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MIXED GENERALIZED FRACTIONAL BROWNIAN MOTION

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Abstract. To extend several known centered Gaussian processes, we introduce a new centered mixed self-similar Gaussian process called the mixed generalized fractional Brownian motion, which could serve as a good model for a larger class of natural phenomena. This process generalizes both the well-known mixed fractional Brownian motion introduced by Cheridito [7] and the generalized fractional Brownian motion introduced by Zili [29]. We study its main stochastic properties, its non-Markovian and non-stationarity characteristics and the conditions under which it is not a semimartingale. We prove the long-range dependence properties of this process.

1. Introduction

Fractional Brownian motion on the whole real line (fBm for short) $B^H = \{B_t^H,\, t \in \mathbb{R}\}$ of Hurst parameter H is the best known centered Gaussian process with long-range dependence. Its covariance function is

$$\mathrm{Cov}(B_t^H,\, B_s^H) = \frac{1}{2}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}], \tag{1.1}$$

where H is a real number in $(0,1)$ and the case $H = \frac{1}{2}$ corresponds to the Brownian motion. It is the unique continuous Gaussian process starting from zero, the self-similarity and stationarity of the increments are two main properties for which fBm enjoyed successes as modeling tool in finance and telecommunications. Researchers have applied fractional Brownian motion to a wide range of problems, such as bacterial colonies, geophysical data, electrochemical deposition, particle diffusion, DNA sequences and stock market indicators [20, 22]. In particular, computer science applications of fBm include modeling network traffic and generating graphical landscapes [21]. The fBm was investigated in many papers (e.g. [2, 12, 16, 17, 18, 19]). The main difference between fBm and regular Brownian motion is that the increments in Brownian motion are independent, increments for fBm are not.

In [4], the authors suggested another kind of extension of the Brownian motion, called the sub-fractional Brownian motion (sfBm for short), which preserves most properties of the fBm, but not the stationarity of the increments. It is a centered

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Gaussian process $\xi^H = \left\{\xi_t^H,\ t \geq 0\right\}$, defined by:

$$\xi_t^H = \frac{B_t^H + B_{-t}^H}{\sqrt{2}}, \quad t \geq 0, \tag{1.2}$$

where $H \in (0,1)$. The case $H = \frac{1}{2}$ corresponds to the Brownian motion.

The sfBm is intermediate between Brownian motion and fractional Brownian motion in the sense that it has properties analogous to those of fBm, self-similarity, not Markovian but the increments on nonoverlapping intervals are more weakly correlated, and their covariance decays polynomially at a higher rate in comparison with fBm (for this reason in [4] is called sfBm). So, the sfBm does not generalize the fBm. The sfBm was investigated in many papers (e.g. [3, 4, 24, 26]).

An extension of the sfBm was introduced by Zili in [28] as a linear combination of a finite number of independent sub-fractional Brownian motions. It was called the mixed sub-fractional Brownian motion (msfBm for short). The msfBm is a centered mixed self-similar Gaussian process and does not have stationary increments. The msfBm do not generalize the fBm.

In [29], Zili introduced new model called the generalized fractional Brownian motion (gfBm for short) which is an extension of both sub-fractional Brownian motion and fractional Brownian motion. A gfBm with parameters $a, b,$ *and* H, is a process $Z^H = \left\{Z_t^H(a,b), t \geq 0\right\}$ defined by

$$Z_t^H(a,b) = aB_t^H + bB_{-t}^H, \quad t \geq 0. \tag{1.3}$$

The gfBm was investigated in [10, 30]. The gfBm generalize the sfBm but not the mixed fractional Brownian motion.

The mixed fractional Brownian motion (mfBm for short) is a linear combination between a Brownian motion and an independent fractional Brownian motion of Hurst parameter H. It was introduced by Cheridito [7] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. The mfBm is a centered Gaussian process starting from zero with covariance function

$$\mathrm{Cov}(N_t^H(a,b), N_s^H(a,b)) = a^2(t \wedge s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{1.4}$$

with $H \in (0,1)$. When $a = 1$ and $b = 0$, the mfBm is the Brownian motion and when $a = 0$ and $b = 1$, is the fBm. We refer also to [1, 7, 9, 25, 27] for further information on this process.

In this paper, we introduce a new stochastic model, which we call the mixed generalized fractional Brownian motion.

Definition 1.1. A *mixed generalized fractional Brownian motion* (mgfBm for short) of parameters a, b, c and $H \in (0,1)$ is a centered Gaussian process

$$M^H(a,b,c) = \{M_t^H(a,b,c),\ t \geq 0\},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the covariance function

$$C(t,s) = a^2(t \wedge s) + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc(t+s)^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H}, \tag{1.5}$$

where $t \wedge s = \frac{1}{2}\left(t + s - |t-s|\right)$.

