \tag{3.47}$$

Substituting from (3.46) into (3.43a) we find

$$\Phi_1^+(t) + \Phi_2^+(t) = \Phi_1^-(t) + \Phi_2^-(t) \quad , \quad t\varepsilon L + L' \quad , \tag{3.48}$$

meaning that $\Phi_1(z) + \Phi_2(z)$ is holomorphic in the entire plane including the real axis. Noting that the stress state at infinity vanishes and assuming that the rotation at infinity is zero, following [2], for large values of $|z|$ it may be shown that

$$\Phi_1(z) = \frac{Q+iP}{2\pi z} + o(1/z)$$

$$\Phi_2(z) = -\frac{Q+iP}{2\pi z} + o(1/z) \tag{3.49a,b}$$

where, in the usual notation $o(1/z)\leq c/z$, c being a positive quantity which depends only on $|z|$ and tends to zero as $|z|\to\infty$. Since $\Phi_1+\Phi_2$ is holomorphic, from (3.49) it is clear that

$$\Phi_1(z) + \Phi_2(z) = 0 \tag{3.50}$$

in the entire plane.

Substituting now from (3.46), (3.47), and (3.50) into the boundary conditions (3.43b) and (3.43c) (after differentiation), we obtain the following Riemann-Hilbert problem for the sectionally holomorphic function $\Phi_2(z)$:

$$\Phi_2^+(t) + \omega\, \Phi_2^-(t) = h(t) \quad , \qquad t \varepsilon L$$

$$\Phi_2^+(t) - \Phi_2^-(t) = p(t) \quad , \qquad t \varepsilon L' \tag{3.51}$$

where

$$\omega = \frac{\mu_1\kappa_2+\mu_2}{\mu_2\kappa_1+\mu_1} \quad , \qquad h(t) = \frac{-2\mu_1\mu_2}{\mu_2\kappa_1+\mu_1}\, f'(t) \quad . \tag{3.52}$$

Referring to [3], the fundamental solution of (3.51) satisfying the related homogeneous equations

$$X^+(t) + \omega X^-(t) = 0 \quad , \qquad t\varepsilon L \quad ,$$

$$X^+(t) - X^-(t) = 0 \quad , \qquad t\varepsilon L'$$

$$L = \sum_1^N L_k \quad , \qquad L_k = (a_k, b_k) \tag{3.53}$$

may be expressed as

$$X(z) = \prod_1^N (z-b_k)^{\alpha_k}(z-a_k)^{\beta_k} \quad ,$$

$$\alpha_k = \frac{1}{2\pi i}\log(-\omega) + A_k = \frac{1}{2} - i\,\frac{\log \omega}{2\pi} + A_k \quad ,$$

$$\beta_k = -\frac{1}{2\pi i}\log(-\omega) + B_k = -\frac{1}{2} + i\,\frac{\log \omega}{2\pi} + B_k \quad , \tag{3.54}$$

where A_k and B_k ($k = 1,..,N$) are again arbitrary (positive, zero, or negative) integers[1]. In the present problem the singular points a_k and b_k correspond to the ends points of interface cracks. Consequently, at these points the stresses and displacement derivatives will have an integrable singularity giving $B_k \doteq 0$, $A_k = -1$, $(k = 1,..,N)$. Here, we will then consider the particular branch of X(z) which is single valued in the plane cut along L and for which

$$\lim_{|z|\to\infty} z^N X(z) = 1 \quad . \tag{3.55}$$

Dividing by (3.51) by $X^+(t)$ and using (3.53) it is found that

$$\left(\frac{\Phi_2(t)}{X(t)}\right)^+ - \left(\frac{\Phi_2(t)}{X(t)}\right)^- = \frac{h(t)}{X^+(t)} \quad , \qquad t\varepsilon L$$

$$\left(\frac{\Phi_2(t)}{X(t)}\right)^+ - \left(\frac{\Phi_2(t)}{X(t)}\right)^- = \frac{p(t)}{X^+(t)} \quad , \qquad t\varepsilon L' \tag{3.56}$$

Noting that the stress state vanishes at infinity, the general solution of (3.56) vanishing at infinity becomes

$$\frac{Q_2(z)}{X(z)} = \frac{1}{2\pi i}\int_L \frac{h(t)dt}{(t-z)X^+(t)} + \frac{1}{2\pi i}\int_{L'} \frac{p(t)dt}{(t-z)X^+(t)} + \sum_0^{N-1} c_n z^n \quad , \tag{3.57}$$

where $c_o,..,c_{N-1}$ are arbitrary constants. From (3.49b), (3.55), and (3.57) it may be shown that

$$c_{N-1} = -\frac{Q+iP}{2\pi} \quad . \tag{3.58}$$

The condition of single-valuedness of displacements gives the remaining N-1 constants, $c_o,..,c_{N-2}$. We recall that in deriving (3.51) the

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[1] In most physical problems A_k and B_k are such that $-1<Re(\alpha_k,\beta_k)<1$. Even though in literature one finds this as a mathematical condition, clearly A_k and B_k must be determined from the physics of the problem.

continuity condition (3.43c) was used in differentiated form. Thus, this condition requires that

$$\int_{b_k}^{a_{k+1}} \frac{\partial}{\partial x} [(u_1+iv_1)^+ - (u_2+iv_2)^-]dx = f(a_{k+1}) - f(b_k)$$

$$(k = 1,..,N-1) \qquad (3.59)$$

or

$$\int_{b_k}^{a_{k+1}} [(\frac{1+\kappa_1}{2\mu_1} + \frac{1+\kappa_2}{2\mu_2})\phi_2^+(t) + (\frac{1}{2\mu_1} + \frac{\kappa_2}{2\mu_2})p(t)]dt = - f(a_{k+1}) + f(b_k)$$

$$(k = 1,..,N-1) \quad . \qquad (3.60)$$

In (3.60) and elsewhere, to obtain the boundary values of the Cauchy integrals the following general Plemelj formulas may be used

$$F(z) = \frac{1}{2\pi i} \int_L \frac{g(t)dt}{t-z} \quad ,$$

$$F^+(x) - F^-(x) = \begin{cases} g(x) \quad , & x\varepsilon L \\ 0 \quad , & x\varepsilon L' \end{cases}$$

$$F^+(x) + F^-(x) = \begin{cases} \frac{1}{\pi i} \int_L \frac{g(t)}{t-x} dt \quad , & x\varepsilon L \\ 2F(x) \quad , & x\varepsilon L' \end{cases} \qquad (3.61a\text{-}c)$$

where L' is the complement of L (on an infinite line or on any closed contour in the complex plane). Thus, (3.57) with (3.58), (3.60), (3.50), (3.45) and (3.44) gives the complete solution of the problem. For example if $L = (-a,a)$ (bonding along a single segment), $p(t) = 0$, and $f(t) = 0$, the solution becomes

$$\Phi_1(z) = - \Phi_2(z) = \frac{Q+iP}{2\pi} \frac{1}{(z^2-a^2)^{1/2}} (\frac{z+a}{z-a})^{i(\log\omega)/2\pi} \qquad (3.62)$$

where ω is given by (3.52).

Observing that the general solution of the problem is of the form

$$\Phi_2(z) = F_1(z) + F_2(z)X(z) \quad , \tag{3.63}$$

as in the previous example, the asymptotic behavior of the stresses and displacements around the points of singularity may easily be investigated (see [16] for details). If the positive constant ω appearing in (3.51) and defined by (3.52) is not unity, from the behavior of the fundamental function $X(z)$ given by (3.54) it is clear that the stresses and displacement derivatives will have a typical oscillating singularity around the end points of the branch cuts which is of the following form:

$$\sigma_{ij}(r,\theta) = \frac{1}{\sqrt{r}} [f_{ij}(\theta)\cos(\gamma \log \frac{r}{\ell}) + g_{ij}(\theta)\sin(\gamma \log \frac{r}{\ell})] + O(1) \quad ,$$

$$\gamma = (\log \omega)/2\pi \quad , \quad (i,j = x,y) \quad , \quad (r<<\ell) \tag{3.64}$$

where r,θ are the polar coordinates around the singular point, f_{ij} and g_{ij} are bounded functions, and the term $O(1)$ again comes from the analytic function $F_1(z)$ in (3.63).

4. REDUCTION TO SINGULAR INTEGRAL EQUATIONS

As pointed out in the previous section, in order to obtain the correct behavior of the singularities and also in most cases to find a simple closed form solution of a given mixed boundary value problem, whenever possible it is always preferable to use the complex potentials and the related complex function theory. However, the technique has its limitations. First, the particular problem may not admit complex potentials. Secondly, the successful application of the technique is severely limited to certain geometries. Finally, there are certain types of boundary conditions which would make the direct use of complex potentials extremely difficult if not impossible (e.g., the boundary conditions described by differential operators containing the unknown function as well as its derivatives[1]). In such cumbersome cases, the method which is perhaps the most general and the easiest to apply is the reduction of the problem to singular integral equations either by using a Green's function formulation or by formulating the problem first in terms of multiple series or multiple integral equations. Reduction of the boundary conditions to integral equation is always possible. The main problems here are the selection of the appropriate auxiliary function (i.e., the new unknown function defined on the boundary) and the proper separation of the dominant parts of the kernel for the correct study of the singular behavior of the solution. In this section this important step of reducing the mixed boundary value problem to a singular integral equation will be discussed by considering some typical examples, and some general remarks will be made regarding the nature of the kernel and the solution.

4.1 Reduction of Dual Series Equations to Singular Integral Equations

Consider the typical simple mixed boundary value problem described by (2.1-2.3) and formulated by the dual series equations (2.6). At the generality that the problem is stated, it is not very fruitful to pursue a technique based on the operational methods to solve the problem and, as stated before, because of the importance of existing singularities, a direct numerical solution is nearly useless. To reduce the problem to an integral equation, the first step is the definition or selection of an

[1] For this type of mixed boundary conditions even the simplest problems such as that in potential theory for a half plane do not seem to have closed form solutions (see the following section).

appropriate auxiliary function. In this problem let this function be

$$\phi(\theta) = \frac{\partial}{\partial\theta} u(1,0) \quad , \qquad \theta\varepsilon L_1 + L_2 = (0,2\pi) \quad . \tag{4.1}$$

Note that

$$\phi(\theta) = \begin{cases} f'(\theta) \quad , & \theta\varepsilon L_2 \quad , \\ \text{unknown} \quad , & \theta\varepsilon L_1 \quad . \end{cases} \tag{4.2}$$

From (2.5) and (4.1) it follows that

$$nA_n = -\frac{1}{\pi} \int_{L_1+L_2} \phi(t) \sin nt \, dt \quad ,$$

$$nB_n = \frac{1}{\pi} \int_{L_1+L_2} \phi(t) \cos nt \, dt \quad . \tag{4.3}$$

Equations (4.3) and (2.5) give the solution once the function $\phi(t)$ and the constant A_o are determined.

In applying the technique described in this section, in order to circumvent the difficulties arising from the divergent series or integrals giving the kernels, for analytical convenience the boundary condition forming the basis of the integral equation will always be expressed in limit form[(1)]. Thus, the boundary condition (2.6a) may be expressed as

$$\lim_{r\to 1-0} \sum_1^\infty n\, r^{n-1}(A_n \cos n\theta + B_n \sin n\theta) = g(\theta) \quad , \qquad \theta\varepsilon L_1 \tag{4.4}$$

Substituting now from (4.3) into (4.4), observing that for r<1 the related infinite series will be uniformly convergent, and hence changing the order

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[(1)]The exception here is, of course, the case in which the related series or integrals are uniformly convergent giving bounded kernels. In that case the limit may be put under summation or integration sign before evaluating the kernels.

of integration and summation, it is found that

$$\lim_{r\to 1-0} \frac{1}{\pi} \int_{L_1+L_2} \phi(t)dt \sum_{1}^{\infty} r^{n-1} \sin n(t-\theta) = -g(\theta) \quad , \quad \theta \varepsilon L_1 \quad . \tag{4.5}$$

First performing the sum as

$$\sum_{1}^{\infty} r^{n-1} \sin nz = \frac{1}{2ir} \sum_{1}^{\infty} [(re^{iz})^n - (re^{-iz})^n] = \frac{\sin z}{1+r^2-2r\cos z} \tag{4.6}$$

and then going to limit, (4.5) becomes

$$\frac{1}{2\pi} \int_{L_1} \phi(t) \cot \frac{t-\theta}{2} dt = -g(\theta) - \frac{1}{2\pi} \int_{L_2} f'(t) \cot \frac{t-\theta}{2} dt, \ \theta \varepsilon L_1 \tag{4.7}$$

giving an integral equation to determine $\phi(\theta)$.

It should be observed that (4.7) is a typical singular integral equation of the following form

$$\frac{1}{\pi} \int_{L_1} \frac{\phi(t)}{t-\theta} dt + \int_{L_1} k(\theta,t)\phi(t)dt = g_o(\theta) \quad , \quad \theta \varepsilon L_1 \tag{4.8}$$

where the kernel

$$k(x,t) = \frac{1}{2\pi} \cot \frac{t-\theta}{2} - \frac{1}{\pi} \frac{1}{t-\theta} \tag{4.9}$$

is bounded everywhere on L_1 (including the end points a_i, b_i, $i = 1,..,N$), and the known function g_o is the right hand side of (4.7). Since the fundamental solution (or the behavior of the singularity) depends only on the dominant part of the integral equation, for the purpose of obtaining this solution tentatively expressing (4.8) as

$$\frac{1}{\pi} \int_{L_1} \frac{\phi(t)}{t-\theta} dt = g_1(\theta) \quad , \quad \theta \varepsilon L_1 \quad , \tag{4.10}$$

defining a sectionally holomorphic function by

$$F(z) = \frac{1}{2\pi i} \int_{L_1} \frac{\phi(t)}{t-z} dt \quad , \tag{4.11}$$

and using the Plemelj formulas

$$F^+(\theta) - F^-(\theta) = \phi(\theta) \quad , \quad \theta\varepsilon L_1 \quad ,$$

$$F^+(\theta) + F^-(\theta) = \frac{1}{\pi i} \int_{L_1} \frac{\phi(t)}{t-\theta}\, dt \quad , \tag{4.12a,b}$$

one obtains the following Riemann-Hilbert problem:

$$F^+(\theta) - F^-(\theta) = 0 \quad , \quad \theta\varepsilon L_2 \quad ,$$

$$F^+(\theta) + F^-(\theta) = - ig_1(\theta) \quad , \quad \theta\varepsilon L_1 \quad . \tag{4.13}$$

Again, following [3] and observing that $L_1 = \sum_1^N L_{1k}$, $L_{1k} = (a_k, b_k)$, the fundamental solution of (4.13) satisfying the related homogeneous Riemann-Hilbert problem is found to be (see (3.5)-(3.10))

$$X(z) = \prod_1^N (z-a_k)^{-1/2+\alpha_k}(z-b_k)^{1/2+\beta_k} \tag{4.14}$$

where the arbitrary integers α_k and β_k $(k = 1,..,N)$ have to be selected in such a way that the solution is consistent with the expected physical behavior. In this problem since at a_k and b_k the "flux" has an integrable singularity, $\alpha_k = 0$ and $\beta_k = -1$, $k = 1,..,N$. Referring, for example, to the solution of (3.5) as given by (3.12) and noting that $X^+(\theta) + X^-(\theta) = 0$, $\theta\varepsilon L_1$, from (4.12a) it is seen that $\phi(\theta) \sim X^+(\theta)$, $\theta\varepsilon L_1$. We now define the <u>fundamental function</u>, $w(\theta)$ of the singular integral equation (4.10) by

$$w(\theta) = \prod_1^N [(\theta-a_k)(b_k-\theta)]^{-1/2} = (-1)^{N/2} X^+(\theta) \quad . \tag{4.15}$$

Thus, the solution of (4.10) or (4. 7) will be of the following form

$$\phi(\theta) = w(\theta)p(\theta) \tag{4.16}$$

where p is an unknown bounded function.

Referring again to (3.12) (and [3]), it is seen that the solution of (4.13) vanishing at infinity (and hence $\phi(\theta)$ obtained from (4.12a)) will contain N arbitrary constants. These constants are determined from the single-valuedness conditions of $u(r,\theta)$. We recall that in deriving (4.7) the boundary condition $u(1,\theta) = f(\theta)$, $\theta \varepsilon L_2$, was used in differentiated form. Therefore $\phi(\theta)$ must satisfy the following conditions:

$$\int_{a_k}^{b_k} \phi(\theta)d\theta = f(b_k) - f(a_k) \quad , \quad (k = 1,..,N) \quad . \tag{4.17}$$

To complete the solution of the problem the constant A_o in (2.5) must be determined. From (2.5), (2.3), and (4.1) it is seen that

$$A_o = \frac{1}{2\pi} \int_{L_1+L_2} u(1,\theta)d\theta = \frac{1}{2\pi} \sum_1^N \int_{b_k}^{a_{k+1}} f(\theta)d\theta$$

$$+ \frac{1}{2\pi} \sum_1^N \int_{a_j}^{b_j} d\theta[f(a_j) + \int_{a_j}^{\theta} \phi(t)dt] \quad , \quad (a_{N+1} = a_1) \quad . \tag{4.18}$$

Using the Dirichlet transformation

$$\int_a^x dy \int_a^y F(x,y,s)f(s)ds = \int_a^x f(s)ds \int_s^x F(x,y,s)dy \tag{4.19}$$

equation (4.18) becomes

$$A_o = \frac{1}{2\pi} \sum_1^N [\int_{b_k}^{a_{k+1}} f(\theta)d\theta - \int_{a_k}^{b_k} t\phi(t)dt + b_k f(b_k) - a_k f(a_k)] \quad ,$$

$$(a_{N+1} = a_1) \quad . \tag{4.20}$$

The solution of (4.13) is given by (3.12) with $g_1 = 2f_1$, $f_2' = 0$, and $N - 1$ as the degree of the polynomial P_k. Thus, using the Plemelj formulas (4.12), the solution of (4.7) or (4.8) may be expressed as

$$\phi(\theta) = 2w(\theta) \sum_{o}^{N-1} c_n \theta^n - \frac{w(\theta)}{\pi} \int_{L_1} \frac{g_o(t)dt}{(t-\theta)w(t)}$$

$$+ \frac{w(\theta)}{\pi} \int_{L_1} \frac{dt}{(t-\theta)w(t)} \int_{L_1} k(t,s)\phi(s)ds \quad , \quad \theta \epsilon L_1 \,. \tag{4.21}$$

Considering now the definition (4.16), it is found that

$$p(\theta) - \frac{1}{\pi} \int_{L_1} n(\theta,s)p(s)ds = 2 \sum_{o}^{N-1} c_n \theta^n - \frac{1}{\pi} \int_{L_1} \frac{g_o(t)dt}{(t-\theta)w(t)} \quad ,$$

$$\theta \epsilon L_1 \tag{4.22}$$

where

$$n(\theta,s) = \int_{L_1} \frac{k(t,s)w(s)}{(t-\theta)w(t)} dt \quad . \tag{4.23}$$

Equation (4.22) may be treated as a Fredholm integral equation [3,5] giving the solution of the problem with (4.17) and (4.20).

It should be remarked that any singular integral equation of the form (4.8) may be "regularized" and reduced to a Fredhom type integral equation by following the foregoing procedure. However, for the solution of (4.8) a considerably simpler numerical technique will be described later in this article. It should also be noted that the particular mixed boundary value problem considered in this section can be solved in closed form. First consider the case of geometric symmetry with respect to the line $\theta = 0$. That is, let us assume that

$$N = 2m \quad , \quad L_1 = M_1 + \bar{M}_1 \quad , \quad M_1 = \sum_{1}^{m} M_{1k} \quad , \quad M_{1k} = (a_k,b_k) \quad ,$$

$$\bar{M}_1 = \sum_{1}^{m} \bar{M}_{1k} \quad , \quad \bar{M}_{1k} = (-b_k,-a_k) \quad . \tag{4.24}$$

Then the problem may be reduced to a singular integral equation with dominant part (i.e., Cauchy kernel) only. For this the problem is first decomposed into symmetric (i.e., $u(r,\theta) = u(r,-\theta)$), and antisymmetric, (i.e., $u(r,\theta) = -u(r,-\theta)$) parts by separating the input functions into even and odd components. In the symmetric problem $g(\theta) = g(-\theta)$, $f(\theta) = f(-\theta)$, $\phi(\theta) = -\phi(-\theta)$, and (4.7) may be expressed as

$$\sum_1^m \frac{1}{2\pi}\int_{a_k}^{b_k} \phi(t)\left[\cot\frac{t-\theta}{2} + \cot\frac{t+\theta}{2}\right]dt = g_o(\theta) = -g(\theta)$$

$$-\frac{1}{2\pi}\int_{M_2} f'(t)\left[\cot\frac{t-\theta}{2} + \cot\frac{t+\theta}{2}\right]dt \quad , \quad \theta\epsilon M_1 \qquad (4.25)$$

where M_2 is the complement of M_1 on $0\leq\theta\leq\pi$. Or, from

$$\cot(a-b)\mp\cot(a+b) = 2\left\{\begin{matrix}\sin 2b \\ \sin 2a\end{matrix}\right\}/(\cos 2b - \cos 2a) \qquad (4.26)$$

it is found that

$$\frac{1}{\pi}\int_{M_1} \frac{\phi(t)\sin t}{\cos\theta - \cos t}\,dt = g_o(\theta) \quad , \quad \theta\epsilon M_1 \quad . \qquad (4.27)$$

If we now define

$$\cos t = \alpha \quad , \quad \phi(t) = \psi(\alpha) \quad , \quad \cos\theta = \beta \quad , \quad g_o(\theta) = G(\beta) \quad ,$$

$$t\epsilon M_1 \rightarrow \alpha\epsilon\Gamma \quad , \quad \Gamma = \sum_1^m \Gamma_k \qquad (4.28)$$

equation (4.27) becomes

$$\frac{1}{\pi}\int_\Gamma \frac{\psi(\alpha)}{\alpha-\beta}\,d\alpha = G(\beta) \quad , \quad \beta\epsilon\Gamma \quad . \qquad (4.29)$$

Similarly for the antisymmetric problem $g(\theta) = -g(-\theta)$, $f(\theta) = -f(-\theta)$, $\phi(\theta) = \phi(-\theta)$ and (4.7) becomes

$$\frac{1}{2\pi}\int_{M_1} \phi(t)\left[\cot\frac{t-\theta}{2} - \cot\frac{t+\theta}{2}\right]dt = g_o(\theta) = -g(\theta)$$

$$-\frac{1}{2\pi}\int_{M_2} f'(t)\left[\cot\frac{t-\theta}{2} - \cot\frac{t+\theta}{2}\right]dt \quad , \quad \theta\epsilon M_1 \quad , \qquad (4.30)$$

with the substitutions (4.28) giving

$$\frac{1}{\pi}\int_{\Gamma}\left(\frac{\psi(\alpha)}{\sqrt{1-\alpha^2}}\right)\frac{d\alpha}{\alpha-\beta}=\left(\frac{G(\beta)}{\sqrt{1-\beta^2}}\right) \quad , \qquad \beta\varepsilon\Gamma \quad . \tag{4.31}$$

In both cases the solution is given by (4.21) with $k(t,s) = 0$ and appropriate changes in notation. m integration constants arising from the solution are again determined from the single-valuedness conditions (4.17). In the symmetric case A_o is given by (4.20) (with, again the appropriate changes) and in the antisymmetric case it is zero.

