MIXED COMPOUND POISSON DISTRIBUTIONS*

By Gord Willmot

Department of Statistics and Actuarial Science, University of Waterloo

Abstract

The distribution of total claims payable by an insurer is considered when the frequency of claims is a mixed Poisson random variable. It is shown how in many cases the total claims density can be evaluated numerically using simple recursive formulae (discrete or continuous).

Mixed Poisson distributions often have desirable properties for modelling claim frequencies. For example, they often have thick tails which make them useful for long-tailed data. Also, they may be interpreted as having arisen from a stochastic process. Mixing distributions considered include the inverse Gaussian, beta, uniform, non-central chi-squared, and the generalized inverse Gaussian as well as other more general distributions.

It is also shown how these results may be used to derive computational formulae for the total claims density when the frequency distribution is either from the Neyman class of contagious distributions, or a class of negative binomial mixtures. Also, a computational formula is derived for the probability distribution of the number in the system for the M/G/1 queue with bulk arrivals.

Keywords

Mixed Poisson, compound Poisson, recursions, Volterra integral equations, inverse Gaussian, Poisson mixtures, Neyman contagious distributions, power mixtures, queues, infinite divisibility, compound distributions.

1. NOTATION AND PRELIMINARIES

It is of interest to obtain the probability distribution of the claims payable by an insurer. To formulate the problem, it is assumed that these total claims on a portfolio of business may be represented as

$$Y = X_1 + X_2 + \cdots + X_N \tag{1.1}$$

where N is a counting random variable representing the number of claims payable by the insurer and $\{X_k;\ k = 1, 2, 3, \dots\}$ is a sequence of independent and identically distributed non-negative random variables (independent of N) representing the size of the single claims.

* This research was supported by the Natural Sciences and Engineering Research Council of Canada. The author is indebted to an anonymous referee for numerous suggestions.

Let

(1.2) $$p_n = Pr\{N=n\}, \qquad n=0,1,2,\ldots$$

and

(1.3) $$P(z)=\sum_{n=0}^{\infty} p_n z^n.$$

Furthermore, let the common cdf of the sequence $\{X_k;\, k=1,2,3,\ldots\}$ be

(1.4) $$F(x)=Pr\{X_k \leq x\}, \qquad x \geq 0$$

and the associated Laplace transform be

(1.5) $$L_X(s)=E(e^{-sX_k}).$$

Similarly, for the random variable Y, let

(1.6) $$G(y)=Pr\{Y \leq y\}, \qquad y \geq 0$$

and

(1.7) $$L_Y(s)=E(e^{-sY}).$$

It is well known

(1.8) $$L_Y(s)=\sum_{n=0}^{\infty} p_n[L_X(s)]^n = P[L_X(s)].$$

To this point, no assumption has been made as to the support of X_k (and hence of T). Two possible cases are considered.

(a) Case 1—X_k has discrete support

In this situation X_k is assumed to be a counting random variable with probabilities

(1.9) $$f_x = Pr\{X_k = x\}, \qquad x=0,1,2,\ldots$$

Clearly, Y is also a counting random variable and so let

(1.10) $$g_y = Pr\{Y=y\}, \qquad y=0,1,2,\ldots$$

It is obvious that in this case the relation (1.8) holds if $L(\cdot)$ is interpreted to be the probability generating function of Y and the X_k.

(b) Case 2—X_k has mixed support

Here it is assumed that X_k is absolutely continuous on $(0,\infty)$ but with a (possible) spike at 0. Thus let

(1.11) $$f_0 = Pr\{X_k = 0\},$$

and

(1.12) $$f(x)=\frac{d}{dx}F(x), \qquad x>0.$$

https://doi.org/10.1017/S051503610001165X Published online by Cambridge University Press

Define the Laplace transform of (1.12) to be

$$\tilde{f}(s)=\int_{0+}^{\infty} e^{-sx}f(x)\,dx = L_X(s)-f_0. \tag{1.13}$$

In this case Y has the same support and so define

$$g_0=Pr\{Y=0\}, \tag{1.14}$$

$$g(y)=\frac{d}{dy}G(y), \qquad y>0 \tag{1.13}$$

and

$$\tilde{g}(s)=\int_{0+}^{\infty} e^{-sy}g(y)\,dy = L_Y(s)-g_0. \tag{1.16}$$

Two choices of support for X_k are made in order to obtain recursive computational formulae for the density $g(y)$ or the probability function g_y for discrete, mixed, and continuous claim amounts. In both cases, the support of X_k is the same as that of Y. This allows for repeated application of the recursive formulae, and thus for extension of the results to contagious type distributions with pgf's of the form $Q_1[Q_2(z)]$ where Q_1 and Q_2 are pgf's. From (1.8), $L_Y(s)=Q_1[L_Z(s)]$ where $L_Z(s)=Q_2[L_X(s)]$ so that, in general, the support of X_k and Y should be the same for repeated application. The continuous case can easily be handled by assuming that $f_0=0$ in the mixed support case.

Separate notation for discrete and continuous portions of the distribution is used rather than a generalized density in order to prevent notational difficifities (e.g., $\lim_{y\to 0} g(y)\neq g_0$). In keeping with this philosophy of ease of interpretation, only discrete sums and Riemann integrals are used.

Computational techniques are (in general) necessary because analytic expressions for compound random variables are only obtainable in the simplest of cases. An important class of frequency distributions is that for which

$$np_n=[(a+b)+a(n-1)]p_{n-1}, \qquad n=2,3,4,\ldots \tag{1.17}$$

Recursive formulae for the total claim distributions with frequency probabiliites satisfying (1.17) were derived by SUNDT and JEWELL (1981) and WILLMOT and PANJER (1985). They showed that if X_k has discrete support, then

$$g_x=h_x+\sum_{y=1}^{x} k_{x,y}g_{x-y}, \qquad x=1,2,3,\ldots \tag{1.18}$$

where

$$h_x=\frac{p_1-(a+b)p_0}{1-af_0}f_x \tag{1.19}$$

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and

$$k_{x,y} = \frac{ax+by}{x(1-af_0)} f_y. \tag{1.20}$$

In the mixed support case,

$$g(x) = h(x) + \int_0^x k(x, y) g(x-y)\, dy, \qquad x > 0 \tag{1.21}$$

where

$$h(x) = \frac{p_1 + (a+b)(g_0 - p_0)}{1 - af_0} f(x) \tag{1.22}$$

and

$$k(x, y) = \frac{ax+by}{x(1-af_0)} f(y). \tag{1.23}$$

These relations, together with

$$g_0 = P(f_0), \tag{1.24}$$

enable one to obtain the distribution of total claims recursively for the class of frequency distributions satifying (1.17). The results are easily extended to more complicated contagious distributions (see WILLMOT and PANJER, 1985) through repeated applications (a finite number of times).

Equation (1.21) is a Volterra integral equation of the second kind and (1.18) the discrete analogue. BAKER (1977) gives a good description of the numerical solution of (1.21) for $g(x)$. STRÖTER (1984) considers these recursions in an insurance situation. Many of the computational formulae of this paper are of the form (1.18) or (1.21).

