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Mixed g -monotone property and quadruple fixed point theorems in partially ordered metric spaces

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Abstract

In this manuscript, we prove some quadruple coincidence and common fixed point theorems for $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ satisfying generalized contractions in partially ordered metric spaces. Our results unify, generalize and complement various known results from the current literature. Also, an application to matrix equations is given.

2000 Mathematics subject Classifications: 46T99; 54H25; 47H10; 54E50.

Keywords: quadruple coincidence point, quadruple common fixed point, ordered set, metric space, generalized contraction

1 Introduction and preliminaries

Existence of fixed points in partially ordered metric spaces was first investigated by Turinici [1], where he extended the Banach contraction principle in partially ordered sets. In 2004, Ran and Reurings [2] presented some applications of Turinici's theorem to matrix equations. Following these initial articles, some remarkable results were reported see, e.g., [3-13].

Gnana Bhashkar and Lakshmikantham in [14] introduced the concept of a coupled fixed point of a mapping $F : X \times X \rightarrow X$ and investigated some coupled fixed point theorems in partially ordered complete metric spaces. Later, Lakshmikantham and Ćirić [15] proved coupled coincidence and coupled common fixed point theorems for nonlinear mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ in partially ordered complete metric spaces. Various results on coupled fixed point have been obtained, since then see, e.g., [6,9,16-33]. Recently, Berinde and Borcut [34] introduced the concept of tripled fixed point in ordered sets.

For simplicity, we denote $\underbrace{X \times X \cdots X \times X}_{k \text{ times}}$ by X^k where $k \in \mathbb{N}$. Let us recall some basic definitions.

Definition 1.1 (See [34]) *Let (X, \leq) be a partially ordered set and $F : X^3 \rightarrow X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$*

$$\begin{aligned}x_1, x_2 \in X, \quad x_1 \leq x_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\y_1, y_2 \in X, \quad y_1 \leq y_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \\z_1, z_2 \in X, \quad z_1 \leq z_2 &\Rightarrow F(x, y, z_1) \leq F(x, y, z_2).\end{aligned}$$

Definition 1.2 *Let $F : X^3 \rightarrow X$. An element (x, y, z) is called a tripled fixed point of F if*

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

Also, Berinde and Borcut [34] proved the following theorem:

Theorem 1.1 *Let (X, \leq, d) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^3 \rightarrow X$ having the mixed monotone property. Suppose there exist $j, r, l \geq 0$ with $j + r + l < 1$ such that*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + rd(y, v) + ld(z, w), \tag{1}$$

for any $x, y, z \in X$ for which $x \leq u, v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

1. if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
2. if a non-increasing sequence $y_n \rightarrow y$, then $y \leq y_n$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z,$$

that is, F has a tripled fixed point.

Recently, Aydi et al. [35] introduced the following concepts.

Definition 1.3 *Let (X, \leq) be a partially ordered set. Let $F : X^3 \rightarrow X$ and $g : X \rightarrow X$. The mapping F is said to have the mixed g -monotone property if for any $x, y, z \in X$*

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \leq gx_2 &\Rightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad gy_1 \leq gy_2 &\Rightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, \quad gz_1 \leq gz_2 &\Rightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{aligned}$$

Definition 1.4 *Let $F : X^3 \rightarrow X$ and $g : X \rightarrow X$. An element (x, y, z) is called a tripled coincidence point of F and g if*

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad \text{and} \quad F(z, y, x) = gz.$$

(gx, gy, gz) is said a tripled point of coincidence of F and g .

Definition 1.5 *Let $F : X^3 \rightarrow X$ and $g : X \rightarrow X$. An element (x, y, z) is called a tripled common fixed point of F and g if*

$$F(x, y, z) = gx = x, \quad F(y, x, y) = gy = y, \quad \text{and} \quad F(z, y, x) = gz = z.$$

Definition 1.6 *Let X be a non-empty set. Then we say that the mappings $F : X^3 \rightarrow X$ and*

$g : X \rightarrow X$ are commutative if for all $x, y, z \in X$

$$g(F(x, y, z)) = F(gx, gy, gz).$$

The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [36]. Very recently, Karapinar used the concept of quadruple fixed point and proved some fixed point theorems on the topic [37]. Following this study, quadruple fixed point is developed and some related fixed point theorems are obtained in [38-41].

Definition 1.7 [38] *Let X be a nonempty set and $F : X^4 \rightarrow X$ be a given mapping. An element $(x, y, z, w) \in X \times X \times X \times X$ is called a quadruple fixed point of F if*

$$F(x, y, z, w) = x, \quad F(y, z, w, x) = y, \quad F(z, w, x, y) = z, \quad \text{and} \quad F(w, x, y, z) = w.$$

Let (X, d) be a metric space. The mapping $\bar{d} : X^4 \rightarrow X$, given by

$$\bar{d}((x, y, z, w), (u, v, h, l)) = d(x, y) + d(y, v) + d(z, h) + d(w, l),$$

defines a metric on X^4 , which will be denoted for convenience by d .

Definition 1.8 [38] Let (X, \leq) be a partially ordered set and $F : X^4 \rightarrow X$ be a mapping. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z and is monotone non-increasing in y and w ; that is, for any $x, y, z, w \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 & \text{ implies } F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 & \text{ implies } F(x, y_2, z, w) \leq F(x, y_1, z, w), \\ z_1, z_2 \in X, \quad z_1 \leq z_2 & \text{ implies } F(x, y, z_1, w) \leq F(x, y, z_2, w), \end{aligned}$$

and

$$w_1, w_2 \in X, \quad w_1 \leq w_2 \text{ implies } F(x, y, z, w_2) \leq F(x, y, z, w_1).$$

In this article, we establish some quadruple coincidence and common fixed point theorems for $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ satisfying nonlinear contractions in partially ordered metric spaces. Also, some interesting corollaries are derived and an application to matrix equations is given.

2 Main results

We start this section with the following definitions.

Definition 2.1 Let (X, \leq) be a partially ordered set. Let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$. The mapping F is said to have the mixed g -monotone property if for any $x, y, z, w \in X$

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \leq gx_2 & \Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w), \\ y_1, y_2 \in X, \quad gy_1 \leq gy_2 & \Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w), \\ z_1, z_2 \in X, \quad gz_1 \leq gz_2 & \Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w) \text{ and} \\ w_1, w_2 \in X, \quad gw_1 \leq gw_2 & \Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2). \end{aligned}$$

Definition 2.2 Let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$. An element (x, y, z, w) is called a quadruple coincidence point of F and g if

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \text{and} \quad F(w, x, y, z) = gw.$$

(gx, gy, gz, gw) is said a quadruple point of coincidence of F and g .

Definition 2.3 Let $F : X^4 \rightarrow X$ and $g : X \rightarrow X$. An element (x, y, z, w) is called a quadruple common fixed point of F and g if

$$\begin{aligned} F(x, y, z, w) = gx = x, \quad F(y, z, w, x) = gy = y, \\ F(z, w, x, y) = gz = z, \quad \text{and} \quad F(w, x, y, z) = gw = w. \end{aligned}$$

Definition 2.4 Let X be a non-empty set. Then we say that the mappings $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are commutative if for all $x, y, z, w \in X$

$$g(F(x, y, z, w)) = F(gx, gy, gz, gw).$$

Let Φ be the set of all functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that:

1. $\varphi(t) < t$ for all $t \in (0, +\infty)$.

$$2. \lim_{r \rightarrow t^+} \phi(r) < t \text{ for all } t \in (0, +\infty).$$

For simplicity, we define the following.

$$M(x, y, z, w, u, v, h, l) = \min \left\{ \begin{array}{l} d(F(x, y, z, w), gx), d(F(x, y, z, w), gu), \\ d(F(u, v, h, l), gu) \end{array} \right\}. \tag{2}$$

Now, we state the first main result of this article.