The mgfBm is completely different from all the extensions mentioned above. The process $M^H(a,b,c)$ is motivated by the fact that this process already introduced for specific values of a, b and c. Indeed $M^H(a,b,0)$ is the mixed fractional Brownian motion and $M^H(0,b,c)$, is the generalized fractional Brownian motion. This why we will name $M^H(a,b,c)$ the mixed generalized fractional Brownian motion. It allows to deal with a larger class of modeled natural phenomena, including those with stationary or non-stationary increments.

Our goal is to study the main stochastic properties of this new model, paying attention to the long-range dependence, self-similarity, increment stationary, Markovity and semi-martingale properties.

2. The Main Properties

Existence of the mixed generalized fractional Brownian motion $M^H(a,b,c)$ for any $H \in (0,1)$ can be shown in the following way: consider the process

$$M_t^H(a,b,c) = aB_t + bB_t^H + cB_{-t}^H, \quad t \geq 0, \tag{2.1}$$

where $B = \{B_t, \, t \in \mathbb{R}\}$ is a Brownian motion and $B^H = \{B_t^H, \, t \in \mathbb{R}\}$ is an independent fractional Brownian motion with Hurst parameter $H \in (0,1)$.

Using (1.1) and since B and B^H are independent we obtain the following lemma.

Lemma 2.1. *For all $s, \ t \geq 0$, the process (2.1) is a centered Gaussian process with covariance function given by (1.5).*

Proof. Let $s, \ t \geq 0$ and $C(t,s) = Cov\left(M_t^H(a,b,c), M_s^H(a,b,c)\right)$. Then

$$\begin{aligned}
C(t,s) &= Cov\big(\left(aB_t + bB_t^H + cB_{-t}^H\right), \left(aB_s + bB_s^H + cB_{-s}^H\right) \big) \\
&= a^2(t \wedge s) + b^2 Cov(B_t^H, B_s^H) + bc Cov(B_t^H, B_{-s}^H) + cb Cov(B_{-t}^H, B_s^H) \\
&\quad + c^2 Cov(B_{-t}^H, B_{-s}^H) \\
&= a^2(t \wedge s) + \frac{b^2}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) + \frac{bc}{2}\left(t^{2H} + s^{2H} - |t+s|^{2H}\right) \\
&\quad + \frac{cb}{2}\left(t^{2H} + s^{2H} - |-(t+s)|^{2H}\right) + \frac{c^2}{2}\left(t^{2H} + s^{2H} - |-(t-s)|^{2H}\right) \\
&= a^2(t \wedge s) + \frac{b^2}{2}t^{2H} + \frac{b^2}{2}s^{2H} - \frac{b^2}{2}|t-s|^{2H} + \frac{bc}{2}t^{2H} + \frac{bc}{2}s^{2H} \\
&= a^2(t \wedge s) + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc|t+s|^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H}.
\end{aligned}$$

Hence the covariance function of the process (2.1) is precisely $C(t,s)$ given by (1.5). Therefore the $M^H(a,b,c)$ exists. □

Remark 2.1. *Some special cases of the mixed generalized fractional Brownian motion:*

1. *If $a = 0, b = 1, c = 0$, then $M^H(0,1,0)$ is a fBm.*
2. *If $a = 0, b = c = \frac{1}{\sqrt{2}}$, then $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is a sfBm.*
3. *If $a = 1, b = 0, c = 0$, then $M^H(1,0,0)$ is a Bm.*
4. *If $a = 0$, then $M^H(0,b,c)$, is a gfBm.*

(5) *If* $c=0$, *then* $M^H(a,b,0)$, *is a mfBm.*
(6) *If* $b=c$, *then* $M^H(a,\frac{b}{\sqrt{2}},\frac{b}{\sqrt{2}})$, *is a smfBm.*

So the mixed generalized fractional Brownian motion is, at the same, a generalization of the fractional Brownian motion, sub-fractional Brownian motion, the sub-mixed fractional Brownian motion, generalized fractional Brownian motion, mixed fractional Brownian motion and of course of the standard Brownian motion.