For example, if

$$m = 1,\ M_1 = (\theta_1,\theta_2) \quad , \quad \alpha_1 = \cos\theta_1 \quad , \quad \alpha_2 = \cos\theta_2 \quad ,$$

$$0<\theta_1<\theta_2<\pi \quad , \quad g(\theta) = g(-\theta) \quad , \quad f(\theta) = 0 \quad , \tag{4.32}$$

it may be shown that

$$\psi(\alpha) = \begin{cases} \dfrac{1}{[(\alpha-\alpha_2)(\alpha_1-\alpha)]^{1/2}}\left[\dfrac{1}{\pi}\displaystyle\int_{\alpha_2}^{\alpha_1}\dfrac{G(\beta)(\beta-\alpha_2)^{1/2}(\alpha_1-\beta)^{1/2}}{\beta-\alpha} + c_o\right] \quad , \\ \qquad\qquad (\alpha_2<\alpha<\alpha_1) \quad , \\ 0 \quad , \quad (-1<\alpha<\alpha_2 \quad , \quad \alpha_1<\alpha<1) \end{cases} \tag{4.33}$$

where c_o and A_o are determined from

$$\int_{\alpha_2}^{\alpha_1}\frac{\psi(\alpha)d\alpha}{\sqrt{1-\alpha^2}} = 0 \quad , \qquad A_o = -\frac{1}{\pi}\int_{\theta_1}^{\theta_2} t\phi(t)dt \quad . \tag{4.34}$$

Furthermore, if $g(\theta) = a_o$ = constant, $\psi(\alpha)$ becomes

$$\psi(\alpha) = a_o(\alpha-\alpha_2)^{-1/2}(\alpha_1-\alpha)^{-1/2}\left(\frac{\alpha_1+\alpha_2}{2} - \alpha + b_o\right) \quad ,$$

$$b_o = \frac{1+\alpha_2}{K(k)}\,\Pi\left(\frac{\alpha_2-\alpha_1}{1+\alpha_1}\,,\,k\right) - \frac{2+\alpha_1+\alpha_2}{2} \quad , \quad k^2 = \frac{2(\alpha_1-\alpha_2)}{(1-\alpha_2)(1+\alpha_1)} \quad ,$$

$$\Pi(\sigma,k) = \int_0^1 \frac{dx}{(1+\sigma x^2)(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} \qquad (4.35)$$

where K(k) is the complete elliptic integral of the first kind.

Consider now the general case (4.7), i.e.

$$\frac{1}{2\pi} \int_{L_1} \phi(t)\cot\frac{t-\theta}{2}\,dt = g_o(\theta) \quad , \qquad \theta\varepsilon L_1 \qquad (4.36)$$

where $g_o(\theta)$ is the right-hand side of (4.7) and $L_1 = \sum_1^N L_{1k}$ has no symmetry. Let the origin is selected in such a way that $\theta = \pi$ is not on L_1. Defining

$$\tan\frac{t}{2} = s \quad , \quad \tan\frac{\theta}{2} = p \quad , \quad \phi(t) = \psi(s) \quad , \quad g_o(\theta) = G(p) \quad ,$$

$$t\varepsilon L_1 \rightarrow s\varepsilon\Gamma_1 \quad , \quad \Gamma_1 = \sum_1^N \Gamma_{1k} \qquad (4.37)$$

equation (4.36) may easily be expressed as

$$\frac{1}{\pi} \int_{\Gamma_1} \psi(s)\,\frac{1+ps}{s-p}\,\frac{ds}{1+s^2} = G(p) \quad , \qquad p\varepsilon\Gamma_1 \quad , \qquad (4.38)$$

or

$$\frac{1}{\pi} \int_{\Gamma_1} \frac{\psi(s)}{s-p}\,ds = G(p) + K \quad , \qquad p\varepsilon\Gamma_1 \quad , \qquad (4.39)$$

$$K = \frac{1}{\pi} \int_{\Gamma_1} \frac{s\psi(s)}{1+s^2}\,ds \qquad (4.40)$$

The singular integral equation (4.39) may be solved in a straightforward manner with (4.40) accounting for the additional constant K. Again, the solution of (4.39) will contain N arbitrary constants which may be determined from (4.17) and (4.20) gives the constant A_o.

4.2 An Example on Triple Series Equations

Consider now the somewhat more general mixed boundary value problem for the unit circle defined by (2.10) and formulated by the triple series equations (2.11). In this problem let the auxiliary function be[(1)]

$$\phi(\theta) = \frac{\partial}{\partial r} u(1,\theta) \quad , \quad \theta \varepsilon L_1 + L_2 + L_3 = L = (0,2\pi) \quad . \tag{4.41}$$

From (2.5) it then follows that

$$nA_n = \frac{1}{\pi} \int_L \phi(t) \cos nt \; dt \quad ,$$

$$nB_n = \frac{1}{\pi} \int_L \phi(t) \sin nt \; dt \tag{4.42}$$

where $\phi(t) = f_1(t)$ is known on L_1. After differentiating (4.11b) and, again for analytical expediency expressing them in limiting form, the remaining boundary conditions (4.11b and c) may be written as

$$\lim_{r\to 1-0} \sum_1^\infty n \, r^n(-A_n \sin n\theta + B_n \cos n\theta) = f_2'(\theta) \quad , \quad \theta \varepsilon L_2$$

$$\lim_{r\to 1-0} \{h_1 A_o + h_1 \sum_1^\infty r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$+ h_2 \sum_1^\infty n \, r^{n-1}(A_n \cos n\theta + B_n \sin n\theta) = f_3(\theta) \quad , \theta \varepsilon L_3 \tag{4.43}$$

Substituting now from (4.42) into (4.43), using (4.6) and [17]

$$\sum_1^\infty \frac{1}{n} \cos nz = -\log(2|\sin \frac{z}{2}|) \quad , \tag{4.44}$$

we obtain

$$\frac{1}{2\pi} \int_L \phi(t) \cot \frac{t-\theta}{2} \, dt = f_2'(\theta) \quad , \quad \theta \varepsilon L_2 \quad ,$$

$$-\frac{h_1}{\pi} \int_L \phi(t) \log(2|\sin \frac{t-\theta}{2}|) dt + h_2 \phi(\theta) = f_3(\theta) - h_1 A_o , \theta \varepsilon L_3 . \tag{4.45a,b}$$

---

(1) See the general remarks and the broad guidelines at the end of Section 4 regarding the selection of the auxiliary (i.e., the new unknown) function.

Just a superficial observation would indicate that the two integral equations in (4.45) are of entirely different character. Hence, near and at the end points of L_2 and L_3 one would expect the function $\phi(t)$ behave quite differently. For closer examination let us assume that

$$\phi(t) = \phi_1(t) \quad , \quad t\varepsilon L_2 \quad , \quad \phi(t) = \phi_2(t) \quad , \quad t\varepsilon L_3 \quad . \tag{4.46}$$

Equations (4.45) may then be expressed as

$$\frac{1}{2\pi} \int_{L_2} \phi_1(t) \cot \frac{t-\theta}{2}\, dt + \frac{1}{2\pi} \int_{L_3} \phi_2(t) \cot \frac{t-\theta}{2}\, dt$$

$$= f_2'(\theta) - \frac{1}{2\pi} \int_{L_1} f_1(t) \cot \frac{t-\theta}{2}\, dt = g_1(\theta) \quad , \quad \theta\varepsilon L_2 \quad ,$$

$$- \frac{h_1}{\pi} \int_{L_2} \phi_1(t) \log(2|\sin \frac{t-\theta}{2}|)dt - \frac{h_1}{\pi} \int_{L_3} \phi_2(t) \log(2|\sin \frac{t-\theta}{2}|)$$

$$+ h_2\phi_2(\theta) = f_3(\theta) - h_1 A_o + \frac{h_1}{\pi} \int_{L_1} f_1(t) \log(2|\sin \frac{t-\theta}{2}|)dt = g_2(\theta) \quad ,$$

$$\theta\varepsilon L_3 \tag{4.47a,b}$$

Separating the dominant parts of the kernels it is seen that (4.47) is of the following general form:

$$\frac{1}{\pi} \int_{L_2} \frac{\phi_1(t)}{t-\theta}\, dt = g_1(\theta) - \int_{L_2} k_{11}(\theta,t)\phi_1(t)dt - \int_{L_3} k_{12}(\theta,t)\phi_2(t)\, dt \quad ,$$

$$\theta\varepsilon L_2 \quad ,$$

$$h_2\phi_2(\theta) - \frac{h_1}{\pi} \int_{L_3} \phi_2(t) \log|t-\theta|dt = g_2(\theta) - \int_{L_2} k_{21}(\theta,t)\phi_1(t)dt$$

$$- \int_{L_3} k_{22}(\theta,t)\phi_2(t)dt \quad , \quad \theta\varepsilon L_3 \tag{4.48a,b}$$

where the kernels $k_{ij}(\theta,t)$, $(i,j = 1,2)$ are bounded in their respective domains.

It is clear that (4.48a) is a typical singular integral equation of the general form (4.8), it has a fundamental function of the form (4.15), and if $L_2 = \sum_1^N L_{2k}$, $L_{2k} = (a_k, b_k)$, $a_k < b_k < a_{k+1}$, its solution will contain N arbitrary constants. On the other hand, (4.48b) is an integral equation of the second kind with a weakly singular but square integrable kernel. Hence, its solution is bounded everywhere on L_3, including the end points (as they are approached from L_3) and is uniquely determined without any reference to any additional conditions. The coupling of the two integral equations is through Fredholm kernels. Therefore the basic singular behavior of the solution will be unaffected[(1)] by coupling. From (4.10) it is clear that if h_1 or h_2 is zero the problem reduces to that considered in the previous section. This can also be seen from (4.47). For $h_1 = 0$ this is quite clear. For $h_2 = 0$ differentiating (4.47b) and observing that

$$-\frac{d}{d\theta}\log\left(2\left|\sin\frac{t-\theta}{2}\right|\right) = \frac{1}{2}\cot\frac{t-\theta}{2} \tag{4.49}$$

(4.47) is seen to reduce to a simple singular integral equation defined on $L_2 + L_3$. It is worthwhile to reemphasize that an integral equation of the second kind with a logarithmic kernel is basically a Fredholm integral equation and has a bounded solution. However, if the integral equation is of the first kind and has a logarithmic kernel, then it is equivalent to a singular integral equation with a simple Cauchy type kernel. Also, the dominant equation

$$A\phi(x) + B\int_L \phi(t)\log|t-x|\,dt = f(x) \quad , \qquad x\varepsilon L \tag{4.50}$$

---

(1) As will be pointed out later in this article, if the coupling is through generalized Cauchy kernels, then the singular behavior of the solution will be affected.

has apparently no closed form solution. Even though at the end points of L the solution of (4.50) is bounded, it may easily be shown that its derivative has a logarithmic singularity. This can be seen by differentiating and modifying (4.50) as follows:

$$\phi'(x) = \frac{B}{A} \int_L \frac{\phi(t)}{t-x}\, dt + \frac{f'(x)}{A}$$

$$= \frac{B}{A} \int_L \frac{\phi(t)-\phi(x)}{t-x}\, dt + \frac{B}{A}\, \phi(x) \int_L \frac{dt}{t-x} + \frac{f'(x)}{A} \;, \quad x \varepsilon L \qquad (4.51)$$

which, in the neighborhood of a typical end point $x \leq c$ becomes

$$\phi'(x) = \frac{B}{A}\, \phi(c) \log|c-x| + G(x) \quad , \quad (x \to c) \qquad (4.52)$$

where $G(c)$ is bounded.

If $L_2 = \sum_1^N L_{2k}$, to complete the solution of the problem the arbitrary constants $c_o, c_1, .., c_{N-1}$ arising from the solution of the singular integral equation (4.47a) and A_o must be determined. Observing that the boundary condition $u(1,\theta) = f_2(\theta)$, $\theta \varepsilon L_2$, was satisfied only in differentiated form and

$$u(1,\theta) = A_o - \frac{1}{\pi} \int_L \phi(t) \log(2|\sin \frac{t-\theta}{2}|)\, dt \quad , \qquad (4.53)$$

$\phi(t)$ must satisfy the following single-valuedness conditions:

$$A_o - \frac{1}{\pi} \int_L \phi(t) \log(2|\sin \frac{t-\theta_k}{2}|)\, dt = f_2(\theta_k) \quad ,$$

$$\theta_k \varepsilon L_k \quad , \quad k = 1, .., N \qquad (4.54)$$

where θ_k is any convenient point on L_{2k}. Thus, equations (4.54) with the flux equilibrium condition

$$\int_{L_1} f_1(t)dt + \int_{L_2} \phi(t)dt + \int_{L_3} \phi_2(t)dt = 0 \qquad (4.55)$$

provide N + 1 algebraic equations to determine $c_o,..,c_{N-1}$, and A_o.

4.3 Reduction of Multiple Integral Equations

To demonstrate the technique of reducing a system of multiple integral equations to that of singular integral equations consider the somewhat general mixed boundary value problem in potential theory which is defined by (2.21) for the half plane y>0. The solution is expressed by (2.17) in terms of the unknown function A(α) which is to be determined from the multiple integral equations (2.22). In this problem let the normal flux

$$\frac{\partial}{\partial y} u(x,o) = \phi(x) = \int_{-\infty}^{\infty} -|\alpha|A(\alpha)e^{-i\alpha x}d\alpha \quad , \quad -\infty<x<\infty \qquad (4.56)$$

be selected as the new unknown function. If $\phi(x)$ is known, the Fourier inversion

$$-|\alpha|A(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)e^{i\alpha t}dt \qquad (4.57)$$

with (2.17) would give the complete solution. From (2.21) it is seen that $\phi(x) = f_1(x)$ is known on L_1 and is unknown on L_2 and L_3. Again for reasons of uniform convergence writing the boundary conditions (2.21b) and (2.21c) in limit form, differentiating (2.21b) and using (4.57) we find

$$\lim_{y\to+0} [\frac{i}{2\pi}\int_o^{\infty} e^{-\alpha(y+ix)}d\alpha \int_{-\infty}^{\infty} \phi(t)e^{i\alpha t}dt$$

$$- \frac{i}{2\pi}\int_{-\infty}^{o} e^{\alpha(y-ix)}d\alpha \int_{-\infty}^{\infty} \phi(t)e^{i\alpha t}dt = f_2'(x) \quad , \quad x\epsilon L_2 \quad ,$$

$$\frac{1}{2\pi} h_1 \lim_{y\to+0} [- \int_o^{\infty} e^{-\alpha(y+ix)} \frac{d\alpha}{\alpha} \int_{-\infty}^{\infty} \phi(t)e^{i\alpha t}dt$$

$$+ \int_{-\infty}^{0} e^{\alpha(y-ix)} \frac{d\alpha}{\alpha} \int_{-\infty}^{\infty} \phi(t) e^{i\alpha t} dt] + h_2\phi(x) = f_3(x)$$

$$x\varepsilon L_3 \quad . \qquad (4.58a,b)$$

Changing the order of integrations and evaluating the inner integrals, (4.58a) becomes

$$\frac{i}{2\pi} \lim_{y\to+0} \int_{-\infty}^{\infty} \phi(t)[\frac{1}{y-i(t-x)} - \frac{1}{y+i(t-x)}]dt = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi(t)}{t-x} dt = f_2'(x) \quad ,$$

$$x\varepsilon L_2 \quad . \qquad (4.59)$$

Similarly, the first term on the left hand side of (4.58b) may be expressed as

$$-\frac{h_1}{\pi} \lim_{y\to+0} \int_{-\infty}^{\infty} \phi(t)dt \int_{0}^{\infty} \frac{e^{-\alpha y}}{\alpha} \cos\alpha(t-x)d\alpha$$

$$= -\frac{h_1}{\pi} \lim_{y\to+0} \int_{-\infty}^{\infty} \phi(t)dt\{\int_{0}^{\infty} \frac{e^{-\alpha y}}{\alpha} d\alpha + \log y$$

$$- \log[y^2 + (t-x)^2]^{1/2}\} = \frac{h_1}{\pi} \int_{-\infty}^{\infty} \phi(t)\log|t-x|dt \qquad (4.60)$$

where the following condition of flux equilibrium is used to eliminate the divergent terms:(1)

$$\int_{-\infty}^{\infty} \phi(t)dt = 0 \quad . \qquad (4.61)$$

Now observing that $L_1 + L_2 + L_3 = (-\infty,\infty)$, $\phi(t) = f_1(t)$, $t\varepsilon L_1$, and defining

$$\phi(t) = \phi_1(t) \quad , \quad t\varepsilon L_2 \quad , \quad \phi(t) = \phi_2(t) \quad , \quad t\varepsilon L_3 \qquad (4.62)$$

(1) Note that if (4.61) is not satisfied and if u is zero at infinity (as assumed in the present problem) then u(x,0) will not be bounded and will tend to infinity as log y, $y\to 0$.

the integral equations (4.58) may be expressed as

$$\frac{1}{\pi}\int_{L_2}\frac{\phi_1(t)}{t-x}\,dt+\frac{1}{\pi}\int_{L_3}\frac{\phi_2(t)}{t-x}\,dt=-f_2'(x)-\frac{1}{\pi}\int_{L_1}\frac{f_1(t)}{t-x}\,dt\quad,\quad x\varepsilon L_2\quad,$$

$$h_2\phi_2(x)+\frac{h_1}{\pi}\int_{L_3}\phi_2(t)\log|t-x|dt+\frac{h_1}{\pi}\int_{L_2}\phi_1(t)\log|t-x|dt$$

$$=f_3(x)-\frac{h_1}{\pi}\int_{L_1}f_1(t)\log|t-x|dt\quad,\quad x\varepsilon L_3\quad. \tag{4.63a,b}$$

In structure the system of equations (4.63) is identical to (4.47) or (4.48). If L_2 is finite with $L_2=\sum_1^N L_{2j}$, $L_{2j}=(a_j,b_j)$, the solution of (4.63a) will again contain N arbitrary constants which may be obtained from (4.61) and the following N-1 single-valuedness conditions:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(t)\log\left|\frac{t-a_{j+1}}{t-bj}\right|dt=f_2(a_{j+1})-f_2(b_j)\quad,\quad j=1,..,N-1\quad. \tag{4.64}$$

In the type of problems under consideration the case of non-zero resultant flux may be particularly important. From (4.60) it is clear that if (4.61) is not satisfied the derivation leading to (4.63) is not valid. Let us now assume that L_2 is finite[(1)] and

$$\int_{-\infty}^{\infty}\phi(t)dt=Q \tag{4.65}$$

(since $u(x,0)=f_2(x)$, $x\varepsilon L_2$ is assumed to be bounded) $u(x,y)\sim\log y$ as $y\to\infty$. However, the components of the flux vector may still be expressed in terms of an unknown function $A(\alpha)$ as follows:

$$\frac{\partial}{\partial x}u(x,y)=-\int_{-\infty}^{\infty}i\alpha A(\alpha)e^{-y|\alpha|-i\alpha x}d\alpha\ ,$$

(1) If L_2 is infinite, by a proper superposition the problem may always be reduced to that in which (4.61) is satisfied.

$$\frac{\partial}{\partial y} u(x,y) = -\int_{-\infty}^{\infty} |\alpha| A(\alpha) e^{-y|\alpha| - i\alpha x} d\alpha \quad . \tag{4.66a,b}$$

Again, defining the auxiliary function $\phi(x)$ by (4.56) and substituting from (4.57) into (4.66a) we obtain

$$\frac{\partial}{\partial x} u(x,0) = \frac{i}{2\pi} \lim_{y \to +0} [\int_{o}^{\infty} e^{-\alpha(y+ix)} d\alpha \int_{-\infty}^{\infty} \phi(t) e^{i\alpha t} dt$$

$$- \int_{-\infty}^{o} e^{\alpha(y-ix)} d\alpha \int_{-\infty}^{\infty} \phi(t) e^{i\alpha t} dt]$$

$$= \frac{i}{2\pi} \lim_{y \to +0} \int_{-\infty}^{\infty} \phi(t) \frac{2i(t-x)}{y^2+(t-x)^2} dt$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t) \frac{dx}{t-x} \quad , \quad (-\infty<x<\infty) \tag{4.67}$$

from which it follows that

$$u(x,0) = f_2(x_o) + \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t) \log\left|\frac{t-x}{t-x_o}\right| dt \quad , \quad (-\infty<x<\infty) \tag{4.68}$$

where x_o is any point on L_2. Thus, the integral equation (4.63a) is still valid and, with the definitions (4.62) and using (4.68), (4.63b) will have to be replaced by

$$h_2\phi_2(x) + \frac{h_1}{\pi} \int_{L_3} \phi_2(t) \log\left|\frac{t-x}{t-x_o}\right| dt + \frac{h_1}{\pi} \int_{L_2} \phi_1(t) \log\left|\frac{t-x}{t-x_o}\right| dt$$

$$= f_3(x) - h_1 f_2(x_o) - \frac{h_1}{\pi} \int_{L_1} f_1(t) \log\left|\frac{t-x}{t-x_o}\right| dt \quad , \quad x \varepsilon L_3 \quad . \tag{4.69}$$

the N integration constants arising from the solution of (4.63a) are determined from (4.64) (which follows now from (4.68) and from the fact that $u(x,0) = f_2(x)$, $x \varepsilon L_2$) and (4.65). Note that under the stated conditions of the problem (4.63b) and (4.69) are always nonhomogeneous. Also note that if $L_2 = 0$, the problem is reduced to

$$h_2\phi_2(x) + \frac{h_1}{\pi} \int_{L_3} \phi_2(t)\log|t-x|dt = f(x) \quad , \quad x\varepsilon L_3 \qquad (4.70)$$

where f(x) is a known function. Equation (4.70) has no closed form solution. In this case it should be noted that $u(x_o,0) = f_2(x_o)$ which appear in (4.69) is an unknown constant and is determined by the flux equilibrium condition (4.65) where x_o now is an arbitrary point on the real axis. On the other hand if $L_3 = 0$ and $L_2 \neq 0$, the problem is formulated by (4.63a) which may easily be solved in closed form.