Some common distributions satisfying (1.17) include the Poisson, negative binomial, binomial, geometric, and the logarithmic series. An additional member of the class of distributions satisfying (1.17) will be introduced in Section 3.

Many authors have looked at claim frequency distributions which satisfy difference equations in order to derive computational formulae for the total claims distribution. PANJER (1981), SUNDT and JEWELL (1981), WILLMOT and PANJER (1985), and others have used this approach.

The purpose of this paper is to present an alternative class of frequency distributions for which recursive techniques are obtainable. Consider the class of mixed Poisson distributions with probabilities

$$p_n = \int_0^\infty \frac{\lambda^n e^{-\lambda}}{n!}\, dU(\lambda), \qquad n = 0, 1, 2, \ldots \tag{1.25}$$

where $U(\cdot)$ is the cdf of a non-negative random variable. These distributions have often been used to model insurance claim numbers in the collective risk theory (BÜHLMANN, 1969). LUNDBERG (1940) and MCFADDEN (1965) provide good descriptions of mixed Poisson processes. Mixed Poisson distributions also arise in some queueing contexts (e.g., M/G/1 queue). They have thicker tails than the Poisson distribution and as such may be more suitable for modelling claim frequencies in some situations. Some basic properties of mixed Poisson distributions are now summarized.

2. BASIC PROPERTIES OF MIXED POISSON RANDOM VARIABLES

From (1.25), it is easily seen that the pgf (1.3) of N satisfies

$$P(z)=\int_0^\infty e^{\lambda(z-1)}\,dU(\lambda). \tag{2.1}$$

If the Laplace transform of the mixing distribution is defined to be

$$\tilde{u}(s)=\int_0^\infty e^{-s\lambda}\,dU(\lambda) \tag{2.2}$$

then, clearly,

$$P(z)=\tilde{u}(1-z). \tag{2.3}$$

If $U(\cdot)$ is a discrete counting distribution, then $P(z)=Q(e^{z-1})$ where Q is a pgf. These discrete mixtures correspond to contagious or compound random variables and will not be dealt with here. See ORD (1972) or DOUGLAS (1980) for a discussion of the relationship between mixing and compounding.

There is one particular relationship between mixing and compounding which is of interest, however. Suppose $U(.)$ is the cdf of an infinitely divisible distribution. Then (MACEDA, 1948) the mixed Poisson distribution (1.25) is also infinitely divisible. See BÜHLMANN and BUZZI (1970) also. This implies (FELLER, 1968) that (1.25) also defines a compound Poisson distribution and so (2.3) may be written as

$$P(z)=e^{\mu[Q(z)-1]} \tag{2.4}$$

where $\mu>0$ is a parameter and $Q(z)$ the pgf of a counting distribution. Furthermore, if one adopts the convention that $Q(0)=0$, then (VAN HARN, 1978) μ and $Q(z)$ are unique. It is often of interest to identify μ and $Q(z)$ explicitly and this can be done in some situations. A well known example is now given.

EXAMPLE 2.1. *The Negative Binomial Distribution.* Suppose $U(\cdot)$ is a gamma distribution, i.e.,

$$dU(\lambda)=\frac{\frac{1}{\beta}\left(\frac{\lambda}{\beta}\right)^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)}\,d\lambda.$$

The Laplace transform of this distribution is

$$\tilde{u}(s)=(1+\beta s)^{-\alpha}.$$

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Thus, by (2.3),

$$P(z)=[1-\beta(z-1)]^{-\alpha}$$

and so

$$p_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!}(1+\beta)^{-\alpha-n}\beta^n, \qquad n=0,1,2,3,\ldots$$

Since $Q(0)=0$ in (2.4),

$$\begin{aligned}\mu &= -\log p_0\\ &= \alpha \log(1+\beta)\end{aligned}$$

and

$$Q(z)=\frac{1}{\mu}\log P(z)+1$$

$$=\frac{\log\left(1-\frac{\beta}{1+\beta}z\right)}{\log\left(1-\frac{\beta}{1+\beta}\right)} \tag{2.5}$$

which is a logarithmic series pgf. For some other mixing distributions, similar results can (and will) be obtained.

It is easy to show that Poisson mixtures are identifiable (see DOUGLAS, 1980). This means that

$$\int e^{\lambda(z-1)}\,dU_1(\lambda)=\int e^{\lambda(z-1)}\,dU_2(\lambda)$$

implies that $U_1(\lambda)=U_2(\lambda)$. Thus, in discussing Poisson mixtures, one can discuss the (unique) mixing function.

The convolution of two mixed Poisson distributions is again a mixed Poisson distribution with mixing distribution which is the convolution of the two component mixing distributions. This is obvious from (2.3), since

$$P(z)=\tilde{u}_1(1-z)\tilde{u}_2(1-z)$$

can arise by convoluting two mixed Poissons, or by mixing over a convolution. Thus, mixed Poisson distributions are closed under convolution.

If $U(\lambda)$ is absolutely continuous with unimodal density, then the corresponding mixed Poisson distribution (1.25) is also unimodal (HOLGATE, 1970).

Many relations are easily obtainable from (2.1). Clearly,

$$\frac{d^k}{dz^k}P(z)=\int_0^\infty \lambda^k e^{\lambda(z-1)}\,dU(\lambda)$$

and so

$$E[N(N-1)\cdots(N-k+1)]=E(\lambda^k). \tag{2.6}$$

In particular,

$$V(N)=E(\lambda^2)+E(\lambda)-[E(\lambda)]^2$$
$$=V(\lambda)+E(N).$$

Thus, mixed Poisson random variables have variance exceeding the mean (unlike the Poisson).

For a thorough discussion of mixed Poisson distributions, see DOUGLAS (1980), HAIGHT (1967), or JOHNSON and KOTZ (1969). Some particular choices of $U(\cdot)$ are now considered.

3. SICHEL'S MIXED POISSON DISTRIBUTIONS

An alternative to the gamma distribution as a mixing distribution is the generalized inverse Gaussian distribution given by

$$dU(x)=\frac{\mu^{-\lambda}x^{\lambda-1}\,e^{-(x^2+\mu^2)/2\beta x}}{2K_\lambda(\mu\beta^{-1})}\,dx \tag{3.1}$$

where $K_\lambda(\cdot)$ is the modified Bessel function of the third kind with index λ. This distribution is discussed in great detail by Jørgensen (1982). The Laplace transform of (3.1) is

$$\tilde{u}(s)=\frac{K_\lambda\{\mu\beta^{-1}(1+2\beta s)^{1/2}\}}{K_\lambda(\mu\beta^{-1})}(1+2\beta s)^{-\lambda/2} \tag{3.2}$$

EMBRECHTS (1983) considers ruin probabilities for GIG claim severities. SICHEL (1971) and ATKINSON and LAM YEH (1982) consider the Poisson mixed over this distribution with pgf (from 2.3)

$$P(z)=\frac{K_\lambda\{\mu\beta^{-1}[1-2\beta(z-1)]^{1/2}\}}{K_\lambda(\mu\beta^{-1})}[1-2\beta(z-1)]^{-\lambda/2}. \tag{3.3}$$

The probabilities are given by

$$p_n=\frac{\mu^n}{n!}\frac{K_{\lambda+n}[\mu\beta^{-1}(1+2\beta)^{1/2}]}{K_\lambda(\mu\beta^{-1})}(1+2\beta)^{-(\lambda+n)/2}, \qquad n=0,1,2,\ldots \tag{3.4}$$

which satisfy

$$(1+2\beta)n(n-1)p_n=2\beta(n-1)(\lambda+n-1)p_{n-1}+\mu^2p_{n-2}, \qquad n=2,3,4,\ldots \tag{3.5}$$

The factorial moments are

$$E[N(N-1)\cdots(N-k+1)]=\mu^k\frac{K_{\lambda+k}(\mu\beta^{-1})}{K_\lambda(\mu\beta^{-1})}$$

using (2.6) and JØRGENSEN (1982, p. 13).