Proposition 2.1. *The mgfBm satisfies the following properties:*

(1) *For all* $t \geq 0$,
$$E\left(M_t^H(a,b,c)\right)^2 = a^2 t + \left(b^2+c^2-(2^{2H}-2)bc\right)t^{2H}.$$
(2) *Let* $0 \leq s < t$ *and* $\alpha(t,s) = E\left(M_t^H(a,b,c) - M_s^H(a,b,c))^2\right)$. *Then*
$$\begin{aligned} E\left(M_t^H(a,b,c) - M_s^H(a,b,c)\right)^2 &= a^2|t-s| - 2^{2H}bc(t^{2H}+s^{2H}) \\ &\quad +(b^2+c^2)|t-s|^{2H} + 2bc|t+s|^{2H}. \end{aligned}$$
(3) *We have for all* $0 \leq s < t$,
$$a^2(t-s) + \gamma_{(b,c,H)}(t-s)^{2H} \leq \alpha(t,s) \leq a^2(t-s) + \nu_{(b,c,H)}(t-s)^{2H},$$
where
$$\gamma_{(b,c,H)} = (b^2+c^2-2bc(2^{2H-1}-1))\,\mathbf{1}_{\mathcal{C}}(b,c,H) + (b^2+c^2)\,\mathbf{1}_{\mathcal{D}}(b,c,H),$$
$$\nu_{(b,c,H)} = (b^2+c^2)\,\mathbf{1}_{\mathcal{C}}(a,b,H) + (b^2+c^2-2bc(2^{2H-1}-1))\,\mathbf{1}_{\mathcal{D}}(b,c,H),$$
$$\mathcal{C} = \{(b,c,H) \in \mathbb{R}^2\times]0,1[;\ (H > \frac{1}{2},\ bc \geq 0)\ or\ (H < \frac{1}{2},\ bc \leq 0)\},$$
and
$$\mathcal{D} = \{(b,c,H) \in \mathbb{R}^2\times]0,1[;\ (H > \frac{1}{2},\ bc \leq 0)\ or\ (H < \frac{1}{2},\ bc \geq 0)\}.$$

Proof. (1) It is a direct consequence of (1.5).

(2) Let $0 \leq s < t$ and $\alpha(t,s) = E\left(M_t^H(a,b,c) - M_s^H(a,b,c)\right)^2$. Then
$$\begin{aligned} \alpha(t,s) &= E\left(M_t^H(a,b,c)\right)^2 + E\left(M_s^H(a,b,c)\right)^2 - 2E\left(M_t^H(a,b,c)M_s^H(a,b,c)\right) \\ &= a^2t + b^2t^{2H} + 2bct^{2H} - 2^{2H}bct^{2H} + c^2t^{2H} + a^2s + b^2s^{2H} + 2bcs^{2H} \\ &\quad -2^{2H}bcs^{2H} + c^2s^{2H} - 2a^2(t\wedge s) - b^2t^{2H} - b^2s^{2H} + b^2|t-s|^{2H} \\ &\quad -bct^{2H} - bcs^{2H} + bc|t+s|^{2H} - cbt^{2H} - cbs^{2H} + cb|t+s|^{2H} - c^2t^{2H} \\ &\quad -c^2s^{2H} + c^2|t-s|^{2H} \\ &= a^2(t+s) - 2^{2H}bc(t^{2H}+s^{2H}) - 2a^2(t\wedge s) + (b^2+c^2)|t-s|^{2H} \\ &\quad +2bc|t+s|^{2H} \\ &= a^2|t-s| - 2^{2H}bc(t^{2H}+s^{2H}) + (b^2+c^2)|t-s|^{2H} + 2bc|t+s|^{2H}. \end{aligned}$$
(3) It is a direct consequence of the second item of Proposition 2.1 and Lemma 3 in [29].

□

Proposition 2.2. *For all* $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ *and* $H \in (0,1) \setminus \{\frac{1}{2}\}$*, the mgfBm is not a self-similar process.*

Proof. This follows from the fact that, for fixed $h > 0$, the processes $\left\{M_{ht}^H(a,b,c),\ t \geq 0\right\}$ and $\left\{h^H M_t^H(a,b,c),\ t \geq 0\right\}$ are Gaussian, centered, but don't have the same covariance function. Indeed

$$\begin{aligned}
C\,(ht,hs) &= a^2(ht \wedge hs) + \frac{b^2}{2}\left((ht)^{2H} + (hs)^{2H} - |ht-hs|^{2H}\right) \\
&\quad + \frac{bc}{2}\left((ht)^{2H} + (hs)^{2H} - |ht+hs|^{2H}\right) \\
&\quad + \frac{cb}{2}\left((ht)^{2H} + (hs)^{2H} - |-(ht+hs)|^{2H}\right) \\
&\quad + \frac{c^2}{2}\left((ht)^{2H} + (hs)^{2H} - |-(ht-hs)|^{2H}\right) \\
&= a^2(ht \wedge hs) + \frac{b^2}{2}(ht)^{2H} + \frac{b^2}{2}(hs)^{2H} - \frac{b^2}{2}|ht-hs|^{2H} \\
&\quad + \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} - \frac{bc}{2}|ht+hs|^{2H} + \frac{bc}{2}(ht)^{2H} + \frac{bc}{2}(hs)^{2H} \\
&\quad - \frac{bc}{2}|ht+hs|^{2H} + \frac{c^2}{2}(ht)^{2H} + \frac{c^2}{2}(hs)^{2H} - \frac{c^2}{2}|ht-hs|^{2H} \\
&= a^2h(t \wedge s) + h^{2H}\frac{(b+c)^2}{2}\left((t)^{2H} + (s)^{2H}\right) - bch^{2H}|t+s|^{2H} \\
&\quad - h^{2H}\frac{(b^2+c^2)}{2}|t-s|^{2H}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
Cov\left(h^H M_t^H(a,b,c), h^H M_s^H(a,b,c)\right) &= h^{2H} Cov\left(M_t^H(a,b,c), M_s^H(a,b,c)\right) \\
&= a^2 h^{2H}(t \wedge s) \\
&\quad + h^{2H}\frac{(b+c)^2}{2}\left(t^{2H} + s^{2H}\right) \\
&\quad - bch^{2H}|t+s|^{2H} \\
&\quad - h^{2H}\frac{(b^2+c^2)}{2}|t-s|^{2H}.
\end{aligned}$$