As pointed out previously the solution of the system of singular integral equations (4.63) will be of the following form

$$\phi_1(x) = w(x)F_1(x) \quad , \quad w(x) = \prod_1^N [(x-a_k)(b_k-x)]^{-1/2} \quad , \quad x\varepsilon L_2 \quad ,$$

$$\phi_2(x) = F_2(x) \quad , \quad x\varepsilon L_3 \qquad (4.71)$$

where F_1 and F_2 are bounded functions and at the end points of L_3 ϕ_2 is bounded and has a behavior similar to (4.52) whereas at the end points of L_2 ϕ_1 is singular.

In order to have a better understanding of the flux distribution $\phi(x)$ on the boundary it may be worthwhile to relate the foregoing results to a simple physical problem. Consider, for example, the anti-plane shear problem for a symmetrically loaded infinite medium shown in Figure 2. Here u(x,y) is the z- component of the displacement vector, $\phi(x) = \sigma_{yz}(x,0)/\mu$ represents the traction at y = 0, and μQ is the resultant force in z- direction, μ being the shear modulus of the medium. The part of the x-axis $L_1 = \Sigma L_{1j}$ corresponds to a series of cracks on which the surface traction $\phi(x) = f_1(x) = \sigma_{yz}/\mu$ is specified. L_2 is clearly the uncut portion of the real axis on which (in this problem) $u(x,0) = f_2(x) = 0$. Physically the part L_3 on which

$$h_1 u(x,0) + h_2 \frac{\partial}{\partial y} u(x,0) = f_3(x) \tag{4.72}$$

corresponds to a series of cracks the surfaces of which are joined through an elastic adhesive layer. If the thickness of the adhesive is sufficiently small it may be represented by a one-dimensional spring and may be modeled by (4.72) with

$$h_1 = \mu_a/h \quad , \quad h_2 = -\mu \quad , \quad f_3(x) = 0 \quad . \tag{4.73}$$

where μ_a and h are, respectively, the shear modulus and the thickness of the adhesive. Thus, the spring force $\mu_a u/h$ will act as a traction on the crack surface along L_3. Physically then the plane is cut along $L_1 + L_3$, which means that the stress $\sigma_{yz}(x,0) = \mu\phi(x)$ must have the expected square root singularity at the end points of L_2 (as x approaches these points from L_2). Note that whether the end point belongs to the intersection of L_2 and L_1 or L_2 and L_3, this singular behavior will remain unchanged. On the other hand in the adhesive layer (i.e., on L_3) the shear stress will be bounded everywhere, including the end points. Further applications of the technique to the mechanics of bonded joints may be found in [18] and [19].

4.4 Reduction of Multiple Series-Multiple Integral Equations

In some mechanics problems because of the geometry of the domain the separation of variables technique may lead to a formulation in which some of the unknown functions are expressed as series of eigenfunctions having a set of undetermined coefficients and some as inversion integrals involving certain undetermined functions. Substitution of these expressions into the mixed boundary conditions invariably gives rise to a system of equations involving both multiple series and multiple integral equations. Once the auxiliary functions are properly selected, reduction of this type of problems to a system of singular integral equations is also rather straightforward. In this section we again select a highly representative and a relatively simple example to demonstrate the technique.

Consider the "load transfer" problem shown in Figure 3. The figure describes either a coupling in which the torque is transmitted from the shaft 1 to the shaft 3 through the sleeve 2, or a gear or pulley in which the external torque acting on the sleeve 2 is not zero. In this problem it is assumed that the shafts are made of the same material with shear modulus μ_1 and the sleeve has the shear modulus μ_2. It is further assumed that (in addition to axial symmetry) $x = 0$ is a plane of symmetry (or antisymmetry) with respect to the external loads and the geometry of the problem. Thus, in both media the circumferential component of the displacement u_i, $(i = 1,2)$ is the only non-zero displacement which satisfies the following differential equation:

$$\frac{\partial^2 u_i}{\partial r^2} + \frac{1}{r}\frac{\partial u_i}{\partial r} - \frac{u_i}{r} + \frac{\partial^2 u_i}{\partial x^2} = 0 \quad , \quad (r<a \; : \; i = 1 \; , \quad a<r<R \; : \; i = 2) \tag{4.74}$$

the nonvanishing stress components are given by

$$\sigma_{ir\theta} = \mu_i\left(\frac{\partial u_i}{\partial r} - \frac{u_i}{r}\right) \quad , \quad \sigma_{i\theta x} = \mu_i \frac{\partial u_i}{\partial x} \quad , \quad (i = 1,2) \tag{4.75}$$

For the sake of generality at this point it is assumed that the end clearance $2b_1$ is not zero and $2c<2b$ (Figure 3). Because of symmetry it is sufficient to consider one half $(x>0)$ of the composite medium only.

In this problem the solution of (4.74) must be found subject to the following conditions:

$$\frac{\partial}{\partial x}[u_2(a+0,x) - u_1(a-0,x)] = f(x) \quad , \quad b_1<x<c \quad , \qquad \text{(a)}$$

$$\sigma_{1r\theta}(a,x) = \sigma_{2r\theta}(a,x) \quad , \quad b_1<x<c \quad , \qquad \text{(b)}$$

$$\sigma_{1r\theta}(a,x) = 0 \quad , \quad x>c \quad , \qquad \text{(c)}$$

$$\sigma_{2r\theta}(a,x) = 0 \quad , \quad 0\leq x<b_1 \quad , \quad c<x<b \quad , \qquad \text{(d)}$$

$$\sigma_{1\theta x}(r,b_1) = 0 \quad , \quad 0\leq r<a \quad , \qquad \text{(e)}$$

$$\sigma_{2\theta x}(r,b) = 0 \quad , \quad a<r<R \quad , \qquad \text{(f)}$$

$$\sigma_{2r\theta}(R,x) = \sigma_o(x) \quad , \text{ or } u_2(R,x) = 0 \quad , \quad 0\leq x<b \quad , \qquad \text{(g)}$$

$$\left.\begin{array}{ll} \frac{\partial}{\partial x} u_2(r,0) = 0 & \text{(symmetric case)} \\ u_2(r,0) = 0 & \text{(antisymmetric case)} \end{array}\right\} \quad a<r<R \qquad \text{(h)}$$

$$\int_{b_1}^{c} 2\pi a^2 p(x)\,dx = T \quad , \qquad \text{(i)(4.76)}$$

where the functions $f(x)$, $\sigma_o(x)$ and the constant T are known. Referring to Figure 3, the following symmetry conditions must be satisfied:

For the symmetric problem:

$$u_i(r,x) = u_2(r,-x) \quad , \quad \sigma_{ir\theta}(r,x) = \sigma_{ir\theta}(r,-x) \quad , \quad i = 1,2 \quad ,$$

$$f(x) = - f(-x) \quad , \quad T_1 = T_2 \quad , \quad \sigma_o(x) = \sigma_o(-x) \quad . \tag{4.77}$$

For the antisymmetric problem:

$$u_i(r,x) = - u_i(r,-x) \quad , \quad \sigma_{ir\theta}(r,x) = - \sigma_{ir\theta}(r,-x) \quad , \quad i = 1,2 \quad ,$$

$$f(x) = f(-x) \quad , \quad T_1 = - T_2 \quad , \quad \sigma_o(x) = - \sigma_o(-x) \quad . \tag{4.78}$$

Considering (4.77) and (4.78) the solution of (4.74) satisfying (4.76e,f, and h) may easily be expressed as

$$u_1(r,x) = \frac{2}{\pi} \int_o^\infty A(\alpha) I_1(\alpha r) \cos\alpha(x-b_1) d\alpha \quad , \quad x>b_1 \quad , \quad 0\leq r<a \quad ,$$

$$u_2(r,x) = \sum_1^\infty [B_n K_1(\alpha_n r) + C_n I_1(\alpha_n r)] \begin{Bmatrix} \cos\alpha_n x \\ \sin\alpha_n x \end{Bmatrix} \quad , \quad 0\leq xb \quad ,$$

$$a<r<R \quad , \tag{4.79a,b}$$

where from (4.75) and (4.76f) it is seen that

$$\alpha_n = \pi n/b \quad \text{for symmetric problem}$$

$$\alpha_n = (2n - 1)\pi/2b \quad \text{for antisymmetric problem} \tag{4.80a,b}$$

In (4.79b) the upper and lower terms in the series stand for symmetric and antisymmetric problems respectively. Substituting now from (4.79) into the mixed boundary conditions (4.76) (a-d) we find

$$\lim_{r\to a+0} \sum^\infty \alpha_n [B_n K_1(\alpha_n r) + C_n I_1(\alpha_n r)] \begin{Bmatrix} - \sin\alpha_n x \\ \cos\alpha_n x \end{Bmatrix}$$

$$+ \lim_{r\to a-0} \frac{2}{\pi} \int_o^\infty \alpha A(\alpha) I_1(\alpha r) \sin\alpha(x-b_1) d\alpha = f(x) \quad , \quad b_1<x<c \quad , \tag{a}$$

$$\mu_2 \sum_1^\infty [- B_n K_2(\alpha_n a) + C_n I_2(\alpha_n a)] \alpha_n \begin{Bmatrix} \cos\alpha_n x \\ \sin\alpha_n x \end{Bmatrix}$$

$$-\frac{2\mu_1}{\pi}\int_o^\infty A(\alpha)\alpha I_2(\alpha a)\cos\alpha(x-b_1)d\alpha = 0 \quad , \quad b_1<x<c \quad , \qquad \text{(b)}$$

$$\frac{2\mu_1}{\pi}\int_o^\infty A(\alpha)\alpha I_2(\alpha a)\cos\alpha(x-b_1)d\alpha = 0 \quad , \quad x>c \quad , \qquad \text{(c)}$$

$$\mu_2\sum_1^\infty[-B_nK_2(\alpha_n a) + C_nI_2(\alpha_n a)]\alpha_n\begin{Bmatrix}\cos\alpha_n x\\ \sin\alpha_n x\end{Bmatrix} = 0 \quad ,$$

$$0\leq x<b_1 \quad , \quad c<x<b \quad . \qquad \text{(d) (4.81)}$$

Note that one set of constants B_n or C_n may be eliminated by using the condition (4.76g). For example,[1]

$$B_nK_1(\alpha_n R) + C_nI_1(\alpha_n R) = 0 \text{ for } u_2(R,x) = 0 \quad ,$$

$$-B_nK_2(\alpha_n R) + C_nI_2(\alpha_n R) = \frac{2}{b\mu_2}\int_o^b \sigma_o(x)\begin{Bmatrix}\cos\alpha_n x\\ \sin\alpha_n x\end{Bmatrix} dx \quad ,$$

$$\text{for } \sigma_{2r\theta}(R,x) = \sigma_o(x) \quad , \qquad \text{(4.82a,b)}$$

With (4.82), (4.81) provides a system of dual series-dual integral equations to determine the set of unknown constants B_n (or C_n) and the unknown function $A(\alpha)$.

In this problem, the contact stress

$$\sigma_{1r\theta}(a,x) = \sigma_{2r\theta}(a,x) = p(x) \qquad \text{(4.83)}$$

suggests itself as being the most appropriate auxiliary function. Thus, from (4.82) and the expressions

$$\sigma_{1r\theta}(a,x) = \frac{2\mu_1}{\pi}\int_o^\infty A(\alpha)\alpha I_2(\alpha a)\cos\alpha(x-b_1)d\alpha = p(x)$$

---

(1) In (4.82b) $\sigma_o(x) = 0$ being the practical case of coupling.

$$(b_1<x<\infty) \quad ,$$

$$\sigma_{2r\theta}(a,x) = \mu_2 \sum_1^{\infty} [- B_n K_2(\alpha_n a) + C_n I_2(\alpha_n a)]\alpha_n \begin{Bmatrix} \cos\alpha_n x \\ \sin\alpha_n x \end{Bmatrix} = p(x)$$

$$(0 \leq x < b) \quad , \qquad (4.84a,b)$$

evaluating $A(\alpha)$, B_n, and C_n in terms of $p(x)$ and substituting into (4.81a) we obtain

$$- \lim_{r \to a+0} \frac{1}{b\mu_2} \int_{b_1}^{c} p(t)dt \sum_1^{\infty} L_n(r)[\sin\alpha_n(t-x) \mp \sin\alpha_n(t+x)]$$

$$- \lim_{r \to a-0} \frac{1}{\pi\mu_1} \int_{b_1}^{c} p(t)dt \int_0^{\infty} \frac{I_1(\alpha r)}{I_2(\alpha a)} [\sin\alpha(t-x) - \sin\alpha(t+x-2b_1)]d\alpha$$

$$= f(x) \quad , \quad b_1<x<c \quad , \qquad (4.85)$$

where the upper and lower signs in the series refer to the symmetric and the antisymmetric cases, respectively, and

$$L_n(r) = \frac{K_1(r\alpha_n)I_1(R\alpha_n) - I_1(r\alpha_n)K_1(R\alpha_n)}{K_2(a\alpha_n)I_1(R\alpha_n) + I_2(a\alpha_n)K_1(R\alpha_n)} \quad \text{for } u_2(R,x) = 0$$

$$L_n(r) = \frac{K_1(r\alpha_n)I_2(R\alpha_n) + I_1(r\alpha_n)K_2(R\alpha_n)}{K_2(a\alpha_n)I_2(R\alpha_n) - I_2(a\alpha_n)K_2(R\alpha_n)} \quad \text{for } \sigma_0(x) = 0 \quad . \qquad (4.86a,b)$$

In deriving (4.85) the conditions (4.81b-d) has been used.

We now observe that for $t = x$ the series and the integral giving the kernels in (4.85) are divergent. These divergent parts may be studied and separated by considering the asymptotic behavior of the terms in the series and of the integrand. In (4.86) note that since $R>a$, letting $r = a+\varepsilon$, for large values of α_n we obtain

$$L_n(r) \cong \frac{K_1(a\alpha_n + \varepsilon\alpha_n)}{K_2(a\alpha_n)} \cong e^{-\varepsilon\alpha_n} \qquad (4.87)$$

where ε is a small positive number and $L_n(r)\to 1$ as $\varepsilon\to 0$ and $\alpha_n\to\infty$. Similarly, note that for large values of α and for a small $\varepsilon = a-r$ we have

$$\frac{I_1(\alpha r)}{I_2(\alpha a)} = \frac{I_1(\alpha a-\alpha\varepsilon)}{I_2(\alpha a)} \cong e^{-\varepsilon\alpha} \quad . \tag{4.88}$$

Thus making use of the following results

$$\lim_{\beta\to 0} \int_{b_1}^{c} p(t)dt\sum_{1}^{\infty} e^{-\beta n}\sin\lambda n = \lim_{\beta\to 0} \int_{b_1}^{c} p(t)dt \frac{\sin\lambda}{2(\cosh\beta-\cos\lambda)}$$

$$= \frac{1}{2}\int_{b_1}^{c} p(t)\cot\frac{\lambda}{2}\,dt \quad ,$$

$$\lim_{\varepsilon\to 0} \int_{b_1}^{c} p(t)dt \int_{o}^{\infty} e^{-\varepsilon\alpha}\sin\lambda\alpha d\alpha = \lim_{\varepsilon\to 0} \int_{b_1}^{c} p(t)dt \frac{\lambda}{\lambda^2+\varepsilon^2}$$

$$= \int_{b_1}^{c} \frac{p(t)}{\lambda}\,dt \quad , \tag{4.89a,b}$$

and adding and subtracting the asymptotic values under the summation and integral signs in (4.85) we obtain

$$\frac{1}{\pi}\int_{b_1}^{c} p(t)dt\left(\frac{1}{t-x} - \frac{1}{t+x-2b_1} + \frac{\pi\mu_1}{2b\mu_2}\left\{\begin{matrix}[\cot\frac{\pi(t-x)}{2b} - \cot\frac{\pi(t+x)}{2b}] \\ [\operatorname{cosec}\frac{\pi(t-x)}{2b} + \operatorname{cosec}\frac{\pi(t+x)}{2b}]\end{matrix}\right\}\right)$$

$$+ \frac{1}{\pi}\int_{b_1}^{c} k(x,t)p(t)dt = -\mu_1 f(x) \ , \quad b_1<x<c \quad , \tag{4.90}$$

$$k(x,t) = \int_{o}^{\infty} \left(\frac{I_1(\alpha a)}{I_2(\alpha a)} - 1\right)[\sin\alpha(t-x) - \sin\alpha(t+x-2b)]d\alpha$$

$$+ \frac{\mu_1\pi}{b_1\mu_2}\sum^{\infty}[L_n(a) - 1][\sin\alpha_n(t-x)\mp\sin\alpha_n(t+x)] \quad , \tag{4.91}$$

where upper and lower signs and kernels again correspond to symmetric and antisymmetric problems, respectively, the kernel $k(x,t)$ given by (4.91) is bounded and continuous for all x and t in the closed interval $[b_1,c]$, and in (4.91), because of uniform convergence, the limit has been put under the integral and the summation signs. The integral equation (4.90) must be solved under (4.76i), the only condition in (4.76) which has not been satisfied.

At a first glance it may appear that (4.90) is a simple singular integral equation of the type discussed in the previous sections. However, a closer examination would indicate that the term $1/(t+x-2b_1)$ in the kernel of (4.90) is not bounded in the closed interval $[b_1,c]$ and becomes unbounded (as $1/x$) as x and t go to b_1 together. Hence, this term would be expected to influence the singular behavior of the solution at the end point b_1. Dominant kernels containing, in addition to the Cauchy kernel $1/(t-x)$, terms such as $1/(t+x-2b_1)$ will be called generalized Cauchy kernels. The properties of the solution of singular integral equations with generalized Cauchy kernels will be discussed in Section 6.

4.5 Remarks on the Selection of Auxiliary Functions

In studying multiple series and multiple integral equations if the objective is their reduction to singular integral equations, the selection of the auxiliary function (i.e., the new unknown function) and the procedure followed in the reduction process appear to be quite straightforward. It may be worthwhile to note that in boundary value problems in mechanics it is always possible to recognize pairs of "complementary functions" on the boundary having basically the same dimension. Such pairs are, for example, the normal and tangential derivatives of the potential, or the potential and the integral of the normal flux along the boundary in potential theory, and the surface tractions and the tangential derivatives of the displacements or the integrals of the tractions and the displacements in solid mechanics. If one considers the structure of the dominant part of a singular integral equation, namely

$$\frac{1}{\pi} \int_L \frac{\phi(t)}{t-x} dt = f(x) \quad , \quad x \varepsilon L \tag{4.92}$$

it is clear that (with proper normalizations) the unknown function $\phi(x)$ and the known function $f(x)$ on the boundary have the same physical dimension and in a correct formulation of the problem they invariably are the complementary pair on the boundary. Thus, if one does not pay any attention to the dimensional consistency in selecting the auxiliary function $\phi(x)$ at the beginning, the resulting integral equation may have a singular kernel with a singularity either weaker or stronger than the Cauchy singularity, $1/(t-x)$. For example, in the former case the kernel is the integral of $1/(t-x)$, i.e., $\log|t-x|$, and the integral equation may be reduced to the standard form by formally differentiating the both sides with respect to x (i.e., the tangential coordinate), indicating that the particular auxiliary function selected is the complement of $f'(x)$ rather than the input function $f(x)$. Analytically, this selection usually does not create any difficulty, since one may easily recover and isolate the logarithmic kernel by following the procedure described in sections 4.1-4.4. However, if the selection is made in such a way that $\phi(x)$ is the complement of the integral of $f(x)$, then technically the dominant kernel is expected to be the x- derivative of $1/(t-x)$, i.e., $1/(t-x)^2$. For a Hölder-continuous $\phi(t)$ since the integral $\int \phi(t)dt/(t-x)^2$ does not exist, this formulation becomes meaningless, and besides it is not possible to recover the strong singularity $1/(t-x)^2$ through a normal procedure outlined in the preceding sections. Clearly, the correct thing to do in such a case is to integrate parts of the multiple series (or integral) equations so that dimensionally consistent auxiliary function can be defined.

One should again emphasize the importance of writing the boundary conditions in limit form on that part of the boundary which will be the support of the resulting integral equations. Without this, one may not be able to change the order of integrations or integration and summation legitimately to evaluate the kernels. Even if this is done, with the limit under the integral or summation sign, the resulting infinite integrals or series giving the kernels are usually divergent or simply meaningless. Consider, for example, the integral equation (4.5) expressing the boundary condition on L_1. If it is not written in limit form the kernel becomes

$$K(\theta,t) = \sum_{1}^{\infty} \sin n(t-\theta) \qquad (4.93)$$

which is not summable. The same thing may be said about the kernels arising from the reduction of multiple integral equations or multiple series-multiple integral equations (see equations (4.58), (4.85), and (4.90)).

