Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods

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Outline

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Five widespread beliefs about mixed finite elements

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- there exist no local flux expressions
- there is no discrete maximum principle
- they cannot work on general polygonal meshes
- they cannot be implemented with one unknown/element
- they are only related to finite difference, finite volume, mimetic finite difference, or MPFA through approximate numerical quadratures

All these beliefs are false!

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Motivations

Motivations of the present work

- recall the rectifications to the five false beliefs
- present a unified framework in which MFEs with one unknown/element can be derived/studied/used
- present a comparative numerical study
- show closeness in building principles of MFE and FD/FV/MFD/MPFA, even on general polygonal meshes

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Model problem and mixed finite elements

A model second-order elliptic problem

$$-\nabla \cdot (\mathbf{S}\nabla p) = g \qquad \text{in } \Omega,$$
$$p = 0 \qquad \text{on } \partial\Omega$$

Mixed finite element method

find $p_h \in \Phi_h$ and $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \phi_h) = (g, \phi_h) \qquad \forall \phi_h \in \Phi_h$$

- Φ_h, $\mathbf{V}_h$: Raviart–Thomas–Nédélec MFE space

Matrix form

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- indefinite, saddle point type
- both fluxes U (1/side) and potentials P (1/element) involved

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Equivalence with nonconforming finite elements

Crouzeix–Raviart nonconforming finite element method

- find $\tilde{\lambda}_h \in \tilde{\Psi}_h$ such that

$$(\mathbf{S}\nabla\tilde{\lambda}_h, \nabla\tilde{\psi}_h) = (g, \tilde{\psi}_h) \qquad \forall \tilde{\psi}_h \in \tilde{\Psi}_h$$

- degrees of freedom: 1 potential/side (vector Λ)
- matrix form

$$\mathbb{Z}\Lambda = E$$

- $\mathbb{Z}$ is symmetric and positive definite

Equivalence of MFEs with nonconforming finite elements

- MFEs $\longrightarrow$ Lagrange multipliers Λ, mixed-hybrid FEM:

$$\mathbb{Z}\Lambda = E$$

- **same matrices and RHS** as in the nonconforming finite element method (when $\mathbf{S}$ and g are piecewise constant)
- $\tilde{\lambda}_h$ from MFEs and $\tilde{\lambda}_h$ from NCFEs **coincide**
- Arnold & Brezzi 1985, Marini 1985, Arbogast & Chen 1995, Chen 1996

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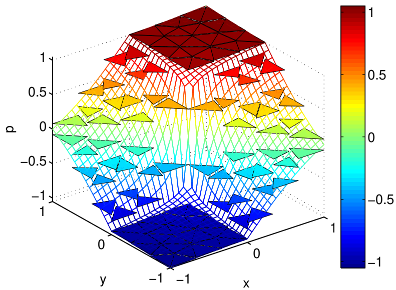
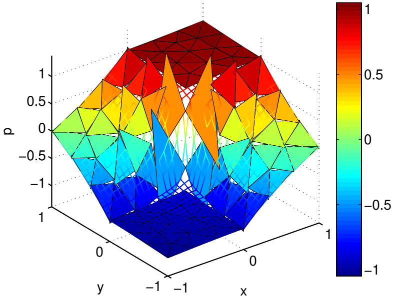
Different representations of the MFE solution



p_h



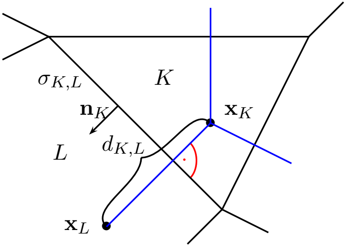
$\tilde{\lambda}_h$

4-point finite volume scheme

4-point finite volume scheme ($\mathbf{S} = \mathbb{I}$)

- find $\bar{p}_h \in \Phi_h$ such that

$$-\sum_{L \in \mathcal{N}(K)} \frac{\bar{p}_h|_L - \bar{p}_h|_K}{d_{K,L}} |\sigma_{K,L}| = (g, 1)_K \qquad \forall K \in \mathcal{T}_h$$



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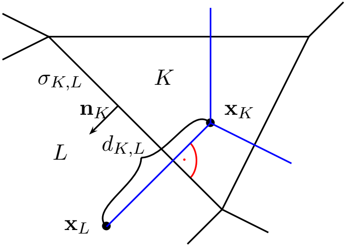