Then the mgfBm is not a self-similar process for all $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. □

Remark 2.2. *As a consequence of Proposition 2.2, we see that:*

1. $M^H(0,b,c)$ *is a self-similar process for all* $(b,c) \in \mathbb{R}^2$.
2. $M^{\frac{1}{2}}(a,b,c)$ *is a self-similar process for all* $(a,b,c) \in \mathbb{R}^3$.

Now, we will study the Markovian property.

Theorem 2.1. *Assume* $H \in (0,1) \setminus \left\{\frac{1}{2}\right\}, a \in \mathbb{R}$ *and* $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$*. Then* $M^H(a,b,c)$ *is not a Markovian process.*

Proof. The process $M^H(a,b,c)$ is a centered Gaussian. Then, if $M_t^H(a,b,c)$ is a Markovian process, according to Revuz and Yor [23], for all $s < t < u$, we would have

$$C(s,u)C(t,t) = C(s,t)C(t,u).$$

We will only prove the theorem in the case where $a \neq 0$, the result with $a = 0$ is known in [29]. For the proof we follow the proof of Proposition 1 given in [29]. Using Proposition 2.1, we get

$$\begin{aligned}
C(s,u) &= a^2 s + \frac{(b+c)^2}{2}(u^{2H} + s^{2H}) - bc|u+s|^{2H} - \frac{(b^2+c^2)}{2}|u-s|^{2H}, \\
C(t,t) &= a^2 t + \left(b^2 + c^2 - (2^{2H} - 2)bc\right) t^{2H}, \\
C(s,t) &= a^2 s + \frac{(b+c)^2}{2}(t^{2H} + s^{2H}) - bc|t+s|^{2H} - \frac{(b^2+c^2)}{2}|t-s|^{2H}, \\
C(t,u) &= a^2 t + \frac{(b+c)^2}{2}(u^{2H} + t^{2H}) - bc|u+t|^{2H} - \frac{(b^2+c^2)}{2}|u-t|^{2H}.
\end{aligned}$$

In the particular case where $1 < s = \sqrt{t} < t < u = t^2$, we have

$$\begin{aligned}
C(\sqrt{t}, t^2) &= a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{4H} + t^{H}) - bc|t^2 + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H}, \\
C(t,t) &= a^2 t + \left(b^2 + c^2 - (2^{2H} - 2)bc\right) t^{2H}, \\
C(\sqrt{t}, t) &= a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{2H} + t^{H}) - bc|t + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t - t^{\frac{1}{2}}|^{2H}, \\
C(t, t^2) &= a^2 t + \frac{(b+c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2 + t|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H}.
\end{aligned}$$

Then by using that,

$$C(\sqrt{t}, t^2)C(t,t) = C(\sqrt{t}, t)C(t, t^2),$$

we have

$$\begin{aligned}
&\left[a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{4H} + t^{H}) - bc|t^2 + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H}\right] \\
&\times \left[a^2 t + \left(b^2 + c^2 - (2^{2H} - 2)bc\right) t^{2H}\right] \\
= &\left[a^2 t^{\frac{1}{2}} + \frac{(b+c)^2}{2}(t^{2H} + t^{H}) - bc|t + t^{\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|t - t^{\frac{1}{2}}|^{2H}\right] \\
&\times \left[a^2 t + \frac{(b+c)^2}{2}(t^{4H} + t^{2H}) - bc|t^2 + t|^{2H} - \frac{(b^2+c^2)}{2}|t^2 - t^{\frac{1}{2}}|^{2H}\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left[a^2 t^{\frac{1}{2}} + t^{4H}\left(\frac{(b+c)^2}{2}(1 + t^{-3H}) - bc|1 + t^{-\frac{3}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|1 - t^{-\frac{3}{2}}|^{2H}\right)\right] \\
&\times \left[a^2 t + \left(b^2 + c^2 - (2^{2H} - 2)bc\right) t^{2H}\right] \\
= &\left[a^2 t^{\frac{1}{2}} + t^{2H}\left(\frac{(b+c)^2}{2}(1 + t^{-H}) - bc|1 + t^{-\frac{1}{2}}|^{2H} - \frac{(b^2+c^2)}{2}|1 - t^{-\frac{1}{2}}|^{2H}\right)\right] \\
&\times \left[a^2 t + t^{4H}\left(\frac{(b+c)^2}{2}(1 + t^{-2H}) - bc|1 + t^{-1}|^{2H} - \frac{(b^2+c^2)}{2}|1 - t^{-1}|^{2H}\right)\right].
\end{aligned}$$