5. NUMERICAL SOLUTION OF SINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND

In the previous sections it was shown that, unless the problem has convective boundary conditions,[1] the mixed boundary value problems in mechanics may invariably be reduced to a system of singular integral

[1] Convective boundary conditions generally reduce to integral equations of the second kind with a logarithmic dominant kernel which have bounded solutions and which may be treated as Fredholm type equations (see Section 4.2 equations (4.48)-(4.49) and Section 4.3 equations (4.63), and (4.70)).

equations of the following general form:

$$\sum_{1}^{N} a_{ij}\phi_j(x) + \Sigma\int_L [\frac{1}{\pi}\frac{b_{ij}}{t-x} + k^s_{ij}(x,t) + k^f_{ij}(x,t)]\phi_j(t)dt$$

$$= f_i(x) \quad , \quad i = 1,..,N \quad , \quad x\varepsilon L_i \qquad (5.1)$$

where $L_i = \sum_{1}^{m_i} L_{ij}$, the matrices (a_{ij}) and (b_{ij}) are nonsingular, the kernels $k^s_{ij}(x,t)$ consists of terms which become unbounded as x and t approach the end points of L_i and which, with the singular terms $b_{ij}/(t-x)$, constitute the generalized Cauchy kernels, $k^f_{ij}(x,t)$ are bounded Fredholm type kernels, and $f_i(x)$ are known functions. For the singular equations with generalized Cauchy kernels there does not seem to be any general method of regularization. The singular behavior of the solution of these equations will be studied in Section 6 where a numerical technique for solving the integral equations will also be discussed. Also, the treatment of the singular integral equations of the second kind will be postponed until Section 7. Thus, in this section we will consider only the singular integral equations of the first kind with simple Cauchy-type singularities which represent by far the largest class of mixed boundary value problems in mechanics. The method will be described for a single equations defined in the normalized interval (-1,1), namely

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x) \quad , \quad -1<x<1 \quad . \qquad (5.2)$$

From the development of the method, it will be clear that the extension of the method to a system of singular integral equations (with unknown functions $\phi_1(x),.., \phi_n(x)$) defined in a single interval $a<x<b$ is quite straightforward. It is also easy to show that if each equation in the system is defined on a (different) union of arcs, the technique developed

for (5.2) may easily be used for the numerical solution of this general system. For example, consider

$$\frac{1}{\pi}\int_L \frac{\phi(t)}{t-x}\,dt + \int_L k(x,t)\phi(t)dt = f(x) \quad , \quad x\varepsilon L$$

$$L = \sum_1^n L_i \quad , \quad L_i = (a_i,b_i) \quad , \quad b_i<a_{i+1} \quad . \qquad (5.3)$$

Defining the following new variables and functions,

$$t_i = \frac{2t}{b_i-a_i} - \frac{b_i+a_i}{b_i-a_i} \quad , \quad a_i<t<b_i \quad , \quad -1<t_i<1 \quad ,$$

$$x_i = \frac{2x}{b_i-a_i} - \frac{b_i+a_i}{b_i-a_i} \quad , \quad a_i<x<b_i \quad , \quad -1<x_i<1 \quad ,$$

$$\phi(t) = \phi_i(t_i) \quad , \quad a_i<t<b_i \quad , \quad -1<t_i<1 \quad ,$$

$$f(x) = f_i(x_i) \quad , \quad a_i<x<b_i \quad , \quad -1<x_i<1 \quad ,$$

$$k(x,t) = k_{ij}(x_i,t_j) \quad , \quad a_i<x<b_i \quad , \quad a_j<t<b_j \quad ,$$

$$-1<x_i<1 \quad , \quad -1<t_j<1 \quad , \quad i = 1,..,n \qquad (5.4)$$

and writing (5.3) on each interval $x\varepsilon L_i$ separately we obtain

$$\sum_{j=1}^n \frac{1}{\pi}\int_{-1}^1 \frac{\frac{b_j-a_j}{2}\,\phi_j(t_j)dt_j}{\frac{b_j-a_j}{2}t_j + \frac{b_j+a_j}{2} - \frac{b_i-a_i}{2}x_i - \frac{b_i+a_i}{2}}$$

$$+ \sum_{j=1}^n \int_{-1}^1 k_{ij}(x_i,t_j)\phi_j(t_j)\,\frac{b_j-a_j}{2}\,dt_j = f_i(x_i) \quad ,$$

$$i = 1,..,n \quad , \quad -1<x_i<1 \quad . \qquad (5.5)$$

It may be noted that all the variables x_i and t_j in (5.5) vary between -1 and 1 hence, the indices i and j in x_i and t_j may be suppressed. Also note that in the first term of (5.5) if $i \neq j$ the kernel is bounded and continuous in the closed interval [-1,1], i.e., for $-1 \leq (x_i, t_j) \leq 1$. Thus (5.5) is equivalent to the following simple system of singular integral equations defined in the normalized interval (-1,1):

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi_i(t)}{t-x}\,dt + \sum_{j=1}^{n}\int_{-1}^{1} h_{ij}(x,t)\phi_j(t)dt = f_i(x) \quad ,$$

$$-1<x<1 \quad , \quad i = 1,..n \quad , \qquad (5.6)$$

where h_{ij} is the sum of k_{ij} and the corresponding nonsingular kernels in the first term of (5.5)[(1)].

Referring to equations (4.10-4.15) and [3] the fundamental function of (5.2) may be expressed as

$$w(x) = (1+x)^{-1/2+\alpha_1}(1-x)^{1/2+\alpha_2} \quad , \quad (-1<x<1) \qquad (5.7)$$

where α_1 and α_2 are (positive, zero, or negative) integers. $-(\alpha_1 + \alpha_2) = \kappa$ is known as the index of the integral equation. The first step in the numerical procedure which will be described in this section is the determination of the integers α_1 and α_2 or the index of the problem. As pointed out earlier, generally this is not possible without referring to the physics of the problem For example, consider the plane contact problem shown in Figure 4 for a rigid stamp with a given profile acting on the elastic for half plane $x_2<0$, $-\infty<x_1<\infty$. Using the Fourier transforms, or the Green's functions, or the complex potentials [1-4] it can easily be shown that in the absence of friction the mixed boundary value problem is reduced to the following simple singular integral equation:

(1) From the analysis and particularly numerical view point another advantage of this procedure is that one now is dealing with a simple fundamental function , for example, of the form $w(t) = (1-t^2)^{\mp 1/2}$ rather than a complicated function defined by (4.14) or (4.15).

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}\,dt = f(x) \quad , \quad -1<x<1 \tag{5.8}$$

where

$$x = \frac{2x_1}{b-a} - \frac{b+a}{b-a} \quad , \quad \phi(x) = -\sigma_{yy}(x_1,0)$$

$$f(x) = -\frac{4\mu}{1+\kappa_o}\frac{\partial}{\partial x_1}v(x_1,0) \quad , \quad a<x_1<b \tag{5.9}$$

with μ and κ_o being the elastic constants ($\kappa_o = 3-4\nu$ for plane strain, $\kappa_o = (3-\nu)/(1+\nu)$ for plane stress, ν: Poisson's ratio). Note that the fundamental function of (5.8) is given by (5.7) and the solution is of the following form [3]

$$\phi(x) = F(x)w(x) \quad , \quad -1<x<1 \tag{5.10}$$

where $F(x)$ is bounded in $-1\leq x\leq 1$. Thus, the singular behavior of the solution is completely determined by that of $w(x)$. In Figure 4a the contact at both ends $x_1 = a$ and $x_1 = b$ is "smooth" and a and b are unknown. Consequently at the end points the contact stress $\phi(x)$ must be bounded (and necessarily zero). Therefore, in (5.7) we have $\alpha_1 = 1$, $\alpha_2 = 0$, and the fundamental function and index become

$$w(x) = (1-x^2)^{1/2} \quad , \quad \kappa = -1 \quad . \tag{5.11}$$

On the other hand, in Figure 4b the leading edges of the stamp are "sharp" and a and b are known. Thus, since the contact stress at these end points is known to be unbounded, from (5.7) and (5.10) it follows that

$$\alpha_1 = 0 \quad , \quad \alpha_2 = -1 \quad , \quad \kappa = 1 \quad , \quad w(x) = (1-x^2)^{-1/2} \quad . \tag{5.12}$$

Similarly for the stamp given in Figure 4c a is known, b us unknown and

$$\alpha_1 = 0 = \alpha_2 \quad , \quad \kappa = 0 \quad , \quad w(x) = (1+x)^{-1/2}(1-x)^{1/2} \quad . \tag{5.13}$$

Also, for the stamp given in Figure 4d b is known, a is unknown and

$$\alpha_1 = 1 \quad , \quad \alpha_2 = -1 \quad , \quad \kappa = 0 \quad , \quad w(x) = (1+x)^{1/2}(1-x)^{-1/2} \quad . \tag{5.14}$$

Referring now to (4.8) and (4.21) the general solution of (5.8) may be expressed as

$$\phi(x) = -\frac{w(x)}{\pi}\int_{-1}^{1}\frac{f(t)dt}{(t-x)w(t)} + Cw(x) \quad , \quad -1<x<1 \tag{5.15}$$

where C is an unknown constant. Although there are very general rules for determining the unknown constants such as C, a, and b [3], as pointed out earlier in this section, since it is always possible to reduce the general singular integral equations to a system defined only in the normalized interval (-1,1), for the numerical methods which will be developed in this section it is sufficient to state the rules for a simple singular integral equation such as (5.8) (or (5.2) for which one simply replaces f(x) by $[f(x) - \int_{-1}^{1}k(x,t)\phi(t)dt]$). One may also note that the statements made here for the singular integral equations of the first kind are also valid for the equations of the second kind without any modification.

(a) $\kappa = -1$ (Figure 4a):

In this case the conditions at infinity require that the constant C must be zero and the following consistency condition must be satisfied [3]:

$$\int_{-1}^{1}[f(x) - \int_{-1}^{1}k(x,t)\phi(t)dt]\frac{dx}{w(x)} = 0 \quad . \tag{5.16}$$

Noting that f(t) contains the constants a and b (see (5.9)), (5.16) provides one equation for the determination of remaining unknowns a and b. The second equation is obtained by considering the following equilibrium condition:

$$-\int_a^b \sigma_{yy}(x_1,0)dx_1 = \frac{b-a}{2}\int_{-1}^{1} \phi(x)dx = P \quad . \tag{5.17}$$

(b) $\kappa = 1$ (Figure 4b):

In this case C is the only unknown constant and is determined by substituting from (5.15) into the equilibrium condition (5.17) (which must be satisfied in all cases)[1].

(c) $\kappa = 0$ (Figure 4c or 4d):

In this case again the conditions at infinity require that the constant C be zero and (5.15) gives the unique solution without any reference to any additional conditions. The problem is solved by assuming that both a and b are known and if, instead of the contact area, the resultant load P is specified, (5.17) is used to relate the two.

The numerical methods used for the solution of singular integral equations may be considered in two separate categories. The first is a rather direct approach which is based on the development Gauss-Jacobi type integration formulas for singular integrals. The second is basically a series solution with Chebyshev or Jacobi polynomials being the related othogonal polynomials used in the series expansion.

5.1 Solution by Gaussian Integration Formulas

The most common numerical technique to solve a Fredholm-type integral equation of the form

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[1] For $\kappa = 1$ there is always a physical condition such as (5.17) to be satisfied. For example, in crack problems instead of equilibrium one has the singlevaluedness condition (in which P is replaced by zero).

$$\int_{-1}^{1} k(x,t)\phi(t)dt = f(x) \quad , \quad -1<x<1 \tag{5.18}$$

is the use of some kind of integration formula to evaluate the integral in terms of some discrete set of values $\phi(t_1),\ldots,\phi(t_n)$ thereby reducing the integral equation to a system of algebraic equations in $\phi(t_i)$. In particular, if it is appropriate [20,21] one may prefer an integration formula of Gaussian type for which (5.18) becomes

$$\sum_{1}^{n} k(x_i,t_j)\phi(t_j)W_j + R_n(x_i) = f(x_i) \quad , \quad i = 1,..,n \quad , \tag{5.19}$$

where W_j, $(j = 1,..,n)$ is the weighting constant of the related integration formula and R_n is the remainder. By selecting n sufficiently large, R_n can be made as small as necessary for the desired accuracy and hence, may be neglected. In (5.19) $t_1,..,t_n$ are the roots of the related orthogonal polynomial. This highly appealing simple technique could be used for the solution of singular integral equations if the Gaussian integration formulas for singular integrals were to be available. Some of these formulas will be developed in this and the following sections.

5.1.1. Gaussian Integration Formula for $\kappa = 1$

For $\kappa = 1$ the fundamental function of (5.2) is given by (5.12) which is the weight of Chebyshev polynomials (of the first kind) $T_n(x)$. Thus, before deriving the integration formula the following property of the Chebyshev polynomials will be proved: Let

$$T_n(t_k) = 0 \quad , \quad k = 1,..,n \, ; \, U_{n-1}(x_r) = 0 \quad , \quad r = 1,..,n-1 \quad . \tag{5.20}$$

Then

$$\sum_{k=1}^{n} \frac{T_j(t_k)}{n(t_k-x_r)} = \begin{cases} 0 \quad , \quad j = 0 \quad , \\ U_{j-1}(x_r) \quad , \quad 0<j<n \quad , \end{cases} \tag{5.21}$$

where the Chebyshev polynomials are defined by

$$T_n(x) = \cos n\theta \quad , \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad , \quad \cos\theta = x \quad . \tag{5.22}$$

To prove (5.21) note that $T_n(x)$ and $U_n(x)$ are polynomials of degree n and consider the following simple fraction expansion:

$$\frac{U_{n-j-1}(x)}{T_n(x)} = \sum_1^n \frac{a_k}{t_k - x} \quad , \quad T_n(t_k) = 0 \quad ,$$

$$a_k = \frac{-U_{n-j-1}(t_k)}{T_n'(t_k)} = \frac{-U_{n-j-1}(t_k)}{nU_{n-1}(t_k)} \quad . \tag{5.23a,b}$$

Using [22]

$$U_{n-j-1}(x) = T_j(x)U_{n-1}(x) - T_n(x)U_{j-1}(x) \tag{5.24}$$

from (5.23) it follows that

$$\sum_{k=1}^{n} \frac{T_j(t_k)}{n(t_k - x)} = -\frac{U_{n-j-1}(x)}{T_n(x)} \quad . \tag{5.25}$$

First part $j = 0$ of (5.21) follows immediately from (5.20) and (5.25). Substituting now from (5.24) into (5.25) we find

$$\sum_{k=1}^{n} \frac{T_j(t_k)}{n(t_k - x)} = U_{j-1}(x) - \frac{T_j(x)U_{n-1}(x)}{T_n(x)} \tag{5.26}$$

which, for $x = x_r$ and $0<j<n$ is reduced to (5.21) by (5.20), thereby completing the proof.

Let the solution of the singular integral equation now be of the form (5.10) with $w(x)$ as given by (5.12). Let us assume that the unknown bounded function $F(x)$ can be approximated to a sufficient degree of accuracy by

$$F(x) \cong \sum_o^p A_j T_j(x) \quad . \tag{5.27}$$

By using the relation [22]

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_j(t)dt}{(t-x)(1-t^2)^{1/2}} = \begin{cases} 0 \quad , \quad j = 0 \\ U_{j-1}(x) \quad , \quad j>0 \end{cases} \quad , \quad (-1<x<1) \tag{5.28}$$

the singular integral in (5.2) or (5.8) may then be expressed as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}\,dt \cong \sum_{j=0}^{p} A_j \frac{1}{\pi}\int_{-1}^{1}\frac{T_j(t)dt}{(t-x)(1-t^2)^{1/2}} = \sum_{1}^{p} A_j U_{j-1}(x) \quad ,$$

$$-1<x<1 \tag{5.29}$$

At $x = x_r$, substituting from (5.21) and (5.27), (5.29) becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x_r}\,dt \cong \sum_{j=1}^{p}\sum_{k=1}^{n}\frac{A_j T_j(t_k)}{n(t_k-x_r)} = \sum_{j=1}^{p}\sum_{k=1}^{n}\frac{A_j T_j(t_k)}{n(t_k-x_r)}$$

$$+ \sum_{k=1}^{n}\frac{A_o T_o(t_k)}{n(t_k-x_r)} = \sum_{k=1}^{n}\frac{F(t_k)}{n(t_k-x_r)} \tag{5.30}$$

where

$$t_k = \cos\frac{(2k-1)\pi}{2n} \quad , \quad k = 1,..,n \; ; \; x_r = \cos\frac{\pi r}{n} \quad , \quad r = 1,..,n-1. \tag{5.31}$$

Note that if the expansion (5.27) is exact, then there is no approximation in (5.30) for any $n>p$. Also note that (5.30) is identical to the following standard Gauss-Chebyshev integration formula for bounded functions [20,21]:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g(x,t)}{(1-t^2)^{1/2}}\,dt \cong \sum_{1}^{n}\frac{1}{n}\,g(x,t_k) \quad , \quad T_n(t_k) = 0 \quad , \tag{5.32}$$

with the important difference that (5.32) is valid for any x whereas (5.30) holds only for certain discrete set of x_r given by (5.31). Using now (5.10), (5.30), and (5.32) the integral equation (5.2) may be reduced to the following system of linear algebraic equations in the unknowns $F(t_1),..,F(t_n)$:

$$\sum_{k=1}^{n} \frac{1}{n} F(t_k)\left[\frac{1}{t_k - x_r} + \pi k(x_r, t_k)\right] = f(x_r) \quad , \quad r = 1,..,n-1$$

$$\sum_{k=1}^{n} \frac{\pi}{n} F(t_k) = A \tag{5.33a,b}$$

where A is a known constant and (5.33b) comes from the additional condition of the problem (such as (5.17)).

5.1.2 Gaussian Integration Formula for $\kappa = -1$

For $\kappa = -1$ the fundamental function of the integral equation is given by (5.11) which is the weight of Chebyshev polynomials of the second kind $U_n(x)$. Thus, we will first prove the following property: Let

$$U_n(t_k) = 0 \quad , \quad T_{n+1}(x_r) = 0 \quad , \tag{5.34}$$

then

$$\sum_{k=1}^{n} \frac{(1-t_k^2)U_j(t_k)}{(n+1)(t_k - x_r)} = - T_{j+1}(x_r) \quad , \quad j<n \quad . \tag{5.35}$$

Proof: Using [22]

$$(1-t^2)U_n'(t) = (n+1)U_{n-1}(t) - ntU_n(t) \quad , \tag{5.36}$$

the following expansion may be obtained

$$\frac{U_{n-j-1}(x)}{U_n(x)} = \sum_{1}^{n} \frac{b_k}{t_k - x} \quad , \quad b_k = - \frac{(1-t_k^2)U_{n-j-1}(t_k)}{(n+1)U_{n-1}(t_k)} \quad , \tag{5.37}$$

Considering the recursion formulas (5.24) and [22]

$$T_n(t) = U_n(t) - tU_{n-1}(t) \quad , \quad U_j(t) = T_j(t) + tU_{j-1}(t) \quad , \tag{5.38}$$

from (5.37) and (5.34) it follows that

$$\sum_{k=1}^{n} \frac{(1-t_k^2)U_j(t_k)}{(n+1)(t_k - x)} = - \frac{U_{n-j-1}(x)}{U_n(x)} = - T_{j+1}(x) + \frac{U_j(x)T_{n+1}(x)}{U_n(x)} \quad , \tag{5.39}$$

Noting that $T_{n+1}(x_r) = 0$, for $x = x_r$ (5.39) would reduce to (5.35) completing the proof.

Consider now the singular integral in (5.2) with the solution expressed as in (5.10) and (5.11). Let the following truncated series represent the unknown bounded function F(t) with sufficient accuracy:

$$F(t) \cong \sum_{o}^{p} B_j U_j(t) \tag{5.40}$$

Using (5.35), (5.40), and the relation [22]

$$\frac{1}{\pi}\int_{-1}^{1} \frac{U_j(t)(1-t^2)^{1/2}}{t-x} dt = - T_{j+1} \quad , \quad -1<x<1 \quad , \tag{5.41}$$

for $x = x_r$ the singular integral may then be expressed as

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t-x_r} dt \cong \sum_{o}^{p} \frac{B_j}{\pi} \int_{-1}^{1} \frac{U_j(t)(1-t^2)^{1/2}}{t-x_r} dt = - \sum_{o}^{p} B_j T_{j+1}(x_r)$$

$$= \sum_{j=0}^{p} \sum_{k=1}^{n} \frac{(1-t_k^2)B_j U_j(t_k)}{(n+1)(t_k-x_r)} = \sum_{k=1}^{n} \frac{(1-t_k^2)F(t_k)}{(n+1)(t_k-x_r)} \quad , \tag{5.42}$$

where

$$t_k = \cos\frac{k\pi}{n+1} \quad , \quad k = 1,..,n \; ; \; x_r = \cos\frac{\pi(2r-1)}{2(n+1)} \quad , \; r = 1,..,n+1. \tag{5.43}$$

One may again note that (5.42) is identical to the following Gaussian integration formula for a bounded function g(x,t)

$$\frac{1}{\pi}\int_{-1}^{1} g(x,t)(1-t^2)^{1/2} dt \cong \sum_{1}^{n} \frac{(1-t_k^2)}{n+1} g(x,t_k) \quad , \quad U_n(t_k) = 0 \tag{5.44}$$

with the important difference that (5.44) is valid for any x whereas (5.42) is valid only for a certain set of $x = x_r$ given by (5.43). Also note that if (5.40) is exact, again there is no approximation in (5.42) for any $n>p$.

Using now (5.10), (5.11), (5.42) and (5.44) the integral equation (5.2) may be reduced to

$$\sum_{k=1}^{n} \frac{1-t_k^2}{n+1} F(t_k)\left[\frac{1}{t_k-x_r} + \pi k(x_r,t_k)\right] = f(x_r) \quad , \quad r = 1,..,n+1 \ . \tag{5.45}$$

Referring to (5.10), (5.40), and (5.41) and using the orthogonality condition

$$\frac{1}{\pi} \int_{-1}^{1} T_k(x) T_r(x) \frac{dx}{(1-x^2)^{1/2}} = \begin{cases} 0 \ , & k \neq r \ , \\ 1 \ , & k = r = 0 \ , \\ 1/2 \ , & k = r > 0 \ , \end{cases} \tag{5.46}$$

the consistency condition for (5.2) may be expressed as

$$\int_{-1}^{1} [f(x) - \int_{-1}^{1} k(x,t)\phi(t)dt] \frac{dx}{(1-x^2)^{1/2}} = \frac{1}{\pi} \int_{-1}^{1} \frac{dx}{(1-x^2)^{1/2}} \int_{-1}^{1} \frac{\phi(t)}{t-x} dt$$

$$= - \int_{-1}^{1} \sum_{o}^{p} B_j T_{j+1}(x) \frac{dx}{(1-x^2)^{1/2}} = 0 \tag{5.47}$$

Thus, (5.45) implies that the consistency condition of the integral equation has been satisfied. Note that in (5.45) there are n unknowns $F(t_k)$ and n+1 equations. If $\kappa = -1$, usually there are two more unknown constants such as a and b in Figure 4a and one more condition such as (5.17) giving altogether n+2 unknowns and n+2 equations.

5.1.3 Gaussian Integration Formula for $\kappa = 0$

Let the fundamental function and the index of the integral equation (5.2) be

$$w(x) = (1-t)^{-1/2}(1+t)^{1/2} \quad , \quad \kappa = 0 \ . \tag{5.48}$$

Expressing again the solution by (5.10) and observing that w(x) is the weight of Jacobi polynomials $P_n^{(-1/2,1/2)}(x)$, it will be assumed that the unknown bounded function F(t) may be approximated to a sufficient degree of accuracy by

$$F(t) \cong \sum_{o}^{p} C_j P_j^{(-1/2,1/2)}(t) \quad , \quad -1<t<1 \quad . \tag{5.49}$$

From (5.10), (5.49), (5.48), and using the following general property of Jacobi polynomials [23]

$$\frac{1}{\pi}\int_{-1}^{1} P^{(\alpha,\beta)}(t)(1-t)^{\alpha}(1+t)^{\beta}\,\frac{dt}{t-x} = -\,\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\pi}\,2^{-\kappa}P_{n-\kappa}^{(-\alpha,-\beta)}(x) \quad ,$$

$$-1<x<1 \quad , \quad \kappa = -(\alpha+\beta) = (-1,0,\text{ or } 1) \quad , \quad -1<(\alpha,\beta)<1 \quad , \tag{5.50}$$

the singular integral in (5.2) may be expressed as

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t-x}\,dt = \sum_{o}^{p} C_j P_j^{(1/2,-1/2)}(x) \quad , \quad -1<x<1 \quad . \tag{5.51}$$

Let

$$P_n^{(-1/2,1/2)}(t_k) = 0 \quad , \quad k = 1,\ldots,n \quad , \tag{5.52}$$

and consider the expansion

$$-\,\frac{P_n^{(1/2,-1/2)}(x)P_j^{(-1/2,1/2)}(x)-P_n^{(-1/2,1/2)}(x)P_j^{(1/2,-1/2)}(x)}{P_n^{(-1/2,1/2)}(x)}$$

$$= \sum_{1}^{n} \frac{c_k}{t_k-x} \quad , \quad j<n \quad . \tag{5.53}$$

Using (5.24) and the relations [21]

$$P_n^{(-1/2,1/2)}(2z^2-1) = \frac{\Gamma(n+1/2)}{n!\,\pi^{1/2}}\,\frac{1}{z}\,T_{2n+1}(z) \quad ,$$

$$P_n^{(1/2,-1/2)}(2z^2-1) = \frac{\Gamma(n+1/2)}{n!\,\pi^{1/2}}\,U_{2n}(z) \quad , \tag{5.54}$$

equation (5.53) may be modified as

$$-\,\frac{\Gamma(j+1/2)}{j!\,\pi^{1/2}}\,\frac{U_{2n-2j-1}(y)}{T_{2n+1}(y)} = \sum_{1}^{n} \frac{c_k}{t_k-x} \quad , \quad x = 2y^2-1 \quad , \tag{5.55}$$

indicating that the expansion in (5.53) is indeed possible.