Theorem 2.1 *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F is continuous and has the mixed g -monotone property. Assume also that there exist $\phi \in \Phi$ and $L \geq 0$ such that*

$$d(F(x, y, z, w), F(u, v, h, l)) \leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\}) + LM(x, y, z, w, u, v, h, l) \tag{3}$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gh$ and $gl \leq gw$. Suppose $F(X^4) \subset g(X)$, g is continuous and commutes with F . If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} gx_0 &\leq F(x_0, y_0, z_0, w_0), & gy_0 &\geq F(y_0, z_0, w_0, x_0), \\ gz_0 &\leq F(z_0, w_0, x_0, y_0), & gw_0 &\geq F(w_0, x_0, y_0, z_0), \end{aligned}$$

then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz \text{ and } F(w, x, y, z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. Let $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} gx_0 &\leq F(x_0, y_0, z_0, w_0), & gy_0 &\geq F(y_0, z_0, w_0, x_0), \\ gz_0 &\leq F(z_0, w_0, x_0, y_0) \text{ and } gw_0 &\geq F(w_0, x_0, y_0, z_0). \end{aligned}$$

Since $F(X^4) \subset g(X)$, then we can choose $x_1, y_1, z_1, w_1 \in X$ such that

$$\begin{aligned} gx_1 &= F(x_0, y_0, z_0, w_0), & gy_1 &= F(y_0, z_0, w_0, x_0), \\ gz_1 &= F(z_0, w_0, x_0, y_0) \text{ and } gw_1 &= F(w_0, x_0, y_0, z_0). \end{aligned} \tag{4}$$

Taking into account $F(X^4) \subset g(X)$, by continuing this process, we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\}$, and $\{w_n\}$ in X such that

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n, w_n), & gy_{n+1} &= F(y_n, z_n, w_n, x_n), \\ gz_{n+1} &= F(z_n, w_n, x_n, y_n), \text{ and } gw_{n+1} &= F(w_n, x_n, y_n, z_n). \end{aligned} \tag{5}$$

We shall show that

$$gx_n \leq gx_{n+1}, \quad gy_{n+1} \leq gy_n, \quad gz_n \leq gz_{n+1}, \text{ and } gw_{n+1} \leq gw_n \text{ for } n = 0, 1, 2, \dots \tag{6}$$

For this purpose, we use the mathematical induction. Since, $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then by (4), we get

$$gx_0 \leq gx_1, \quad gy_1 \leq gy_0, \quad gz_0 \leq gz_1, \text{ and } gw_1 \leq gw_0$$

that is, (6) holds for $n = 0$.

We presume that (6) holds for some $n > 0$. As F has the mixed g -monotone property and $gx_n \leq gx_{n+1}$, $gy_{n+1} \leq gy_n$, $gz_n \leq gz_{n+1}$ and $gw_{n+1} \leq gw_n$, we obtain

$$\begin{aligned} gx_{n+1} &= F(x_n, \gamma_n, z_n, w_n) \leq F(x_{n+1}, \gamma_n, z_n, w_n) \\ &\leq F(x_{n+1}, \gamma_n, z_{n+1}, w_n) \leq F(x_{n+1}, \gamma_{n+1}, z_{n+1}, w_n) \\ &\leq F(x_{n+1}, \gamma_{n+1}, z_{n+1}, w_{n+1}) = gx_{n+2}, \end{aligned}$$

$$\begin{aligned} gy_{n+2} &= F(\gamma_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) \leq F(\gamma_{n+1}, z_n, x_{n+1}, w_{n+1}) \\ &\leq F(\gamma_n, z_n, x_{n+1}, w_{n+1}) \leq F(\gamma_n, z_n, x_n, w_{n+1}) \\ &\leq F(\gamma_n, z_n, x_n, w_n) = gy_{n+1}, \end{aligned}$$

$$\begin{aligned} gz_{n+1} &= F(z_n, \gamma_n, x_n, w_n) \leq F(z_{n+1}, \gamma_n, x_n, w_n) \\ &\leq F(z_{n+1}, \gamma_{n+1}, x_n, w_n) \leq F(z_{n+1}, \gamma_{n+1}, x_{n+1}, w_n) \\ &\leq F(z_{n+1}, \gamma_{n+1}, x_{n+1}, w_{n+1}) = gz_{n+2}, \end{aligned}$$

and

$$\begin{aligned} gw_{n+2} &= F(w_{n+1}, x_{n+1}, \gamma_{n+1}, z_{n+1}) \leq F(w_{n+1}, x_n, \gamma_{n+1}, z_{n+1}) \\ &\leq F(w_n, x_n, \gamma_{n+1}, z_{n+1}) \leq F(w_n, x_n, \gamma_n, z_{n+1}) \\ &\leq F(w_n, x_n, \gamma_n, z_n) = gw_{n+1}. \end{aligned}$$

Thus, (6) holds for any $n \in \mathbb{N}$. Assume for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad gy_n = gy_{n+1}, \quad gz_n = gz_{n+1}, \quad \text{and} \quad gw_n = gw_{n+1}$$

then, by (5), $(x_n, \gamma_n, z_n, w_n)$ is a quadruple coincidence point of F and g . From now on, assume for any $n \in \mathbb{N}$ that at least

$$gx_n \neq gx_{n+1} \quad \text{OR} \quad gy_n \neq gy_{n+1} \quad \text{OR} \quad gz_n \neq gz_{n+1} \quad \text{OR} \quad gw_n \neq gw_{n+1}. \quad (7)$$

By (2) and (5), it is easy that

$$\begin{aligned} M(x_{n-1}, \gamma_{n-1}, z_{n-1}, w_{n-1}, x_n, \gamma_n, z_n, w_n) &= M(\gamma_n, z_n, w_n, x_n, \gamma_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \\ &= M(z_{n-1}, \gamma_{n-1}, x_{n-1}, z_n, \gamma_n, x_n) \\ &= M(w_n, x_n, \gamma_n, z_n, w_{n-1}, x_{n-1}, \gamma_{n-1}, z_{n-1}) = 0 \quad \text{for all } n \geq 1. \end{aligned} \quad (8)$$

Due to (3) and (8), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, \gamma_{n-1}, z_{n-1}, w_{n-1}), F(x_n, \gamma_n, z_n, w_n)) \\ &\leq \phi(\max\{d(gx_{n-1}, gx_n), d(g\gamma_{n-1}, g\gamma_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}) \\ &\quad + LM(x_{n-1}, \gamma_{n-1}, z_{n-1}, w_{n-1}, x_n, \gamma_n, z_n, w_n) \\ &= \phi(\max\{d(gx_{n-1}, gx_n), d(g\gamma_{n-1}, g\gamma_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}), \end{aligned} \quad (9)$$

$$\begin{aligned} d(gy_n, gy_{n+1}) &= d(F(\gamma_n, z_n, w_n, x_n), F(\gamma_{n-1}, z_{n-1}, w_{n-1}, x_{n-1})) \\ &\leq \phi(\max\{d(g\gamma_{n-1}, g\gamma_n), d(gx_{n-1}, gx_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}) \\ &\quad + LM(\gamma_n, z_n, w_n, w_n, \gamma_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) \\ &= \phi(\max\{d(g\gamma_{n-1}, g\gamma_n), d(gx_{n-1}, gx_n), d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n)\}), \end{aligned} \quad (10)$$

$$\begin{aligned}
 d(gz_n, gz_{n+1}) &= d(F(z_{n-1}, w_{n-1}, x_{n-1}, \gamma_{n-1}), F(z_n, w_n, x_n, \gamma_n)) \\
 &\leq \phi(\max\{d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(g\gamma_{n-1}, g\gamma_n)\}) \\
 &\quad + LM(z_{n-1}, w_{n-1}, x_{n-1}, \gamma_{n-1}, z_n, w_n, x_n, \gamma_n) \\
 &= \phi(\max\{d(gz_{n-1}, gz_n), d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(g\gamma_{n-1}, g\gamma_n)\})
 \end{aligned} \tag{11}$$

and

$$\begin{aligned}
 d(gw_n, gw_{n+1}) &= d(F(w_n, x_n, \gamma_n, z_n), F(w_{n-1}, x_{n-1}, \gamma_{n-1}, z_{n-1})) \\
 &\leq \phi(\max\{d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(g\gamma_{n-1}, g\gamma_n), d(gz_{n-1}, gz_n)\}) \\
 &\quad + LM(w_n, x_n, \gamma_n, z_n, w_{n-1}, x_{n-1}, \gamma_{n-1}, z_{n-1}) \\
 &= \phi(\max\{d(gw_{n-1}, gw_n), d(gx_{n-1}, gx_n), d(g\gamma_{n-1}, g\gamma_n), d(gz_{n-1}, gz_n)\}).
 \end{aligned} \tag{12}$$

Having in mind that $\phi(t) < t$ for all $t > 0$, so from (9)-(12) we obtain that

$$\begin{aligned}
 0 &< \max\{d(gx_n, gx_{n+1}), d(g\gamma_n, g\gamma_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\} \\
 &\leq \phi(\max\{d(gz_{n-1}, gz_n), d(g\gamma_{n-1}, g\gamma_n), d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n)\}) \\
 &< \max\{d(gz_{n-1}, gz_n), d(g\gamma_{n-1}, g\gamma_n), d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n)\}.
 \end{aligned} \tag{13}$$