In general, it is much more convenient if λ is a half integer, in other words, if $\lambda=m+\frac{1}{2}$ where m is a (positive or negative) integer. In this case (JØRGENSEN, 1982, p. 170),

$$K_{m+1/2}(x)=\sqrt{\pi/(2x)}\,e^{-x}\sum_{i=0}^{m}\frac{(m+i)!}{(m-i)!\,i!}(2x)^{-i}, \qquad m=0,1,2,\ldots, \tag{3.7}$$

https://doi.org/10.1017/S051503610001165X Published online by Cambridge University Press

and so considerable simplification in the above formulae is thus possible. A recursive formula for the total claims distribution with claim frequency distribution given by (3.4) was derived by WILLMOT and PANJER (1985) using (3.5). However, it is not convenient unless the claim severity distribution admits a certain form. A more general technique which places few restrictions on the claim severity distribution and works when λ is a half integer will be derived. If λ is not of this form the distribution (3.4) is more difficult to deal with in any event.

Before deriving this recursive algorithm, however, it is expedient to consider the special case when $\lambda = -\frac{1}{2}$. This distribution, which is important in its own right, was introduced by HOLLA (1967) and studied extensively by SICHEL (1971). Known as the *Poisson-Inverse Gaussian* distribution, it is a Poisson mixture with mixing distribution

$$dU(x) = \mu(2\pi\beta x^3)^{-1/2} e^{-(x-\mu)^2/2\beta x}. \tag{3.8}$$

Formula (3.8) and subsequent formulae are obtainable by substituting $\lambda = -\frac{1}{2}$ into the previous formulae and using JØRGENSEN (1982, pp. 170–171). The Laplace transform of (3.8) is

$$\tilde{u}(s) = e^{-(\mu/\beta)\{(1+2\beta s)^{1/2}-1\}} \tag{3.9}$$

and the pgf of the resulting Poisson mixture is

$$P(z) = e^{-(\mu/\beta)\{[1-2\beta(z-1)]^{1/2}-1\}}. \tag{3.10}$$

The probabilities are given by

$$p_n = p_0 \frac{\mu^n}{n!} \sum_{k=0}^{n-1} \frac{(n-1+k)!}{(n-1-k)!k!} \left(\frac{\beta}{2\mu}\right)^k (1+2\beta)^{-(n+k)/2}, \qquad n = 1, 2, 3, \ldots \tag{3.11}$$

where

$$p_0 = e^{-(\mu/\beta)\{(1+2\beta)^{1/2}-1\}}. \tag{3.12}$$

The explicit formula (3.11) is not particularly useful, however, since (3.5) becomes

$$(1+2\beta)n(n-1)p_n = 2\beta(n-1)(n-\tfrac{3}{2})p_{n-1} + \mu^2 p_{n-2}, \qquad n = 2, 3, 4, \ldots \tag{3.13}$$

and so the probabilities may be calculated using (3.13) together with (3.12) and

$$p_1 = \mu(1+2\beta)^{-1/2} p_0. \tag{3.14}$$

The factorial moments are

$$E[N(N-1)\cdots(N-k+1)] = \mu^k \sum_{m=0}^{k-1} \frac{(k-1+m)!}{(k-1-m)!m!} \left(\frac{\beta}{2\mu}\right)^m. \tag{3.15}$$

The mean and variance are

$$E(N) = \mu, \tag{3.16}$$

$$V(N) = \mu(1+\beta). \tag{3.17}$$

This distribution, like the negative binomial, has 2 parameters. The skewness of the negative binomial distribution, in terms of the mean μ and variance σ^2, may be expressed as $\sigma^{-3}\{3\sigma^2-2\mu+2\mu^{-1}(\sigma^2-\mu)^2\}$. The corresponding quantity for the Poisson-Inverse Gaussian distribution is $\sigma^{-3}\{3\sigma^2-2\mu+3\mu^{-1}(\sigma^2-\mu)^2\}$ which exceeds that of the negative binomial. Thus, this distribution has a thicker right tail and may be more suitable for modelling claim frequencies as a result.

Also, like the negative binomial, it has a convolutive property. If N_i has pgf (3.10) with parameters β and μ_i and $N_1, N_2, \ldots, N_p$ are independent, then $\sum_{i=1}^p N_i$ has pgf (3.10) with parameters β and $\sum_{i=1}^p \mu_i$. Hence it is closed under convolution. Rewriting (3.10) as

$$P(z)=e^{2\mu(z-1)\{1+[1+2\beta(1-z)]^{1/2}\}^{-1}},$$

it is easy to see that

$$\lim_{\beta\to 0} P(z)=e^{\mu(z-1)} \tag{2.18}$$

and so the Poisson-Inverse Gaussian has a limiting Poisson form obtained when $\beta\to 0$.

It is clear from (3.9) that the inverse Gaussian distribution is infinitely divisible and so (3.10) may be put in the form

$$e^{\lambda[Q(z)-1]}. \tag{3.19}$$

Imposing the condition that $Q(0)=0$, it is clear that

$$\lambda=\frac{\mu}{\beta}[(1+2\beta)^{1/2}-1] \tag{3.20}$$

and

$$Q(z)=\frac{[1-2\beta(z-1)]^{1/2}-(1+2\beta)^{1/2}}{1-(1+2\beta)^{1/2}}. \tag{3.21}$$

The pgf $Q(z)$ is that of an *extended truncated negative binomial* distribution as defined by ENGEN (1974). The coefficient of z^n in (3.21) is

$$q_n=\frac{\frac{1}{2}\Gamma(n-\frac{1}{2})(1+2\beta)^{1/2}\left(\dfrac{2\beta}{1+2\beta}\right)^n}{n!\Gamma(\frac{1}{2})[(1+2\beta)^{1/2}-1]}, \qquad n=1,2,3,\ldots \tag{3.22}$$

The distribution (3.22) satisfies (1.17) with $a=2\beta(1+2\beta)^{-1}$ and $b=-3\beta(1+2\beta)^{-1}$.