Referring now to the results in Section 5.1.1 and to (5.54) and using [23]

$$\frac{d}{dt} P_n^{(\alpha,\beta)}(t_k) = \frac{n+\beta}{1+t_k} P_n^{(\alpha+1,\beta-1)}(t_k) = -\frac{n+\alpha}{1-t_k} P_n^{(\alpha-1,\beta+1)}(t_k) ,$$

$$P_n^{(\alpha,\beta)}(t_k) = 0 \quad , \quad k = 1,..,n, \tag{5.56}$$

the coefficients c_k are found to be

$$c_k = \frac{2(1+t_k)}{2n+1} P_j^{(-1/2,1/2)}(t_k) \quad . \tag{5.57}$$

If we let

$$P_n^{(1/2,-1/2)}(x_r) = 0 \quad , \quad r = 1,..,n \tag{5.58}$$

from (5.53) and (5.57) it follows that

$$\sum_{k=1}^{n} \frac{2(1+t_k)}{2n+1} P_j^{(-1/2,1/2)}(t_k) \frac{1}{t_k-x_r} = P_j^{(1/2,-1/2)}(x_r) \quad . \tag{5.59}$$

Finally, from (5.49), (5.51) and (5.59) we find

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(t)}{t-x_r} dt = \frac{1}{\pi} \int_{-1}^{1} \frac{F(t)}{t-x_r} \left(\frac{1+t}{1-t}\right)^{1/2} dt = \sum_{k=1}^{n} \frac{2(1+t_k)}{2n+1} \frac{F(t_k)}{t_k-x_r} , \tag{5.60}$$

where

$$P_n^{(-1/2,1/2)}(t_k) = 0 \quad , \quad t_k = \cos\left(\frac{2k-1}{2n+1}\pi\right) \quad , \quad k = 1,..,n$$

$$P_n^{(1/2,-1/2)}(x_r) = 0 \quad , \quad x_r = \cos\left(\frac{2r\pi}{2n+1}\right) \quad , \quad r = 1,..,n \tag{5.61}$$

Equation (5.60) too is identical to the corresponding Gauss-Jacobi integration formula for a bounded function $g(x,t)$ given by [21]

$$\frac{1}{\pi} \int_{-1}^{1} g(x,t) \left(\frac{1+t}{1-t}\right)^{1/2} dt \cong \sum_{k=1}^{n} \frac{2(t_k+1)}{2n+1} g(x,t_k) \tag{5.62}$$

with again the important difference that (5.62) is valid for any x whereas (5.60) is valid only for $x = x_r$, r = 1,..,n.

Using the integration formulas (5.60) and (5.62), the singular integral equation (5.2) may easily be reduced to the following system of linear algebraic equations in $F(t_1),..,F(t_n)$:

$$\sum_{k=1}^{n} \frac{2(1+t_k)}{2n+1} F(t_k)\left[\frac{1}{t_k-x_r} + \pi k(x_r,t_k)\right] = f(x_r) \quad , \quad r = 1,..,n \tag{5.63}$$

where t_k and x_r are given by (5.61).

If the weight function is $w(x) = (1-t)^{1/2}/(1+t)^{1/2}$ following a similar procedure, it can be shown that

$$\frac{1}{\pi}\int_{-1}^{1} \frac{F(t)}{t-x}\left(\frac{1-t}{1+t}\right)^{1/2} dt \cong \sum_{k=1}^{n} \frac{2(1-t_k)}{2n+1}\,\frac{F(t_k)}{t_k-x_r} \quad , \tag{5.64}$$

and (5.2) is reduced to

$$\sum_{k=1}^{n} \frac{2(1-t_k)}{2n+1} F(t_k)\left[\frac{1}{t_k-x_r} + \pi k(x_r,t_k)\right] = f(x_r) \quad , \quad r = 1,..,n \tag{5.65}$$

where

$$P_n^{(1/2,-1/2)}(t_k) = 0 \quad , \quad t_k = \cos\left(\frac{2k\pi}{2n+1}\right) \quad , \quad k = 1,..,n$$

$$P_n^{(-1/2,1/2)}(x_r) = 0 \quad , \quad x_r = \cos\left(\frac{2r-1}{2n+1}\pi\right), \quad r = 1,..,n \quad . \tag{5.66}$$

5.2 Solution by Orthogonal Polynomials

In the previous sections it was indicated that the fundamental function of the singular integral equation (5.2) is of the general form

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta} \quad , \quad \kappa = -(\alpha+\beta) = (0,\mp 1) \quad . \tag{5.67}$$

Noting that (5.67) is the weight of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, it is natural to look for a solution of (5.2) in the following series form:

$$\phi(t) = w(t)\sum_{o}^{\infty} C_n P_n^{(\alpha,\beta)}(t) \quad , \quad -1<t<1 \quad . \tag{5.68}$$

Thus, substituting from (5.68) and (5.50) into (5.2) we obtain

$$\sum_{n=0}^{\infty} C_n[-\frac{2^{-\kappa}}{\sin\pi\alpha} P_{n-\kappa}^{(-\alpha,-\beta)}(x) + k_n(x)] = f(x) \quad , \quad -1<x<1 \quad ,$$

$$k_n(x) = \int_{-1}^{1} k(x,t)P_n^{(\alpha,\beta)}(t)w(t)dt \quad . \tag{5.69}$$

Equation (5.69) can further be reduced to an infinite algebraic system in C_n as follows:

$$-\frac{2^{-\kappa}}{\sin\pi\alpha} \theta_k(-\alpha,-\beta)C_{k+\kappa} + \sum_{j=1}^{\infty} c_{kj}C_j = c_k \quad , \quad k = 0,1,.. \tag{5.70}$$

where for $\kappa = -1$, $C_{-1} = 0$ and

$$c_{kj} = \int_{-1}^{1} P_k^{(-\alpha,-\beta)}(x)k_j(x)(1-x)^{-\alpha}(1+x)^{-\beta}dx \quad ,$$

$$c_k = \int_{-1}^{1} f(x)P_k^{(-\alpha,-\beta)}(x)(1-x)^{-\alpha}(1+x)^{-\beta}dx \quad . \tag{5.71}$$

the constants θ_k come from the orthogonality condition

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)P_k^{(\alpha,\beta)}(t)w(t)dt = \begin{cases} 0 \quad , & n \neq k \\ \theta_k(\alpha,\beta) \quad , & n = k \quad , \end{cases}$$

$$k = 0,1,2,..,$$

$$\theta_o(\alpha,\beta) = \int_{-1}^{1} w(t)dt = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \quad ,$$

$$\theta_k(\alpha,\beta) = \frac{2^{\alpha+\beta+1}\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k!\Gamma(k+\alpha+\beta+1)} \quad , \quad k = 1,2,... \tag{5.72}$$

The infinite system (5.70) may be solved by the method of reduction [24]. Note that if $\kappa = 1$ in the reduction letting $k = 0,..,n$ in (5.70) involves the unknowns $C_o, .., C_{n+1}$, that is, there is one more unknown than the number of equations. The additional equation is provided by the following physical condition (e.g., (5.17)):

$$\int_{-1}^{1} \phi(x)dx = A = C_o\theta_o(\alpha,\beta) \tag{5.73}$$

where A is a known constant. On the other hand if $\kappa = -1$, from (5.69) and (5.71) it is clear that the first equation in (5.70), (i.e., k = 0) is equivalent to the consistency condition (5.16). It should further be noted that the integrations necessary for the evaluation of the constants c_{kj} and c_k in (5.71) are of Gauss-Jacobi type may easily be evaluated by using [20]

$$\int_{-1}^{1} g(x)(1-x)^{-\alpha}(1+x)^{-\beta}dx \cong \sum_{k=1}^{n} W_k g(x_k) \quad ,$$

$$P_n^{(-\alpha,-\beta)}(x_k) = 0 \quad , \quad k = 1,..,n \quad ,$$

$$W_k = -\frac{(2n-\alpha-\beta+2)\Gamma(n-\alpha+1)\Gamma(n-\beta+1)}{(n+1)!(n-\alpha-\beta+1)\Gamma(n-\alpha-\beta+1)} *$$

$$* \frac{2^{-\alpha-\beta}}{\frac{d}{dx}P_n^{(-\alpha,-\beta)}(x_k)P_{n+1}^{(-\alpha,-\beta)}(x_k)} \quad . \tag{5.74}$$

In the special case of $\kappa = 1$, $w(x) = (1-x^2)^{-1/2}$ the related orthogonal polynomials are $T_n(x)$, $(n = 0,1,2,..)$, and we have

$$\phi(x) = (1-x^2)^{-1/2} \sum_{o}^{\infty} A_n T_n(x) \quad , \tag{5.75}$$

$$\frac{1}{\pi}\int_{-1}^{1} U_n(t)U_k(t)(1-t^2)^{1/2}dt = \begin{cases} 0 & , \quad n \neq k \\ \pi/2 & , \quad n = k \end{cases} \tag{5.76}$$

$$\frac{\pi^2}{2} A_{k+1} + \sum_{n=0}^{\infty} a_{kn}A_n = a_k \quad , \quad k = 0,1,.., \tag{5.77}$$

$$a_k = \int_{-1}^{1} f(x)U_k(x)(1-x^2)^{1/2}dx \quad ,$$

$$a_{kn} = \int_{-1}^{1} U_k(x)(1-x^2)^{1/2}dx \int_{-1}^{1} k(x,t)T_n(t)(1-t^2)^{-1/2}dt$$

$$\int_{-1}^{1} \phi(x)dx = A = \pi A_o \qquad (5.78)$$

where (5.28) and (5.46) have been used. Truncating (5.77) at the Nth term and retaining first N equations, (5.77) and (5.78) give the N+1 unknown constants $A_o, A_1, .., A_n$.

In the other special case $\kappa = -1$, $w(x) = (1-x^2)^{1/2}$ it may easily be shown that

$$\phi(x) = (1-x^2)^{1/2} \sum_{o}^{\infty} B_n U_n(x) \quad , \qquad (5.79)$$

$$-\frac{\pi^2}{2} B_{k-1} + \sum_{n=0}^{\infty} b_{kn} B_n = b_k \quad , \quad k = 0,1,... \qquad (5.80)$$

$$b_k = \int_{-1}^{1} f(x)T_k(x)(1-x^2)^{-1/2} \quad ,$$

$$b_{kn} = \int_{-1}^{1} T_k(x)(1-x^2)^{-1/2}dx \int_{-1}^{1} k(x,t)U_n(t)(1-t^2)^{1/2}dt$$

where (5.41) and (5.46) have been used. Again in (5.80) the first equation (k = 0) corresponds to the consistency condition (5.16).

Some typical applications of the numerical methods described in this section may be found in [25-27] which also include extensive references to the solution of mixed boundary value problems in mechanics obtained by applying these methods.

6. INTEGRAL EQUATIONS WITH GENERALIZED CAUCHY KERNELS

As pointed out earlier in this article, in some mixed boundary value problems formulated in terms of singular integral equations, in addition to Cauchy type singularities the kernel may contain terms which become unbounded as both variables in the kernel approach an end point. For example, in the torsion problem shown in Figure 3 and formulated by the integral equation (4.90) if b>c, separating all the singular terms, (4.90) may be expressed as

$$\frac{1}{\pi}\int_{b_1}^{c} p(t)dt\left[\left(1+\frac{\mu_1}{\mu_2}\right)\frac{1}{t-x}-\frac{1}{t+x-2b_1}\right]$$

$$+\int_{b_1}^{c} k_{1f}(x,t)p(t)dt = -\mu_1 f(x) \quad , \quad b_1<x<c \quad , \tag{6.1}$$

where $k_{1f}(x,t)$ is a Fredholm type kernel which is bounded in the closed interval $b_1\le(x,t)\le c$ and is the sum of $k(x,t)$ (see, 4.91) and the nonsingular terms in cotangent or cosecant kernels appearing in (4.90). Note that in (6.1) the term $1/(t+x-2b_1)$ becomes unbounded at the end point b_1.

Similarly, if c = b the kernel in (4.90) contains additional "singular" terms which may be separated by observing that

$$\frac{\pi}{2b}\cot\frac{\pi(t-x)}{2b} = \frac{1}{t-x}+h_1(x,t) \quad ,$$

$$\frac{\pi}{2b}\cot\frac{\pi(t+x)}{2b} = \frac{1}{t+x-2b}+h_2(x,t) \quad ,$$

$$\frac{\pi}{2b}\operatorname{cosec}\frac{\pi(t-x)}{2b} = \frac{1}{t-x}+h_3(x,t) \quad ,$$

$$\frac{\pi}{2b}\operatorname{cosec}\frac{\pi(t+x)}{2b} = -\frac{1}{t+x-2b}+h_4(x,t) \quad , \tag{6.2a-d}$$

where $h_1,..,h_4$ are bounded in the closed interval $b_1 \leq (x,t) \leq b$. Thus, substituting from (6.2) into (4.90) we find

$$\frac{1}{\pi}\int_{b_1}^{b} p(t)dt[(1+\frac{\mu_1}{\mu_2})\frac{1}{t-x} - \frac{1}{t+x-2b_1} - \frac{\mu_1}{\mu_2}\frac{1}{t+x-2b}]$$

$$+\int_{b_1}^{b} k_{2f}(x,t)p(t)dt = -\mu_1 f(x) \quad , \quad b_1<x<b \qquad (6.3)$$

where, again k_{2f} is a bounded kernel. As seen from (6.1) and (6.3), the integral equations for the symmetric and the antisymmetric problems have the same dominant parts but different Fredholm kernels. In (6.3) in addition to the Cauchy singularity, the dominant part of the kernel contains terms which become unbounded at both ends, b_1 and b. The dominant kernels of this type are defined as generalized Cauchy kernels.

In the torsion problem under consideration there is one more limiting case which is worth considering. This is the case of $b_1 = 0$, (e.g., a broken shaft). Thus, for $b_1 = 0$, $c = b$, observing that in addition to (6.2a and c), around the end points one may write

$$\frac{\pi}{2b}\cot\frac{\pi(t+x)}{2b} = \frac{1}{t+x} + \frac{1}{t+x-2b} + h_5(x,t) \quad ,$$

$$\frac{\pi}{2b}\operatorname{cosec}\frac{\pi(t+x)}{2b} = \frac{1}{t+x} - \frac{1}{t+x-2b} + h_6(x,t) \quad , \qquad (6.4)$$

the integral equation (4.90) may be expressed as

$$\frac{1}{\pi}\int_{o}^{b} p(t)dt[(1+\frac{\mu_1}{\mu_2}) - (1 \pm \frac{\mu_1}{\mu_2})\frac{1}{t+x} - \frac{\mu_1}{\mu_2}\frac{1}{t+x-2b}]$$

$$+\int_{o}^{b} k_{3f}(x,t)p(t)dt = -\mu_1 f(x) \quad , \quad 0<x<b \quad , \qquad (6.5)$$

where h_5, h_6, and k_{3f} are bounded in $0 \leq (x,t) \leq b$, and the upper and lower signs in (6.5) correspond to the symmetric and the antisymmetric problems, respectively. Structurally, (6.5) is identical to (6.3), however, at $x = 0$ their solutions may have quite different behavior.

6.1 A Plane Elasticity Problem for Nonhomogeneous Media

Perhaps the most typical application of the singular integral equations with a generalized Cauchy kernel arises in the study of crack problems in nonhomogeneous elastic solids. Consider, for example, the three-dimensional problem in which part of periphery of a plane crack extends to the bimaterial interface in a nonhomogeneous medium. Consider the cross-section shown in Figure 5a. In the neighborhood of the point of interest O let the interface be a smooth surface and, for simplicity, let the crack plane be perpendicular to the interface. Through a proper superposition one can separate the singular or the perturbation part of the problem in which statically self-equilibrating crack surface tractions are the only external loads. From the viewpoint of the stress state around the point O, the perturbation problem in turn may be assumed as having three components, namely, the in-plane extension (mode I), the in-plane shear (mode II), and the anti-plane shear (mode III) with the corresponding surface tractions shown in Figure 5a. Clearly, the behavior of the solution of the anti-plane component of this problem around O will be identical to that of the torsion problem shown in Figure 3 at $x = 0 = b_1$, $r = a$. This behavior should be completely characterized by the first two terms of the genralized Cauchy kernel appearing in (6.5).

Similarly, the singular behavior of the solution under in-plane loading conditions will be equivalent to that of the plane strain problem for two semi-infinite media having a finite crack perpendicular to and ending at the interface shown in Figure 5b. Formulation of this problem is straightforward. For example, referring to [28] for details, in the case of in-plane tension problem, i.e., for the following symmetric loading conditions:

$$\sigma_{1\theta\theta}(r,\pi) = f(r) \quad , \quad \sigma_{1r\theta}(r,\pi) = 0 \quad , \quad 0<r<b \quad , \tag{6.6}$$

the integral equation of the problem may be expressed as

$$\frac{1}{\pi}\int_o^b \left[\frac{1}{t-r} + \frac{c_1}{t+r} + \frac{c_2 r}{(t+r)^2} + \frac{c_3 r^2}{(t+r)^3}\right]\phi(t)dt = \frac{1+\kappa_1}{2\mu_1} f(r) \quad ,$$

$$0<r<b \quad , \tag{6.7}$$

where

$$\phi(r) = -\frac{\partial}{\partial r}\,[u_{1\theta}(r,\pi+0) - u_1\ (r,\pi-0)] \quad ,$$

$$c_1 = \frac{1}{2} - m(1+\kappa_1)/[2(m+\kappa_2)] - 3(1-m)/[2(1+m\kappa_1)] \quad ,$$

$$c_2 = 6(1-m)/(1+m\kappa_1) \quad , \quad c_3 = 4(m-1)/(1+m\kappa_1) \quad , \quad m = \mu_2/\mu_1 \quad , \tag{6.8}$$

and μ_i and κ_i are the elastic constants ($i = 1,2$, $\kappa_i = 3-\nu_i$ for plane strain, $\kappa_i = (3-\nu_i)/(1+\nu_i)$ for plane stress, ν_i: Poisson's ratio). For the in-plane shear problem one obtains the same (generalized Cauchy) kernel as that shown in (6.7). Thus in the general problem of more complex geometry and loading conditions the system of three integral equations representing the problem will have the dominant kernels given in (6.5) and (6.7) (in uncoupled form) and coupling will be through Fredholm type kernels only.

Referring to Figure 5 if we now assume that the crack crosses the boundary and extends into the medium 2 (dashed lines), it is again not difficult to argue that the dominant parts of the related system of integral equations will be the same as that of the idealized through crack problem for two elastic half planes shown in Figure 5b. The derivation of integral equations is again straightforward. For example, in the symmetric problem selecting the unknown functions

$$\phi_1(r) = -\frac{\partial}{\partial r}\left[u_{1\theta}(r,\pi+0) - u_{1\theta}(r,\pi-0)\right] \quad , \quad 0<r<b_1 \quad ,$$

$$\phi_2(r) = \frac{\partial}{\partial r}\left[u_{2\theta}(r,+0) - u_{2\theta}(r,-0)\right] \quad , \quad 0<r<b_2 \quad , \tag{6.9}$$

for the following crack surface tractions

$$\sigma_{1\theta\theta}(r,\pi) = f_1(r) \quad , \quad \sigma_{1r\theta}(r,\pi) = 0 \quad , \quad 0<r<b_1 \quad ,$$

$$\sigma_{2\theta\theta}(r,0) = f_2(r) \quad , \quad \sigma_{2r\theta}(r,0) = 0 \quad , \quad 0<r<b_2 \quad , \tag{6.10}$$

the integral equations for the perturbation problem may be expressed as (see [29] for details)

$$\frac{1}{\pi}\int_0^{b_i}\frac{\phi_i(t)}{t-r}\,dt + \sum_{j=1}^{2}\frac{1}{\pi}\int_0^{b_j} k_{ij}(r,t)\phi_j(t)\,dt = \frac{1+\kappa_i}{2\mu_i}f_i(r) \quad ,$$

$$i = 1,2;\; 0<r<b_i \quad , \tag{6.11}$$

where

$$k_{ii}(r,t) = \sum_{k=1}^{3}\frac{c_{ik}r^{k-1}}{(t+r)^k} \quad , \quad i = 1,2 \quad ,$$

$$k_{ij}(r,t) = \sum_{k=1}^{2}\frac{d_{ik}r^{k-1}}{(t+r)^k} \quad , \quad i,j = 1,2;\; i \neq j \quad , \tag{6.12a,b}$$

$$c_{11} = 1/2 - m_1(1+\kappa_1)/[2(m_1+\kappa_2)] - 3(1-m_1)/[2(1+m_1\kappa_1)] \quad ,$$

$$c_{12} = 6(1-m_1)/(1+m_1\kappa_1) \quad , \quad c_{13} = 4(m_1-1)/(1+m_1\kappa_1) \quad ,$$

$$c_{21} = 1/2 - m_2(1+\kappa_2)/[2(m_2+\kappa_1)] - 3(1-m_2)/[2(1+m_2\kappa_2)] \quad ,$$

$$c_{22} = 6(1-m_2)/(1+m_2\kappa_2) \quad , \quad c_{23} = 4(m_2-1)/(1+m_2\kappa_2) \quad ,$$

$$d_{11} = 3(1+\kappa_1)/[2(m_2+\kappa_1)] - (1+\kappa_1)/[2(1+m_2\kappa_2)] \quad ,$$

$$d_{12} = (1+\kappa_1)/(1+m_2\kappa_2) - (1+\kappa_1)/(m_2+\kappa_1) \quad ,$$

$$d_{21} = 3(1+\kappa_2)/[2(m_1+\kappa_2)] - (1+\kappa_2)/[2(1+m_1\kappa_1)] \quad ,$$

$$d_{22} = (1+\kappa_2)/(1+m_1\kappa_1) - (1+\kappa_2)/(m_1+\kappa_2) \quad ,$$

$$m_1 = \mu_2/\mu_1 \quad , \quad m_2 = \mu_1/\mu_2 \quad . \tag{6.13}$$

From (6.11) and (6.12) it is seen that the integral equations contain dominant kernels only which are of generalized Cauchy type.

