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Mixed Hodge polynomials of character varieties

With an appendix by Nicholas M. Katz

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Abstract. We calculate the E-polynomials of certain twisted $\mathrm{GL}(n,\mathbb{C})$-character varieties $\mathcal{M}_n$ of Riemann surfaces by counting points over finite fields using the character table of the finite group of Lie-type $\mathrm{GL}(n,\mathbb{F}_q)$ and a theorem proved in the appendix by N. Katz. We deduce from this calculation several geometric results, for example, the value of the topological Euler characteristic of the associated $\mathrm{PGL}(n,\mathbb{C})$-character variety. The calculation also leads to several conjectures about the cohomology of $\mathcal{M}_n$: an explicit conjecture for its mixed Hodge polynomial; a conjectured curious hard Lefschetz theorem and a conjecture relating the pure part to absolutely indecomposable representations of a certain quiver. We prove these conjectures for $n = 2$.

Contents

1 Introduction

Let $g \geq 0$ and $n > 0$ be integers. Let $\zeta_n \in \mathbb{C}$ be a primitive n-th root of unity. Abbreviating $[A, B] = ABA^{-1}B^{-1}$ and denoting the identity matrix

$I_n \in \mathrm{GL}(n, \mathbb{C})$ we define

$$\mathcal{M}_n := \{A_1, B_1, \dots, A_g, B_g \in \mathrm{GL}(n, \mathbb{C})| \\ [A_1, B_1] \dots [A_g, B_g] = \zeta_n I_n\}//\mathrm{GL}(n, \mathbb{C}) \quad (1.1.1)$$

an affine GIT quotient by the conjugation action of $\mathrm{GL}(n, \mathbb{C})$. It is a twisted character variety of a genus g closed Riemann surface Σ; its points can be thought of twisted homomorphisms of $\pi_1(\Sigma) \to \mathrm{GL}(n, \mathbb{C})$ modulo conjugation. It is a non-singular affine variety of dimension $d_n := n^2(2g-2)+2$ by Theorem 2.2.5.

One of the goals of this paper is to find the Poincaré polynomial $P(\mathcal{M}_n; t) = \sum_i b_i(\mathcal{M}_n) t^i$ which encodes the Betti numbers $b_i(\mathcal{M}_n)$ of $\mathcal{M}_n$. They were calculated for $n = 2$ by Hitchin [40] and for $n = 3$ by Gothen [24]. To be precise, Hitchin and Gothen work with a certain moduli space of Higgs bundles on Σ, which is known to be diffeomorphic to $\mathcal{M}_n$ by non-abelian Hodge theory [40,64]. On the other hand, the Poincaré polynomial of the $\mathrm{U}(n)$-character variety $\mathcal{N}_n^d$ of Σ, where $\mathrm{GL}(n, \mathbb{C})$ is replaced by $\mathrm{U}(n)$ in the above definition and $\zeta_n = \exp(\frac{d}{n} 2\pi i)$, were obtained by Harder–Narasimhan [28], using the Weil conjectures proved by Deligne [10], and by Atiyah–Bott [2] using gauge theory. An explicit closed formula for the Poincaré polynomial of the $\mathrm{U}(n)$-character varieties was given by Zagier [67].

Other motivations to study the cohomology of character varieties are discussed in [31], which also announces many of the results of this paper. Character varieties appear in the Geometric Langlands program of Beilinson–Drinfeld [3]. Recently, many new ideas relating physics, in particular mirror symmetry, to the Geometric Langlands program have been discussed by Kapustin–Witten in [46]. One can expect [31,33] that the results of this paper will have analogues for $\mathrm{SL}(n, \mathbb{C})$ character varieties reflecting the expected relationship between certain Hodge numbers of $\mathrm{PGL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ character varieties dictated by mirror symmetry considerations.

In this paper we study the mixed Hodge polynomials $H(\mathcal{M}_n; q, t)$ and uncover a surprising amount of structure governing them. The mixed Hodge polynomial is a common deformation of the Poincaré polynomial $P(\mathcal{M}_n; t) = H(\mathcal{M}_n; 1, t)$ and the so-called E-polynomial $E(\mathcal{M}_n; q) = q^{d_n} H(\mathcal{M}_n; 1/q, -1)$ and is defined using Deligne's construction of mixed Hodge structures on the cohomology of a complex algebraic variety [8,9] (see Subsect. 2.1).

We explicitly calculate the E-polynomial of $\mathcal{M}_n$ in terms of a generating function using arithmetic algebraic geometry. One key result used in this calculation is Theorem 6.1.2.3 of Katz in the appendix, which basically says that if the number of points of a variety over every finite field $\mathbb{F}_q$ is a polynomial in q then this polynomial agrees with the E-polynomial of the variety.

Another ingredient is a well-known character formula, Theorem 2.3.2, which counts the number of solutions of certain equations in a finite group.

Similar counting formulas go back to the birth of character theory of finite groups by Frobenius [18] in 1896. Combining these and Corollary 3.5.1 we get

$$E(\mathcal{M}_n; q) = \sum_{\chi \in \operatorname{Irr}(\operatorname{GL}(n,\mathbb{F}_q))} \frac{|\operatorname{GL}(n, \mathbb{F}_q)|^{2g-2}}{\chi(I_n)^{2g-1}} \chi(\zeta_n I_n). \tag{1.1.2}$$

The character table of $\operatorname{GL}(n, \mathbb{F}_q)$ was determined by Green [22] in 1955. Using Green's results, the evaluation of the formula (1.1.2) is carried out in Sect. 3. The calculation makes non-trivial use of the inclusion–exclusion principle for the poset of finite set partitions. The end result is an expression for the E-polynomials in terms of an explicit generating function in Theorem 3.5.2. An important consequence of our Theorem 3.5.2 is that the number of points of the variety $\mathcal{M}_n$ over the finite field $\mathbb{F}_q$ is a polynomial in q. For example, for $n = 2$ we prove in Corollary 3.6.1 that

$$\begin{aligned} E(\mathcal{M}_2; q)/(q-1)^{2g} &= (q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} \\ &\quad - \frac{1}{2}q^{2g-2}(q-1)^{2g-2} - \frac{1}{2}q^{2g-2}(q+1)^{2g-2}. \end{aligned} \tag{1.1.3}$$