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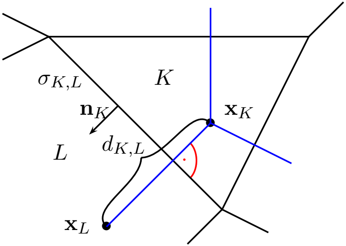

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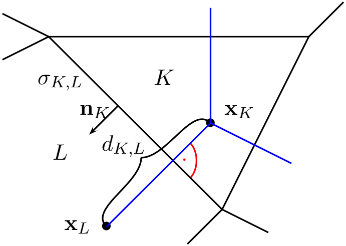

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Equivalence with 4-point finite volumes

Equivalence of MFEs with 4-point finite volumes

- let $g = 0$, $\mathbf{S} = \mathbb{I}$, and $\mathcal{T}_h$ consist of **equilateral simplices**: then p_h from MFEs and $\bar{p}_h$ from FVs **coincide**
- $g \neq 0$, $\mathbf{S} \neq \mathbb{I}$, or $\mathcal{T}_h$ not consisting of equilateral simplices: p_h from MFEs and $\bar{p}_h$ from FVs **do not coincide** anymore
- **conclusion: MFEs and FVs are different**
- this **conclusion is almost completely wrong**

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Equivalence with 4-point finite volumes

- in MFEs (Marini 1985):

$$p_h|_K = \tilde{\lambda}_h(\mathbf{x}_K) + \frac{g_K}{2d|K|}((\mathbf{x}-\mathbf{x}_K)^t\mathbf{S}_K^{-1}(\mathbf{x}-\mathbf{x}_K),1)_K$$

  - $\mathbf{x}_K$ is the **barycenter**
  - p_h represents the **mean value** of the potential
  - **influence of the source term g**
- in FVs, if $g = 0$ (Younès, Mose, Ackerer, & Chavent 1999–2004):

$$\bar{p}_h|_K = \tilde{\lambda}_h(\mathbf{z}_K)$$

  - $\mathbf{z}_K$ is the **circumcenter**
  - $\bar{p}_h$ represents the **point value** of the potential
  - **no influence of the source term g**
- **MFEs and FVs are equivalent when g is constant**
  - holds on **arbitrary simplicial meshes** (not necessarily Delaunay)!
  - holds for **full matrix S**!

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Links to MFDs and MPFAs

Links to the mimetic finite difference and multi-point flux-approximation methods

- using approximate numerical integration
  - Klausen & Winther, 2006
  - Wheeler & Yotov, 2006
  - Aavatsmark, Eigestad, Klausen, Wheeler, & Yotov, 2007
  - Droniou, Eymard, Gallouët, & Herbin, 2010
  - Bause Hoffmann, & Knabner, 2010
  - . . . Brezzi, da Veiga, Lipnikov, Manzini, Shashkov . . .

Outline

Discrete maximum principle in MFEs

Discrete maximum principle in MFEs (S $= \mathbb{I}$)

- DMP for the Lagrange multipliers λ_σ (values of $\tilde{\lambda}_h$ in side barycenters) whenever $\mathcal{T}_h$ is acute (equivalence with the NCFE method)
- DMP in 2D for the values $\bar{p}_K$ (values of $\tilde{\lambda}_h$ in circumcenters) whenever $\mathcal{T}_h$ is Delaunay and the source g is constant (equivalence with the FV method)
- DMP not necessarily for the original values p_K (recall that p_K = value of $\tilde{\lambda}_h$ in the barycenter + a small influence of the source term)

Discrete maximum principle in MFEs

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- DMP for the Lagrange multipliers λ_σ (values of $\tilde{\lambda}_h$ in side barycenters) whenever $\mathcal{T}_h$ is acute (equivalence with the NCFE method)
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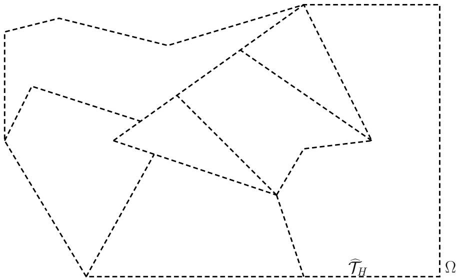
Outline