To derive a recursive formula for the total claims distribution with claim frequency distribution given by (3.11) one can use the relation (3.13) or, more generally (3.5), as was mentioned earlier. A more general technique is to note that, from (3.19) and (1.8), one can write

$$L_Y(s)=e^{\lambda\{Q[L_X(s)]-1\}}. \tag{3.23}$$

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Hence, by the above results, one can compute the distribution with transform $Q[L_X(s)]$ using (1.18) or (1.21) with $a = 2\beta(1+2\beta)^{-1}$ and $b = -3\beta(1+2\beta)^{-1}$ since (1.17) is satisfied by the distribution (3.22). Then this distribution may be used in a second application of (1.18) or (1.21) as the severity distribution with Poisson frequencies to get the total claims distribution with transform (3.23). It should be noted that for certain claim severity distributions (eg. gamma or inverse Gaussian) the recursion given by WILLMOT and PANJER (1985) is more efficient than the one proposed here, needing only one application of (1.18) or (1.21) in these cases. Otherwise, it is normally less convenient.

Hence, one can obtain the total claims distribution for the claim frequency distribution (3.4) in the special case $\lambda = -\frac{1}{2}$. This may be used to derive a recursive formula in the case when λ is a half integer. From (1.8), define the total claims distribution (for fixed λ) by

$$L_Y(s, \lambda) = P_\lambda[L_X(s)] \tag{3.24}$$

where $P_\lambda(\cdot)$ is given by (3.3), It is easily shown from the elementary properties of the modified Bessel function of the third kind (JØRGENSEN, 1982, p. 170) that

$$\frac{\partial}{\partial x} K_\lambda(x) = \frac{\lambda}{x} K_\lambda(x) - K_{\lambda+1}(x). \tag{3.25}$$

Using this fact, it is easily shown that

$$\frac{\partial}{\partial s} L_Y(s, \lambda) = \mu \frac{K_{\lambda+1}(\mu\beta^{-1})}{K_\lambda(\mu\beta^{-1})} L'_X(s) L_Y(s, \lambda+1). \tag{3.26}$$

Thus there is a close relationship between the total claims distribution having (3.4) as claim frequencies and those having (3.4) but with λ replaced by $\lambda+1$. If X_k has discrete support, then the L's in (3.26) may be interpreted as pgf's, and so the coefficients of s^{x-1} on each side must be equal, yielding (in an obvious notation)

$$_\lambda g_x = \frac{\mu}{x} \frac{K_{\lambda+1}(\mu\beta^{-1})}{K_\lambda(\mu\beta^{-1})} \sum_{y=1}^{x} y f_y \cdot {}_{\lambda+1}g_{x-y}, \qquad x = 1, 2, 3, \ldots \tag{3.27}$$

and

$$_\lambda g_0 = P_\lambda(f_0). \tag{3.28}$$

Similarly, if X_k has mixed support, (3.26) may be rewritten as

$$\frac{\partial}{\partial s} \tilde{g}_\lambda(s) = \mu \frac{K_{\lambda+1}(\mu\beta^{-1})}{K_\lambda(\mu\beta^{-1})} \tilde{f}'(s)[{}_{\lambda+1}g_0 + \tilde{g}_{\lambda+1}(s)] \tag{3.29}$$

which yields on inversion

$$g_\lambda(x) = \left[\mu \frac{K_{\lambda+1}(\mu\beta^{-1})}{K_\lambda(\mu\beta^{-1})} {}_{\lambda+1}g_0\right] f(x) + \frac{\mu}{x} \frac{K_{\lambda+1}(\mu\beta^{-1})}{K_\lambda(\mu\beta^{-1})} \int_0^\alpha y f(y) g_{\lambda+1}(x-y)\, dy. \tag{3.30}$$

Thus, if one begins with $\lambda = -\frac{1}{2}$ and computes the total claims distribution for the Poisson mixed over the inverse Gaussian case, one can use (3.28) and (3.27) or (3.30) repetitively to obtain the total claims distribution with the claim frequency probabilities (3.4) for $\lambda = -\frac{3}{2}, -\frac{5}{2}, \ldots$.

To compute the distribution for $\lambda = n + \frac{1}{2}$ where n is a non-negative integer, one could use the above technique to obtain the distribution when $\lambda = -n - \frac{1}{2}$, then note that, if $\lambda > 0$,

$$L_Y(s, \lambda) = \{1 - 2\beta[L_X(s) - 1]\}^{-\lambda} L_Y(s, -\lambda). \tag{3.31}$$

The first term on the right hand side is a compound negative binomial transform whose associated distribution can be computed using (1.18) or (1.21). Denoting this distribution by ${}_1g_x$ or $g_1(x)$ and the distribution with transform $L_Y(s, -\lambda)$ by ${}_2g_x$ or $g_2(x)$, the required distribution can be computed from (4.19) or (4.20).

Hence one can compute the total claims distribution for $\lambda = n + \frac{1}{2}$ for any integral n. Clearly, the closer λ is to $-\frac{1}{2}$ the better, and the technique is not practical for $|\lambda|$ large. However, the result does not depend on the severity distribution to as great an extent as that of WILLMOT and PANJER (1985).

4. POISSON-BETA, POISSON-UNIFORM, AND THE NEYMAN CLASS

Consider the distribution obtained when the Poisson mean is mixed over the beta distribution given by

$$dU(x) = \frac{\beta(\mu - x)^{\beta - 1}}{\mu^\beta}\, dx, \qquad 0 < x < \mu. \tag{4.1}$$

Thus,

$$\begin{aligned} p_n &= \beta \int_0^\mu \frac{\lambda^n e^{-\lambda}}{n!} \cdot \frac{(\mu - \lambda)^{\beta - 1}}{\mu^\beta}\, d\lambda \\ &= \frac{\mu^n}{n!}\Gamma(\beta + 1) \sum_{k=0}^{\infty} \frac{\Gamma(n + k + 1)}{\Gamma(\beta + n + k + 1)} \frac{(-\mu)^k}{k!}, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{4.2}$$

and is discussed by JOHNSON and KOTZ (1969, p. 227) and BEALL and RESCIA (1953). A more general beta mixture was discussed by WILLMOT and PANJER (1985). However, the recursion obtained is not very convenient for some severity distributions. The above mixture is useful in obtaining results for more general severity distributions for certain choices of β. These results are quite simple to apply to get computational formulae for the Neyman class of frequency distributions, and for the Poisson-uniform mixture ($\beta = 1$ in (4.2)). The formulae place little restriction on the severity distribution.

From (4.2), the pgf is

$$P(z) = M[1, \beta + 1, \mu(z - 1)] \tag{4.3}$$

where $M(\cdot)$ is the confluent hypergeometric function (see JOHNSON and KOTZ, 1969, p. 8). The relation (4.3) can be expanded as

$$M[1, \beta + 1, \mu(z - 1)] = \Gamma(\beta + 1) \sum_{n=0}^{\infty} \frac{[\mu(z - 1)]^n}{\Gamma(\beta + n + 1)}, \tag{4.4}$$

from which it is clear that

$$E(N)=\mu(\beta+1)^{-1}. \tag{4.5}$$

To derive a computational formula for the class (4.2), consider an arbitrary frequency distribution $\{q_n;\ n=0,1,2,\ldots\}$ with pgf $Q(z)$ and mean $Q'(1)<\infty$. Then, if

$$r_n=\frac{\sum_{k=n+1}^{\infty} q_k}{Q'(1)}, \qquad n=0,1,2,\ldots \tag{4.6}$$

it is easily shown (see JOHNSON and KOTZ, 1969, p. 261) that $\{r_n;\ n=0,1,2,\ldots\}$ also defines a distribution with pgf