An interesting topological outcome of our calculation is the precise value of the Euler characteristic of our character varieties. The variety $\mathcal{M}_n$ is cohomologically a product of $(\mathbb{C}^\times)^{2g}$ and the $\operatorname{PGL}(n, \mathbb{C})$-character variety $\tilde{\mathcal{M}}_n := \mathcal{M}_n//(\mathbb{C}^\times)^{2g}$, which is defined as the quotient of $\mathcal{M}_n$ by the natural action of the torus $(\mathbb{C}^\times)^{2g}$ on $\mathcal{M}_n$. Therefore, the Euler characteristic of $\mathcal{M}_n$ is 0 due to the fact that the Euler characteristic of the torus $(\mathbb{C}^\times)^{2g}$ is 0. However the Euler characteristic of $\tilde{\mathcal{M}}_n$ is more interesting (see Subsect. 3.7):

Corollary 1.1.1. *Let $g > 1$. The Euler characteristic of the* $\operatorname{PGL}(n, \mathbb{C})$-*character variety $\tilde{\mathcal{M}}_n$ is $\mu(n)n^{2g-3}$, where μ is the Möbius function.*

The last result of the first part of this paper is a formula for the number of points on the untwisted $\operatorname{GL}(n)$ character variety (which is defined by replacing $\zeta_n I$ by I in the definition (1.1.1)) over a finite field $\mathbb{F}_q$. Our explicit generating function formula in Theorem 3.8.1 could be interesting to compare with recent work of Liebeck–Shalev [49] studying asymptotics of the same quantities.

The second part of this paper concerns the mixed Hodge polynomial of $\mathcal{M}_n$. In Conjecture 4.2.1 we give a formula for it as a natural t-deformation of our calculation of the E-polynomial of $\mathcal{M}_n$. Here we only give our conjecture in the case $g = 1$. In this case we know *a priori* that our character variety is $\mathcal{M}_1 = (\mathbb{C}^\times)^2$, the 2-torus (see Theorem 2.2.17). Therefore our conjecture becomes a purely combinatorial statement.

Conjecture 1.1.2. The following combinatorial identity holds:

$$\sum_{\lambda}\prod\frac{(z^{2a+1}-w^{2l+1})^2}{(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})}T^{|\lambda|} = \exp\left(\sum_{k\geq 1}\frac{(z^k-w^k)^2}{(z^{2k}-1)(1-w^{2k})(1-T^k)}\frac{T^k}{k}\right),$$

where the sum on the left hand side is over all partitions λ, and the product is over all boxes in the Ferrers diagram of λ, and a and l are its arm and leg-length, as defined in Subsect. 2.4.

This yet unproven identity is reminiscent of the Macdonald identities and the Weyl–Kac character formula; it is conceivable that it has a representation theory interpretation. For example, the corresponding formula in the $g = 0$ case will be proved as Theorem 4.3.1 using a result of Garsia–Haiman [20] obtained from the study of Macdonald polynomials. This presently mysterious link between mixed Hodge polynomials of character varieties and Macdonald polynomials is further developed in [34, 35]. In particular, the main conjecture of [35] says that the mixed Hodge polynomials of character varieties of Riemann surfaces with semisimple conjugacy classes at the punctures are governed by Macdonald polynomials in a simple way, its structure resembling a topological quantum field theory.

We study many implications of our main Conjecture 4.2.1 and prove several consistency results. Because we have an explicit description of the cohomology ring of $\mathcal{M}_2$ given in [37, 39] we are able to determine the mixed Hodge polynomial in the $n = 2$ case and confirm all our conjectures.

Rather than giving a full description of our conjectures for general n here, we present instead the corresponding theorems in the $n = 2$ case.

Theorem 1.1.3. *The mixed Hodge polynomials of $\tilde{\mathcal{M}}_2$ and $\mathcal{M}_2$ are given by*

$$\begin{aligned} H(\tilde{\mathcal{M}}_2; q, t) &= \frac{H(\mathcal{M}_2; q, t)}{(qt+1)^{2g}} \\ &= \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} \\ &\quad - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)}. \end{aligned} \tag{1.1.4}$$

By setting $t = -1$ in this formula we recover the E-polynomial in (1.1.3). Thus, by purely cohomological calculations on $\mathcal{M}_2$ we derive formula (1.1.3), which reflects the structure of irreducible characters of $\mathrm{GL}(2, \mathbb{F}_q)$. For example, the four terms above correspond to the four types of irreducible characters of $\mathrm{GL}(2, \mathbb{F}_q)$. Looking at the other specialization $q = 1$ gives a pleasant formula for the Poincaré polynomial $P(\mathcal{M}_2, t) = H(\mathcal{M}_2; 1, t)$, which agrees with Hitchin's calculation [40].

Changing q by $1/qt^2$ in the right hand side of (1.1.4) interchanges the first two terms and fixes the other two. This implies the following

Corollary 1.1.4. *The mixed Hodge polynomial of $\tilde{\mathcal{M}}_2$ satisfies the following* curious Poincaré duality*:*

$$H\big(\tilde{\mathcal{M}}_2; 1/qt^2, t\big) = (qt)^{-\dim \tilde{\mathcal{M}}_2} H(\tilde{\mathcal{M}}_2; q, t). \tag{1.1.5}$$

In fact, we can give a geometrical interpretation of this combinatorial observation. First, $H^2(\tilde{\mathcal{M}}_2)$ is one dimensional generated by a class α. Define the Lefschetz map $L : H^i(\tilde{\mathcal{M}}_2) \to H^{i+2}(\tilde{\mathcal{M}}_2)$ by $x \mapsto \alpha \cup x$. As it respects mixed Hodge structures and α has weight 4 it defines a map on the graded pieces of the weight filtration $L : Gr_l^W H^i(\tilde{\mathcal{M}}_2) \to Gr_{l+4}^W H^{i+2}(\tilde{\mathcal{M}}_2)$. In Subsect. 5.3 we prove the following *curious hard Lefschetz*

Theorem 1.1.5. *The Lefschetz map*

$$L^l : Gr^W_{6g-6-2l} H^{i-l}(\tilde{\mathcal{M}}_2) \longrightarrow Gr^W_{6g-6+2l} H^{i+l}(\tilde{\mathcal{M}}_2)$$

is an isomorphism.