General polygonal meshes

A general polygonal mesh $\widehat{\mathcal{T}}_H$



- nonconvex and non star-shaped elements in $\widehat{\mathcal{T}}_H$
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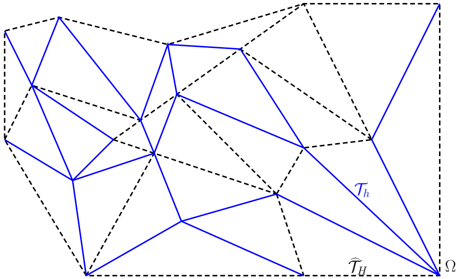

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MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

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- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
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Local flux expression from the Lagrange multipliers

Nonconforming finite element method

find $\tilde{\lambda}_h \in \tilde{\Psi}_h$ such that

$$(\mathbf{S}\nabla\tilde{\lambda}_h, \nabla\tilde{\psi}_h) = (g, \tilde{\psi}_h) \qquad \forall \tilde{\psi}_h \in \tilde{\Psi}_h$$

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there holds (Marini 1985)

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A new element value

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- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
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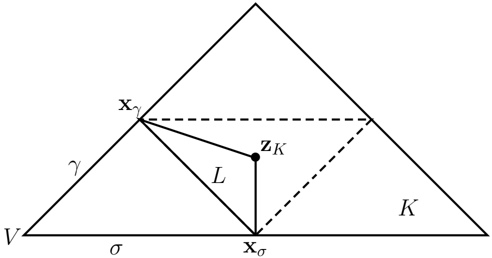

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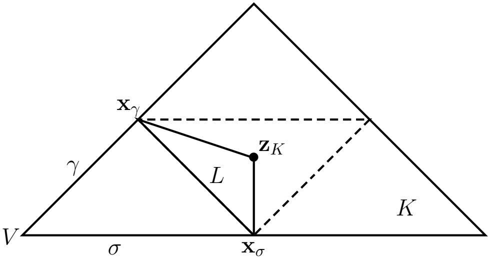

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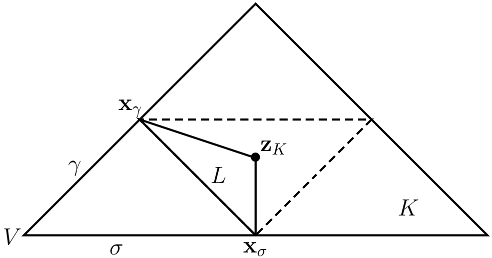

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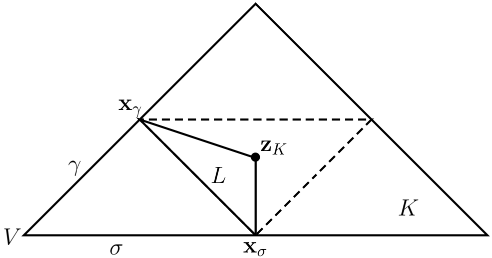

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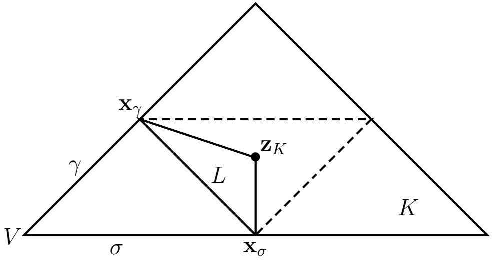

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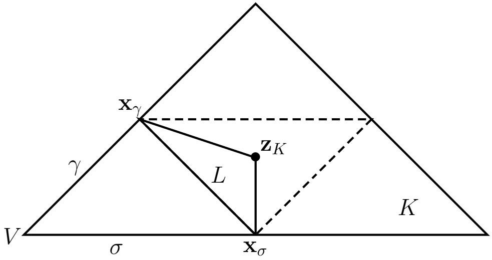

Definition of a local problem

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- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\rm int}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
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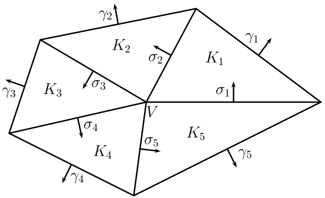