6.2 The Fundamental Functions

The integral equations (6.5), (6.7), and (6.11) are some special cases of the following system of singular integral equations with generalized Cauchy kernels:

$$\frac{1}{\pi}\sum_{n=1}^{N}\int_{a_n}^{b_n}\Big[\frac{A_{mn}}{t-x} + \sum_{k=0}^{K_n} B_{mnk}(x-a_n)^k \frac{d^k}{dx^k}(t-z_{1n})^{-1}$$

$$+ \sum_{j=0}^{J_n} C_{mnj}(b_n-x)^j \frac{d^j}{dx^j}(t-z_{2n})^{-1}\Big]\phi_n(t)dt$$

$$+ \sum_{n=1}^{N}\int_{a_n}^{b_n} k_{mn}(x,t)\phi_n(t)dt = f_m(x) \quad , \quad m = 1,..,N \quad , \quad a_m<x<b_m \quad , \tag{6.14}$$

where

$$z_{1n} = a_n + (x-a_n)e^{i\theta_n} \quad , \quad 0<\theta_n<2\pi \quad ,$$

$$z_{2n} = b_n + (b_n-x)e^{i\omega_n} \quad , \quad -\pi<\omega_n<\pi \quad , \quad n = 1,..,N \quad , \tag{6.15}$$

A_{mn}, B_{mnk}, C_{mnj} are known constants, $f_1,..,f_N$ are known input functions,

$k_{mn}(x,t)$ are Fredholm kernels, and $\phi_1,..,\phi_N$ are the unknown functions. In most practical problems $\theta_n = \pi$ and $\omega_n = 0$. However, occasionally one may encounter cases in which $\theta_n = \mp \pi/2$, $\omega_n = \mp \pi/2$ with the related terms in the kernels of the form [28]

$$(t-a_n)/[(t-a_n)^2 + (x-a_n)^2] \quad , \quad (b_n-t)/[(b_n-t)^2 + (b_n-x)^2] \quad . \qquad (6.16)$$

Referring to Figure 6a and (6.15) note that as the variable x varies on the line of integration L_n, the variables z_{1n} and z_{2n} vary on lines L_{1n} and L_{2n}, respectively. In some problems the kernels may contain terms of the form (6.16) as well as, for example, (6.12). This means that in Figure 6a at a given end point there may be more than one auxiliary line L_{1n} or L_{2n}, also meaning that there may be additional terms in (6.14) and (6.15) defined by additional angles θ_n or ω_n.

In this type of problems another point which requires special emphasis is whether the cuts $L_n = (a_n,b_n)$, $(n = 1,..,N)$ intersect each other or not. The only type of intersection which is physically meaningful is for two adjacent cuts to have a common end point (Figure 6b). A physical example for this would be the through crack problem shown in Figure 5b. In such problems, aside from the necessary changes in the limits of integrations, the integral equations (6.14) remain unchanged. However, as will be pointed out in this section, one may have to be somewhat more careful in determining the fundamental functions. In the following, first it will be assumed that all end points are distinct, then an example for coinciding ends will be considered.

To find the fundamental functions of (6.14) we let

$$\phi_n(t) = g_n(t)w_n(t) = g_n(t)(b_n-t)^{\alpha_n}(t-a_n)^{\beta_n}$$

$$= g_n(t)e^{-\pi i\alpha_n}(t-b_n)^{\alpha_n}(t-a_n)^{\beta_n} \quad , \quad a_n<t<b_n \quad ,$$

$$-1<\mathrm{Re}(\alpha_n,\beta_n)<0 \quad , \quad n = 1,..,N \quad , \quad g_n(a_n) \neq 0 \quad , \quad g_n(b_n) \neq 0 \ , \qquad (6.17)$$

where the function $g_n(t)$, $(n = 1,..,N)$ is Hölder-continuous in $a_n \leq t \leq b_n$, $(t-b_n)^{\alpha_n}(t-a_n)^{\beta_n}$ is any definite branch which varies continuously in $a_n<t<b_n$, and α_n and β_n are unknown constants which have to be determined. Consider now the following sectionally holomorphic functions:

$$F_n(z) = \frac{1}{\pi}\int_{a_n}^{b_n}\frac{\phi_n(t)}{t-z}\,dt = \frac{e^{-\pi i\alpha_n}}{\pi}\int_{a_n}^{b_n}(t-b_n)^{\alpha_n}(t-a_n)^{\beta_n}\frac{g_n(t)}{t-z}\,dt \qquad (6.18)$$

Examining the singular behavior of $F_n(z)$ around the end points (see, e.g., [3], Chapter 4) and separating the principal parts we obtain

$$F_n(z) = -\,g_n(a_n)(b_n-a_n)^{\alpha_n}\frac{e^{-\pi i\beta_n}}{\sin\pi\beta_n}(z-a_n)^{\beta_n}$$
$$+\,g_n(b_n)(b_n-a_n)^{\beta_n}\frac{1}{\sin\pi\beta_n}(z-b_n)^{\alpha_n} + G_n(z) \quad ,$$
$$n = 1,..,N \quad . \qquad (6.19)$$

The functions $G_n(z)$ is bounded everywhere except possibly at the ends a_n, b_n near which it may have the following behavior:

$$|G_n(z)| < \frac{A_n}{|z-a_n|^{p_n}} \quad , \quad p_n < -\mathrm{Re}(\beta_n) \quad , \quad |G_n(z)| < \frac{B_n}{|z-b_n|^{r_n}} \quad ,$$
$$r_n<-\mathrm{Re}(\alpha_n) \qquad (6.20)$$

where A_n, p_n, B_n, and r_n are real constants.

Using the Plemelj formulas (3.61), from (6.19) we find

$$\frac{1}{\pi}\int_{a_n}^{b_n}\frac{\phi_n(t)}{t-x}\,dt = \frac{1}{2}\,[F_n^+(x) + F_n^-(x)]$$

$$= - g_n(a_n)(b_n-a_n)^{\alpha_n}\cot(\pi\beta_n)(x-a_n)^{\beta_n}$$

$$+ g_n(b_n)(b_n-a_n)^{\beta_n}\cot(\pi\alpha_n)(b_n-x)^{\alpha_n} + H_n(x) \quad ,$$

$$n = 1,..,N \quad , \quad a_n<x<b_n \quad , \qquad (6.21)$$

where $H_n(x)$ is bounded in $a_n<x<b_n$, and at the end points its behavior is similar to that of $G_n(z)$ which is given by (6.20). From the definition of $F_n(z)$, (6.18), and from Figure 6a it is clear that for $z = z_{1n} \neq a_n$ and $z = z_{2n} \neq b_n$ i.e., on L_{1n} and L_{2n}, $F_n(z)$ is holomorphic. Therefore

$$\frac{1}{\pi}\int_{a_n}^{b_n} \frac{\phi_n(t)}{t-z_{sn}} dt = F_n(z_{sn}) \quad , \quad z_{sn}\varepsilon L_{sn} \quad , \quad s = 1,2, \; n = 1,..,N \quad , \qquad (6.22)$$

or, substituting from (6.19), using (6.15) and separating the principal parts

$$\frac{1}{\pi}\int_{a_n}^{b_n} \frac{\phi_n(t)}{t-z_{1n}} dt = - g_n(a_n)(b_n-a_n)^{\alpha_n} \frac{e^{i\beta_n(\theta_n-\pi)}}{\sin\pi\beta_n} (x-a_n)^{\beta_n}$$

$$+ h_n(x) \quad , \quad a_n<x<b_n \quad ,$$

$$\frac{1}{\pi}\int_{a_n}^{b_n} \frac{\phi_n(t)}{t-z_{2n}} dt = g_n(b_n)(b_n-a_n)^{\beta_n} \frac{e^{i\omega_n\alpha_n}}{\sin\pi\alpha_n} (b_n-x)^{\alpha_n} + s_n(x) \quad ,$$

$$a_n<x<b_n \quad , \qquad (6.23)$$

where $h_n(x)$ is bounded in $a_n<x\leq b_n$, $s_n(x)$ is bounded in $a_n\leq x<b_n$, and the behavior of h_n and s_n near the ends a_n and b_n, respectively, is determined by that of $G_n(z)$ as given by (6.20).

Since $F_n(z)$ is holomorphic on L_{1n} and L_{2n} (see (6.18) and Figure 6a), using (6.23) we may also write

$$\frac{1}{\pi}\int_{a_n}^{b_n} \phi_n(t)(x-a_n)^k \frac{d^k}{dx^k} (t-z_{1n})^{-1}dt = (x-a_n)^k \frac{d^k}{dx^k} F_n(z_{1n})$$

$$= - g_n(a_n)(b_n-a_n)^{\alpha_n} \frac{e^{i\beta_n(\theta_n-\pi)}}{\sin\pi\beta_n} \beta_n(\beta_n-1)\cdots(\beta_n-k+1)(x-a_n)^{\beta_n}$$

$$+ (x-a_n)^k \frac{d^k}{dx^k} h_n(x) \quad , \quad k = 1,..,K_n \quad , \quad a_n<x<b_n \quad ,$$

$$\frac{1}{\pi} \int_{a_n}^{b_n} \phi_n(t)(b_n-x)^j \frac{d^j}{dx^j} (t-z_{2n})^{-1}dt = (b_n-x)^j \frac{d^j}{dx^j} F_n(z_{2n})$$

$$= (-1)^j g_n(b_n)(b_n-a_n)^{\beta_n} \frac{e^{i\omega_n\alpha_n}}{\sin\pi\alpha_n} \alpha_n(\alpha_n-1)\cdots(\alpha_n-j+1)(b_n-x)^{\alpha_n}$$

$$+ (b_n-x)^j \frac{d^j}{dx^j} s_n(x) \quad , \quad j = 1,..,J_n \quad , \quad a_n<x<b_n \quad , \quad n = 1,..,N \quad . \tag{6.24}$$

Equations (6.21), (6.23) and (6.24) give all the terms in the dominant part of the integral equations (6.14). Thus, by substituting from (6.21), (6.23) and (6.24) into (6.14), observing that for the mth equation $a_m<x<b_m$, multiplying both sides first by $(x-a_m)^{-\beta_m}$ and letting $x\to a_m$, then by $(b_m-x)^{-\alpha_m}$ and letting $x\to b_m$, and also observing that $g(a_m) \neq 0$, $g(b_m) \neq 0$, $0\le r_n<Re(-\alpha_n)$, $0\le p_n<Re(-\beta_n)$, we obtain

$$A_{mm} \cos\pi\beta_m + e^{i\beta_m(\theta_m-\pi)} [B_{mmo} + \sum_{k=1}^{K_m} B_{mmk}\beta_m (\beta_m-1)\cdots(\beta_m-k+1)]$$

$$= 0 \quad , \quad m = 1,..,N \quad ,$$

$$A_{mm} \cos\pi\alpha_m + e^{i\omega_m\alpha_m} [C_{mmo} + \sum_{j=1}^{J_m} C_{mmj}(-1)^j\alpha_m(\alpha_m-1)\cdots(\alpha_m-k+1)]$$

$$= 0 \quad , \quad m = 1,..,N \quad , \tag{6.25a,b}$$

The characteristic equations (6.25) provide two sets of (highly nonlinear) algebraic equations to determine the unknown constants α_m and β_m, $(m = 1,..,N)$. With α_m and β_m determined, (6.17) gives the fundamental functions $w_m(t)$, $(m = 1,..,N)$.

In the foregoing analysis it was assumed that the 2N end points a_n, b_n are distinct. In practice this may not always be the case. To give an example in which some of the end points may coincide, consider the case of, for example, $b_1 = a_2$ in (6.14) (Figure 6b). As seen from Figure 6b (see also Figure 5), physically the problem really has 2N-1 points of singularity, and mathematically there are only 2N-1 irregular points. This implies that in the expressions of the fundamental functions there will be only 2N-1 exponents, i.e., $\alpha_1 = \beta_2$. Hence, the fundamental functions of (6.14) given by (6.17) will have to be modified as

$$w_1(t) = (b_1-t)^{\alpha_1}(t-a_1)^{\beta_1} \quad , \quad a_1<t<b_1 \quad ;$$

$$w_2(t) = (b_2-t)^{\alpha_2}(t-b_1)^{\alpha_1} \quad , \quad b_1<t<b_2 \quad ,$$

$$w_n(t) = (b_n-t)^{\alpha_n}(t-a_n)^{\beta_n} \quad , \quad a_n<t<b_n \quad , \quad n = 3,..,N \tag{6.26}$$

The solutions of (6.14) are still of the form $\phi_n(t) = g_n(t)w_n(t)$, and the conditions on α_n, β_n, $g_n(a_n)$, and $g_n(b_n)$, $(n = 1,..,N)$ shown in (6.17) are still valid. Also with $a_2 = b_1$ and $\beta_2 = \alpha_1$ the definition of $F_n(z)$, (6.18) and the asymptotic expressions (6.21), (6.23), and (6.24) are still valid. Thus, proceeding as before, one obtains (6.25a) for m = 1,3,4,..,N and (6.25b) for m = 2,3..,N giving altogether 2N-2 algebraic equations for 2N-2 unknowns $\beta_1,\beta_3,..,\beta_N$ and $\alpha_2,\alpha_3,..,\alpha_N$.

Writing (6.14) for m = 1 and letting $x \to b_1$ and for m = 2 and letting $x \to a_2 = b_1$ one obtains two homogeneous linear algebraic equations for $g_1(b_1)$ and $g_2(b_1)$ of the form

$$\sum_1^2 c_{ij}(\alpha_1)g_j(b_1) = 0 \quad , \quad i = 1,2 \quad . \tag{6.27}$$

Since $g_j(b_1) \neq 0$ (6.27) gives the following characteristic equation to determine α_1:

$$|c_{ij}(\alpha_1)| = 0 \quad . \tag{6.28}$$

Note that $g_1(b_1)$ and $g_2(b_1)$ are not independent and are related through (6.27). This provides the additional condition which was eliminated by letting $a_2 = b_1$ and which is necessary for the unique solution of (6.14)[(1)].

As an example for coinciding end points consider the crack problem formulated by (6.11) and (6.12) and shown by Figure 5a (with dashed lines included). Note that this problem was formulated by using polar coordinates. Hence in the terminology of this section, $a_1 = 0 = a_2$, $\beta_1 = \beta_2 = \beta$, and the fundamental functions and the solution may be expressed as

$$\phi_n(t) = g_n(t)w_n(t) = g_n(t)(b_n-t)^{\alpha_n}t^{\beta} = g_n(t)e^{-\pi i\alpha_n}(t-b_n)^{\alpha_n}t^{\beta} \quad ,$$

$$0<t<b_n \quad , \quad -1<\mathrm{Re}(\alpha_n,\beta)<0 \quad , \quad g_n(b_n) \neq 0 \quad , \quad g_n(0) \neq 0 \quad , \quad n = 1,2. \tag{6.29}$$

Consider now the following sectionally holomorphic functions

$$F_n(z) = \frac{1}{\pi}\int_0^{b_n}\frac{\phi_n(t)}{t-z}\,dt = \frac{e^{-\pi i\alpha_n}}{\pi}\int_0^{b_n}(t-b_n)^{\alpha_n}t^{\beta}\frac{g_n(t)}{t-z}\,dt \quad n = 1,2 \;, \tag{6.30}$$

where, noting that (6.11) is derived in polar coordinates, the complex variable $z = r+i\rho$ for $n = 1$ and $n = 2$ is defined in such a way that in each case the cut $(0<r<b_n)$ lies along the positive real axis. Repeating the analysis

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(1) For example, in collinear crack problems letting $a_2=b_1$ essentially eliminates one crack while the number N of integral equations and unknown functions remain the same. Physically, since there are N-1 cuts, there will be only N-1 single-valuedness conditions. However, the general solution of N integral equations will have N arbitrary constants. Thus for a unique solution one more condition is needed which is provided by (6.27) (see [29]).

given previously, we obtain

$$F_n(z) = - g_n(0)b_n^{\alpha_n} \frac{e^{-\pi i\beta}}{\sin\pi\beta} z^\beta + g_n(b_n)b_n^\beta \frac{1}{\sin\pi\alpha_n} (z-b_n)^{\alpha_n} + G_n(z) \quad ,$$

$$\frac{1}{\pi}\int_o^{b_n} \frac{\phi_n(t)}{t-r} dt = - g_n(0)b_n^{\alpha_n}\cot\pi\beta r^\beta + g_n(b_n)b_n^\beta \cot\pi\alpha_n(b_n-r)^{\alpha_n} + H_n(r) \quad ,$$

$$\frac{1}{\pi}\int_o^{b_n} \frac{\phi_n(t)}{t+r} dt = F_n(-r) = - g(0)b_n^{\alpha_n} \frac{1}{\sin\pi\beta} r^\beta + u_{on}(r) \quad ,$$

$$\frac{1}{\pi}\int_o^{b_n} \frac{r\phi_n(t)}{(t+r)^2} dt = - r \frac{d}{dr} F_n(-r) = g_n(0)b_n^{\alpha_n} \frac{\beta}{\sin\pi\beta} r^\beta + u_{1n}(r) \quad ,$$

$$\frac{1}{\pi}\int_o^{b_n} \frac{r^2\phi_n(t)}{(t+r)^3} dt = \frac{r^2}{2}\frac{d^2}{dr^2} F_n(-r) = - g_n(0)b_n^{\alpha_n} \frac{\beta(\beta-1)}{2\sin\pi\beta} r^\beta + u_{2n}(r) \quad ,$$

$$0<r<b_n \quad , \quad n = 1,2 \quad , \tag{6.31}$$

where the behavior of $H_n(r)$ and $u_{jn}(r)$, $(n = 1,2;\ j = 0,1,2)$ around $r = 0$ is similar to that of $G_n(z)$ around $a_n = 0$ as given by (6.20), otherwise they are bounded functions.

If we substitute from (6.31) into (6.11), multiply both sides first by $(b_n-r)^{-\alpha_n}$ and let $r\to b_n$, then by $r^{-\beta}$ and let $r\to 0$ we obtain

$$\cot\pi\alpha_n = 0 \quad , \quad n = 1,2 \quad , \tag{6.32}$$

$$[\cos\pi\beta + c_{11} - \beta c_{12} + \beta(\beta-1)c_{13}/2][g_1(0)b_1^{\alpha_1}/\sin\pi\beta]$$
$$+ (d_{11} - \beta d_{12})[g_2(0)b_2^{\alpha_2}/\sin\pi\beta] = 0 \quad ,$$

$$(d_{12} - \beta d_{22})[g_1(0)b_1^{\alpha_1}/\sin\pi\beta]$$
$$+ [\cos\pi\beta + c_{21} - \beta_{22} + \beta(\beta-1)c_{23}/2][g_2(0)b_2^{\alpha_2}/\sin\pi\beta] = 0 \; . \tag{6.33}$$

Equations (6.32) are the well-known characteristic equations for crack tip singularities which are fully imbedded in a homogeneous medium and give $\alpha_1 = \alpha_2 = -1/2$. Writing the determinant of the linear system (6.33) one obtains the third characteristic equation giving β as follows:

$$[\cos\pi\beta + c_{11} - \beta c_{12} + \beta(\beta-1)c_{13}/2][\cos\pi\beta + c_{21} - \beta c_{22}$$

$$+ \beta(\beta-1)c_{23}/2] - (d_{11}-\beta d_{12})(d_{21}-\beta d_{22}) = 0 \quad . \tag{6.34}$$

It should be noted that as expected (6.34) is identical to the characteristic equation giving the stress singularity at the apex of two bonded quarter planes by using an entirely different method (see, e.g., [30]).

Going now back to the other examples discussed in this section, in the torsion problem for $c<b$ the dominant part of the integral equation is given by (6.1) which is a very special case of (6.14). Thus defining the solution and the fundamental function by

$$p(x) = g(x)w(x) \quad , \quad w(x) = (c-x)^{\alpha}(x-b_1)^{\beta} \quad , \quad b_1<x<c \quad , \tag{6.35}$$

and applying the procedure described in this section we find

$$\cos\pi\alpha = 0 \quad , \quad \cos\pi\beta = \mu_2/(\mu_1+\mu_2) \quad . \tag{6.36}$$

If $c = b$, there is a contribution to the generalized Cauchy kernel at both ends, the integral equation is given by (6.3), and characteristic equations are found to be(1)

$$\cos\pi\alpha = \mu_1/(\mu_1+\mu_2) \quad , \quad \cos\pi\beta = \mu_2/(\mu_1+\mu_2) \quad . \tag{6.37}$$

In the case of the "broken shaft" that is, for $b_1 = 0$, $c = b$, the integral

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(1) In this case note that in (6.14) $N=1$, $K_1=0=J_1$, $\theta_1=\pi$, $\omega_1=0$.

equation is given by (6.5) from which the characteristic equations are obtained as

$$\cos\pi\alpha = \mu_1/(\mu_1+\mu_2) \quad , \quad \cos\pi\beta = (\mu_2\pm\mu_1)/(\mu_1+\mu_2) \quad , \tag{6.38a,b}$$

where the upper sign is for the symmetric and lower sign is for the antisymmetric loading. Note that for symmetric loading (for which $\sigma_{\theta r}(a,x) = p(x) = p(-x)$, Figure 3) $\beta = 0$, that is the stress state at ($r = a$, $x = 0$) is bounded, whereas for antisymmetric loading the characteristic equation (6.38b) is identical to that of a semi-infinite crack perpendicular to a bimaterial interface under anti-plane shear loading.

For the plane problem of a crack perpendicular to and terminating at the interface of two bonded half planes (Figure 5), the integral equation is given by (6.7). Thus, by defining,

$$\phi(t) = g(t)w(t) \quad , \quad w(t) = (b-t)^{\alpha}t^{\beta} \quad , \tag{6.39}$$

the characteristic equations are found to be

$$\cos\pi\alpha = 0 \quad , \quad 2\gamma_1\cos\pi(\beta+1) + \gamma_2(\beta+1)^2 - \gamma_3 = 0 \tag{6.40a,b}$$

$$\gamma_1 = (m+\kappa_2)(1+m\kappa_1) \quad , \quad \gamma_2 = 4(m+\kappa_2)(1-m) \quad , \quad m = \mu_2/\mu_1 \quad ,$$

$$\gamma_3 = (1-m)(m+\kappa_2) + (1+m\kappa_1)(m+\kappa_2) - m(1+\kappa_1)(1+\kappa_1) \quad .$$

Again (6.40a) is the well-known result giving $\alpha = -1/2$ and (6.40b) is identical to the characteristic equation for a semi-infinite crack perpendicular to and terminating at a bimaterial interface under in-plane loading (see, for example, [28]).