The agreement of the dimensions of these two isomorphic vector spaces is equivalent to (1.1.5).

Interestingly, this theorem implies (see Remark 4.2.8) a theorem of [30] that the Lefschetz map $L^k : H^{\tilde{d}_2/2-k}(\tilde{\mathcal{M}}_2) \to H^{\tilde{d}_2/2+k}(\tilde{\mathcal{M}}_2)$ is injective; where $\tilde{d}_2 = \dim \tilde{\mathcal{M}}_2 = 6g-6$. As it is explained in [30] this weak version of hard Lefschetz applied to toric hyperkähler varieties yields new inequalities for the h-numbers of matroids. See also [36] for the original argument on toric hyperkähler varieties. Theorem 1.1.5 can also be thought of as an analogue of the Faber conjecture [13] on the cohomology of the moduli space of curves, which is another non-compact variety whose cohomology ring is conjectured to satisfy a certain form of the hard Lefschetz theorem.

For any smooth variety X there is an important subring of $H^*(X)$, namely the so-called pure ring $PH^*(X) \cong \bigoplus_k W_k H^k(X)$. We denote by $PP(X; t)$ the Poincaré polynomial of the pure ring. We can obtain $PP(X; t)$ from $H(X; q, t)$ by taking the monomials which are powers of qt^2. In the case of $\mathcal{M}_2$ the pure ring is generated by a single class $\beta \in H^4(\mathcal{M}_2)$ and with one relation $\beta^g = 0$, the so-called Newstead relation. Consequently $PP(\mathcal{M}_2; t) = 1 + t^4 + \cdots + t^{4g-4}$. This implies the following:

Theorem 1.1.6. *Let $A_n(q)$ be the number of absolutely indecomposable g-tuples of n by n matrices over the finite field $\mathbb{F}_q$ modulo conjugation. Then for $n = 2$ we have*

$$PP(\mathcal{M}_2; \sqrt{q}) = q^{d_n/2} A_2(1/q).$$

In Subsect. 4.4 we conjecture the same for any n. The function $A_n(q)$ is an instance of the A-polynomial defined by Kac [45] for any quiver. The quiver here is S_g, g loops on one vertex. Kac showed that the A function is always a polynomial and conjectured it has non-negative coefficients. When the dimension vector is indivisible this has been proved by Crawley-Boevey and Van den Bergh [5] by giving a cohomological interpretation of the A-polynomial. For S_g the result of [5] only applies in the $n = 1$ case, since all other dimension vectors are divisible. Our Theorem 1.1.6 shows a cohomological interpretation for $A_2(q)$. For general n Theorem 4.4.1 together with our main Conjecture 4.2.1 will then give a conjectural cohomological interpretation of $A_n(q)$, implying Kac's conjecture for S_g.

Theorem 1.1.6 implies that the middle dimensional cohomology $H^{6g-6}(\tilde{\mathcal{M}}_2)$ has a trivial pure part. It follows that the middle dimensional compactly supported cohomology also has trivial pure part. This implies the following theorem, which will be proved in Corollary 5.4.1.

Theorem 1.1.7. *The intersection form on middle dimensional compactly supported cohomology $H_c^{6g-6}(\tilde{\mathcal{M}}_2)$ is trivial or equivalently the forgetful map $H_c^{6g-6}(\tilde{\mathcal{M}}_2) \to H^{6g-6}(\tilde{\mathcal{M}}_2)$ is* 0.

This was the main result of [29] and was interpreted there as the vanishing of "topological L^2 cohomology" for $\tilde{\mathcal{M}}_2$. It is surprising that we can deduce this result only from the knowledge [37,39] of the structure of the ordinary cohomology ring $H^*(\tilde{\mathcal{M}}_2)$ and the study of its mixed Hodge structure. In fact, we only need to know that the famous Newstead relation $\beta^g = 0$ holds in $H^*(\tilde{\mathcal{M}}_2)$. See [33] for a more detailed discussion on the background and various ramifications of Theorem 1.1.7.

The structure of the paper is as follows. In Sect. 2 we collect various facts which we will need later. In Subsect. 2.1 we define and list properties of the mixed Hodge polynomials obtained from Deligne's mixed Hodge structure. In Subsect. 2.2 we define and prove the basic properties of the character varieties we study. In Subsect. 2.3 we derive a classical character formula for the number of solutions of a certain equation over finite groups. In Subsect. 2.4 we collect the definitions and notations for partitions which will be used throughout the paper. In Subsect. 2.5 we introduce a formalism to handle various formal infinite products. Then in Sect. 3 we calculate the E-polynomial of our variety. In Sect. 4 we formulate our main conjecture on the mixed Hodge polynomial of our character varieties and derive various consequences, several of which we can test for consistency. We also relate our conjectured mixed Hodge polynomial to Kac's A-polynomial in Subsect. 4.4. Finally in Sect. 5 we prove all our conjectures in the $n = 2$ case.

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2 Preliminaries

2.1 Mixed Hodge polynomials. Motivated by the (then still unproven) Weil Conjectures and Grothendieck's "yoga of weights", which drew cohomological conclusions about complex varieties from the truth of those conjectures, Deligne in [8,9] proved the existence of mixed Hodge structures on the cohomology of a complex algebraic variety.

Proposition 2.1.1 (Deligne [8,9])**.** *Let X be a complex algebraic variety. For each j there is an increasing weight filtration*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2j} = H^j(X, \mathbb{Q})$$

and a decreasing Hodge filtration

$$H^j(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{m+1} = 0$$

such that the filtration induced by F on the complexification of the graded pieces $Gr_l^W := W_l/W_{l-1}$ of the weight filtration endows every graded piece with a pure Hodge structure of weight l, or equivalently for every $0 \leq p \leq l$ we have

$$Gr_l^{W^{\mathbb{C}}} = F^p Gr_l^{W^{\mathbb{C}}} \oplus \overline{F^{l-p+1} Gr_l^{W^{\mathbb{C}}}}. \qquad (2.1.1)$$

We now list properties of this mixed Hodge structure, which we will need in this paper. From now on we use the notation $H^*(X)$ for $H^*(X, \mathbb{Q})$.

Theorem 2.1.2. *1. The map $f^* : H^*(Y) \to H^*(X)$, induced by an algebraic map $f : X \to Y$, strictly preserves mixed Hodge structures.*

2. A field automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ induces an isomorphism $H^*(X) \cong H^*(X^\sigma)$, which preserves the mixed Hodge structure.