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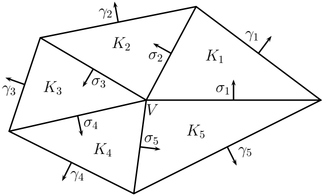

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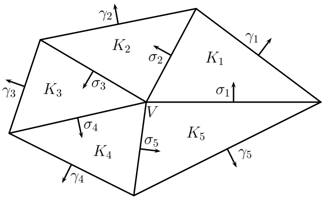

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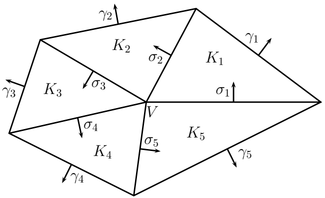

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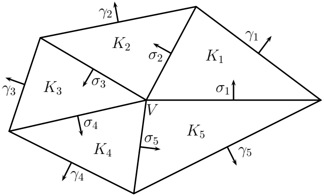

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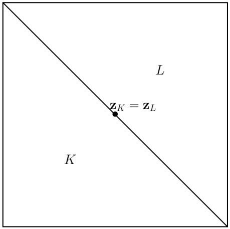
S-circumcenter as the evaluation point

S-circumcenter as the point $\mathbf{z}_K$

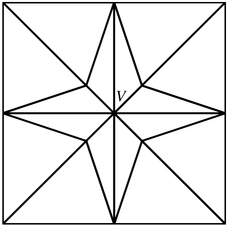
- circumcenter when $\mathbf{S}_K = \mathbb{I}s_K$
- the approach of Younès, Mose, Ackerer, & Chavent, 1999
- $\mathbb{M}_V$ gets diagonal
- no local linear system needs to be solved
- two-point flux expression (on arbitrary triangular grids and full-matrix piecewise constant $\mathbf{S}$)
- impossible in 3D (except particular cases)
- $\mathbb{M}_V$ can explode (modifications necessary):



Barycenter as the evaluation point

Barycenter as the point $\mathbf{z}_K$

- this is the approach of Vohralík, 2004/2006
- $\mathbb{M}_V$ is not diagonal (unless barycenter = circumcenter)
- a local linear system needs to be solved
- multi-point flux expression
- works generally in *d* space dimensions
- $\mathbb{M}_V$ can get singular (modifications necessary):



Changing adaptively the evaluation point

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- change $\mathbf{z}_K$ according to the local geometry and diffusion tensor
- ensure the well-posedness of the local problems
- influence the properties of the local matrices $\mathbb{M}_V$
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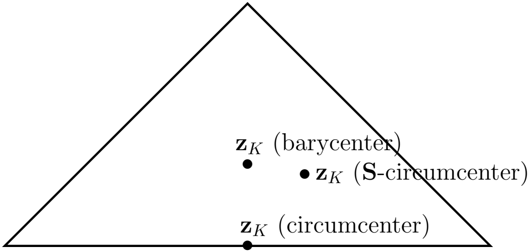
Examples of the different evaluation points

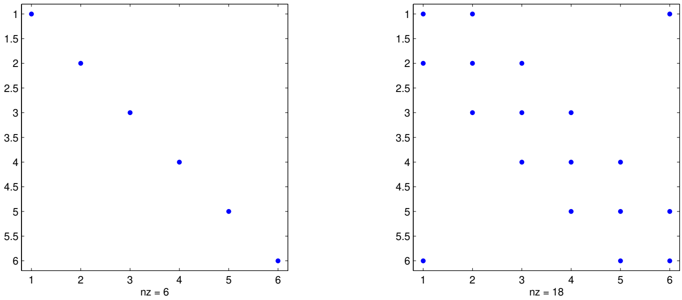
Examples of the different evaluation points $\mathbf{z}_K$

- $\mathbf{S} = \begin{pmatrix} 0.7236 & 0.3804 \\ 0.3804 & 0.4764 \end{pmatrix}$