$$R(z)=\frac{Q(z)-1}{Q'(1)[z-1]}. \tag{4.7}$$

Substitution of $L_X(s)$ in (4.7) in the place of z and cross multiplying yields, (with $L_R(s)=R[L_X(s)]$, $L_Q(s)=Q[L_X(s)]$),

$$Q'(1)L_R(s)=Q'(1)L_R(s)L_X(s)+1-L_Q(s). \tag{4.8}$$

Clearly, both $L_R(\cdot)$ and $L_Q(\cdot)$ are transforms of compound random variables. If X_k has discrete support then so do the random variables with transforms $L_R(\cdot)$ and $L_Q(\cdot)$. Interpreting them all as pgf's the coefficients of s^x in (4.8) must be equal, yielding, (in an obvious notation),

$$_Rg_0=\frac{1-{}_Qg_0}{Q'(1)(1-f_0)}, \tag{4.9}$$

$$_Rg_x=-\frac{_Qg_x}{Q'(1)(1-f_0)}+\frac{1}{1-f_0}\sum_{y=1}^{x} f_y\cdot{}_Rg_{x-y}, \qquad x=1,2,3,\ldots. \tag{4.10}$$

The equation (4.10) is of the same form as (1.18). Thus, the coefficients $_Rg_x$ are obtainable from those of $_Qg_x$.

If the X_k's have mixed support, then using (1.13) and (1.16) together with (4.9) and (4.8), one obtains

$$\tilde{g}_R(s)[Q'(1)(1-f_0)]=Q'(1)\,{}_Rg_0\tilde{f}(s)-\tilde{g}_Q(s)+Q'(1)\tilde{f}(s)\tilde{g}_R(s) \tag{4.11}$$

which may be inverted to yield

$$g_R(x)=\frac{Q'(1)\,{}_Rg_0 f(x)-g_Q(x)}{Q'(1)(1-f_0)}+\frac{1}{1-f_0}\int_0^x f(y)g_R(x-y)\,dy \tag{4.12}$$

which is of the form (1.21). Again the subscripts on the $g(\cdot)$ functions refer to the underlying frequency distribution in each case. The importance of these results lies in the ability to obtain the total claims distribution with claim frequencies $\{r_n;\ n=0,1,2,\ldots\}$ form the total claims distribution with claim frequencies $\{q_n;\ n=0,1,2,\ldots\}$.

These results are now used to obtain recursive formulae for the total claims distribution with claim frequencies pgf (4.3). It is easily shown from (4.4) that

$$M[1, 1, \mu(z-1)] = e^{\mu(z-1)}, \tag{4.13}$$

and that, for $\beta \geq 0$,

$$M[1, \beta+2, \mu(z-1)] = \frac{M[1, \beta+1, \mu(z-1)]-1}{\mu(\beta+1)^{-1}(z-1)}. \tag{4.14}$$

Thus (4.13) is a Poisson pgf and (4.14) is a relation of the form (4.7). Hence the probabilities of random variable with pgf's of the form (4.3) with successive β values are related by an equation of the form (4.6). For example, from (4.13) and (4.14),

$$M[1, 2, \mu(z-1)] = \sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty} \frac{\mu^{k-1}e^{-\mu}}{k!}\right) z^k. \tag{4.15}$$

More importantly, a recursive calculation can be obtained to get the total claims distribution with frequencies of claims having pgf's of the form (4.3) with β any positive integer. Simply obtain the total claims distribution for the Poisson frequencies (4.13) from (1.18) or (1.21). Then, use (4.9) and (4.10) or (4.12) to obtain the total claims distribution with claim frequencies pgf $M[1, 2, \mu(z-1)]$ and then $M[1, 3, \mu(z-1)]$, etc., until $M[1, \beta+1, \mu(z-1)]$ is reached. In each application, $M[1, \gamma+1, \mu(z-1)]$ is interpreted as $Q(z)$ and $M[1, \gamma+2, \mu(z-1)]$ as $R(z)$ in formulae (4.10) and (4.12).

EXAMPLE 4.1. *The Neyman Class.* BEALL and RESCIA (1953) studied frequency distributions with pgf's of the form

$$P(z) = e^{\lambda\{M[1,\beta+1,\mu(z-1)]-1\}}. \tag{4.16}$$

These are compound Poisson distributions with "severities" pgf (4.3). Hence for integral β one may use the above technique to obtain the distribution with transform $M\{1, \beta+1, \mu[L_X(s)-1]\}$. This then becomes the severity distribution in the compound Poisson recursion (with mean λ) using (1.18) or (1.21). Neyman's Type A, B, and C correspond to $\beta = 0$, 1 or 2 respectively.

A more general situation corresponding to (4.2) occurs when the range of values of λ is $(\sigma, \sigma+\mu)$. In this case, it is easily seen that (4.3) becomes

$$P(z) = e^{\sigma(z-1)}M[1, \beta+1, \mu(z-1)]. \tag{4.17}$$

Thus, from (1.8),

$$L_Y(s) = e^{\sigma[L_X(s)-1]}M\{1, \beta+1, \mu[L_X(s)-1]\} \tag{4.18}$$

which is the product of a compound Poisson transform and a compound Poisson-beta transform as discussed in this section. This implies, that in the discrete case (in an obvious notation)

$$g_x = \sum_{y=0}^{x} {}_1g_y \cdot {}_2g_{x-y}, \qquad x = 0, 1, 2, \ldots \tag{4.19}$$

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and in the mixed support case,

$$(4.26)\quad g(x) = {}_1g_0 \cdot g_2(x) + {}_2g_0 \cdot g_1(x) + \int_0^x g_1(y) g_2(x-y)\,dy, \qquad x>0.$$

EXAMPLE 4.2. *The Poisson Uniform.* If the claim frequencies are Poisson mixed over a $U(\sigma, \sigma+\mu)$ distribution, the pgf is seen from (4.2) and (4.17) to be the special case of (4.17) when $\beta = 1$. Thus the technique just described and formulae (4.19) and (4.20) are applicable.

There are clearly situations when the above technique is not useful (e.g., if β is large). However, in some situations (the two examples, for instance) the recursive technique may be appropriate and the exact distribution may be obtainable with just a few recursive computations.

5. LINEAR COMBINATIONS

Suppose W_i has Laplace transform $\tilde{u}_i(s)$, $i = 1, 2, 3, \ldots, n$ and $W_1, W_2, \ldots, W_n$ are independent. Then if a Poisson mean is mixed over the linear combination $b_1 W_1 + b_2 W_2 + \cdots + b_n W_n$, where $b_i \geqslant 0$, the pgf of the mixture is

$$(5.1)\qquad P(z) = \prod_{i=1}^{n} \tilde{u}_i[b_i(1-z)],$$

and so, by (1.8),

$$(5.2)\qquad L_Y(s) = \prod_{i=1}^{n} \tilde{u}_i\{b_i[1-L_X(s)]\}.$$

The expression (5.2) is clearly of the same form as (4.18) or (3.31), and so, if n is not too large, the distribution can be obtained by convoluting two at a time using (4.19) or (4.20).