3. The Künneth isomorphism

$$H^*(X \times Y) \cong H^*(X) \otimes H^*(Y)$$

is compatible with mixed Hodge structures.

4. The cup product

$$H^k(X) \times H^l(X) \longrightarrow H^{k+l}(X)$$

is compatible with mixed Hodge structures.

5. *If X is smooth* $W_{j-1}H^j(X)$ *is trivial.*
6. *If X is smooth the* pure part $PH^*(X) := \bigoplus_k W_k H^k(X) \subset H^*(X)$ *is a subring.*
7. *If X is smooth and* $i : X \to Y$ *is a smooth compactification of* X*, then* $Im(i^*) = PH^*(X)$.

Using Deligne's [9, 8.3.8] construction of mixed Hodge structure on relative cohomology one can define [7] (for a general discussion of this cf. Note 11 on p. 141 of [19]) a well-behaved mixed Hodge structure on compactly supported cohomology $H_c^*(X) := H_c^*(X, \mathbb{Q})$. Its basic properties are as follows (for proofs see [58]):

Theorem 2.1.3. *1. The forgetful map*

$$H_c^k(X) \longrightarrow H^k(X)$$

is compatible with mixed Hodge structures.

2. *For a smooth connected X we have Poincaré duality*

$$H^k(X) \times H_c^{2d-k}(X) \longrightarrow H_c^{2d}(X) \cong \mathbb{Q}(-d) \tag{2.1.2}$$

is compatible with mixed Hodge structures, where $\mathbb{Q}(-d)$ *is the pure mixed Hodge structure on* $\mathbb{Q}$ *with weight* $2d$ *and Hodge filtration* $F^d = \mathbb{Q}$ *and* $F^{d+1} = 0$.

3. *In particular, for a smooth X,* $W^{j+1}H_c^j(X) \cong H_c^j(X)$.

Definition 2.1.4. Define the *mixed Hodge numbers* by

$$h^{p,q;j}(X) := \dim_{\mathbb{C}} \left(Gr_p^F Gr_{p+q}^W H^j(X)^{\mathbb{C}} \right),$$

and the *compactly supported mixed Hodge numbers* by

$$h_c^{p,q;j}(X) := \dim_{\mathbb{C}} \left(Gr_p^F Gr_{p+q}^W H_c^j(X)^{\mathbb{C}} \right).$$

Form the *mixed Hodge polynomial*:

$$H(X; x, y, t) := \sum h^{p,q;j}(X) x^p y^q t^j,$$

the *compactly supported mixed Hodge polynomial*:

$$H_c(X; x, y, t) := \sum h_c^{p,q;j}(X) x^p y^q t^j,$$

and the *E-polynomial* of X:

$$E(X; x, y) := H_c(X; x, y, -1).$$

(2.1.2) implies the following

Corollary 2.1.5. *For a smooth connected X of dimension d we have*

$$H_c(X; x, y, t) = (xyt^2)^d H(X; 1/x; 1/y; 1/t).$$

Remark 2.1.6. By definition $E(X; 1, 1) = H_c(X; 1, 1, -1)$ is the Euler characteristic of X.

Remark 2.1.7. For our varieties $\mathcal{M}_n$ we will find in Corollary 4.1.11 that only Hodge type (p, p) can be non-trivial in the mixed Hodge structure, in other words $h^{p,q;j} = 0$ unless $p = q$. Hence, $H(\mathcal{M}_n; x, y, t)$ only depends on xy and t. To simplify our notation we will denote by

$$H(\mathcal{M}_n; q, t) := H(\mathcal{M}_n; \sqrt{q}, \sqrt{q}, t) \tag{2.1.3}$$

and

$$E(\mathcal{M}_n; q) := E(\mathcal{M}_n; \sqrt{q}, \sqrt{q}).$$

It is in fact the E-polynomial which could sometimes be calculated using arithmetic algebraic geometry. Here we explain a theorem of Katz (for details see the appendix). The setup is the following. Let X be a variety over $\mathbb{C}$. By a *spreading out* of X we mean a separated scheme $\mathcal{X}$ over a finitely generated $\mathbb{Z}$-algebra with an embedding $\varphi : R \hookrightarrow \mathbb{C}$, such that the extension of scalars $\mathcal{X}_\varphi \cong X$. We say that X has *polynomial count*[1] if there is a polynomial $P_X(t) \in \mathbb{Z}[t]$ and a spreading out $\mathcal{X}$ such that for every homomorphism $\phi : R \to \mathbb{F}_q$ to a finite field, the number of $\mathbb{F}_q$-points of the scheme $\mathcal{X}_\phi$ is

$$\#\mathcal{X}_\phi(\mathbb{F}_q) = P_X(q).$$

Then we have the following (cf. Theorem 6.1.2.3)

Theorem 2.1.8 (Katz)**.** *Let X be a variety over $\mathbb{C}$. Assume X has polynomial count with count polynomial $P_X(t) \in \mathbb{Z}[t]$, then the E-polynomial of X is given by:*

$$E(X; x, y) = P_X(xy).$$

Remark 2.1.9. Informally this means that if we can count the number of solutions of the equations defining our variety over $\mathbb{F}_q$, and this number turns out to be some universal polynomial evaluated at q, then this polynomial determines the E-polynomial of the variety.

In fact it is enough for this to be true for all finite fields of all but finitely many characteristics. We illustrate this in a simple example.

[1] A similar property for smooth and proper schemes was studied in [65].