It follows that

$$\max \left\{ \begin{array}{l} d(gx_n, gx_{n+1}), d(g\gamma_n, g\gamma_{n+1}), \\ d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1}) \end{array} \right\} < \max \left\{ \begin{array}{l} d(gz_{n-1}, gz_n), d(g\gamma_{n-1}, g\gamma_n), \\ d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n) \end{array} \right\}. \tag{14}$$

Thus, $\{\max\{d(gx_n, gx_{n+1}), d(g\gamma_n, g\gamma_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\}\}$ is a positive decreasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow +\infty} \max\{d(gx_n, gx_{n+1}), d(g\gamma_n, g\gamma_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\} = r.$$

Suppose that $r > 0$. Letting $n \rightarrow +\infty$ in (13), we obtain that

$$0 < r \leq \lim_{n \rightarrow +\infty} \phi \left(\max \left\{ \begin{array}{l} d(gz_{n-1}, gz_n), d(g\gamma_{n-1}, g\gamma_n), \\ d(gx_{n-1}, gx_n), d(gw_{n-1}, gw_n) \end{array} \right\} \right) = \lim_{t \rightarrow r^+} \phi(t) < r. \tag{15}$$

It is a contradiction. We deduce that

$$\lim_{n \rightarrow +\infty} \max\{d(gx_n, gx_{n+1}), d(g\gamma_n, g\gamma_{n+1}), d(gz_n, gz_{n+1}), d(gw_n, gw_{n+1})\} = 0. \tag{16}$$

We shall show that $\{gx_n\}$, $\{g\gamma_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the metric space (X, d) . Assume the contrary, that is, one of the sequence $\{gx_n\}$, $\{g\gamma_n\}$, $\{gz_n\}$ or $\{gw_n\}$ is not a Cauchy, that is,

$$\lim_{n, m \rightarrow +\infty} d(gx_m, gx_n) \neq 0 \quad \text{or} \quad \lim_{n, m \rightarrow +\infty} d(g\gamma_m, g\gamma_n) \neq 0$$

or

$$\lim_{n, m \rightarrow +\infty} d(gz_m, gz_n) \neq 0 \quad \text{or} \quad \lim_{n, m \rightarrow +\infty} d(gw_m, gw_n) \neq 0.$$

This means that there exists $\varepsilon > 0$, for which we can find subsequences of integers (m_k) and (n_k) with $n_k > m_k > k$ such that

$$\max\{d(gx_{m_k}, gx_{n_k}), d(g\gamma_{m_k}, g\gamma_{n_k}), d(gz_{m_k}, gz_{n_k}), d(gw_{m_k}, gw_{n_k})\} \geq \varepsilon. \tag{17}$$

Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (17). Then

$$\max\{d(gx_{m_k}, gx_{n_k-1}), d(gy_{m_k}, gy_{n_k-1}), d(gz_{m_k}, gz_{n_k-1}), d(gw_{m_k}, gw_{n_k-1})\} < \varepsilon. \quad (18)$$

By triangular inequality and (18), we have

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &\leq d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &< \varepsilon + d(gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Thus, by (16) we obtain

$$\lim_{k \rightarrow +\infty} d(gx_{m_k}, gx_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gx_{m_k}, gx_{n_k-1}) \leq \varepsilon. \quad (19)$$

Similarly, we have

$$\lim_{k \rightarrow +\infty} d(gy_{m_k}, gy_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gy_{m_k}, gy_{n_k-1}) \leq \varepsilon, \quad (20)$$

$$\lim_{k \rightarrow +\infty} d(gz_{m_k}, gz_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gz_{m_k}, gz_{n_k-1}) \leq \varepsilon, \quad (21)$$

and

$$\lim_{k \rightarrow +\infty} d(gw_{m_k}, gw_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gw_{m_k}, gw_{n_k-1}) \leq \varepsilon. \quad (22)$$

Again by (18), we have

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &\leq d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &\leq d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) \\ &\quad + d(gx_{m_k}, gx_{n_k-1}) + d(gx_{n_k-1}, gx_{n_k}) \\ &< d(gx_{m_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) + \varepsilon + d(gx_{n_k-1}, gx_{n_k}). \end{aligned}$$

Letting $k \rightarrow +\infty$ and using (16), we get

$$\lim_{k \rightarrow +\infty} d(gx_{m_k}, gx_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gx_{m_k-1}, gx_{n_k-1}) \leq \varepsilon, \quad (23)$$

$$\lim_{k \rightarrow +\infty} d(gy_{m_k}, gy_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gy_{m_k-1}, gy_{n_k-1}) \leq \varepsilon, \quad (24)$$

$$\lim_{k \rightarrow +\infty} d(gz_{m_k}, gz_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gz_{m_k-1}, gz_{n_k-1}) \leq \varepsilon \quad (25)$$

and

$$\lim_{k \rightarrow +\infty} d(gw_{m_k}, gw_{n_k}) \leq \lim_{k \rightarrow +\infty} d(gw_{m_k-1}, gw_{n_k-1}) \leq \varepsilon. \quad (26)$$

Using (17) and (23)-(26), we have

$$\begin{aligned} &\lim_{k \rightarrow +\infty} \max\{d(gx_{m_k}, gx_{n_k}), d(gy_{m_k}, gy_{n_k}), d(gz_{m_k}, gz_{n_k}), d(gw_{m_k}, gw_{n_k})\} \\ &= \lim_{k \rightarrow +\infty} \max\{d(gx_{m_k-1}, gx_{n_k-1}), d(gy_{m_k-1}, gy_{n_k-1}), d(gz_{m_k-1}, gz_{n_k-1}), d(gw_{m_k-1}, gw_{n_k-1})\} \\ &= \varepsilon. \end{aligned} \quad (27)$$

By (16), it is easy to see that

$$\begin{aligned}
 & \lim_{k \rightarrow +\infty} M(x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}) \\
 &= \lim_{k \rightarrow +\infty} M(\gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{m_k-1}) \\
 &= \lim_{k \rightarrow +\infty} M(z_{m_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{m_k-1}) \\
 &= \lim_{k \rightarrow +\infty} M(w_{n_k-1}, x_{n_k-1}, \gamma_{m_k-1}, z_{n_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}) = 0.
 \end{aligned} \tag{28}$$

Now, using inequality (3), we obtain

$$\begin{aligned}
 d(gx_{m_k}, gx_{n_k}) &= d(F(x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}), F(x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1})) \\
 &\leq \phi(\max\{d(x_{m_k-1}, x_{n_k-1}), d(\gamma_{m_k-1}, \gamma_{n_k-1}), d(z_{m_k-1}, z_{n_k-1}), d(w_{m_k-1}, w_{n_k-1})\}) \\
 &\quad + LM(x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}),
 \end{aligned} \tag{29}$$

$$\begin{aligned}
 d(g\gamma_{n_k}, g\gamma_{m_k}) &= d(F(\gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}), F(\gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{m_k-1})) \\
 &\leq \phi(\max\{d(\gamma_{m_k-1}, \gamma_{n_k-1}), d(z_{m_k-1}, z_{n_k-1}), d(w_{m_k-1}, w_{n_k-1}), d(x_{m_k-1}, x_{n_k-1})\}) \\
 &\quad + LM(\gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{m_k-1}),
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 d(gz_{m_k}, gz_{n_k}) &= d(F(z_{m_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}), F(z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{n_k-1})) \\
 &\leq \phi(\max\{d(z_{m_k-1}, z_{n_k-1}), d(w_{m_k-1}, w_{n_k-1}), d(x_{m_k-1}, x_{n_k-1}), d(\gamma_{m_k-1}, \gamma_{n_k-1})\}) \\
 &\quad + LM(z_{m_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{n_k-1})
 \end{aligned} \tag{31}$$

and

$$\begin{aligned}
 d(gw_{n_k}, gw_{m_k}) &= d(F(w_{n_k-1}, x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}), F(w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1})) \\
 &\leq \phi(\max\{d(w_{m_k-1}, w_{n_k-1}), d(x_{m_k-1}, x_{n_k-1}), d(\gamma_{m_k-1}, \gamma_{n_k-1}), d(z_{m_k-1}, z_{n_k-1})\}) \\
 &\quad + LM(w_{n_k-1}, x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}).
 \end{aligned} \tag{32}$$

From (29)-(32), we deduce that

$$\begin{aligned}
 & \max\{d(gx_{m_k}, gx_{n_k}), d(g\gamma_{m_k}, g\gamma_{n_k}), d(gz_{m_k}, gz_{n_k}), d(gw_{m_k}, gw_{n_k})\} \\
 &\leq \phi(\max\{d(x_{m_k-1}, x_{n_k-1}), d(\gamma_{m_k-1}, \gamma_{n_k-1}), d(z_{m_k-1}, z_{n_k-1}), d(gw_{m_k}, gw_{n_k})\}) \\
 &\quad + LM(x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}) \\
 &\quad + LM(\gamma_{n_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{m_k-1}, z_{m_k-1}, w_{m_k-1}, x_{m_k-1}) \\
 &\quad + LM(z_{m_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, \gamma_{n_k-1}) \\
 &\quad + LM(w_{n_k-1}, x_{n_k-1}, \gamma_{n_k-1}, z_{n_k-1}, w_{m_k-1}, x_{m_k-1}, \gamma_{m_k-1}, z_{m_k-1}).
 \end{aligned} \tag{33}$$

Letting $k \rightarrow +\infty$ in (33) and having in mind (27) and (28), we get that

$$0 < \varepsilon \leq \lim_{t \rightarrow \varepsilon^+} \phi(t) < \varepsilon,$$

it is a contradiction. Thus, $\{gx_n\}$, $\{g\gamma_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in (X, d) .