Hence

$$
\begin{aligned}
a^4t^{\frac{3}{2}} \quad &+ \quad a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{2H+\frac{1}{2}} \\
&+ \quad a^2\left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right) \\
&\times \quad t^{4H+1} \\
&+ \quad \left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right) \\
&\times \quad \left(b^2+c^2-(2^{2H}-2)bc\right)t^{6H} \\
= a^4t^{\frac{3}{2}} \quad &+ \quad a^2\left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right) \\
&\times \quad t^{4H+\frac{1}{2}} \\
&+ \quad a^2\left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right) \\
&\times \quad t^{2H+1} \\
&+ \quad \left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right) \\
&\times \quad \left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right)t^{6H}.
\end{aligned}
$$

Take t^{6H} as a common factor, we get

$$
\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}} \\
& +a^2\left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right)t^{-2H+1} \\
& +\left(\frac{(b+c)^2}{2}(1+t^{-3H})-bc|1+t^{-\frac{3}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{3}{2}}|^{2H}\right) \\
& \times\left(b^2+c^2-(2^{2H}-2)bc\right) \\
= \quad & a^2\left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right)t^{-2H+\frac{1}{2}} \\
& +a^2\left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{1}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{1}{2}}|^{2H}\right)t^{-4H+1} \\
& +\left(\frac{(b+c)^2}{2}(1+t^{-H})-bc|1+t^{-\frac{1}{2}}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-\frac{1}{2}}|^{2H}\right) \\
& \times\left(\frac{(b+c)^2}{2}(1+t^{-2H})-bc|1+t^{-1}|^{2H}-\frac{(b^2+c^2)}{2}|1-t^{-1}|^{2H}\right).
\end{aligned}
$$

Therefore

$$\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}} \\
& +a^2\Big[\frac{1}{2}(b+c)^2\left(1+t^{-3H}\right)-bc\Big(1+2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big)\Big]t^{-2H+1} \\
& +\Big[\frac{1}{2}(b+c)^2\left(1+t^{-3H}\right)-bc\Big(1+2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{3}{2}}+H(2H-1)t^{-3}+\circ(t^{-3})\Big)\Big] \\
& \times[b^2+c^2-(2^{2H}-2)bc] \\
= \;& a^2\Big[\frac{1}{2}(b+c)^2\left(1+t^{-2H}\right)-bc\left(1+2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right) \\
& -\frac{1}{2}(b^2+c^2)\left(1-2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right)\Big]t^{-2H+\frac{1}{2}} \\
& +a^2\Big[\frac{1}{2}(b+c)^2\left(1+t^{-H}\right)-bc\Big(1+2Ht^{-1/2}+H(2H-1)t^{-1}+\circ(t^{-1})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{1}{2}}+H(2H-1)t^{-1}+\circ(t^{-1})\Big)\Big]t^{-4H+1} \\
& +\Big[\frac{1}{2}(b+c)^2\left(1+t^{-H}\right)-bc\Big(1+2Ht^{-\frac{1}{2}}+H(2H-1)t^{-1}+\circ(t^{-1})\Big) \\
& -\frac{1}{2}(b^2+c^2)\Big(1-2Ht^{-\frac{1}{2}}+H(2H-1)t^{-1}+\circ(t^{-1})\Big)\Big] \\
& \times\Big[\frac{1}{2}(b+c)^2\left(1+t^{-2H}\right)-bc\left(1+2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right) \\
& -\frac{1}{2}(b^2+c^2)\left(1-2Ht^{-1}+H(2H-1)t^{-2}+\circ(t^{-2})\right)\Big].
\end{aligned}$$

First case: $0<H<\frac{1}{2}$, $a\neq 0$ and $b+c\neq 0$. By Taylor's expansion we get, as $t\to\infty$,

$$\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}}+a^2\frac{1}{2}(b+c)^2t^{-5H+1} \\
& +\frac{1}{2}(b+c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-3H} \\
\approx \;& a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}}+a^2\frac{1}{2}(b+c)^2t^{-5H+1}+\frac{1}{4}(b+c)^4t^{-3H}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}}+\frac{1}{2}(b+c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-3H} \\
\approx \;& a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}}+\frac{1}{4}(b+c)^4t^{-3H},
\end{aligned}$$

which is true if and only if

$$\frac{(b-c)^2}{2} - (2^{2H}-2)bc = 0 \quad \text{and} \quad a^2\frac{(b-c)^2}{2} - a^2(2^{2H}-2)bc = 0.$$