Example 2.1.10. Fix a non-zero integer $m \in \mathbb{Z}$ and let $\mathcal{X}$ be the scheme over $\mathbb{Z}$ determined by the equation

$$xy = m. \tag{2.1.4}$$

The extension of scalars $\mathcal{X}_\phi$ of $\mathcal{X}$ determined by a ring homomorphism $\phi : \mathbb{Z} \to \mathbb{F}_q$ is given by the same equation (2.1.4) now viewed over $\mathbb{F}_q$. It is easy to count solutions to (2.1.4). Let p be the characteristic of $\mathbb{F}_q$ (so that q is a power of p). Then

$$\#\mathcal{X}_\phi(\mathbb{F}_q) = \begin{cases} 2q-1 & p \mid m \\ q-1 & \text{otherwise.} \end{cases} \tag{2.1.5}$$

Therefore $\mathcal{X}/\mathbb{Z}$ is fiberwise polynomial-count but not strongly polynomial-count (for precise definitions see the appendix). This is not a contradiction to Theorem 6.1.4 of the appendix; if we extend scalars to $\mathbb{Z}[\frac{1}{m}]$ then we eliminate the primes dividing m and find that in all cases $\#\mathcal{X}_\phi(\mathbb{F}_q) = q-1$, hence $\mathcal{X}$ has polynomial count. In fact, $\mathcal{X}/\mathbb{Z}[\frac{1}{m}]$ is just isomorphic to $\mathbb{G}_m/\mathbb{Z}[\frac{1}{m}]$.

Example 2.1.11. To illustrate Katz's theorem further, we consider the variety $X = \mathbb{C}^\times$. First we determine its mixed Hodge polynomial (cf. proof of Theorem 9.1.1 in [9]). The only question is to decide the Hodge numbers on the one-dimensional $H^1(\mathbb{C}^*)$. Because $h^{0,1;1} = h^{1,0;1}$ and $h^{2,0;1} = h^{0,2;1}$ we must have $h^{1,1;1}(\mathbb{C}^\times) = 1$ and the mixed Hodge polynomial is

$$H(\mathbb{C}^\times; x, y, t) = 1 + xyt. \tag{2.1.6}$$

Consequently, the compactly supported mixed Hodge polynomial is

$$H_c(\mathbb{C}^\times; x, y, t) = t + xyt^2,$$

by Corollary 2.1.5. Therefore the E-polynomial is

$$E(\mathbb{C}^\times; x, y) = xy - 1.$$

We can obtain the variety $\mathbb{C}^\times$ by extension of scalars $\mathbb{Z} \subset \mathbb{C}$ from the group scheme $\mathbb{G}_m$ over $\mathbb{Z}$. The counting polynomial of this scheme is the polynomial $P_{\mathbb{C}^\times}(q) = q - 1 = \#\mathbb{G}_m(\mathbb{F}_q)$, which is consistent with Katz's theorem above.

2.2 Character varieties. Here we define the character varieties and list their basic properties.

Let $g \geq 0, n > 0$ be integers. Let $\mathbb{K}$ be an algebraically closed field with $\zeta_n \in \mathbb{K}$ a primitive n-th root of unity. The existence of such ζ_n is equivalent to the condition

$$\mathrm{char}(\mathbb{K}) \nmid n \tag{2.2.1}$$

which we henceforth assume. Examples to bear in mind are $\mathbb{K} = \mathbb{C}$ and the algebraic closure of a finite field $\mathbb{K} = \overline{\mathbb{F}_q}$, where $q = p^r$ is a prime power, with $p \nmid n$.

Denote by $I_n \in \mathrm{GL}(n, \mathbb{K})$ the identity matrix, $[A, B] := ABA^{-1}B^{-1} \in \mathrm{SL}(n, \mathbb{K})$, the commutator. The group $\mathrm{GL}(n, \mathbb{K})$ acts by conjugation on $\mathrm{GL}(n, \mathbb{K})^{2g}$:

$$\begin{aligned}\sigma : \mathrm{GL}(n, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K})^{2g} &\longrightarrow \mathrm{GL}(n, \mathbb{K})^{2g}\\ (h, (A_1, B_1, \ldots, A_g, B_g)) &\longmapsto \big(h^{-1}A_1h, h^{-1}B_1h, \ldots, h^{-1}A_gh, h^{-1}B_gh\big).\end{aligned}$$

As the center of $\mathrm{GL}(n, \mathbb{K})$ acts trivially, this action induces an action

$$\bar{\sigma} : \mathrm{PGL}(n, \mathbb{K}) \times \mathrm{GL}(n, \mathbb{K})^{2g} \longrightarrow \mathrm{GL}(n, \mathbb{K})^{2g} \tag{2.2.2}$$

of $\mathrm{PGL}(n, \mathbb{K})$. Let $\mu : \mathrm{GL}(n, \mathbb{K})^{2g} \to \mathrm{SL}(n, \mathbb{K})$ be given by

$$\mu(A_1, B_1, \ldots, A_g, B_g) := [A_1, B_1] \ldots [A_g, B_g].$$

We define

$$\mathcal{U}_n := \mu^{-1}(\zeta_n I_n). \tag{2.2.3}$$

Clearly the $\mathrm{PGL}(n, \mathbb{K})$-action (2.2.2) will leave the affine variety $\mathcal{U}_n$ invariant. Thus we have a $\mathrm{PGL}(n, \mathbb{K})$ action on $\mathcal{U}_n$:

$$\bar{\sigma} : \mathrm{PGL}(n, \mathbb{K}) \times \mathcal{U}_n \longrightarrow \mathcal{U}_n. \tag{2.2.4}$$

The categorical quotient

$$\pi_n : \mathcal{U}_n \longrightarrow \mathcal{M}_n \tag{2.2.5}$$

exists by [56, Theorem1.1] (cf. also [57, §3]) in the sense of geometric invariant theory [56]. Explicitly we have

$$\mathcal{M}_n = \mathrm{Spec}\big(\mathbb{K}[\mathcal{U}_n]^{\mathrm{PGL}(n,\mathbb{K})}\big)$$

and π_n is induced by the obvious embedding of $\mathbb{K}[\mathcal{U}_n]^{\mathrm{PGL}(n,\mathbb{K})} \subset \mathbb{K}[\mathcal{U}_n]$. We call $\mathcal{M}_n$ a *twisted* $\mathrm{GL}(n, \mathbb{K})$-*character variety* of a closed Riemann surface of genus g. We will use the notation $\mathcal{M}_n/\mathbb{K}$ for the variety $\mathcal{M}_n$, when we want to emphasize the ground field $\mathbb{K}$.

Example 2.2.1. When $g = 0$ $\mathcal{M}_n$ is clearly empty, unless $n = 1$ when it is a point.