Since (X, d) is complete, there exist $x, y, z, w \in X$ such that

$$\lim_{n \rightarrow +\infty} gx_n = x, \quad \lim_{n \rightarrow +\infty} g\gamma_n = y, \quad \lim_{n \rightarrow +\infty} gz_n = z, \quad \text{and} \quad \lim_{n \rightarrow +\infty} gw_n = w. \tag{34}$$

From (34) and the continuity of g , we have

$$\lim_{n \rightarrow +\infty} g(gx_n) = gx, \quad \lim_{n \rightarrow +\infty} g(g\gamma_n) = gy, \quad \lim_{n \rightarrow +\infty} g(gz_n) = gz, \quad \text{and} \quad \lim_{n \rightarrow +\infty} g(gw_n) = gw. \tag{35}$$

From (5) and the commutativity of F and g , we have

$$g(gx_{n+1}) = g(F(x_n, \gamma_n, z_n, w_n)) = F(gx_n, g\gamma_n, gz_n, gw_n), \tag{36}$$

$$g(g\gamma_{n+1}) = g(F(\gamma_n, z_n, w_n, x_n)) = F(g\gamma_n, gz_n, gw_n, gx_n), \tag{37}$$

$$g(gz_{n+1}) = g(F(z_n, w_n, x_n, \gamma_n)) = F(gz_n, gw_n, gx_n, g\gamma_n), \tag{38}$$

and

$$g(gw_{n+1}) = g(F(w_n, x_n, \gamma_n, z_n)) = F(gw_n, gx_n, g\gamma_n, gz_n). \tag{39}$$

Now we shall show that $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gw = F(w, x, y, z)$.

By letting $n \rightarrow +\infty$ in (36) - (39), by (34), (35) and the continuity of F , we obtain

$$\begin{aligned} gx &= \lim_{n \rightarrow +\infty} g(gx_{n+1}) = \lim_{n \rightarrow +\infty} F(gx_n, g\gamma_n, gz_n, gw_n) \\ &= F(\lim_{n \rightarrow +\infty} gx_n, \lim_{n \rightarrow +\infty} g\gamma_n, \lim_{n \rightarrow +\infty} gz_n, \lim_{n \rightarrow +\infty} gw_n) \\ &= F(x, y, z, w), \end{aligned} \tag{40}$$

$$\begin{aligned} gy &= \lim_{n \rightarrow +\infty} g(g\gamma_{n+1}) = \lim_{n \rightarrow +\infty} F(g\gamma_n, gz_n, gw_n, gx_n) \\ &= F(\lim_{n \rightarrow +\infty} g\gamma_n, \lim_{n \rightarrow +\infty} gz_n, \lim_{n \rightarrow +\infty} gw_n, \lim_{n \rightarrow +\infty} gx_n) \\ &= F(y, z, w, x), \end{aligned} \tag{41}$$

$$\begin{aligned} gz &= \lim_{n \rightarrow +\infty} g(gz_{n+1}) = \lim_{n \rightarrow +\infty} F(gz_n, gw_n, gx_n, g\gamma_n) \\ &= F(\lim_{n \rightarrow +\infty} gz_n, \lim_{n \rightarrow +\infty} gw_n, \lim_{n \rightarrow +\infty} gx_n, \lim_{n \rightarrow +\infty} g\gamma_n) \\ &= F(z, w, x, y), \end{aligned} \tag{42}$$

and

$$\begin{aligned} gw &= \lim_{n \rightarrow +\infty} g(gw_{n+1}) = \lim_{n \rightarrow +\infty} F(gw_n, gx_n, g\gamma_n, gz_n) \\ &= F(\lim_{n \rightarrow +\infty} gw_n, \lim_{n \rightarrow +\infty} gx_n, \lim_{n \rightarrow +\infty} g\gamma_n, \lim_{n \rightarrow +\infty} gz_n) \\ &= F(w, x, y, z). \end{aligned} \tag{43}$$

We have proved that F and g have a quadruple coincidence point. This completes the proof of Theorem 2.1.

In the following theorem, we omit the continuity hypothesis of F . We need the following definition.

Definition 2.5 Let (X, \leq) be a partially ordered metric set and d be a metric on X . We say that (X, d, \leq) is regular if the following conditions hold:

- (i) if non-decreasing sequence $a_n \rightarrow a$, then $a_n \leq a$ for all n ,
- (ii) if non-increasing sequence $b_n \rightarrow b$, then $b \leq b_n$ for all n .

Theorem 2.2 Let (X, \leq) be a partially ordered set and d be a metric on X such that (X, d, \leq) is regular. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property. Assume that there exist $\phi \in \Phi$ and $L \geq 0$ such that

$$d(F(x, y, z, w), F(u, v, h, l)) \leq \phi(\max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\}) + LM(x, y, z, w, u, v, h, l)$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gh$, and $gl \leq gw$. Also, suppose $F(X^4) \subset g(X)$ and $(g(X), d)$ is a complete metric space. If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0), gy_0 \geq F(y_0, z_0, w_0, x_0), gz_0 \leq F(z_0, w_0, x_0, y_0)$ and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz \quad \text{and} \quad F(w, x, y, z) = gw$$

that is, F and g have a quadruple coincidence point.

Proof. Proceeding exactly as in Theorem 2.1, we have that $\{gx_n\}, \{gy_n\}, \{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the complete metric space $(g(X), d)$. Then, there exist $x, y, z, w \in X$ such that

$$gx_n \rightarrow gx, \quad gy_n \rightarrow gy, \quad gz_n \rightarrow gz, \quad \text{and} \quad gw_n \rightarrow gw. \tag{44}$$

Since $\{gx_n\}, \{gz_n\}$ are non-decreasing and $\{gy_n\}, \{gw_n\}$ are non-increasing, then since (X, d, \leq) is regular we have

$$gx_n \leq gx, \quad gy_n \geq gy, \quad gz_n \leq gz, \quad gw_n \geq gw$$

for all n . If $gx_n = gx, gy_n = gy, gz_n = gz$, and $gw_n = gw$ for some $n \geq 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n, gy \leq gy_{n+1} \leq gy_n = gy, gz = gz_n \leq gz_{n+1} \leq gz = gz_n$, and $gw \leq gw_{n+1} \leq gw_n = gw$, which implies that

$$gx_n = gx_{n+1} = F(x_n, y_n, z_n, w_n), \quad gy_n = gy_{n+1} = F(y_n, z_n, w_n, x_n),$$

and

$$gz_n = gz_{n+1} = F(z_n, w_n, x_n, y_n), \quad gw_n = gw_{n+1} = F(w_n, w_n, y_n, z_n),$$

that is, (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g . Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n \geq 0$. By (3), consider now

$$\begin{aligned} d(gx, F(x, y, z, w)) &\leq d(gx, gx_{n+1}) + d(gx_{n+1}, F(x, y, z, w)) \\ &= d(gx, gx_{n+1}) + d(F(x_n, y_n, z_n, w_n), F(x, y, z, w)) \\ &\leq d(gx, gx_{n+1}) + \phi \left(\max \left\{ \begin{aligned} &d(gx_n, gx), d(gy_n, gy), \\ &d(gz_n, gz), d(gw_n, gw) \end{aligned} \right\} \right) + LM(x_n, y_n, z_n, w_n, x, y, z, w) \\ &< d(gx, gx_{n+1}) + \max\{d(gx_n, gx), d(gy_n, gy), d(gz_n, gz), d(gw_n, gw)\} + LM(x_n, y_n, z_n, w_n, x, y, z, w). \end{aligned} \tag{45}$$

Taking $n \rightarrow \infty$ and using (44), the quantity $M(x_n, y_n, z_n, w_n, x, y, z, w)$ tends to 0 and so the right-hand side of (45) tends to 0, hence we get that $d(gx, F(x, y, z, w)) = 0$. Thus, $gx = F(x, y, z, w)$. Analogously, one finds

$$F(x, y, z, w) = gx, \quad F(z, w, x, y) = gz, \quad \text{and} \quad F(w, x, y, z) = gw.$$

Thus, we proved that F and g have a quartet coincidence point. This completes the proof of Theorem 2.2.