However, it is easy to check that

$$\frac{(b-c)^2}{2} - (2^{2H}-2)bc > 0 \quad \text{and} \quad a^2\frac{(b-c)^2}{2} - a^2(2^{2H}-2)bc > 0,$$

for fixed c, b and every real a.
Second case: $0 < H < \frac{1}{2}$, $a \neq 0$ and $b + c = 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} + a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-2H-\frac{1}{2}} \\ & +\left[-2Hbc+(b^2+c^2)H\right] \times \left[b^2+c^2-(2^{2H}-2)bc\right]t^{-\frac{3}{2}} \\ \approx\ & a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-2H-\frac{1}{2}} + a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-4H+\frac{1}{2}} \\ & +\left[-2Hbc+(b^2+c^2)H\right]^2 t^{-\frac{3}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} \\ & +\left[-2Hbc+(b^2+c^2)H\right] \times \left[b^2+c^2-(2^{2H}-2)bc\right]t^{-\frac{3}{2}} \\ \approx\ & a^2\left[-2Hbc+(b^2+c^2)H\right]t^{-4H+\frac{1}{2}} + \left[-2Hbc+(b^2+c^2)H\right]^2 t^{-\frac{3}{2}}, \end{aligned}$$

which is true if and only if $b = c = 0$. This is a contradiction.
Third case: $\frac{1}{2} < H < 1$, $a \neq 0$ and $b - c \neq 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} + a^2H(b-c)^2t^{-2H-\frac{1}{2}} \\ & +H(b-c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-\frac{3}{2}} \\ \approx\ & a^2H(b-c)^2t^{-2H-\frac{1}{2}} + a^2H(b-c)^2t^{-4H+\frac{1}{2}} + H^2(b-c)^4t^{-\frac{3}{2}}. \end{aligned}$$

Then

$$\begin{aligned} & a^2[b^2+c^2-(2^{2H}-2)bc]t^{-4H+\frac{1}{2}} + H(b-c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-\frac{3}{2}} \\ \approx\ & a^2H(b-c)^2t^{-4H+\frac{1}{2}} + H^2(b-c)^4t^{-\frac{3}{2}}, \end{aligned}$$

which is true if and only if

$$\left[b^2(1-H) + c^2(1-H) + (2-2^{2H}+2H)bc\right] = 0.$$

However, it is easy to check that $b^2(1-H) + c^2(1-H) + (2-2^{2H}+2H)bc > 0$ for fixed c, b and every real a.
Fourth case: $\frac{1}{2} < H < 1$, $a \neq 0$ and $b - c = 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\begin{aligned} & a^2\left(b^2+c^2-(2^{2H}-2)bc\right)t^{-4H+\frac{1}{2}} + \frac{1}{2}(b+c)^2[b^2+c^2-(2^{2H}-2)bc]t^{-3H} \\ \approx\ & a^2\frac{1}{2}(b+c)^2t^{-4H+\frac{1}{2}} + \frac{1}{4}(b+c)^4t^{-3H}, \end{aligned}$$

which is true if and only if $2 - 2^{2H} = 0$. This contradicts the fact that $H \neq \frac{1}{2}$. □

Let us check the mixed self-similarity property of the mgfBm. This property was introduced in [27] for the mfBm and investigated to show the Hölder continuity of the mfBm. See also [11] for the sfBm case.

Proposition 2.3. *For any $h>0$, $\left\{M^H_{ht}(a,b,c)\right\} \stackrel{\triangle}{=} \left\{M^H_t(ah^{\frac{1}{2}}, bh^H, ch^H)\right\}$, where $\stackrel{\triangle}{=}$ "to have the same law".*

Proof. For fixed $h>0$, the processes $\{M^H_{ht}(a,b,c)\}$ and $\{M^H_t(ah^{\frac{1}{2}}, bh^H, ch^H)\}$ are Gaussian and centered. Therefore, one only have to prove that they have the same covariance function. But, for any $s,\ t \geq 0$, since B and B^H are independent, then

$$\begin{aligned} C\left(ht,hs\right) &= a^2 h(t\wedge s) + \frac{(b^2+c^2)}{2}\left[h^{2H}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)\right] \\ &\quad + bc\left[h^{2H}\left(t^{2H}+s^{2H}-|t+s|^{2H}\right)\right] \\ &= Cov\left(M^H_t(ah^{\frac{1}{2}}, bh^H, ch^H), M^H_s(ah^{\frac{1}{2}}, bh^H, ch^H)\right). \end{aligned}$$

□

Proposition 2.4. *For all $a\in\mathbb{R}$ and $(b,c)\in\mathbb{R}^2\setminus\{(0,0)\}$, the increments of the mgfBm are not stationary.*