Example 2.2.2. When $n = 1$, $\mathrm{SL}(1, \mathbb{K})$ and $\mathrm{PGL}(1, \mathbb{K})$ are trivial, and so $\mathcal{M}_1 = \mathrm{GL}(1, \mathbb{K})^{2g} \cong (\mathbb{K}^\times)^{2g}$ is a torus. The mixed Hodge polynomial of $\mathcal{M}_n/\mathbb{C}$ then is

$$H(\mathcal{M}_1/\mathbb{C}; x, y, t) = (1 + xyt)^{2g} \tag{2.2.6}$$

by Theorem 2.1.2.3 and (2.1.6).

Remark 2.2.3. It will be important for us to have a spreading out $\mathcal{X}_n/R$ of the variety $\mathcal{M}_n/\mathbb{C}$ over a finitely generated $\mathbb{Z}$-algebra R. Clearly $\mathcal{U}_n$ can be defined to be an affine scheme over $R := \mathbb{Z}[\zeta_n]$. Using Seshadri's extension of geometric invariant theory quotients for schemes [63], we can take the categorical quotient by the conjugation action of the reductive group scheme $\mathrm{PGL}(n, R)$. Explicitly, $\mathcal{X}_n = \mathrm{Spec}\left(R[\mathcal{U}_n]^{\mathrm{PGL}(n,R)}\right)$. As the embedding $\phi : R \to \mathbb{C}$ is a flat morphism, [63, Lemma 2] implies that

$$R[\mathcal{U}_n]^{\mathrm{PGL}(n,R)} \otimes_R \mathbb{C} = \mathbb{C}[\mathcal{U}_n]^{\mathrm{PGL}(n,\mathbb{C})},$$

thus $\mathcal{X}_n$ is the required spreading out of $\mathcal{M}_n/\mathbb{C}$. Roughly speaking the scheme $\mathcal{X}_n/R$ will be our bridge between the varieties $\mathcal{M}_n/\mathbb{C}$ and $\mathcal{M}_n/\overline{\mathbb{F}_q}$.

We have the following immediate

Corollary 2.2.4. *The mixed Hodge polynomial $H(\mathcal{M}_n/\mathbb{C}; x, y, t)$ does not depend on the choice of the primitive n-th root of unity $\zeta_n \in \mathbb{C}$.*

Proof. As $\mathcal{M}_n/\mathbb{C}$ can be obtained by base change from $\mathcal{X}_n/R$ with $\phi : R \to \mathbb{C}$, we see that the Galois conjugate $\mathcal{M}_n^\sigma$ for any field automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ can be obtained from the same scheme $\mathcal{X}_n/R$ by extension of scalars $\sigma\phi : R \to \mathbb{C}$. Now Theorem 2.1.2.2 implies the corollary. □

Theorem 2.2.5. *The variety $\mathcal{M}_n$ is non-singular.*

Proof. Because of Example 2.2.1 we can assume $g > 0$ for the rest of the proof.

We first pr ove that the affine subvariety $\mathcal{U}_n \subset (\mathrm{GL}(n, \mathbb{K}))^{2g}$ is non-singular. By definition it is enough to show that at a solution $s = (A_1, B_1, \dots, A_g, B_g)$ of the equation

$$[A_1, B_1] \dots [A_g, B_g] = \zeta_n I_n \tag{2.2.7}$$

the derivative of μ on the tangent spaces

$$d\mu_s : T_s(\mathrm{GL}(n, \mathbb{K})^{2g}) \longrightarrow T_{\zeta_n I_n}\mathrm{SL}(n, \mathbb{K})$$

is surjective. So take $(X_1, Y_1, \dots, X_g, Y_g) \in T_s(\mathrm{GL}(n, \mathbb{K})^{2g}) \cong \mathfrak{gl}(n, \mathbb{K})^{2g}$. Then differentiate μ to get:

$$\begin{aligned}
&d\mu_s(X_1, Y_1, \dots, X_g, Y_g) \\
&= \sum_{i=1}^{g} [A_1, B_1] \dots [A_{i-1}, B_{i-1}] X_i B_i A_i^{-1} B_i^{-1} [A_{i+1}, B_{i+1}] \dots [A_g, B_g] \\
&+ \sum_{i=1}^{g} [A_1, B_1] \dots [A_{i-1}, B_{i-1}] A_i Y_i A_i^{-1} B_i^{-1} [A_{i+1}, B_{i+1}] \dots [A_g, B_g] \\
&- \sum_{i=1}^{g} [A_1, B_1] \dots [A_{i-1}, B_{i-1}] A_i B_i A_i^{-1} X_i A_i^{-1} B_i^{-1} [A_{i+1}, B_{i+1}] \dots [A_g, B_g] \\
&- \sum_{i=1}^{g} [A_1, B_1] \dots [A_{i-1}, B_{i-1}] A_i B_i A_i^{-1} B_i^{-1} Y_i B_i^{-1} [A_{i+1}, B_{i+1}] \dots [A_g, B_g],
\end{aligned}$$

where we used the product rule for matrix valued functions, in particular that $d\nu_A(X) = -A^{-1}XA^{-1}$ for the derivative of the function $\nu : \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$ defined by $\nu(A) = A^{-1}$ at $A \in \mathrm{GL}(n, \mathbb{K})$ and $X \in T_A\mathrm{GL}(n, \mathbb{K}) \cong \mathfrak{gl}(n, \mathbb{K})$. Using (2.2.7) for each of the four terms, we get:

$$d\mu_s(X_1, Y_1, \ldots, X_g, Y_g) = \sum_{i=1}^{g}(f_i(X_i) + g_i(Y_i)), \tag{2.2.8}$$

where we define linear maps $f_i : \mathfrak{gl}(n, \mathbb{K}) \to \mathfrak{sl}(n, \mathbb{K})$ and $g_i : \mathfrak{gl}(n, \mathbb{K}) \to \mathfrak{sl}(n, \mathbb{K})$ by

$$\begin{aligned} f_i(X) = \zeta_n[A_1, B_1]\ldots[A_{i-1}, B_{i-1}]\big(XA_i^{-1} - A_iB_iA_i^{-1}X\,B_i^{-1}A_i^{-1}\big) \\ \times ([A_1, B_1]\ldots[A_{i-1}, B_{i-1}])^{-1} \end{aligned}$$

and

$$\begin{aligned} g_i(X) = \zeta_n[A_1, B_1]\ldots[A_{i-1}, B_{i-1}] \\ \times \left(A_iY_iB_i^{-1}A_i^{-1} - A_iB_iA_i^{-1}B_i^{-1}Y_iA_iB_i^{-1}A_i^{-1}\right) \\ \times ([A_1, B_1]\ldots[A_{i-1}, B_{i-1}])^{-1}. \end{aligned}$$