Examples of the local matrices

Examples of the local matrices $\mathbb{M}_V$



S-circumcenter barycenter/opt. evaluation point

Outline

Expressing the Lagrange multipliers Λ or the fluxes U

Expressing the Lagrange multipliers Λ or the fluxes U

- local problems give $\Lambda_V^{\mathrm{int}} = (\mathbb{M}_V)^{-1}(\widetilde{G}_V - \mathbb{J}_V \bar{P}_V)$
- for every vertex V, we have one expression for Λ_V^{int}
- run through all vertices and combine the (weighted) inverses of the local condensation matrices
- this gives

$$\Lambda = \widetilde{\mathbb{M}}^{\mathrm{inv}} \widetilde{G} - \mathbb{M}^{\mathrm{inv}} \bar{P}$$

- similarly

$$U = \widetilde{\mathbb{O}}^{\mathrm{inv}} G - \mathbb{O}^{\mathrm{inv}} \bar{P}$$

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- recall $U = \widetilde{\mathbb{O}}^{\mathrm{inv}} G - \mathbb{O}^{\mathrm{inv}} \bar{P}$
- put this into $\mathbb{B}U = G$
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$$\bar{\mathbb{S}} \bar{P} = \bar{H}$$

  with

$$\bar{\mathbb{S}} = -\mathbb{B}\mathbb{O}^{\mathrm{inv}}, \quad \bar{H} = G - \mathbb{B}\widetilde{\mathbb{O}}^{\mathrm{inv}} G$$

- $\mathbf{z}_K$ = **S**-circumcenter gives the FV method (Younès, Mose, Ackerer, & Chavent, 1999)
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- using $\mathbf{z}_K$ = **S**-circumcenter, we name it the MFEC method
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Outline

Model problem

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- $\Omega = (0,1) \times (0,1)$
- inhomogeneous Dirichlet boundary condition given by $p(x,y) = 0.1y + 0.9$
- $K \in \mathcal{T}_h$:

$$\mathbf{S}|_K = \begin{pmatrix} \cos(\theta_K) & -\sin(\theta_K) \\ \sin(\theta_K) & \cos(\theta_K) \end{pmatrix} \begin{pmatrix} s_K & 0 \\ 0 & \nu s_K \end{pmatrix} \begin{pmatrix} \cos(\theta_K) & \sin(\theta_K) \\ -\sin(\theta_K) & \cos(\theta_K) \end{pmatrix}$$

- homogeneous isotropic case, $s_K = 1$ for all $K \in \mathcal{T}_h$, $\nu = 1$
- anisotropic case, $s_K = 1$ for all $K \in \mathcal{T}_h$, $\theta_K \in \{\frac{\pi}{5}, \frac{3\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{5}, \frac{\pi}{3}\}$, $\nu = 0.2$
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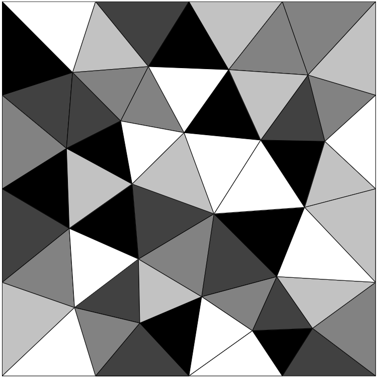
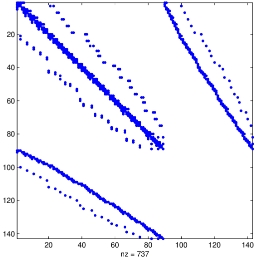
Initial mesh