Corollary 2.1 *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F is continuous and has the mixed g -monotone property. Assume also that there exist $\phi \in \Phi$ a non-decreasing function and $L \geq 0$ such that*

$$d(F(x, y, z, w), F(u, v, h, l)) \leq \phi \left(\frac{d(gx, gu) + d(gy, gv) + d(gz, gh) + d(gw, gl)}{4} \right) + LM(x, y, z, w, u, v, h, l),$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gw$, and $gl \leq gw$. Suppose $F(X^4) \subset g(X)$, g is continuous and commutes with F .

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \text{and} \quad F(w, x, y, z) = gw.$$

Proof. It suffices to remark that

$$\frac{d(gx, gu) + d(gy, gv) + d(gz, gh) + d(gw, gl)}{4} \leq \max \left\{ \begin{array}{l} d(gx, gu), d(gu, gv), \\ d(gz, gh), d(gw, gl) \end{array} \right\}.$$

Then, we apply Theorem 2.1, since ϕ is assumed to be non-decreasing.

Similarly, as an easy consequence of Theorem 2.2 we have the following corollary.

Corollary 2.2 Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d, \leq) is regular. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property. Assume also that there exist $\phi \in \Phi$ a non-decreasing function and $L \geq 0$ such that

$$d(F(x, y, z, w), F(u, v, h, l)) \leq \phi \left(\frac{d(gx, gu) + d(gy, gv) + d(gz, gh) + d(gw, gl)}{4} \right) + LM(x, y, z, w, u, v, h, l),$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gw$, and $gl \leq gw$. Also, suppose $F(X^4) \subset g(X)$ and $(g(X), d)$ is a complete metric space.

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \text{and} \quad F(w, x, y, z) = gw.$$

Corollary 2.3 Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F is continuous and has the mixed g -monotone property. Assume that there exist $k \in [0, 1)$ and $L \geq 0$ such that

$$d(F(x, y, z, w), F(u, v, h, l)) \leq k \max \left\{ \begin{array}{l} d(gx, gu), d(gy, gv), \\ d(gz, gh), d(gw, gl) \end{array} \right\} + LM(x, y, z, w, u, v, h, l),$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gw$, and $gl \leq gw$. Suppose $F(X^4) \subset g(X)$, g is continuous and commutes with F .

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0)$, $gy_0 \geq F(y_0, z_0, w_0, x_0)$, $gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad f(y, z, w, x) = gy, \quad f(z, w, x, y) = gz, \quad \text{and } F(w, x, y, z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Theorem 2.1.

Corollary 2.4 *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d, \leq) is regular. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property. Assume that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$d(F(x, y, z, w), F(u, v, h, l)) \leq k \max \left\{ \begin{array}{l} d(gx, gu), d(gy, gv), \\ d(gz, gh), d(gw, gl) \end{array} \right\} + LM(x, y, z, w, u, v, h, l),$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gw$, and $gl \leq gw$. Suppose $F(X^4) \subset g(X)$ and $(g(X), d)$ is a complete metric space.

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0), gy_0 \geq F(y_0, z_0, w_0, x_0), gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \text{and } F(w, x, y, z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Theorem 2.2.

Corollary 2.5 *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F is continuous and has the mixed g -monotone property. Assume that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$d(F(x, y, z, w), F(u, v, h, l)) \leq \frac{k}{4} \left\{ \begin{array}{l} d(gx, gu) + d(gy, gv) + \\ d(gz, gh) + d(gw, gl) \end{array} \right\} + LM(x, y, z, w, u, v, h, l),$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gw$, and $gl \leq gw$. Also, suppose $F(X^4) \subset g(X)$ and $(g(X), g)$ is continuous and commutes with F .

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0), gy_0 \geq F(y_0, z_0, w_0, x_0), gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = gx, \quad F(y, z, w, x) = gy, \quad F(z, w, x, y) = gz, \quad \text{and } F(w, x, y, z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Corollary 2.1.

Corollary 2.6 *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d, \leq) is regular. Suppose $F : X^4 \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property. Assume that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$d(F(x, y, z, w), F(u, v, h, l)) \leq \frac{k}{4} \left\{ \begin{array}{l} d(gx, gu) + d(gy, gv) + \\ d(gz, gh) + d(gw, gl) \end{array} \right\} + LM(x, y, z, w, u, v, h, l),$$

for any $x, y, z, w, u, v, h, l \in X$ for which $gx \leq gu, gv \leq gy, gz \leq gw$, and $gl \leq gw$. Suppose $F(X^4) \subset g(X)$ and $(g(X), d)$ is a complete metric space.

If there exist $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0, w_0), gy_0 \geq F(y_0, z_0, w_0, x_0), gz_0 \leq F(z_0, w_0, x_0, y_0)$, and $gw_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$F(x, \gamma, z, w) = gx, \quad F(y, z, w, r) = gy, \quad F(z, w, x, \gamma) = gz, \quad \text{and } F(w, x, \gamma, z) = gw.$$

Proof. It suffices to take $\varphi(t) = kt$ in Corollary 2.2.

Remark 1 • Corollary 2.4 of Karapinar [39] is a particular case of Corollary 2.5 by taking $L = 0$ and $g = I_X$ the identity on X .

- Corollary 2.4 of Karapinar [39] is a particular case of Corollary 2.6 by taking $L = 0$ and $g = I_X$.
- Theorem 2.6 of Berinde and Karapinar [40] is a particular case of Corollary 2.1 by taking $L = 0$.
- Theorem 2.6 of Berinde and Karapinar [40] is a particular case of Corollary 2.1 by taking $L = 0$.

Now, we shall prove the existence and uniqueness of quadruple common fixed point. For a product X^4 of a partial ordered set (X, \leq) , we define a partial ordering in the following way: For all $(x, y, z, w), (u, v, r, h) \in X^4$

$$(x, \gamma, z, w) \leq (u, v, r, h) \Leftrightarrow x \leq u, \quad \gamma \geq v, \quad z \leq r \text{ and } w \geq h \tag{46}$$

We say that (x, y, z, w) and (u, v, r, l) are comparable if

$$(x, \gamma, z, w) \leq (u, v, r, l) \quad \text{or} \quad (u, v, r, l) \leq (x, \gamma, z, w).$$

Also, we say that (x, y, z, w) is equal to (u, v, r, l) if and only if $x = u, y = v, z = r$ and $w = l$.

Theorem 2.3 In addition to hypotheses of Theorem 2.1, suppose that for all $(x, y, z, w), (u, v, r, l) \in X^4$, there exists $(a, b, c, d) \in X^4$ such that

$$(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$$

is comparable to

$$(F(x, \gamma, z, w), F(y, z, w, x), F(z, w, x, \gamma), F(w, x, \gamma, z)) \text{ and } (F(u, v, r, l), F(v, r, l, u), F(r, l, u, v), F(l, u, v, r)).$$

Then, F and g have a unique quadruple common fixed point (x, y, z, w) such that

$$\begin{aligned} x &= gx = F(x, \gamma, z, w), & \gamma &= g\gamma = F(y, z, w, x), \\ z &= gz = F(z, w, x, \gamma), & \text{and } w &= gw = F(w, x, \gamma, z). \end{aligned}$$

Proof. The set of quadruple coincidence points of F and g is not empty due to Theorem 2.1. Assume, now, (x, y, z, w) and (u, v, r, l) are two quadruple coincidence points of F and g , that is,

$$\begin{aligned} F(x, \gamma, z, w) &= gx, & F(u, v, r, l) &= gu, \\ F(y, z, w, x) &= g\gamma, & F(v, r, l, u) &= gv, \\ F(z, w, x, \gamma) &= gz, & F(r, l, u, v) &= gr, \\ F(w, x, \gamma, z) &= gw, & F(l, u, v, r) &= gl. \end{aligned} \tag{47}$$

We shall show that $(gx, g\gamma, gz, gw)$ and (gu, gv, gr, gl) are equal. By assumption, there exists $(a, b, c, d) \in X^4$ such that $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$

is comparable to $(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$ and $(F(u, v, r, l), F(v, r, l, u), F(r, l, u, v), F(l, u, v, r))$.