Assume that $Z \in \mathfrak{sl}(n, \mathbb{K})$ such that

$$\mathrm{Tr}(Zd\mu_s(X_1, Y_1, \ldots, X_g, Y_g)) = 0 \tag{2.2.9}$$

for all X_i and Y_i. By (2.2.8) this is equivalent to

$$\mathrm{Tr}(Zf_i(X_i)) = \mathrm{Tr}(Zg_i(X_i)) = 0$$

for all i and $X_i \in \mathrm{GL}(n, \mathbb{K})$. We show by induction on i that this implies that Z commutes with A_i and B_i. Assume we have already proved this for $j < i$ and calculate

$$0 = \mathrm{Tr}(Zf_i(X_i)) = \mathrm{Tr}\left(\left(A_i^{-1}Z - B_i^{-1}A_i^{-1}ZA_iB_iA_i^{-1}\right)X_i\right)$$

for all X_i, thus Z commutes with $A_iB_iA_i^{-1}$. Similarly we have

$$0 = \mathrm{Tr}(Zg_i(X_i)) = \mathrm{Tr}\left(\left(B_i^{-1}A_i^{-1}ZA_i - A_iB_i^{-1}A_i^{-1}ZA_iB_iA_i^{-1}B_i^{-1}\right)X_i\right),$$

which implies that Z commutes with $A_iB_iA_iB_i^{-1}A_i^{-1}$. Thus Z commutes with A_i and B_i. The next lemma proves that this implies that Z has to be central. Because Z was traceless, we also get $Z = 0$ by (2.2.1). Thus there is no non-zero Z such that (2.2.9) holds for all X_i and Y_i. Again because of (2.2.1) this implies that $d\mu$ is surjective at any solution s of (2.2.7). Thus $\mathcal{U}_n$ is non-singular.

Lemma 2.2.6. *Suppose* $Z \in \mathfrak{gl}(n, \mathbb{K})$ *commutes with each of the* $2g$ *matrices* $A_1, B_1, \ldots, A_g, B_g \in \mathrm{GL}(n, \mathbb{K})$, *which solve* (2.2.7). *Then* Z *is central.*

Proof. If Z is not central and λ is an eigenvalue then $E = \ker(Z - \lambda I_n)$ is a proper subspace of $\mathbb{K}^n$. Because A_i and B_i all commute with Z, they preserve E. Let $\tilde{A}_i = A_i|_E$ and $\tilde{B}_i = B_i|_E$. Then restricting (2.2.7) to E we get

$$[\tilde{A}_1, \tilde{B}_1] \dots [\tilde{A}_g, \tilde{B}_g] = \zeta_n I_E.$$

As the determinant of a commutator is 1, the determinant of the right hand side has to be 1. But this implies $\zeta_n^{\dim E} = 1$, which is a contradiction as ζ_n is a primitive n-th root of unity. The lemma follows. □

This lemma also proves that if $g \in \mathrm{GL}(n, \mathbb{K})$ is not central then it acts *set-theoretically freely* on the solution space of (2.2.7). We can also deduce the following more general

Corollary 2.2.7. *The action $\bar{\sigma}$ of* $\mathrm{PGL}(n, \mathbb{K})$ *on $\mathcal{U}_n$, defined in* (2.2.4)*, is* scheme-theoretically free *(*[56, Definition 0.8 (iv)]*).*

Proof. The statement says that the map $\Psi := (\bar{\sigma}, p_2) : \mathrm{PGL}(n, \mathbb{K}) \times \mathcal{U}_n \to \mathcal{U}_n \times \mathcal{U}_n$ is a closed immersion. We prove it by an argument similar to the proof of [60, Lemma 6.5].

To prove this consider the map $\Phi : \mathcal{U}_n \times \mathcal{U}_n \to \mathrm{Hom}_{\mathbb{K}}(\mathfrak{gl}(n, \mathbb{K}), \mathfrak{gl}(n, \mathbb{K})^{2g})$, defined by

$$\begin{aligned}\Phi((A_1, B_1, \dots, A_g, B_g), &(\tilde{A}_1, \tilde{B}_1, \dots, \tilde{A}_g, \tilde{B}_g))(h) \\ &= (hA_1 - \tilde{A}_1 h, \dots, hB_g - \tilde{B}_g h).\end{aligned}$$

We show that $\Phi(x, y)$ has a non-trivial kernel if and only if $(x, y) \in \mathcal{U}_n \times \mathcal{U}_n$ is in the image of Ψ, i.e., there is an $\bar{h} \in \mathrm{PGL}(n, \mathbb{K})$ such that $y = \bar{h}x$. The if part is clear. For the other direction assume that $\Phi(x, y)$ has a non-trivial kernel, i.e., $0 \neq h \in \mathfrak{gl}(n, \mathbb{K})$, such that $\Phi(x, y)(h) = 0$. Then the matrices $A_1, B_1, \dots, A_g, B_g$, which solve (2.2.7), will leave $\ker(h)$ invariant. As in the proof of Lemma 2.2.6 this implies that $\ker(h)$ is trivial i.e., h is invertible. So indeed $\ker(\Phi(x, y)) \neq 0$ implies that there exists $\bar{h} \in \mathrm{PGL}(n, \mathbb{K})$ such that $y = \bar{h}x$. We also get that in this case $\dim(\ker \Phi(x, y)) = 1$ as the action $\bar{\sigma}$ is set-theoretically free.

Now we fix a basis for $\mathfrak{gl}(n, \mathbb{K})$, and take the closed subscheme given by the vanishing of all $n^2 \times n^2$ minors in the entries of the matrices $\Phi(x, y) \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{gl}(n, \mathbb{K}), \mathfrak{gl}(n, \mathbb{K})^{2g})$. This shows that the image of Ψ is a closed subscheme Z of $\mathcal{U}_n \times \mathcal{U}_n$.