EXAMPLE 5.1. *Non-Central Chi-Squared.* The Laplace transform of a non-central chi-square distribution with m degrees of freedom and non-centrality parameter λ is (see JOHNSON and KOTZ, 1970*b*)

$$(5.3)\qquad \tilde{u}(s) = (1+2s)^{-m/2} \exp\left\{\frac{\lambda}{2}[(1+2s)^{-1} - 1]\right\}.$$

This is clearly the convolution of a gamma and a compound Poisson distribution. Hence, the pgf of a Poisson mixed over this distribution is

$$(5.4)\qquad P(z) = [1-2(z-1)]^{-m/2} \exp\left\{\frac{\lambda}{2}[1-2(z-1)]^{-1} - \frac{\lambda}{2}\right\}.$$

Clearly, the total claims distribution can be obtained as the convolution of a compound negative binomial and a compound Poisson (with "severity" distribution itself a compound geometric). Since the non-central chi-squared is infinitely divisible, it follows that $P(z)$ can be put in the form (2.4) with

$$\mu=\frac{m}{2}\log 3+\frac{\lambda}{2} \tag{5.5}$$

$$Q(z)=\mu^{-1}\left\{-\frac{m}{2}\log\left(1-\tfrac{2}{3}z\right)+\frac{\lambda}{2}[1-2(z-1)]^{-1}\right\} \tag{5.6}$$

Thus, $Q(z)$ is a weighted average of a logarithmic (2.5) and a geometric random variable.

EXAMPLE 6.2. *Reciprocal Inverse Gaussian.* The reciprocal of an inverse Gaussian random variable (see JOHNSON and KOTZ, 1970*a*, p. 149) has density of the form

$$dU(x)=\left(\frac{\lambda}{2\pi x}\right)^{1/2}\exp\left\{-\frac{\lambda\mu^2}{2x}\left(1-\frac{x}{\mu}\right)^2\right\}dx \tag{5.7}$$

and Laplace tranform

$$\tilde{u}(s)=\left(1+\frac{2s}{\lambda}\right)^{-1/2}\exp\left\{\lambda\mu\left[1-\left(1+\frac{2s}{\lambda}\right)^{1/2}\right]\right\} \tag{5.8}$$

which is the Laplace transform of the convolution of a gamma and an inverse Gaussian distribution. The pgf of a Poisson mixed over this distribution is

$$P(z)=\left[1-\frac{2}{\lambda}(z-1)\right]^{-1/2}\exp\left(\lambda\mu\left\{1-\left[1-\frac{2}{\lambda}(z-1)\right]^{1/2}\right\}\right) \tag{5.9}$$

which is the convolution of a negative binomial and Poisson-inverse Gaussian distribution (Section 3). The total claims distribution is thus the convolution of a compound negative binomial and a compound Poisson-inverse Gaussian distribution.

6. POWER MIXTURES

In this section a more general set of mixed Poisson distributions is derived, when the mixing distribution is from a class of power mixtures. The pgf of a mixed Poisson distribution over this distribution will be derived, and in some cases the total claims distribution can be calculated fairly simply.

Consider a Markov process $\{X_m;\, m=0,1,2,\ldots\}$ where X_m, $m>0$ are non-negative, absolutely continuously distributed random variables. Let $X_0=x_0>0$ with certainty and

$$f_m(x_m\,|\,x_{m-1}),\qquad m=1,2,3,\ldots \tag{6.1}$$

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be the conditional density of X_m given X_{m-1}. Suppose that X_m, $m>0$, can take on any positive value with nonzero probability so that (6.1) is well defined. Assume that there exist functions $h_m(s)$, $m=1,2,3,\ldots$ such that

$$\int_0^\infty e^{-sx_m} f_m(x_m \mid x_{m-1})\, dx_m = e^{-x_{m-1}h_m(s)}, \qquad m=1,2,3,\ldots. \tag{6.2}$$

Let k be a fixed positive integer. It is of interest to consider the marginal distribution of X_k. Since the process is Markovian, the pdf of X_k is

$$u(x_k) = \int_{R_{k-1}^+} \left\{ \prod_{m=1}^{k} f_m(x_m \mid x_{m-1}) \right\} \left\{ \prod_{m=1}^{k-1} dx_m \right\} \tag{6.3}$$

where $R_{k-1}^+ = \{(x_1, x_2, \ldots, x_{k-1}) \mid x_i > 0;\ i=1,2,3,\ldots,k-1\}$. The Laplace transform of X_k has a fairly simple form. For notational simplicity, define the set of composite functions

$$\begin{aligned} h_m^*(s) &= h_m[h_{m+1}^*(s)], \qquad m=1,2,3,\ldots,k-1 \\ &= h_k(s), \qquad m=k \end{aligned} \tag{6.4}$$

where $h_m(s)$ is defined in (6.2). Thus (6.4) defines the composite functions recursively. The Laplace transform of (6.3) is

$$\tilde{u}(s) = \int_{R_k^+} e^{-sx_k} \left\{ \prod_{m=1}^{k} [f_m(x_m \mid x_{m-1})\, dx_m] \right\} \tag{6.5}$$

$$= e^{-x_0 h_1^*(s)}. \tag{6.6}$$

It is of interest to derive a useful form for the pgf of a Poisson random variable mixed over the density (6.3). Rather than simply replace s by $1-z$ in (6.6), it is of use to examine the form of the functions $h_m(s)$ in (6.2). By assumption, x_m can take on any positive value, so that it is clear from (6.2) that $f_m(x_m \mid x_{m-1})$ defines an infinitely divisible density (assume true for $m=1$ also). Hence if a Poisson mean is mixed over this density, the resulting distribution is compound Poisson (Section 2). Thus, noting that μX is infinitely divisible if X is, setting $x_{m-1}=1$, and using (6.2) and (2.3), it is clear that there exist $\lambda_m(\mu)$ and $Q_m(z,\mu)$ such that

$$h_m[\mu(1-z)] = \lambda_m(\mu)[1-Q_m(z,\mu)], \qquad m=1,2,3,\ldots. \tag{6.7}$$

The function $\lambda_m(\mu)$ is a constant and $Q_m(z,\mu)$ a pgf, both depending on μ. Assuming that $Q_m(0,\mu)=0$, they are unique, and in this case,

$$\lambda_m(\mu) = h_m(\mu), \tag{6.8}$$

and

$$Q_m(z,\mu) = 1 - \frac{h_m[\mu(1-z)]}{h_m(\mu)}. \tag{6.9}$$

The pgf of the Poisson mixture will be expressed in terms of (6.8) and (6.9), rather than $h_m(\cdot)$. Again, for simplicity, let

$$\lambda_m^* = \lambda_m(\lambda_{m+1}^*), \qquad m = 1, 2, 3, \ldots, k-1 \tag{6.10}$$
$$= \lambda_k(1), \qquad m = k$$

where $\lambda_m(\mu)$ is defined by (6.8). Also let

$$Q_m^*(z) = Q_m[Q_{m+1}^*(z), \lambda_{m+1}^*], \qquad m = 1, 2, 3, \ldots, k-1 \tag{6.11}$$
$$= Q_k(z, 1), \qquad m = k$$

where $Q_m(z, \mu)$ is defined by (6.9). Substituting $1-z$ for s in (6.6) yields the required pgf. It is

$$P(z) = e^{-x_0 h_1(h_2(\cdots(h_k(1-z))\cdots))} \tag{6.12}$$
$$= e^{\lambda_1^* x_0 [Q_1^*(z)-1]}.$$

From (6.12), the mixed Poisson random variable is also a compound Poisson random variable, as it must be. However, the functions λ_1^* and $Q_1^*(z)$ are obtainable recursively using (6.10) and (6.11) from each of the component mixing distributions. Thus, for a given power-mixed density of the form (6.3), one can obtain $h_m(s)$ from (6.2) for each m, then $\lambda_m(\mu)$ and $Q_m(z, \mu)$ from (6.8) and (6.9), and finally λ_1^* and $Q_1^*(z)$ from (6.10) and (6.11) recursively. The usefulness of this approach is most evident when $Q_m(z, \mu)$ admits a simple parametric representation. This is often the case, as is now demonstrated for the following distributions.