Define sequences $\{ga_n\}$, $\{gb_n\}$, $\{gc_n\}$, and $\{gd_n\}$ such that

$$\begin{aligned}
 a_0 &= a, b_0 = b, c_0 = c, d_0 = d \text{ and for any } n \geq 1 \\
 ga_n &= F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}), \\
 gb_n &= F(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1}), \\
 gc_n &= F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}), \\
 gd_n &= F(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}),
 \end{aligned} \tag{48}$$

for all n . Further, set $x_0 = x, y_0 = y, z_0 = z, w_0 = w$ and $u_0 = u, v_0 = v, r_0 = r, l_0 = l$ and on the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, $\{gw_n\}$ and $\{gu_n\}$, $\{gv_n\}$, $\{gr_n\}$, $\{gl_n\}$. Then, it is easy that

$$\begin{aligned}
 gx_n &= F(x, y, z, w), & gu_n &= F(u, v, r, l), \\
 gy_n &= F(y, z, w, x), & gv_n &= F(v, r, l, u), \\
 gz_n &= F(z, w, x, y), & gr_n &= F(r, l, u, v), \\
 gw_n &= F(w, x, y, z), & gl_n &= F(l, u, v, r)
 \end{aligned} \tag{49}$$

for all $n \geq 1$. Since

$$\begin{aligned}
 (F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z)) &= (gx_1, gy_1, gz_1, gw_1) \\
 &= (gx, gy, gz, gw)
 \end{aligned}$$

is comparable to

$$(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c)) = (ga_1, gb_1, gc_1, gd_1),$$

then it is easy to show $(gx, gy, gz, gw) \geq (ga_1, gb_1, gc_1, gd_1)$. Recursively, we get that

$$(ga_n, gb_n, gc_n, gd_n) \leq (gx, gy, gz, gw) \text{ for all } n. \tag{50}$$

From (2) and (47), it is obvious that

$$\begin{aligned}
 M(a_n, b_n, c_n, d_n, x, y, z, w) &= M(y, z, w, x, b_n, c_n, d_n, a_n) \\
 &= M(c_n, d_n, a_n, b_n, z, w, x, y) = M(w, x, y, z, d_n, a_n, b_n, c_n) = 0.
 \end{aligned} \tag{51}$$

By (50), (51), and (3), we have

$$\begin{aligned}
 d(ga_{n+1}, gx) &= d(F(a_n, b_n, c_n, d_n), F(x, y, z, w)) \\
 &\leq \phi(\max\{d(gx, ga_n), d(gy, gb_n), d(gz, gc_n), d(gw, gd_n)\}) \\
 &\quad + LM(a_n, b_n, c_n, d_n, x, y, z, w) \\
 &= \phi(\max\{d(gx, ga_n), d(gy, gb_n), d(gz, gc_n), d(gw, gd_n)\}),
 \end{aligned} \tag{52}$$

$$\begin{aligned}
 d(gy, gb_{n+1}) &= d(F(y, z, w, x), F(b_n, c_n, d_n, a_n)) \\
 &\leq \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}) \\
 &\quad + LM(y, z, w, x, b_n, c_n, d_n, a_n) \\
 &= \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}),
 \end{aligned} \tag{53}$$

$$\begin{aligned}
 d(gc_{n+1}, gz) &= d(F(c_n, d_n, a_n, b_n), F(z, w, x, y)) \\
 &\leq \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}) \\
 &\quad + LM(c_n, d_n, a_n, b_n, z, w, x, y) \\
 &= \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\})
 \end{aligned} \tag{54}$$

and

$$\begin{aligned}
 d(gw, gd_{n+1}) &= d(F(w, x, y, z), F(d_n, a_n, b_n, c_n)) \\
 &\leq \phi(\max\{d(ga_n, gx), d(gb_n, gy), d(gc_n, gz), d(gd_n, gw)\}) \\
 &\quad + LM(w, x, y, z, d_n, a_n, b_n, c_n) \\
 &= \phi(\max\{d(gd_n, gw), d(ga_n, gx), d(gb_n, gy), d(gc_n, gz)\}).
 \end{aligned} \tag{55}$$

From (52)-(55), it follows that

$$\max \left\{ \begin{array}{l} d(gz, gc_{n+1}), d(gy, gb_{n+1}), \\ d(gx, ga_{n+1}), d(gw, gd_{n+1}) \end{array} \right\} \leq \phi \left(\max \left\{ \begin{array}{l} d(gz, gc_n), d(gy, gb_n), \\ d(gx, ga_n), d(gw, gd_n) \end{array} \right\} \right). \tag{56}$$

Therefore, for each $n \geq 1$,

$$\max \left\{ \begin{array}{l} d(gz, gc_n), d(gy, gb_n), \\ d(gx, ga_n), d(gw, gd_n) \end{array} \right\} \leq \phi^n \left(\max \left\{ \begin{array}{l} d(gz, gc_0), d(gy, gb_0), \\ d(gx, ga_0), d(gw, gd_0) \end{array} \right\} \right). \tag{57}$$

It is known that $\phi(t) < t$ and $\lim_{r \rightarrow t^+} \phi(r) < t$ imply $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for each $t > 0$. Thus, from (57)

$$\lim_{n \rightarrow \infty} \max\{d(gz, gc_n), d(gy, gb_n), d(gx, ga_n), d(gw, gd_n)\} = 0.$$

This yields that

$$\lim_{n \rightarrow \infty} d(gx, ga_n) = 0, \quad \lim_{n \rightarrow \infty} d(gy, gb_n) = 0, \quad \lim_{n \rightarrow \infty} d(gz, gc_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gw, gd_n) = 0. \tag{58}$$

Analogously, we may show that

$$\lim_{n \rightarrow \infty} d(gu, ga_n) = 0, \quad \lim_{n \rightarrow \infty} d(gv, gb_n) = 0, \quad \lim_{n \rightarrow \infty} d(gr, gc_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(gl, gd_n) = 0. \tag{59}$$

Combining (58) and (59) yields that (gx, gy, gz, gw) and (gu, gv, gr, gl) are equal.

Since $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, and $gw = F(w, x, y, z)$, by commutativity of F and g we have

$$\begin{aligned}
 gx' &= g(gx) = g(F(x, y, z, w)) = F(gx, gy, gz, gw), \\
 gy' &= g(gy) = g(F(y, z, w, x)) = F(gy, gz, gw, gx), \\
 gz' &= g(gz) = g(F(z, w, x, y)) = F(gz, gw, gx, gy)
 \end{aligned}$$

and

$$gw' = g(gw) = g(F(w, x, y, z)) = F(gw, gx, gy, gz)$$

where $gx = x'$, $gy = y'$, $gz = z'$, and $gw = w'$. Thus, (x', y', z', w') is a quadruple coincidence point of F and g . Consequently, (gx', gy', gz', gw') and (gx, gy, gz, gw) are equal. We deduce

$$gx' = gx = x', \quad gy' = gy = y' \quad \text{and} \quad gz' = gz = z', \quad gw' = gw = w'.$$

Therefore, (x', y', z', w') is a quadruple common fixed of F and g . Its uniqueness follows easily from (3).

Example 2.1 Let $X = \mathbb{R}$ be endowed with the usual ordering and the usual metric, which is complete.

Let $g: X \rightarrow X$ and $F: X^4 \rightarrow X$ be defined by

$$g(x) = \frac{3}{4}x, \quad F(x, y, z, w) = \frac{x - y + z - w}{8}, \quad \text{for all } x, y, z, w \in X$$

Take $\varphi: [0, \infty) \rightarrow [0, \infty)$ be given by $\varphi(t) = \frac{2}{3}t$ for all $t \in [0, \infty)$.

We will check that the contraction (3) is satisfied for all $x, y, z, w, u, v, h, l \in X$ satisfying $gx \leq gu, gv \leq gy, gz \leq gh, \text{ and } gl \leq gw$. In this case, we have

$$\begin{aligned} d(F(x, y, z, w), F(u, v, h, l)) &= \frac{u - x}{8} + \frac{y - v}{8} + \frac{h - z}{8} + \frac{w - l}{8} \\ &\leq \frac{1}{2}[\max\{(u - x), (y - v), (h - z), (w - l)\}] \\ &= \frac{2}{3} \max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\} \\ &\leq \varphi(\max\{d(gx, gu), d(gy, gv), d(gz, gh), d(gw, gl)\}) \\ &\quad + LM(x, y, z, w, u, v, h, l), \end{aligned}$$

for arbitrary $L \geq 0$.