Proof. Let $a\in\mathbb{R}$ and $(b,c)\in\mathbb{R}^2\setminus\{(0,0)\}$. For a fixed $t\geq 0$ consider the processes $\{P_t,\ t\geq 0\}$ define by $P_t = M^H_{t+s}(a,b,c) - M^H_s(a,b,c)$. Using Proposition 2.1, we get

$$\begin{aligned} Cov(P_t,P_t) &= E\left(M^H_{t+s}(a,b,c) - M^H_s(a,b,c)\right)^2 \\ &= a^2(t+s+s) - 2^{2H}bc((t+s)^{2H}+s^{2H}) - 2a^2 s \\ &\quad +(b^2+c^2)|t+s-s|^{2H} + 2bc|t+s+s|^{2H} \\ &= a^2(t+2s) - 2^{2H}bc((t+s)^{2H}+s^{2H}) - 2a^2 s + (b^2+c^2)t^{2H} \\ &\quad +2bc|t+2s|^{2H}. \end{aligned}$$

Using Proposition 2.1, we get

$$Cov(M^H_t(a,b,c), M^H_t(a,b,c)) = a^2 t + \left(b^2+c^2-(2^{2H}-2)bc\right)t^{2H}.$$

Since both processes are centered Gaussian, the inequality of covariance functions implies that P_t does not have the same distribution as $M^H_t(a,b,c)$. Thus, the incremental behavior of $M^H(a,b,c)$ at any point in the future is not the same. Hence the increments of $M^H(a,b,c)$ are not stationary. □

Remark 2.3. *As a consequence of Proposition 2.4, we see that:*

(1) *the increments of $M^H(0,b,c)$ are not stationary for all $(b,c)\in\mathbb{R}^2\setminus\{(0,0)\}$.*
(2) *the increments of $M^H(a,b,0)$ are stationary for all $(a,b)\in\mathbb{R}^2$.*

Proposition 2.5.
(1) *Let $H\in(0,1)$. The mgfBm admits a version whose sample paths are almost Hölder continuous of order strictly less than $\frac{1}{2}\wedge H$.*
(2) *When b or c not zero and $H\in(0,1)\setminus\{\frac{1}{2}\}$ the mgfBm is not a semimartingale.*

Proof. (1) Let $s,\ t \geq 0$ and $\alpha = 2$. The proof follows by Kolmogorov criterion from Lemma 3 in [29] and using Proposition 2.1 we get

$$\begin{aligned} E\left(|M_t^H(a,b,c) - M_s^H(a,b,c)|^{\alpha}\right) &= a^2|t-s| - 2^{2H}bc(t^{2H}+s^{2H}) \\ &\quad +(b^2+c^2)|t-s|^{2H} + 2bc|t+s|^{2H} \\ &\leq C_\alpha |t-s|^{\alpha(\frac{1}{2}\wedge H)}, \end{aligned}$$

where $C_\alpha = \left(a^2 + \nu(b,c,H)\right)$ and $\nu_{(b,c,H)}$ is given in Lemma 3 in [29].

(2) Suppose first that $H < \frac{1}{2}$. We get from Proposition 2.1

$$\alpha(t,s) \geq \gamma_{(b,c,H)}(t-s)^{2H}.$$

Since $2H < 1$ and $\gamma_{(b,c,H)} > 0$ then the assumption of Corollary 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale. Suppose now that $H > \frac{1}{2}$. We get from Proposition 2.1

$$a^2(t-s) + \gamma_{(b,c,H)}(t-s)^{2H} \leq \alpha(t,s) \leq (a^2 + \nu_{(b,c,H)})(t-s)^{1\wedge 2H},$$

then

$$\gamma_{(b,c,H)}(t-s)^{2H} \leq \alpha(t,s) \leq (a^2 + \nu_{(b,c,H)})(t-s)^{2H}.$$

Since $1 < 2H < 2$ and $\nu_{(b,c,H)} > 0$ then the assumption of Lemma 2.1 in [5] is satisfied, and consequently the mgfBm is not a semi-martingale.

□

3. Long-range Dependence of the mgfBm Increments

Definition 3.1. We say that the increments of a stochastic process X are *long-range dependent* if for every integer $p \geq 1$, we have

$$\sum_{n\geq 1} R_X(p,p+n) = \infty,$$

where

$$R_X(p,p+n) = E\left((X_{p+1} - X_p)(X_{p+n+1} - X_{p+n})\right).$$

This property was investigated in many papers (e.g. [3, 6, 8, 12, 20]).