(i) *Gamma*

$$f_m(x_m \mid x_{m-1}) = \frac{\alpha(\alpha x_m)^{x_{m-1}-1} e^{-\alpha x_m}}{\Gamma(x_{m-1})} \tag{6.13}$$

$$h_m(s) = \log\left(1 + \frac{s}{\alpha}\right) \tag{6.14}$$

$$\lambda_m(\mu) = \log\left(1 + \frac{\mu}{\alpha}\right) \tag{6.15}$$

$$Q_m(z, \mu) = \frac{\log\left(1 - \dfrac{\mu}{\alpha+\mu} z\right)}{\log\left(1 - \dfrac{\mu}{\alpha+\mu}\right)} \tag{6.16}$$

It is clear from (2.5) that $Q_m(z, \mu)$ is a logarithmic pgf.

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(ii) *Inverse Gaussian*

$$f_m(x_m \mid x_{m-1}) = \left(\frac{x_{m-1}^2}{2\pi\beta x_m^3}\right)^{1/2} e^{-(x_m - x_{m-1})^2/2\beta x_m} \tag{6.17}$$

$$h_m(s) = \frac{1}{\beta}[(1+2\beta s)^{1/2} - 1] \tag{6.18}$$

$$\lambda_m(\mu) = \frac{1}{\beta}[(1+2\beta\mu)^{1/2} - 1] \tag{6.19}$$

$$Q_m(z, \mu) = \frac{[1+2\beta\mu(1-z)]^{1/2} - (1+2\beta\mu)^{1/2}}{1-(1+2\beta\mu)^{1/2}}. \tag{6.20}$$

Comparison of (6.20) with (3.21) and (3.22) shows that the distribution with pgf (6.20) satisfies the recursion (1.17).

Similar results may be obtained for other distributions, such as mixtures over linear combinations like the non-central chi-squared distribution.

The advantages of the above approach are two-fold. First, it allows for explicit identification of the pgf for some complicated mixing distributions. Secondly, if (6.12) is the pgf of the claim frequency distribution, then from (1.8) the total claim transform is of the form

$$L_Y(s) = e^{\lambda[P_1(P_2(\cdots P_k[L_X(s)]))]} \tag{6.21}$$

where $P_m(\cdot)$ is a pgf of the form $Q_m(z, \mu^*)$ for a particular choice of μ^*. Hence, if a recursive formula exists for compound distributions with claim frequency pgf $Q_m(z, \mu)$, one can repeatedly apply the recursion to $Q_k[L_X(s)]$, $Q_{k-1}[L_X(s)]$, etc. $(k+1)$ times to obtain the distribution of total claims. Since (6.16) and (6.20) are members of the class (1.17), recursions do in fact exist, for the pgf's (6.16) and (6.20), and they are given by (1.18) and (1.21). Thus, for example, if all $f_m(x_m \mid x_{m-1})$ are gamma or inverse Gaussian (or non-central chi-squared) densities, then repeated recursions are simple to apply. An example of this is now given.

Example 6.1. *Inverse Gaussian-Exponential.* Consider the distribution obtained by assuming that the mean of an inverse Gaussian distribution has an exponential distribution. Then $k=2$, $f_2(x_2 \mid x_1)$ has pdf (6.17) and $f_1(x_1 \mid 1)$ has the pdf (6.13). The density corresponding to (6.3) is

$$\begin{aligned} u(x_2) &= \int_0^\infty \left(\frac{x_1^2}{2\pi\beta x_2^3}\right)^{1/2} e^{-(x_2-x_1)^2/2\beta x_2} \alpha\, e^{-\alpha x_1}\, dx_1 \\ &= \alpha\left(\frac{\beta}{2\pi x_2}\right)^{1/2} e^{-x_2/2\beta} \\ &\quad + \alpha(1-\alpha\beta)\, e^{-\alpha x_2 + (\alpha^2\beta x_2/2)}\left\{1 - \Phi\left[\left(\frac{x_2}{\beta}\right)^{1/2}(\alpha\beta - 1)\right]\right\} \end{aligned} \tag{6.22}$$

where Φ is the standard normal cdf. The Laplace transform of (6.22) is

$$\tilde{u}(s)=\left\{1+\frac{1}{\alpha\beta}[(1+2\beta s)^{1/2}-1]\right\}^{-1}. \tag{6.23}$$

If the Poisson mean has the distribution (6.22), the pgf is given by (6.12) with

$$\begin{aligned}\lambda_1^* &= \lambda_1[\lambda_2(1)]\\ &=\log\left\{1+\frac{1}{\alpha\beta}[(1+2\beta)^{1/2}-1]\right\},\\ x_0 &= 1,\end{aligned} \tag{6.24}$$

and

$$Q_1^*(z)=Q_1[Q_2(z,1),\lambda_2(1)]. \tag{6.25}$$

Here $\lambda_1(\mu)$ is given by (6.15), $Q_1(z,\mu)$ by (6.16), $\lambda_2(\mu)$ by (6.19) and $Q_2(z,\mu)$ by (6.20). Thus, (6.21) becomes

$$L_Y(s)=e^{\lambda_1^*\{Q_1[Q_2(L_X(s),1),\lambda_2(1)]-1\}}. \tag{6.26}$$

To evaluate this distribution, one application of (1.18) or (1.21) yields the distribution with transform $Q_2[L_X(s),1]$, and a second $L_Y(s)$ since, in this particular case, $e^{\lambda_1^*[Q_1^*(z)-1]}$ is a negative binomial pgf.

7. NEGATIVE BINOMIAL MIXTURES

Consider the negative binomial distribution from example 2.1 where α has a distribution with cdf $U(\alpha)$. Then the pgf of the mixture is

$$\begin{aligned}P(z)&=\int_0^\infty e^{\alpha\log(1+\beta)[Q_1(z)-1]}\,dU(\alpha)\\ &=\int_0^\infty e^{\alpha[Q_1(z)-1]}\,dU_1(\alpha)\end{aligned} \tag{7.1}$$

where $Q_1(z)$ is the logarithmic series pgf from example 2.1 and $U(\alpha)$ is the cdf of $\alpha\log(1+\beta)$. Hence all results for Poisson mixing apply if the mixing random variable is multiplied by the constant $\log(1+\beta)$ (which involves merely a change in parameters of the mixing distribution for all situations described in this paper) and $L_X(s)$ is replaced by

$$L_Z(s)=\frac{\log\left[1-\dfrac{\beta}{1+\beta}L_X(s)\right]}{\log\left(1-\dfrac{\beta}{1+\beta}\right)}. \tag{7.2}$$

This latter transform is merely that of a compound logarithmic series and the associated distribution can be calculated by (1.18) or (1.21), if need be.