It is obvious that the other hypotheses of Theorem 2.3 are satisfied. We deduce that $(0, 0, 0, 0)$ is the unique quadruple common fixed point of F and g .

3 Application to matrix equations

In this section, we study the existence and uniqueness of solutions (X, Y, Z, T) to the system of matrix equations

$$\begin{cases} X = Q + A_1^* X A_1 - B_1^* Y B_1 + A_2^* Z A_2 - B_2^* T B_2 \\ Y = Q + A_1^* Y A_1 - B_1^* Z B_1 + A_2^* T A_2 - B_2^* X B_2 \\ Z = Q + A_1^* Z A_1 - B_1^* T B_1 + A_2^* X A_2 - B_2^* Y B_2 \\ T = Q + A_1^* T A_1 - B_1^* X B_1 + A_2^* Y A_2 - B_2^* Z B_2, \end{cases} \quad (60)$$

where $A_1, A_2, B_1, B_2 \in \mathcal{M}(n)$: the set of all $n \times n$ matrices, $Q \in \mathcal{P}(n)$: the set of all $n \times n$ positive definite matrices, and $\mathcal{H}(n)$ is the set of all $n \times n$ Hermitian matrices.

We endow $\mathcal{H}(n)$ with the partial order \preceq given by

$$M, N \in \mathcal{H}(n), \quad M \preceq N \Leftrightarrow N - M \in \mathcal{P}(n).$$

For a fixed $P \in \mathcal{P}(n)$, we consider

$$\|H\|_{1,P} = \text{tr}(P^{\frac{1}{2}} H P^{\frac{1}{2}}).$$

for all $H \in \mathcal{H}(n)$, where tr is the trace operator. The space $\mathcal{H}(n)$ equipped with the metric induced by $\|\cdot\|_{1,P}$ is a complete metric space for any positive definite matrix P (see [42]).

The following lemma will be useful for our application.

Lemma 3.1 Let $A \succcurlyeq 0$ and $B \succcurlyeq 0$ be $n \times n$ matrices. Then, we have

$$0 \leq \text{tr}(AB) = \text{tr}(BA) \leq \|A\| \text{tr}(B),$$

where $\|\cdot\|$ is the spectral norm.

Theorem 3.1 Suppose that there exists $P \in \mathcal{P}(n)$ such that

$$k = 4 \max\{\|P^{-\frac{1}{2}}A_1^*PA_1P^{-\frac{1}{2}}\|, \|P^{-\frac{1}{2}}A_2^*PA_2P^{-\frac{1}{2}}\|, \|P^{-\frac{1}{2}}B_1^*PB_1P^{-\frac{1}{2}}\|, \|P^{-\frac{1}{2}}B_2^*PB_2P^{-\frac{1}{2}}\|\} < 1. \quad (61)$$

Suppose also that

$$0 \preccurlyeq \sum_{i=1}^2 A_i^*QA_i \text{ and } Q \preccurlyeq \sum_{i=1}^2 B_i^*QB_i. \quad (62)$$

Then, the system (60) has one and only one solution $(X_1, X_2, X_3, X_4) \in (\mathcal{H}(n))^4$.

Proof. Consider the mappings $F : (\mathcal{H}(n))^4 \rightarrow \mathcal{H}(n)$ and $g : \mathcal{H}(n) \rightarrow \mathcal{H}(n)$ defined by

$$F(X_1, X_2, X_3, X_4) = Q + A_1^*X_1A_1 - B_1^*X_2B_1 + A_2^*X_3A_2 - B_2^*X_4B_2 \text{ and } gX = X,$$

for all $X, X_i \in \mathcal{H}(n) \ i = 1, \dots, 4$.

For all $X_i, Y_i \in \mathcal{H}(n) \ i = 1, \dots, 4$ with $gX_1 \preccurlyeq gY_1, gY_2 \preccurlyeq gX_2, gX_3 \preccurlyeq gY_3$ and $gY_4 \preccurlyeq gX_4$, by using Lemma 3.1, we have

$$\begin{aligned} & \|F(Y_1, Y_2, Y_3, Y_4) - F(X_1, X_2, X_3, X_4)\|_{1,P} \\ &= \|A_1^*(Y_1 - X_1)A_1 - B_1^*(Y_2 - X_2)B_1 + A_2^*(Y_3 - X_3)A_2 - B_2^*(Y_4 - X_4)B_2\|_{1,P} \\ &= \text{tr} \left[P^{\frac{1}{2}} \left(A_1^*(Y_1 - X_1)A_1 - B_1^*(Y_2 - X_2)B_1 + A_2^*(Y_3 - X_3)A_2 - B_2^*(Y_4 - X_4)B_2 \right) P^{\frac{1}{2}} \right] \\ &= \text{tr}[A_1PA_1^*(Y_1 - X_1)] + \text{tr}[B_1PB_1^*(X_2 - Y_2)] + \text{tr}[A_2PA_2^*(Y_3 - X_3)] + \text{tr}[B_2PB_2^*(X_4 - Y_4)] \\ &= \text{tr}[A_1PA_1^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(Y_1 - X_1)P^{\frac{1}{2}}P^{-\frac{1}{2}}] + \text{tr}[B_1PB_1^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(X_2 - Y_2)P^{\frac{1}{2}}P^{-\frac{1}{2}}] \\ &+ \text{tr}[A_2PA_2^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(Y_3 - X_3)P^{\frac{1}{2}}P^{-\frac{1}{2}}] + \text{tr}[B_2PB_2^*P^{-\frac{1}{2}}P^{\frac{1}{2}}(X_4 - Y_4)P^{\frac{1}{2}}P^{-\frac{1}{2}}] \\ &\leq \|P^{-\frac{1}{2}}A_1PA_1^*P^{-\frac{1}{2}}\| \text{tr}(P^{\frac{1}{2}}(Y_1 - X_1)P^{\frac{1}{2}}) + \|P^{-\frac{1}{2}}B_1PB_1^*P^{-\frac{1}{2}}\| \text{tr}(P^{\frac{1}{2}}(X_2 - Y_2)P^{\frac{1}{2}}) \\ &+ \|P^{-\frac{1}{2}}A_2PA_2^*P^{-\frac{1}{2}}\| \text{tr}(P^{\frac{1}{2}}(Y_3 - X_3)P^{\frac{1}{2}}) + \|P^{-\frac{1}{2}}B_2PB_2^*P^{-\frac{1}{2}}\| \text{tr}(P^{\frac{1}{2}}(X_4 - Y_4)P^{\frac{1}{2}}) \\ &= \|P^{-\frac{1}{2}}A_1PA_1^*P^{-\frac{1}{2}}\| \|Y_1 - X_1\|_{1,P} + \|P^{-\frac{1}{2}}B_1PB_1^*P^{-\frac{1}{2}}\| \|X_2 - Y_2\|_{1,P} \\ &+ \|P^{-\frac{1}{2}}A_2PA_2^*P^{-\frac{1}{2}}\| \|Y_3 - X_3\|_{1,P} + \|P^{-\frac{1}{2}}B_2PB_2^*P^{-\frac{1}{2}}\| \|X_4 - Y_4\|_{1,P} \\ &\leq \frac{k}{4} (\|gY_1 - gX_1\|_{1,P} + \|gX_2 - gY_2\|_{1,P} + \|gY_3 - gX_3\|_{1,P} + \|gX_4 - gY_4\|_{1,P}). \end{aligned}$$

Thus, we proved that the contractive condition given in Corollary 2.5 is satisfied for all $L \geq 0$. Moreover, from (62), we have letting $gQ \preccurlyeq F(Q, 0, Q, 0)$ and $g0 \succcurlyeq F(0, Q, 0, Q)$. Applying Corollary 2.5, F and g have a coupled coincidence point (and so a quadrupled fixed point since g is the identity on $\mathcal{H}(n)$). Then, there exist $X_1, X_2, X_3, X_4 \in \mathcal{H}(n)$ such that

$$\begin{aligned} F(X_1, X_2, X_3, X_4) &= X_1, & F(X_2, X_3, X_4, X_1) &= X_2, \\ F(X_3, X_4, X_1, X_2) &= X_3 & \text{and} & & F(X_4, X_1, X_2, X_3) &= X_4. \end{aligned}$$

On the other hand, for all $X, Y \in \mathcal{H}(n)$ there is a greatest lower bound and a least upper bound, hence it is obvious that the hypotheses of Theorem 2.3 hold, so the

uniqueness of that quadrupled fixed point of F , which is also the unique solution of the system (60).