6.3 Numerical Method for Solving the Integral Equations with Generalized Cauchy Kernels

As shown in Section 5 of this article, a general system of singular integral equations such as (6.14) can always be expressed by means of a simple system in which both variables x and t vary in the normalized interval (-1,1). Furthermore it was also indicated that, once the method of solution is developed for a single equation, it can easily be extended to a system consisting of any number of equations. The numerical method for solving the singular integral equations with generalized Cauchy kernels will therefore be described for a single equation of the following form only:

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{t-x}+k_s(x,t)+k_f(x,t)\right]\phi(t)dt = f(x) \quad , \quad -1<x<1 \quad , \tag{6.41}$$

where f is a known function, k_f is a known Fredholm kernel, $k_s(x,t)$ becomes unbounded as x and t appraoch the end points ∓ 1, is otherwise bounded, and with $1/(t-x)$ form the generalized Cauchy kernel. Generally, (6.41) must be solved under an additional (physical, such as an equilibrium or a single-valuedness) condition of the form

$$\int_{-1}^{1}\phi(t)dt = A \quad , \tag{6.42}$$

where A is a known constant. As indicated previously, the unknown and the fundamental functions may be expressed as

$$\phi(t) = g(t)w(t) \quad , \quad w(t) = (1-t)^{\alpha}(1+t)^{\beta} \tag{6.43}$$

where α and β are known constants with $-1<\mathrm{Re}(\alpha,\beta)<0$ and $g(t)$ is an unknown function which is bounded in $-1\le t\le 1$. Observing that $w(t)$ is the weight function of Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$, the integral equation may be solved by using a numerical method based on a Gauss-Jacobi integration formula which is similar to the methods described in Section 5. In this case the related integration formula is [20]

$$\int_{-1}^{1} G(x,t)(1-t)^{\alpha}(1+t)^{\beta}dt \cong \sum_{1}^{n} W_k G(x,t_k) \quad , \quad -1<\mathrm{Re}(\alpha,\beta)<1 \tag{6.44}$$

where t_k are the roots of

$$P_n^{(\alpha,\beta)}(t_k) = 0 \quad , \quad k = 1,..,n \tag{6.45}$$

and the weighting constants are given by

$$W_k = -\frac{(2n+\alpha+\beta+2)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)2^{\alpha+\beta}}{(n+1)!(n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)}(t_k)\frac{d}{dt}P_n^{(\alpha,\beta)}(t_k)} . \tag{6.46}$$

Analogous to the numerical integration methods developed in Section 5, the integral equation (6.41) and the condition (6.42) may now be expressed as

$$\sum_{k=1}^{n} g(t_k)W_k\left[\frac{1}{t_k-x_j} + k_s(x_j,t_k) + k_f(x_j,t_k)\right] \cong \pi f(x_j) \quad , \quad j = 1..,n-1 \quad ,$$

$$\sum_{1}^{n} g(t_k)W_k = A \quad , \tag{6.47a,b}$$

where

$$-1<\mathrm{Re}(\alpha,\beta)<0 \quad , \quad P_n^{(\alpha,\beta)}(t_k) = 0 \quad , \quad k = 1,..,n \quad ,$$

$$P_{n-1}^{(\alpha+1,\beta+1)}(x_j) = 0 \quad , \quad j = 1,..,n-1 \quad , \tag{6.48}$$

and W_k are given by (6.46). Equations (6.47) provide n equations to determine $g(t_1),..,g(t_n)$.

If at one or both ends the solution is required to be bounded (hence, necessarily zero), (6.46) and (6.47a) will still be valid, the points t_k describing the location and the number of the unknowns $g(t_k)$ will still be obtained from (6.45) and the locations x_j giving the number of equations will be obtained from

$$\begin{aligned}
&(a)\quad 0<\mathrm{Re}(\alpha)<1 \;,\quad 0<\mathrm{Re}(\beta)<1 \;:\; P_{n+1}^{(\alpha-1,\beta-1)}(x_j) = 0 \;,\quad j = 1,..,n+1 \;,\\
&(b)\quad 0<\mathrm{Re}(\alpha)<1 \;,\quad -1<\mathrm{Re}(\beta)<0 \;:\; P_{n}^{(\alpha-1,\beta+1)}(x_j) = 0 \;,\quad j = 1,..,n \;,\\
&(c)\quad -1<\mathrm{Re}(\alpha)<0 \;,\quad 0<\mathrm{Re}(\beta)<1 \;:\; P_{n}^{(\alpha+1,\beta-1)}(x_j) = 0 \;,\quad j = 1,..,n \;,
\end{aligned} \tag{6.49}$$

In none of the cases given by (6.49) the condition (6.42) or (6.47b) is part of the formulation of the problem. In (6.49a) the additional equation (provided by $x_1,..,x_{n+1}$) is equivalent to the consistency condition of the integral equation and in (6.49b and c) the unique solution is obtained by simply solving (6.47a) with $j = 1,..,n$. As explained in detail in Section 5, the (physical) problem may, however, have additional conditions and unknown constants which may be handled in a straightforward manner.

In the application of the numerical methods described in Section 5 and in this section it is essential that special attention is paid to the convergence of the calculated results. At least one technique regarding the evaluation of the limit value (as $n\to\infty$) of certain calculated results is described in [25] and [27]. In the interest of space no numerical results will be presented in this article. However, there are certain techniques related to the methods described in this article which are quite useful for extracting relevant physical information from the solution. Description of such techniques also falls outside the scope of this article. Among these one may mention the methods used to evaluate such physical quantities as the stress intensity factors, the strain energy release rate, and the crack opening displacement directly in terms of the (calculated) density functions of the integral equations (see, for example, [28], [29], [31]-[34]). The detailed numerical results of the torsion problem discussed in this section may be found in [35] and [36]. In addition to [28, 29, 35, 36], further applications of the singular integral equations with generalized Cauchy kernels may be found in [37-39].

7. SINGULAR INTEGRAL EQUATIONS OF THE SECOND KIND

Some relatively very simple mixed boundary value problems in mechanics give rise to singular integral equations which are of the second kind. For example consider the following basic formulas for the elastic half plane $-\infty<x<\infty$, $y<0$ relating the surface tractions and the displacement derivatives:

$$\frac{4\mu}{1+\kappa}\frac{\partial}{\partial x}u(x,0) = \gamma\sigma_{yy}(x,0) + \frac{1}{\pi}\int_{-\infty}^{\infty}\sigma_{xy}(t,0)\frac{dt}{t-x}\ , \quad -\infty<x<\infty\ ,$$

$$\frac{4\mu}{1+\kappa}\frac{\partial}{\partial x}v(x,0) = -\gamma\sigma_{xy}(x,0) + \frac{1}{\pi}\int_{-\infty}^{\infty}\sigma_{yy}(t,0)\frac{dt}{t-x}\ , \quad -\infty<x<\infty\ ,$$

$$\frac{1+\gamma}{2\mu}\sigma_{yy}(x,0) = -\gamma\frac{\partial}{\partial x}u(x,0) - \frac{1}{\pi}\int_{\infty}^{\infty}\frac{\partial}{\partial t}v(t,0)\frac{dt}{t-x}\ , \quad -\infty<x<\infty\ ,$$

$$\frac{1+\gamma}{2\mu}\sigma_{xy}(x,0) = \gamma\frac{\partial}{\partial x}v(x,0) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}u(t,0)\frac{dt}{t-x}\ , \quad -\infty<x<\infty \qquad (7.1a\text{-}d)$$

where u, v are, respectively, the x, y- components of the displacement vector, σ_{ij}, (i,j = x,y) are the stress components and $\gamma = (\kappa-1)/(\kappa+1)$. From (7.1) it is clear that if the displacement vector is specified on part of the boundary y = 0 and the traction vector is specified on the remainder, (7.1a and b) or (7.1c and d) give the integral equations of the problem which are coupled and are of the second kind. It is also clear that instead of a half plane if one is dealing with an axisymmetric or plane problem for a domain bounded by a "smooth" surface the system of integral equations of the related mixed boundary value problem will have a dominant part which is identical to that given by (7.1).

As a very simple example consider the plane contact problem with friction shown in Figure 7 in which a rigid stamp of given profile is pressed against an elastic half plane. Assuming the coefficient of friction η to be constant and defining

$$\sigma_{yy}(t,0) = -\,p(t) \quad , \quad \sigma_{xy}(t,0) = -\,\eta p(t) \quad , \quad a<t<b \quad ,$$

$$\frac{4\mu}{1+\kappa}\frac{\partial}{\partial x}\,v(x,0) = f(x) \quad , \quad (a<x<b) \quad , \quad \omega = \eta\gamma \tag{7.2}$$

Equation (7.1b) gives the integral equation of the problem as follows:

$$\omega p(x) - \frac{1}{\pi}\int_a^b \frac{p(t)}{t-x}\,dt = f(x) \quad , \quad a<x<b \tag{7.3}$$

Here the input function f(x) is, aside from a constant multiplier, the derivative of the function describing the stamp profile (i.e., $f(x) = F'(x)$, where $y = F(x)$ is the stamp profile). At the end points if the contact is smooth, the constants a and b defining the contact area are unknown. In this case the solution (or the input function) has to be such that the following consistency condition is satisfied:

$$\int_a^b [\omega p(x) - \frac{1}{\pi}\int_a^b \frac{p(t)}{t-x}\,dt]\,\frac{dx}{w(x)} = \int_a^b \frac{f(x)}{w(x)}\,dx = 0 \tag{7.4}$$

where w(x) is the fundamental function of the singular integral equation (7.3). Also, if the load P is specified the contact pressure must satisfy the following equilibrium condition

$$\int_a^b p(t)dt = P \quad . \tag{7.5}$$

In solving the problem the conditions (7.4) and (7.5) account for the unknowns a and b.

For an elastic medium with a more complicated geometry the foregoing formulation basically remains the same with the only difference that the integral equation (7.3) now contains a Fredholm kernel which comes from the change in the geometry. For example, in the symmetric contact problem for an infinite elastic wedge of angle $2\theta_o$ shown in Figure 8, using the standard Mellin transform technique the integral equation for the contact pressure may be obtained as

$$\omega p(r) - \frac{1}{\pi}\int_a^b \left[\frac{1}{t-r} + k(r,t)\right]p(t)dt = \dot{f}(r) \quad , \quad a<r<b \quad , \tag{7.6}$$

where

$$p(r) = -\sigma_{\theta\theta}(r,\theta_o) \quad , \quad \frac{1+\kappa}{4\mu} f(r) = \frac{\partial}{\partial r} u_\theta(r,\theta_o) = F'(r) \quad , \tag{7.7}$$

$$k(r,t) = \frac{1}{r\,\log(t/r)} - \frac{1}{t-r} - \frac{\pi}{r}\,\frac{\sin^2\theta_o - \eta\sin\theta_o\cos\theta_o}{2\theta_o+\sin 2\theta_o}$$
$$+ \int_o^\infty \left(1 - \frac{\sinh 2\theta_o y}{D(y)}\right)\eta\,\frac{\cos\rho y}{r}\,dy$$
$$- \int_o^\infty \left(1 + \frac{\eta\sin 2\theta_o + \cos 2\theta_o - \cosh 2\theta_o y}{D(y)}\right)\frac{\sin\rho y}{r}\,dy \quad ,$$

$$D(y) = \sinh 2\theta_o y + y\,\sin 2\theta_o \quad , \quad \rho = \log(t/r) \quad . \tag{7.8}$$

The problem is formulated in polar coordinates and F(r) describes the profile of rigid stamp.

7.1 The Fundamental Function

The dominant part of the singular integral equations of the second kind is of the following general form

$$A\phi(x) + \frac{B}{\pi}\int_a^b \frac{\phi(t)}{t-x}\,dt = f_o(x) \quad , \quad a<x<b \quad , \tag{7.9}$$

where the bounded function f_o may contain the part of the integral equation with the Fredholm kernel. Defining

$$F(z) = \frac{1}{\pi}\int_a^b \frac{\phi(t)}{t-z}\,dt \quad , \tag{7.10}$$

and using the Plemelj formulas (3.61), (7.9) may be reduced to the following Riemann-Hilbert problem for the sectionally holomorphic function $F(z)$:

$$F^+(x) - \frac{A-iB}{A+iB}F^-(x) = 2if_o(x)/(A+iB) \quad . \tag{7.11}$$

Considering the corresponding homogeneous equation

$$X^+(x) - \frac{A-iB}{A+iB}X^-(x) = 0 \quad , \tag{7.12}$$

the fundamental solution $X(z)$ and the fundamental function $w(x)$ of (7.9) may be obtained as [3]

$$X(z) = (z-b)^\alpha(z-a)^\beta \quad , \quad w(x) = (b-x)^\alpha(x-a)^\beta \quad ,$$

$$\alpha = \frac{1}{2\pi i}\log\left(\frac{A-iB}{A+iB}\right) + N \quad , \quad \beta = -\frac{1}{2\pi i}\log\left(\frac{A-iB}{A+iB}\right) + M \tag{7.13}$$

where N and M are arbitrary (positive, zero, or negative) integers[(1)]. From (7.11) and (7.12) we have

$$\left(\frac{F(x)}{X(x)}\right)^+ - \left(\frac{F(x)}{X(x)}\right)^- = \frac{2if_o(x)}{(A+iB)X^+(x)} \quad , \quad a<x<b \quad , \tag{7.14}$$

the solution of which is

$$\frac{F(z)}{X(z)} = \frac{1}{\pi}\int_a^b \frac{f_o(t)/(A+iB)}{(t-z)X^+(t)}\,dt + C \tag{7.15}$$

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(1) Note that if A and B are real then the exponents α and β are also real. For an example of complex exponents see Section 3.3. Also, see [40] for the equivalence of the two formulation.

where C is an arbitrary constant. From (7.10) and (7.15) the solution of (7.9) may be expressed as

$$\phi(x) = - \frac{B}{A^2+B^2} \frac{w(x)}{\pi} \int_a^b \frac{f_o(t)dt}{(t-x)w(t)} + C_o w(x) + \frac{A}{A^2+B^2} f_o(x) \quad ,$$

$$a<x<b \quad , \tag{7.16}$$

where (7.13) is used to replace $X^+(x)$ by $w(x)$ and

$$C_o = - BCe^{\pi i\alpha}/(A-iB) \quad . \tag{7.17}$$

The index of the integral equation is again defined by

$$\kappa = -(\alpha+\beta) = -(N+M) \quad . \tag{7.18}$$

The general remarks made in Section 5 in connection with the contact problem shown in Figure 4 and the singular integral equation of the first kind regarding the determination of the arbitrary integers N and M or the index of the problem and the constant C_o which appears in the solution (7.16) are valid for the integral equations of the second kind also and will not be repeated here.

It should be pointed out that the fundamental function of the problem can also be determined directly by applying the method described in Section 6.2. For this we let the solution of (7.9) be

$$\phi(x) = g(x)w(x) \quad , \quad w(x) = (b-x)^{\alpha}(x-a)^{\beta} \quad . \tag{7.19}$$

From (7.10), (7.19), (6.17) and (6.18) it is clear that (6.21) is still valid. Thus, substituting from (7.19) and (6.21) into (7.9) and multiplying both sides first by $(x-a)^{-\beta}$ and letting $x\to a$ and then by $(b-x)^{-\alpha}$ and letting $x\to b$ we obtain

$$\cot\pi\beta = A/B \quad , \quad \cot\pi\alpha = - A/B \quad . \tag{7.20}$$

From

$$\cot\pi\theta = \cot\pi(\theta+K) \quad , \quad K = 0,\mp 1,\ldots,$$

$$\frac{1}{2\pi i}\log\left(\frac{A-iB}{A+iB}\right) = -\frac{1}{\pi}\,\mathrm{Arccot}(A/B) \quad , \tag{7.21}$$

it is seen that (7.13) and (7.20) are identical.

To develop a numerical method for the solution of the singular integral equations of the second kind, it will again be assumed that $\phi(t) = g(t)w(t)$. At a first glance the exact solution given by (7.16) appears to be not of this form. However, the integral in (7.16) can be evaluated and it can be shown that the expression for $\phi(t)$ indeed has no regular terms. To do this first observe that

$$w(x)\int_a^b \frac{f_o(t)dt}{(t-x)w(t)} = X^+(x)\int_a^b \frac{f_o(t)dt}{(t-x)X^+(t)} \quad ,$$

$$\Phi(z) = \int_a^b \frac{f_o(t)dt}{(t-z)X^+(t)} = \frac{1}{1-X^+/X^-}\left[\int_a^b \frac{f_o(t)dt}{(t-z)X^+(t)} + \int_b^a \frac{f_o(t)dt}{(t-z)X^-(t)}\right]$$

$$= \frac{A+iB}{2iB}\int_S \frac{f_o(\tau)d\tau}{(\tau-z)X(\tau)} \quad , \tag{7.22}$$

where S is the contour shown in Figure 9 and (7.12) and (7.13) have been used. Now if $f_o(\tau)$ is holomorphic outside S and continuous up to S, the contour integral may be evaluated as follows:

$$\frac{1}{2\pi i}\int_S \frac{f_o(\tau)d\tau}{(\tau-z)X(\tau)} = \frac{f_o(z)}{X(z)} - P(z) \tag{7.23}$$

where $P(z)$ is such that for large $|z|$

$$\frac{f_o(z)}{X(z)} = P(z) + O(1/z) \tag{7.24}$$

On the other hand using the Plemelj formula, from (7.22) it follows that

$$\int_a^b \frac{f_o(t)dt}{(t-x)X^+(t)} = \frac{1}{2}\,[\Phi^+(x) + \Phi^-(x)] \tag{7.25}$$

Thus, substituting from (7.22)-(7.25) into (7.16) and observing that

$$(A+iB)X^+(x) = e^{\pi i\alpha}w(x)(A^2+B^2)^{1/2}e^{-\pi i\alpha} \tag{7.26}$$

we obtain

$$\phi(x) = [(A^2+B^2)^{-1/2}P(x) + C_o]w(x) \quad . \tag{7.27}$$

7.2 Solution by Orthogonal Polynomials

Consider the following integral equation in the normalized interval (-1,1)

$$A\phi(x) + \frac{B}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}\,dt + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x) \quad , \quad -1<x<1 \tag{7.28}$$

Assume that the fundamental function and the index of the problem have been determined and are given by (see Section 7.1)

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta} \quad , \quad \kappa = -(\alpha+\beta) = (0,+1,-1) \quad . \tag{7.29}$$

Observing that w(t) is the weight of Jacobi polynomials, we may express the solution of (7.28) as

$$\phi(t) = g(t)w(t) \quad , \quad g(t) = \sum_{o}^{\infty} c_n P_n^{(\alpha,\beta)}(t) \quad , \quad -1<t<1 \tag{7.30}$$

where the coefficients c_n are unknown. Consider now the following property of Jacobi polynomials [23, 41, 26]:

$$AP_n^{(\alpha,\beta)}(x)w(x) + \frac{B}{\pi}\int_{-1}^{1}P_n^{(\alpha,\beta)}(t)\,\frac{w(t)}{t-x}\,dt$$

$$= -\,2^{-\kappa}\,\frac{B}{\sin\pi\alpha}\,P_{n-\kappa}^{(-\alpha,-\beta)}(x) \quad , \quad -1<x<1 \quad ,$$

$$\mathrm{Re}(\alpha)>-1 \quad , \quad \mathrm{Re}(\beta)>-1 \quad , \quad \mathrm{Re}(\alpha) \neq (0,1,..) \tag{7.31}$$

From (7.30), (7.31), and (7.28) it follows that

$$\sum_{n=0}^{\infty} c_n [- \frac{2^{-\kappa} B}{\sin\pi\alpha} P_{n-\kappa}^{(-\alpha,-\beta)}(x) + k_n(x)] = f(x) \quad , \quad -1<x<1 \quad ,$$

$$k_n(x) = \int_{-1}^{1} k(x,t) P_n^{(\alpha,\beta)}(t) w(t) dt \quad . \tag{7.32}$$

An effective way of solving for c_n in (7.32) would be to expand both sides into series of Jacobi polynomials $P_j^{(-\alpha,-\beta)}(x)$, and to compare the coefficients. Thus, using the orthogonality relations (5.72) we obtain

$$- \frac{2^{-\kappa} B}{\sin\pi\alpha} \theta_j(-\alpha,-\beta) c_{j+\kappa} + \sum_{n=0}^{\infty} d_{jn} c_n = F_j \quad , \quad j = 0,1.. \tag{7.33}$$

$$d_{jn} = \int_{-1}^{1} P_j^{(-\alpha,-\beta)}(x) k_n(x) dx/w(x) \quad ,$$

$$F_j = \int_{-1}^{1} P_j^{(-\alpha,-\beta)}(x) f(x) dx/w(x) \quad . \tag{7.34}$$

It is again worthwhile to consider the following three cases separately.

(a) $\kappa = 1$

In this case solving (7.33) by the method of reduction [24], if the series is truncated at $n = N$, the system of $N+1$ linear equation will contain the unknowns $c_0, \ldots, c_{N+1}$. The additional equation to solve the problem is provided by the following condition:

$$\int_{-1}^{1} \phi(t) dt = A_0 \tag{7.35}$$

which, substituting from (7.30) and using the orthogonality conditions, may be expressed as [(1)]

[(1)] Note that for $\kappa = 1$ the arbitrary constant C_0 appearing in the exact solution (7.16) or (7.27) is also determined from (7.35).

$$c_o\theta_o(\alpha,\beta) = A_o \quad . \tag{7.36}$$

(b) $\kappa = 0$

In this case, truncated at $n = N$, (7.33) has N+1 equations and N+1 unknowns and gives the unique solution.

(c) $\kappa = -1$

In this case observing that $P_o^{(\alpha,\beta)}(t) = 1$, the first equation ($j = 0$) in (7.33) may be expressed as[1]

$$F_o - \sum_o^\infty d_{on}c_n = \int_{-1}^1 f(x)\frac{dx}{w(x)} - \int_{-1}^1 \frac{dx}{w(x)}\sum_o^\infty c_n k_n(x)$$

$$= \int_{-1}^1 \frac{dx}{w(x)}\left[f(x) - \int_{-1}^1 k(x,t)\phi(t)dt\right] = 0 \tag{7.37}$$

which is the consistency condition of the integral equation and is seen to be automatically satisfied if this technique is used. If the boundary value has a symmetry with respect to x in the sense that, for example, in Figure 7 $a = -b$, then there is only one additional unknown, b and (7.33) and (7.35) provide N+2 equations to determine $c_o,..,c_N$, and b. However, if the problem has no symmetry (which is always the case when the friction is involved), both a and b are unknown. Theoretically, the consistency condition with (7.35) provide the additional equations to determine a and b. In this case an extra equation may be gained by writing (7.33) as

$$-\frac{2B}{\sin\pi\alpha}\theta_j(-\alpha,-\beta)c_{j-1} + \sum_{n=0}^{N-1} d_{jn}c_n = F_j \quad , \quad j = 0,..,N \quad , \tag{7.38}$$

which, with (7.35) provide N+2 equations to determine $c_o,..,c_{N-1}$, a, and b.