Theorem 3.1. *For every $a \in \mathbb{R}$ and $(b,c) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the increments of $M^H(a,b,c)$ are long-range dependent if and only if $H > \frac{1}{2}$ and $b \neq c$.*

Proof. For all $n \geq 1$ and $p \geq 1$, we have

$$
\begin{aligned}
& R_M(p,p+n) \\
= \; & E\big(\left(M^H_{p+1}(a,b,c) - M^H_p(a,b,c)\right) \left(M^H_{p+n+1}(a,b,c) - M^H_{p+n}(a,b,c)\right) \big) \\
= \; & E\left(M^H_{p+1}(a,b,c) M^H_{p+n+1}(a,b,c)\right) - E\left(M^H_{p+1}(a,b,c) M^H_{p+n}(a,b,c)\right) \\
& -E\left(M^H_{p}(a,b,c) M^H_{p+n+1}(a,b,c)\right) + E\left(M^H_{p}(a,b,c) M^H_{p+n}(a,b,c)\right) \\
= \; & C(p+1,p+n+1) - C(p+1,p+n) - C(p,p+n+1) + C(p,p+n) \\
= \; & a^2(p+1) + \frac{(b+c)^2}{2}\left((p+1)^{2H} + (p+n+1)^{2H}\right) - bc(2p+n+2)^{2H} \\
& -\frac{(b^2+c^2)}{2} n^{2H} - a^2(p+1) - \frac{(b+c)^2}{2}\left((p+1)^{2H} + (p+n)^{2H}\right) \\
& +bc(2p+n+1)^{2H} + \frac{(b^2+c^2)}{2}|n-1|^{2H} - a^2 p \\
& -\frac{(b+c)^2}{2}\left(p^{2H} + (p+n+1)^{2H}\right) + bc(2p+n+1)^{2H} \\
& +\frac{(b^2+c^2)}{2}|n+1|^{2H}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
R_M(p,p+n) \; = \; & \frac{(b^2+c^2)}{2}\left((n+1)^{2H} - 2n^H + (n-1)^{2H}\right) \\
& -bc\left((2p+n+2)^{2H} - 2(2p+n+1)^{2H} + (2p+n)^{2H}\right).
\end{aligned}
$$

Then for every integer $p \geq 1$, by Taylor's expansion, as $n \to \infty$, we have

$$
\begin{aligned}
R_M(p,p+n) \; = \; & \frac{b^2+c^2}{2} n^{2H}\left[\left(1+\frac{1}{n}\right)^{2H} - 2 + \left(1-\frac{1}{n}\right)^{2H}\right] \\
& -bcn^{2H}\left[\left(1+\frac{2p+2}{n}\right)^{2H} - 2\left(1+\frac{2p+1}{n}\right) + \left(1+\frac{2p}{n}\right)^{2H}\right] \\
= \; & H(2H-1)n^{2H-2}(b-c)^2 \\
& -4H(2H-1)(H-1)bc(2p+1)n^{2H-3}(1+\circ(1)).
\end{aligned}
$$

If $b \neq c$, we see that as $n \to \infty$,

$$R_M(p,p+n) \approx H(2H-1)n^{2H-2}(b-c)^2.$$

Then

$$\sum_{n\geq 1} R_M(p,p+n) = \infty \quad \Leftrightarrow \; 2H-2 > -1 \quad \Leftrightarrow H > \frac{1}{2}.$$

If $b = c$, then, as $n \to \infty$,

$$R_M(p,p+n) \approx 4H(2H-1)(H-1)a^2(2p+1)n^{2H-3}.$$

For every $H \in (0,1)$, we have $2H-3 < -1$ and, consequently,

$$\sum_{n\geq 1} R_M(p,p+n) < \infty.$$

□

Remark 3.1. (1) *For all* $a \in \mathbb{R}$ *and* $b \in \mathbb{R} \setminus \{0\}$*, the increments of* $M^H(a, b, 0)$ *are long-range dependent if and only if* $H > \frac{1}{2}$.

(2) *If* $b = c = \frac{1}{\sqrt{2}}$*, the increments of* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *are short-range dependent if and only if* $H \in (0, 1)$*. But if* $b \neq c$*, the increments of* $M^H(0, b, c)$ *are long-range dependent if and only if* $H > \frac{1}{2}$.

(3) *From* [4]*, the increments of* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ *on intervals* $[u, u + r], [u + r, u + 2r]$ *are more weakly correlated than those of* $M^H(0, 1, 0)$.

(4) *From* [30]*, If* $H > \frac{1}{2}$, $b^2 + c^2 = 1$ *and* $bc \geq 0$*, the increments of* $M^H(0, b, c)$ *are more weakly correlated than those of* $M^H(0, 1, 0)$*, but more strongly correlated than those of* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(5) *From* [30]*, If* $H \geq \frac{1}{2}$, $(bc \leq 0$ *and* $(b-c)^2 \leq 1)$ *or* $(bc \geq 0$ *and* $b^2 + c^2 \leq 1)$*, the increments of* $M^H(0, b, c)$ *are more strongly correlated than those of both* $M^H(0, 1, 0)$ *and* $M^H(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

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