8. QUEUEING APPLICATIONS

In this section it is shown how the results previously derived can apply in a queuing context. Consider the M/G/1 queue with Poisson arrivals at rate λ of a batch of customers, the number of which are distributed according to a distribution with pgf $Q(z)$. Suppose also that there is one server serving customers according to a distribution with Laplace transform $\tilde{u}(s)$. Then, as is well known (see e.g., KLEINROCK, 1975), the pgf of the number in the system at the departure instants when equilibrium has been reached is given by

$$P(z)=\frac{(1-\rho)(1-z)R(z)}{R(z)-z} \tag{8.1}$$

where $R(z)=\tilde{u}\{\lambda[1-Q(z)]\}$ in (8.1) is easily recognized as being that of a mixed compound Poisson distribution of the type discussed in this paper and $\rho=-\lambda Q'(1)\tilde{u}'(0)$. The service distribution is the mixing distribution in this case, and as in section 7, the presence of the factor λ causes little difficulty. Hence, any of the mixing distributions discussed in this paper can be interpreted as service distributions and the coefficients $\{r_x;\, x=0,1,2,\dots\}$ defined by

$$R(z)=\sum_{x=0}^{\infty} r_x z^x \tag{8.2}$$

easily obtained. Letting $P(z)$ be defined by (1.3), one can rewrite (8.1) as

$$P(z)R(z)-zP(z)=(1-\rho)(1-z)R(z).$$

The coefficients of z^x on both sides must be equal, yielding

$$r_0p_0=(1-\rho)r_0$$

and

$$\sum_{y=0}^{x} r_y p_{x-y}-p_{x-1}=(1-\rho)(r_x-r_{x-1}), \qquad x=1,2,3,\dots$$

These equations may be rewritten as

$$p_0=(1-\rho), \tag{8.3}$$

$$p_x=\frac{(1-\rho)}{r_0}(r_x-r_{x-1})+\frac{p_{x-1}}{r_0}-\sum_{y=1}^{x}\frac{r_y}{r_0}p_{x-y}, \qquad x=1,2,3,\dots \tag{8.4}$$

which is of the form (1.18). Using the fact that $r_0=\tilde{u}(\lambda)$ if $Q(0)=0$, the probability distribution of the number in the system for this queue may be calculated recursively using (8.3) and (8.4). The M/G/1 queue results correspond to the case when $Q(z)=z$. The relations (8.3) and (8.4) hold in general for both queues, but the results are most useful when the mixed compound Poisson coefficients $\{r_x, x=0,1,2,\dots\}$ are easily obtainable (as for the distributions in this paper).

https://doi.org/10.1017/S051503610001165X Published online by Cambridge University Press

REFERENCES

ATKINSON, A. C. and LAM YEH (1982). Inference for Sichel's Compound Poisson Distribution. *Journal of the American Statistical Association* 77, 153–158.

BAKER, C. T. (1977) *The Numerical Treatment of Integral Equations.* Clarendon Press: Oxford.

BEALL, G. and RESCIA, R. (1953) A Generalization of Neyman's Contagious Distributions. *Biometrics* **9**, 354–386.

BÜHLMANN, H. (1970) *Mathematical Methods in Risk Theory.* Springer-Verlag: New York.

BÜHLMANN, H. and BUZZI, R. (1970) On a Transformation of the Weighted Compound Poisson Process. *Astin Bulletin* **6**, 42–46.

DOUGLAS, J. B. (1980) *Analysis with Standard Contagious Distributions.* International Co-operative Publishing House: Fairland, Maryland.

EMBRECHTS, P. (1983) A Property of the Generalized Inverse Gaussian Distribution with Some Applications. *Journal of Applied Probability* **20**, 537–544.

ENGEN, S. (1974) On Species Frequency Models. *Biometrika* **61**, 263–270.

FELLER, W. (1968) *An Introduction to Probability Theory and Its Applications*, vol. 1 (3rd ed.). John Wiley: New York.

HAIGHT, F. (1967) *Handbook of the Poisson Distribution.* John Wiley: New York.

HOLGATE, P. (1970) The Modality of Some Compound Poisson Distributions. *Biometrika* **57**, 666–667.

HOLLA, M. S. (1967) On A Poisson-Inverse Gaussian Distribution. *Metrika* **11**, 115–121.

JOHNSON, N. L. and KOTZ, S. (1969). Distributions in Statistics: Discrete Distributions. John Wiley: New York.

JOHNSON, N. L. and KOTZ, S. (1970*a*) *Distributions in Statistics: Continuous Univariate Distributions*, 1. John Wiley: New York.

JOHNSON, N. L. and KOTZ, S. (1970*b*) *Distributions in Statistics: Continuous Univariate Distributions* 2. John Wiley: New York.

JØRGENSEN, B. (1982) *Statistical Properties of the Generalized Inverse Gaussian Distribution.* Lecture Notes in Statistics 9. Springer-Verlag: New York.

KLEINROCK, L. (1975) *Queueing Systems, Volume 1, Theory.* John Wiley, New York.

LUNDBERG, O. (1940) *On Random Processes and Their Application to Sickness and Accident Statistics.* Almquist and Wiksell: Uppsala.

MACEDA, E. C. (1948) On the Compound and Generalized Poisson Distributions. *Annals of Mathematical Statistics* **19**, 414–416.

MCFADDEN, J. A. (1965) the Mixed Poisson Process. *Sankhya A* **27**, 83–92.

ORD, J. (1972) *Families of Frequency Distributions.* Charles Griffin: London.

PANJER, H. H. (1981) Recursive Evaluation of A Family of Compound Distributions. *Astin Bulletin* **12**, 22–26.

SICHEL, H. S. (1971) On A Family of Discrete Distributions Particularly Suited to Represent Long Tailed Frequency Data. *Proceedings of the Third Symposium on Mathematical Statistics*, ed. N. F. Loubscher. Pretoria: CSIR.

STRÖTER, B. (1984) The Numerical Evaluation of the Aggregate Claim Density Function Via Integral Equations. *Blätter der Deutschen Gesellschaft für Versicherungs-mathematik* **17**, 1–14.

SUNDT, B. and JEWELL, W. (1981) Further Results on Recursive Evaluation of Compound Distributions. *Astin Bulletin* **12**, 27–39.

VAN HARN, K. (1978) Classifying Infinitely Divisible Distributions By Functional Equations. *Math. Centre Tracts* 103: Math. Centre, Amsterdam.

WILLMOT, G. E. and PANJER, H. H. (1985) *Difference Equation Approaches in Evaluation of Compound Distributions.* To appear.

GORD WILLMOT

Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1