It is seen that if $k(x,t)$ is zero and $f(x)$ is a polynomial of finite degree, by expressing it in terms of a series of Jacobi polynomials $P_k^{(-\alpha,-\beta)}(x)$

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[1] Note that (7.32) starts with $c_oP_1^{(-\alpha,-\beta)}(x)$. Therefore in (7.33) $c_{-1} = 0$.

and using (7.31) the solution of the integral equation (7.28) may be obtained in closed form by simple observation. Such solutions of some simple examples regarding the contact problem shown in Figure 7 may be found in [26]. The solution of the wedge problem shown in Figure 8 with and without a crack initiating at the apex will appear elsewhere. The application of the technique described in this section to singular integral equations of the second kind with complex coefficients may be found in [31], [33], [42] and [43]. Extension of the technique to a system of equations in which A, B, and k(x,y) are square matrices is given in [44] and [25].

7.3 Solution by Gauss-Jacobi Integration Formulas

In order to solve the singular integral equation of the second kind (7.28) in a direct way one needs an integration formula for the dominant part

$$K(\phi) = A\phi(x) + \frac{B}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t-x}\,dt \quad . \tag{7.39}$$

Following closely the procedure outlined in Section 5 (see also [26]) such an integration formula can indeed be developed [45]. Analogous to (5.53) first consider the expansion

$$\frac{P_{n-\kappa}^{(-\alpha,-\beta)}(x)P_j^{(\alpha,\beta)}(x) - P_n^{(\alpha,\beta)}(x)P_{j-\kappa}^{(-\alpha,-\beta)}(x)}{P_n^{(\alpha,\beta)}(x)} = -\sum_{k=1}^{n} \frac{b_k}{t_k-x} \tag{7.40}$$

where

$$P_n^{(\alpha,\beta)}(t_k) = 0 \quad , \quad k = 1,..,n \quad , \tag{7.41}$$

$$b_k = [P_{n-\kappa}^{(-\alpha,-\beta)}(t_k)P_j^{(\alpha,\beta)}(t_k)] / \frac{d}{dt}P_n^{(\alpha,\beta)}(t_k) \quad . \tag{7.42}$$

If we now select x_r as the roots of

$$P_{n-\kappa}^{(-\alpha,-\beta)}(x_r) = 0 \quad , \quad r = 1,..,n-\kappa \quad , \tag{7.43}$$

from (7.40) it follows that

$$P_{j-\kappa}^{(-\alpha,-\beta)}(x_r) = \sum_{k=1}^{n} \frac{P_{n-\kappa}^{(-\alpha,-\beta)}(t_k) P_j^{(\alpha,\beta)}(t_k)}{\frac{d}{dt} P_n^{(\alpha,\beta)}(t_k)} \frac{1}{t_k - x_r} \tag{7.44}$$

We next consider (7.30) and assume that g(t) can be approximated to a sufficient degree of accuracy by

$$g(t) \cong \sum_{j=0}^{p} c_j P_j^{(\alpha,\beta)}(t) \quad . \tag{7.45}$$

From (7.30), (7.31), (7.39), (7.44) and (7.45) for $n>p$ it then follows that

$$K[\phi(x_r)] \cong -\frac{2^{-\kappa}B}{\sin\pi\alpha} \sum_{j=0}^{p} c_j P_{j-\kappa}^{(-\alpha,-\beta)}(x_r)$$

$$= \frac{-2^{-\kappa}B}{\sin\pi\alpha} \sum_{k=1}^{n} \frac{P_{n-\kappa}^{(-\alpha,-\beta)}(t_k)}{\frac{d}{dt} P_n^{(\alpha,\beta)}(t_k)} \frac{g(t_k)}{t_k - x_r} \tag{7.46}$$

Again, the only approximation in (7.46) is due to the truncation in (7.45). Using the properties of Jacobi polynomials (7.46) can be put into the standard form [45]

$$K[\phi(x_r)] \cong \sum_{k=1}^{n} W_k \frac{Bg(t_k)}{t_k - x_r} \quad , \tag{7.47}$$

$$W_k = \frac{2^{\alpha+\beta}(2n+\alpha+\beta)\Gamma(n+\alpha)\Gamma(n+\beta)}{\pi n!\Gamma(n+\alpha+\beta+1)P_{n-1}^{(\alpha,\beta)}(t_k)\frac{d}{dt}P_n^{(\alpha,\beta)}(t_k)}$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\pi n!\Gamma(n+\alpha+\beta+1)(1-t_k^2)[\frac{d}{dt}P_n^{(\alpha,\beta)}(t_k)]^2}$$

$$= -\frac{2^{\alpha+\beta}(2n+\alpha+\beta+2)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\pi(n+1)!\Gamma(n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)}(t_k)\frac{d}{dt}P_n^{(\alpha,\beta)}(t_k)} \tag{7.48}$$

Using (7.47) the integral equation (7.28) may now be expressed as

$$\sum_{k=1}^{n} W_k\left[\frac{Bg(t_k)}{t_k-x_r} + \pi k(x_r,t_k)\right] = f(x_r) \quad , \quad r = 1,..,n-\kappa \quad , \tag{7.49}$$

where t_k and x_r are given by (7.41) and (7.43). For $\kappa = 1$, (7.49) gives n-1 equations to determine $g(t_1),..,g(t_n)$. The nth equation is obtained from (7.35) which becomes

$$\sum_{k=1}^{n} W_k g(t_k) = A_o/\pi \quad . \tag{7.50}$$

For $\kappa = 0$ (7.49) contains n equations and n unknowns and gives a unique solution. In this case (7.35) may be used to determine the unknown end value (e.g., a or b in Figure 7).

For $\kappa = -1$ (7.49) and (7.35) give n+2 equations to determine $g(t_1),..,g(t_n)$, and the unknown end values a and b. It can again be shown that by using this method the consistency condition of the integral equation

$$\int_{-1}^{1} [f(x) - \int_{-1}^{1} k(x,t)\phi(t)dt] \frac{dx}{w(x)} = \int_{-1}^{1} K[\phi(x)]\frac{dx}{w(x)} = 0 \tag{7.51}$$

is automatically satisfied. This can be seen by observing that $1/w(x)$ is the weight of $P_n^{(-\alpha,-\beta)}(x)$, $\kappa = -1$, $P_o^{(-\alpha,-\beta)}(x) = 1$,

$$K[\phi(x)] = -\frac{2B}{\sin\pi\alpha}\sum_{j=0}^{p} c_j P_{j+1}^{(-\alpha,-\beta)}(x) \quad , \tag{7.52}$$

and by using the orthogonality conditions for the Jacobi polynomials.

As an example consider the wedge problem shown by Figure 8 and formulated by (7.6-7.8). Assume that the rigid wedge has flat faces with sharp corners at the ends a and b. Thus, (7.6) must be solved for $f(r) = 0$ and under the condition that

$$\int_a^b p(r)dr = P = P_o/[2(\sin\theta_o - \eta\cos\theta_o)] = A_o \quad . \tag{7.53}$$

Let the coefficient of friction be $\eta = 0.5$. From (7.13) it then follows that

$$\alpha = -0.545167 \quad , \quad \beta = -0.454833 \quad , \quad \kappa = 1 \quad ,$$

$$p(r) = g(r)w(r) \quad , \quad w(r) = (b-r)^{\alpha}(r-a)^{\beta} \quad . \tag{7.54}$$

The solution is obtained in a very straightforward manner by considering (7.6-7.8) with (7.49), (7.50), and (7.54).

Of particular interest in problems of this type is the strength of stress singularity around the singular points a and b. This is known as the stress intensity factor and, in this case, may be defined by

$$k(a) = \lim_{r \to a} \sqrt{2}\,(r-a)^{-\beta} p(r) \quad ,$$

$$k(b) = \lim_{r \to b} \sqrt{2}\,(b-r)^{-\alpha} p(r) \quad . \tag{7.55}$$

Some sample numberical results are shown in Table 7.1 where $c = (b-a)/2$ and P is defined by (7.53), P_o being the total wedging force.

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TABLE 1

Stress intensity factors for the contact problem in an elastic wedge (Figure 8).

$\frac{b+a}{b-a}$	$\theta_o(^o)$	$\frac{k(a)}{P/(\pi c^{1+\beta})}$	$\frac{k(b)}{P/(\pi c^{1+\alpha})}$
10	150.0	1.012	0.989
	165.0	1.031	0.971
	172.5	1.037	0.967
4	150.0	1.055	0.957
	165.0	1.108	0.916
	172.5	1.122	0.903
2	150.0	1.155	0.906
	165.0	1.281	0.825
	172.5	1.316	0.803
4/3	150.0	1.346	0.852
	165.0	1.611	0.726
	172.5	1.688	0.689

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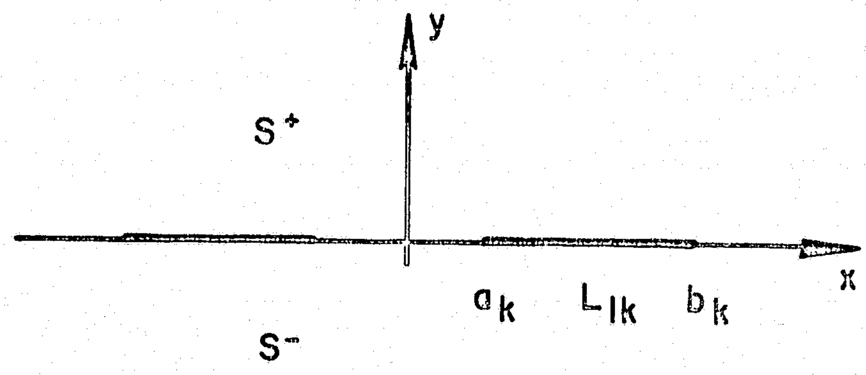

Figure 1. Plane with collinear cuts.

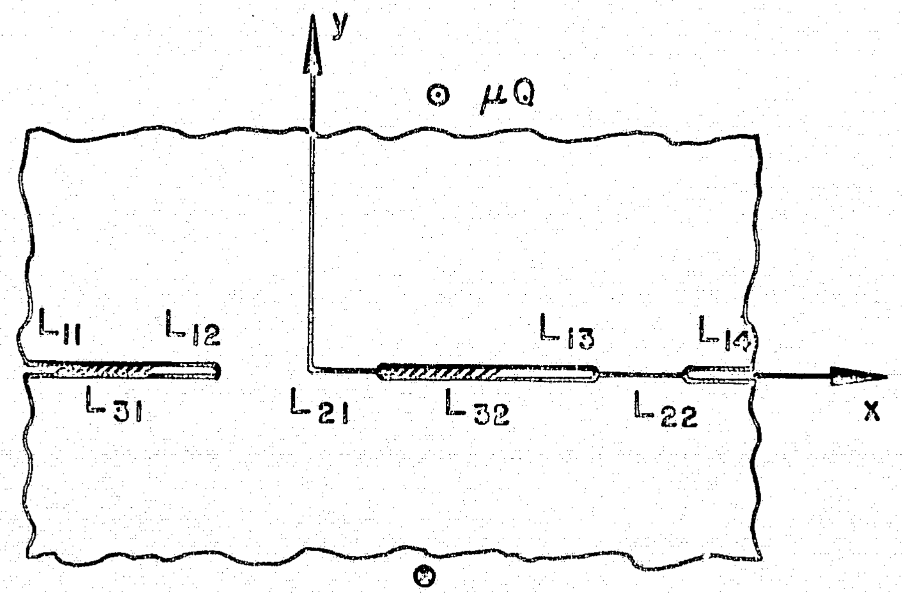

Figure 2. Plane with collinear cracks under antiplane shear loading.

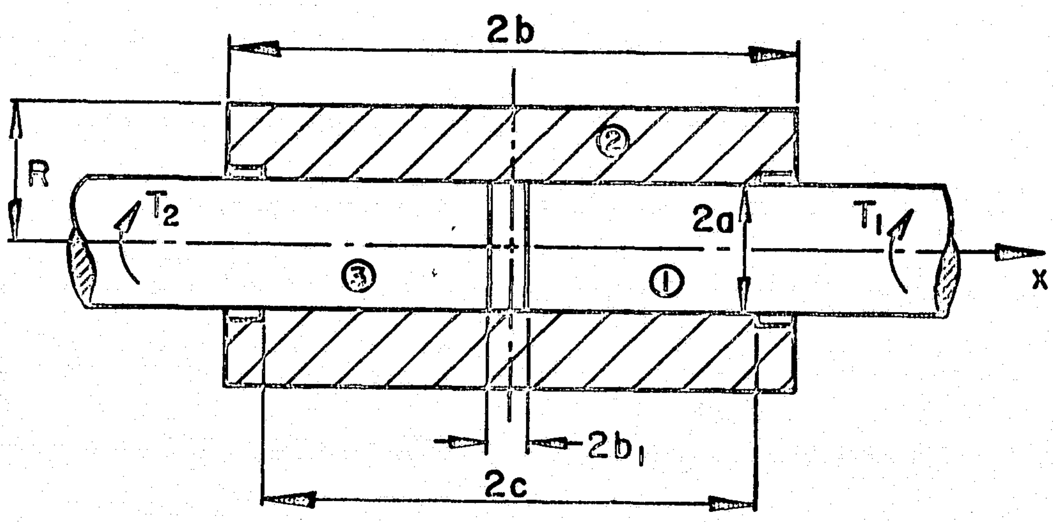

Figure 3. The torsion problems for two shafts coupled through an elastic sleeve.

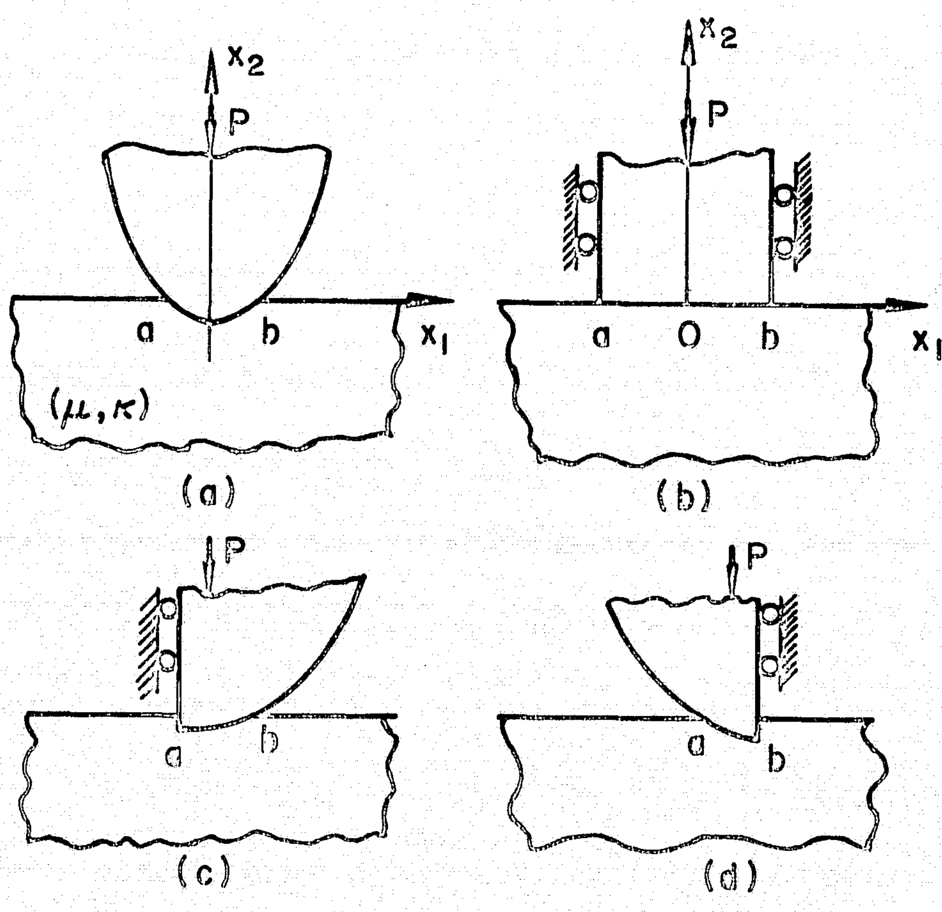

Figure 4. Contact problem for an elastic half plane.

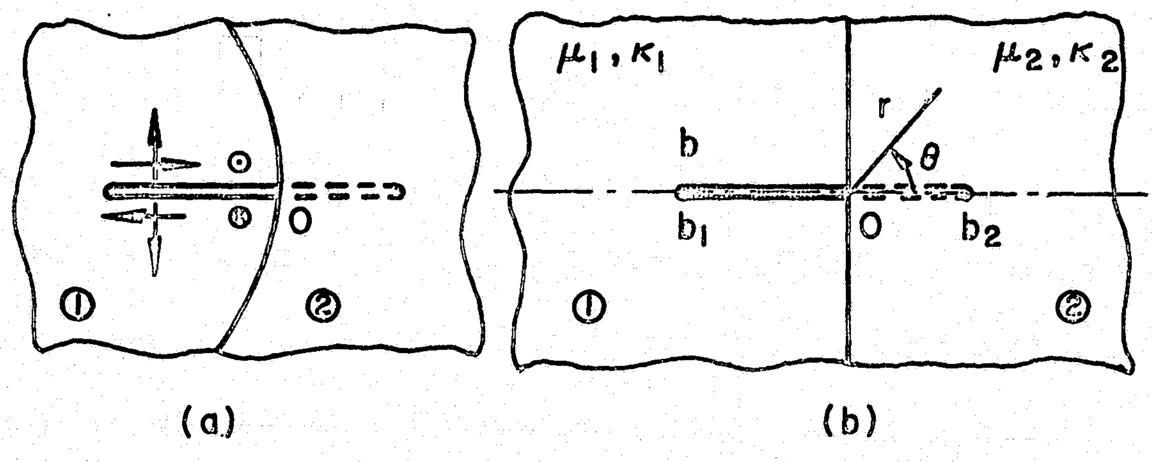

Figure 5. Bonded elastic materials containing a crack terminating at or going through the interface.

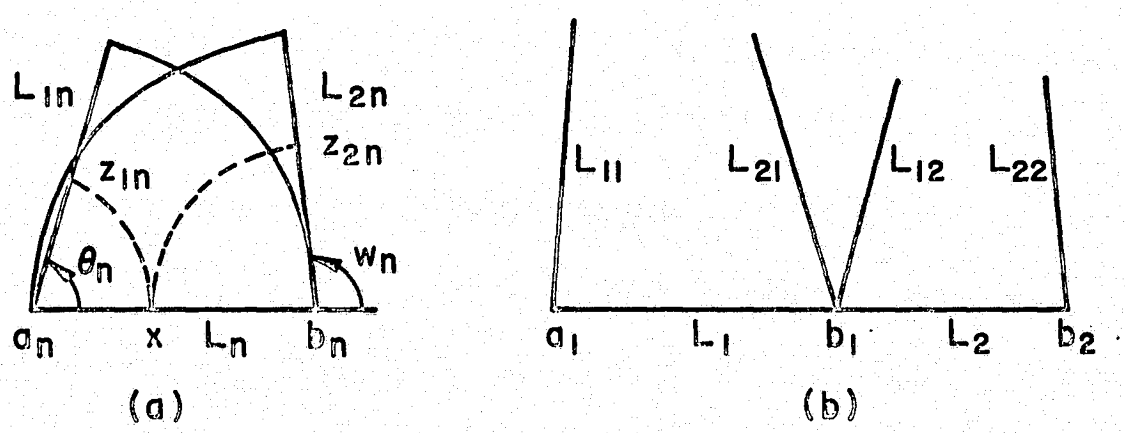

Figure 6. Cuts for generalized Cuachy kernel.

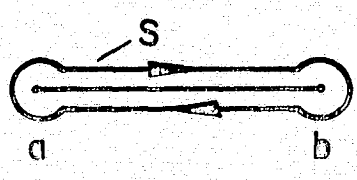

Figure 9. Contour for evaluating the integral in equation (7.22).

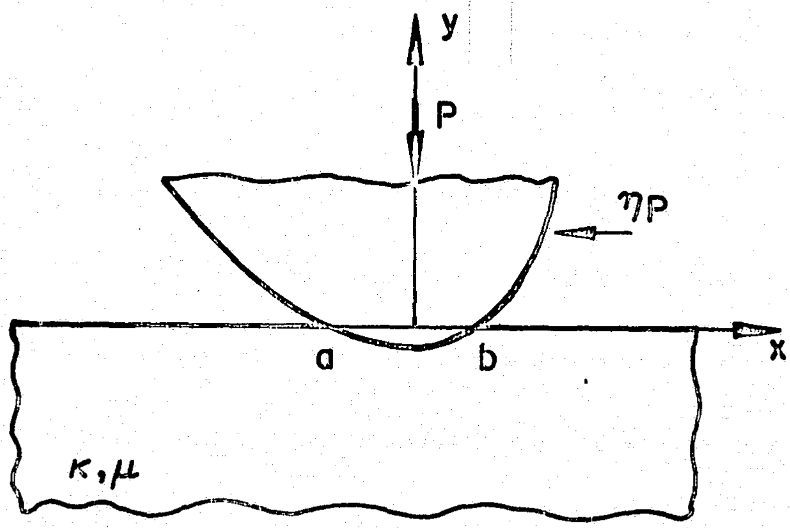

Figure 7. The contact problem for an elastic half plane.

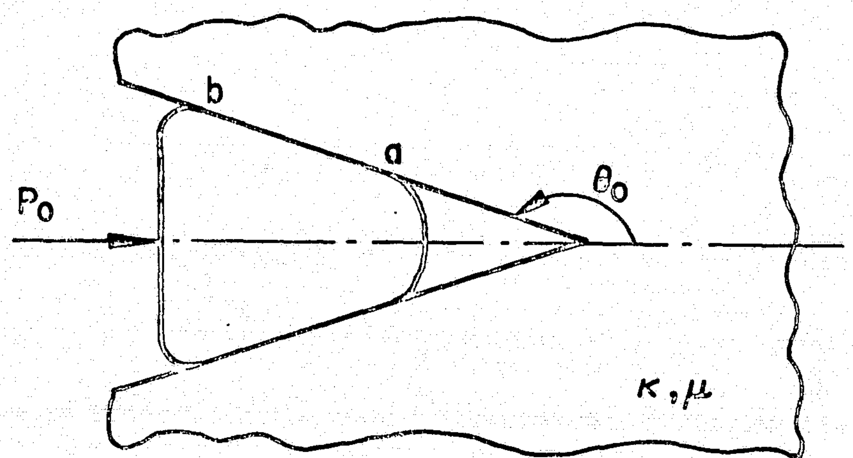

Figure 8. The contact problem for an elastic wedge in the presence of friction.