Initial mesh and the distribution of the inhomogeneities and anisotropies

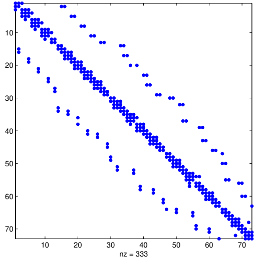


Matrices of the different methods

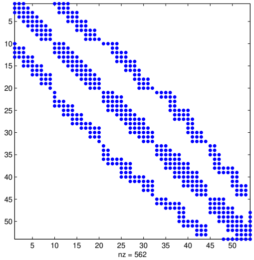
System matrix sparsity patterns



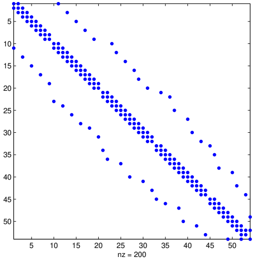
MFE



NCFE



MFEB
MFEO
CMFE



FV
MFEC

Results, homogeneous isotropic case

							DS	CG/ Bi-CGStab		PCG/ PBi-CGStab		
Meth.	Un.	Mat.	St.	Nonz.	CN	CNS	CPU	CPU	Iter.	CPU	IC/ ILU	Iter.
MFEB	13824	NPD	14	177652	7564	7580	0.27	4.86	324.5	0.81	0.36	9.0
MFEC	13824	NNS	4	55040	11256	11056	0.09	2.23	372.0	0.42	0.19	6.5
MFEO	13824	NPD	14	177652	7531	7558	0.28	4.08	270.0	0.80	0.41	7.5
CMFE	13824	NPD	14	177652	7397	7380	0.27	4.70	312.0	0.83	0.39	8.5
FV	13824	SPD	4	55040	65722	8898	0.07	3.09	1098.0	0.42	0.17	17.0
NCFE	20608	SPD	5	102528	14064	9944	0.14	2.92	620.0	1.11	0.56	19.0

Results, anisotropic case

							DS	CG/ Bi-CGStab		PCG/ PBi-CGStab		
Meth.	Un.	Mat.	St.	Nonz.	CN	CNS	CPU	CPU	Iter.	CPU	IC/ ILU	Iter.
MFEB	13824	NPD	14	177652	14489	11203	0.28	6.61	448.0	0.98	0.59	6.5
MFEC	13824	NID	4	55040	2401279	416769	0.08	—	—	0.45	0.20	7.0
MFEO	13824	NPD	14	177652	13401	10767	0.27	6.51	440.5	0.95	0.41	10.0
CMFE	13824	NPD	14	177652	9276	7758	0.28	5.27	350.5	0.84	0.38	9.0
FV	13824	SID	4	55040	247055	239934	0.09	—	—	0.45	0.20	7.0
NCFE	20608	SPD	5	102528	25393	16969	0.18	4.03	850.0	1.12	0.41	30.0

Results, inhomogeneous case

							DS	CG/ Bi-CGStab		PCG/ PBi-CGStab		
Meth.	Un.	Mat.	St.	Nonz.	CN	CNS	CPU	CPU	Iter.	CPU	IC/ ILU	Iter.
MFEB	13824	NPD	14	177652	819248	740706	0.28	13.33	897.5	1.05	0.62	6.5
MFEC	13824	NNS	4	55040	903789	763849	0.09	5.34	947.5	0.47	0.20	7.5
MFEO	13824	NPD	14	177652	820367	739957	0.28	12.45	790.5	1.05	0.56	8.0
CMFE	13824	NPD	14	177652	2500730	478974	0.28	102.27	6842.5	1.01	0.41	10.5
FV	13824	SPD	4	55040	16387758	497974	0.07	39.41	14101.0	0.44	0.17	16.0
NCFE	20608	SPD	5	102528	4797335	670623	0.18	52.42	11226.0	1.22	0.64	16.0

Outline

Conclusions and future work

Conclusions

- mixed finite elements: one method with
  - saddle point / symmetric pos. definite / nonsymmetric pos. definite / symmetric indefinite / nonsymmetric indef. matrix
  - U and P unknowns / Λ unknowns / P unknowns
  - narrow stencil and two-point flux expressions / wider stencil and multi-point flux expressions
  - discrete maximum principle for values in some points but not in some others
- no free parameter to choose, no stabilization, the best method if your criterion is min. complementary energy
- close relations in building principles between MFE/FD/FV/MFD/MPFA, even on general polygonal meshes

Work in progress

- a general principle for nonconforming finite elements
- extensions to all order MFE schemes
- multigrid solvers

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Conclusions and future work

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- mixed finite elements: one method with
  - saddle point / symmetric pos. definite / nonsymmetric pos. definite / symmetric indefinite / nonsymmetric indef. matrix
  - U and P unknowns / Λ unknowns / P unknowns
  - narrow stencil and two-point flux expressions / wider stencil and multi-point flux expressions
  - discrete maximum principle for values in some points but not in some others
- no free parameter to choose, no stabilization, the best method if your criterion is min. complementary energy
- close relations in building principles between MFE/FD/FV/MFD/MPFA, even on general polygonal meshes

Work in progress

- a general principle for nonconforming finite elements
- extensions to all order MFE schemes
- multigrid solvers

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Thank you for your attention!