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Authors' contributions

All authors have contributed in obtaining the new results presented in this article. All authors read and approve the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 19 December 2011 Accepted: 1 May 2012 Published: 1 May 2012

References

1. Turinici, M: Abstract comparison principles and multivariable Gronwall-Bellman in-equalities. *J Math Anal Appl.* **117**, 100–127 (1986). doi:10.1016/0022-247X(86)90251-9
2. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some application to matrix equations. *Proc Am Math Soc.* **132**, 1435–1443 (2004). doi:10.1090/S0002-9939-03-07220-4
3. Abbas, M, Sintunavarat, W, Kumam, P: Coupled fixed point in partially ordered G-metric spaces. *Fixed Point Theory Appl.* **2012**, 31 (2012). doi:10.1186/1687-1812-2012-31
4. Altun, I, Simsek, H: Some fixed point theorems on ordered metric spaces and application. *Fixed Point Theory Appl.* **2010** (2010). Article ID 621492
5. Aydi, H: Coincidence and common fixed point results for contraction type maps in partially ordered metric spaces. *Int J Math Anal.* **5**(13):631–642 (2011)
6. Choudhury, BS, Metiya, N, Kundu, A: Coupled coincidence point theorems in ordered metric spaces. *Ann Univ Ferrara.* **57**, 1–16 (2011). doi:10.1007/s11565-011-0117-5
7. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonlinear Anal.* **73**, 2524–2531 (2010). doi:10.1016/j.na.2010.06.025
8. Harjani, J, Sadarangani, K: Fixed point theorems for weakly contractive mappings in partially ordered sets. *Nonlinear Anal.* **71**, 3403–3410 (2009). doi:10.1016/j.na.2009.01.240
9. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. *Nonlinear Anal.* **74**, 983–992 (2011). doi:10.1016/j.na.2010.09.055
10. Nashine, HK, Samet, B: Fixed point results for mappings satisfying (ψ, Φ) -weakly contractive condition in partially ordered metric spaces. *Nonlinear Anal.* **74**, 2201–2209 (2011). doi:10.1016/j.na.2010.11.024
11. Nashine, HK, Shatanawi, W: Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces. *Comput Math Appl.* **62**, 1984–1993 (2011). doi:10.1016/j.camwa.2011.06.042
12. Nieto, JJ, López, RR: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order.* **22**, 223–239 (2005). doi:10.1007/s11083-005-9018-5
13. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. *Nonlinear Anal.* **7**(12):4508–4517 (2010)
14. Bhaskar, TG, Lakshmikantham, V: Fixed point theory in partially ordered metric spaces and applications. *Nonlinear Anal.* **65**, 1379–1393 (2006). doi:10.1016/j.na.2005.10.017
15. Lakshmikantham, V, Ćirić, LjB: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal.* **70**, 4341–4349 (2009). doi:10.1016/j.na.2008.09.020
16. Abbas, M, Khan, MA, Radenović, S: Common coupled fixed point theorem in cone metric space for w -compatible mappings. *Appl Math Comput.* **217**, 195–202 (2010). doi:10.1016/j.amc.2010.05.042
17. Aydi, H: Some coupled fixed point results on partial metric spaces. *Int J Math Math Sci.* **2011** (2011). Article ID 647091, 11
18. Aydi, H, Samet, B, Vetro, C: Coupled fixed point results in cone metric spaces for \tilde{w} -compatible mappings. *Fixed Point Theory Appl.* **2011**, 27 (2011). doi:10.1186/1687-1812-2011-27
19. Aydi, H, Karapinar, E, Shatanawi, W: Coupled fixed point results for (ψ, Φ) -weakly contractive condition in ordered partial metric spaces. *Comput Math Appl.* **62**, 4449–4460 (2011). doi:10.1016/j.camwa.2011.10.021
20. Aydi, H, Postolache, M, Shatanawi, W: Coupled fixed point results for (ψ, Φ) -weakly contractive mappings in ordered G-metric spaces. *Comput Math Appl.* **63**, 298–309 (2012). doi:10.1016/j.camwa.2011.11.022
21. Aydi, H, Damjanović, B, Samet, B, Shatanawi, W: Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces. *Math Comput Model.* **54**, 2443–2450 (2011). doi:10.1016/j.mcm.2011.05.059
22. Aydi, H, Abbas, M, Postolache, M: Coupled coincidence points for hybrid pair of mappings via mixed monotone property. *J Adv Math Stud.* **5**(1):118–126 (2012)
23. Cho, YJ, Rhoades, BE, Saadati, R, Samet, B, Shatanawi, W: Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type. *Fixed Point Theory Appl.* **2012**, 8 (2012). doi:10.1186/1687-1812-2012-8
24. Karapinar, E: Couple fixed point on cone metric spaces. *Gazi Univ J Sci.* **24**(1):51–58 (2011)
25. Karapinar, E: Coupled fixed point theorems for nonlinear contractions in cone metric spaces. *Comput Math Appl.* **59**(12):3656–3668 (2010). doi:10.1016/j.camwa.2010.03.062
26. Sintunavarat, W, Cho, YJ, Kumam, P: Common fixed point theorems for c -distance in ordered cone metric spaces. *Comput Math Appl.* **62**, 1969–1978 (2011). doi:10.1016/j.camwa.2011.06.040

27. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. *Fixed Point Theory Appl.* **2011**, 81 (2011). doi:10.1186/1687-1812-2011-81
28. Shatanawi, W, Samet, B, Abbas, M: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. *Math Comput Model.* **55**, 680–687 (2012). doi:10.1016/j.mcm.2011.08.042
29. Shatanawi, W: Coupled fixed point theorems in generalized metric spaces. *Hacetatepe J Math Stat.* **40**(3):441–447 (2011)
30. Shatanawi, W, Abbas, M, Nazir, T: Common coupled coincidence and coupled fixed point results in two generalized metric spaces. *Fixed point Theory Appl.* **2011**, 80 (2011). doi:10.1186/1687-1812-2011-80
31. Shatanawi, W, Samet, B: On (ψ, Φ) -weakly contractive condition in partially ordered metric spaces. *Comput Math Appl.* **62**, 3204–3214 (2011). doi:10.1016/j.camwa.2011.08.033
32. Shatanawi, W: Fixed point theorems for nonlinear weakly C-contractive mappings in metric spaces. *Math Comput Model.* **54**, 2816–2826 (2011). doi:10.1016/j.mcm.2011.06.069
33. Shatanawi, W: Partially ordered cone metric spaces and coupled fixed point results. *Comput Math Appl.* **60**, 2508–2515 (2010). doi:10.1016/j.camwa.2010.08.074
34. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. *Nonlinear Anal.* **74**(15):4889–4897 (2011). doi:10.1016/j.na.2011.03.032
35. Aydi, H, Karapinar, E, Postolache, M: Tripled coincidence point theorems for weak φ -contractions in partially ordered metric spaces. *Fixed Point Theory Appl.* **2012**, 44 (2012). doi:10.1186/1687-1812-2012-44
36. Samet, B, Vetro, C: Coupled fixed point, F -invariant set and fixed point of N -order. *Ann Funct Anal.* **1**(2):46–56 (2010)
37. Karapinar, E: Quartet fixed point for nonlinear contraction. <http://arxiv.org/abs/1106.5472>
38. Karapinar, E, Luong, NV: Quadruple fixed point theorems for nonlinear contractions. *Comput Math Appl.* doi:10.1016/j.camwa.2012.02.061
39. Karapinar, E: Quadruple fixed point theorems for weak Φ -contractions. *ISRN Mathematical Analysis* **2011** (2011). Article ID 989423, 16
40. Karapinar, E, Berinde, V: Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Banach J Math Anal.* **6**(1):74–89 (2012)
41. Karapinar, E: A new quartet fixed point theorem for nonlinear contractions. *JP J Fixed Point Theory Appl.* **6**(2):119–135 (2011)
42. Gohberg, I, Goldberg, S: *Basic Operator Theory*. Birkhäuser, Boston, MA (1981)

doi:10.1186/1687-1812-2012-71

Cite this article as: Mustafa et al.: Mixed g -monotone property and quadruple fixed point theorems in partially ordered metric spaces. *Fixed Point Theory and Applications* 2012 **2012**:71.

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