Moreover on the Zariski open subscheme of Z where a given $(n^2 - 1) \times (n^2 - 1)$ minor of $\Psi(x, y)$ is non-zero, we can solve algebraically for the unique $\bar{h} \in \mathrm{PGL}(n, \mathbb{K})$ such that $y = \bar{h}x$, giving us locally an inverse $Z \to \mathcal{U}_n \times \mathcal{U}_n$ to Ψ; showing that Ψ is an isomorphism onto its image. The corollary follows. □

In particular Ψ is a closed map. Consequently the action is closed so [56, Amplification 1.3, Proposition 0.9] imply

Corollary 2.2.8. *The categorical quotient* $(\mathcal{M}_n, \pi_n)$ *is a geometric quotient and* π_n *in* (2.2.5) *is a* $\mathrm{PGL}(n, \mathbb{K})$*-principal bundle, in particular* π_n *is flat.*

Because the geometrical fibres of the flat morphism π_n are non-singular (they are all isomorphic to $\mathrm{PGL}(n, \mathbb{K})$) π_n is a smooth morphism by [55, Theorem III.10.3']. By [26, Corollary 17.16.3] a smooth surjective morphism locally has an étale section, so étale-locally the principal bundle $\pi_n : \mathcal{U}_n \to \mathcal{M}_n$ is trivial. As $\mathcal{U}_n$ is non-singular, we get that $\mathcal{M}_n$ is also non-singular. □

We will see in Corollary 3.5.5 that our varieties $\mathcal{U}_n$ and $\mathcal{M}_n$ are connected. Here we can determine their dimension.

Corollary 2.2.9. *For* $g > 0$ *the dimension of (each connected component of)* $\mathcal{M}_n$ *is* $d_n := n^2(2g - 2) + 2$.

Proof. From the previous proof we see that the dimension of (each connected component of) $\mathcal{U}_n$ is

$$\dim(\mathrm{GL}(n, \mathbb{K})^{2g}) - \dim(\mathrm{SL}(n, \mathbb{K})) = 2gn^2 - (n^2 - 1).$$

Because π_n is flat we have to subtract $\dim(\mathrm{PGL}(n, \mathbb{K})) = n^2 - 1$ from this to get the dimension of $\mathcal{M}_n$ proving the claim. □

Definition 2.2.10. The torus $(\mathbb{K}^\times)^{2g}$ acts on $\mathcal{U}_n \subset \mathrm{GL}(n, \mathbb{K})^{2g}$ by the following formula:

$$\begin{aligned} \tau : (\mathbb{K}^\times)^{2g} \times \mathrm{GL}(n, \mathbb{K})^{2g} &\to \mathrm{GL}(n, \mathbb{K})^{2g} \\ ((\lambda_1, \dots, \lambda_{2g}), (A_1, B_1, \dots, A_g, B_g)) &\mapsto (\lambda_1 A_1, \lambda_2 B_1, \dots, \lambda_{2g-1} A_g, \lambda_{2g} B_g). \end{aligned} \tag{2.2.10}$$

This action commutes with the action $\bar{\sigma}$. Thus $(\mathbb{K}^\times)^{2g}$ acts on $\mathcal{M}_n$. We call the categorical quotient

$$\tilde{\mathcal{M}}_n := \mathcal{M}_n // (\mathbb{K}^\times)^{2g} \cong \mathcal{U}_n // (\mathrm{PGL}(n, \mathbb{K}) \times (\mathbb{K}^\times)^{2g})$$

the twisted $\mathrm{PGL}(n, \mathbb{K})$-*character variety* of the genus g Riemann surface Σ.

Remark 2.2.11. $\tilde{\mathcal{M}}_n$ could be considered as a component of the variety of homomorphisms of $\pi_1(\Sigma)$ into $\mathrm{PGL}(n, \mathbb{K})$ modulo conjugation, this motivates its name.

Theorem 2.2.12. *The variety* $\tilde{\mathcal{M}}_n$ *is an orbifold. Each connected component of* $\tilde{\mathcal{M}}_n$ *has dimension* $\tilde{d}_n = (n^2 - 1)(2g - 2)$. *Moreover when* $\mathbb{K} = \mathbb{C}$ *its cohomology satisfies*

$$H^*(\mathcal{M}_n/\mathbb{C}) = H^*(\tilde{\mathcal{M}}_n/\mathbb{C}) \otimes H^*(\mathcal{M}_1/\mathbb{C}),$$

and the mixed Hodge polynomial satisfies:

$$H(\mathcal{M}_n/\mathbb{C}; x, y, t) = H(\tilde{\mathcal{M}}_n/\mathbb{C}; x, y, t)(1 + xyt)^{2g}. \tag{2.2.11}$$

Proof. Let $\mu' : \mathrm{SL}(n,\mathbb{K})^{2g} \to \mathrm{SL}(n,\mathbb{K})$ be given by

$$\mu'(A_1, B_1, \ldots, A_g, B_g) := [A_1, B_1] \ldots [A_g, B_g].$$

We define

$$\mathcal{U}'_n := (\mu')^{-1}(\zeta_n I_n). \tag{2.2.12}$$

The $\mathrm{PGL}(n,\mathbb{K})$-action (2.2.2) on $\mathrm{SL}(n,\mathbb{K})^{2g} \subset \mathrm{GL}(n,\mathbb{K})^{2g}$ will leave the affine variety $\mathcal{U}'_n$ invariant. We have the categorical quotient

$$\pi'_n : \mathcal{U}'_n \longrightarrow \mathcal{M}'_n, \tag{2.2.13}$$

defining the *twisted* $\mathrm{SL}(n,\mathbb{K})$*-character variety* $\mathcal{M}'_n$. Exactly as in the $\mathrm{GL}(n,\mathbb{K})$ case we can argue that $\mathcal{U}'_n$ and $\mathcal{M}'_n$ are non-singular, π'_n is a $\mathrm{PGL}(n,\mathbb{K})$-principal bundle and the components of $\mathcal{M}'_n$ have dimension

$$\dim(\mathrm{SL}(n,\mathbb{K})^{2g}) - \dim \mathrm{SL}(n,\mathbb{K}) - \dim \mathrm{PGL}(n,\mathbb{K}) = (n^2-1)(2g-2).$$

We denote by μ_n the group scheme of n-th roots of unity. $\mu_n^{2g} \subset (\mathbb{K}^\times)^{2g}$ acts on $\mathcal{U}'_n \subset \mathcal{U}_n \subset \mathrm{GL}(n,\mathbb{K})^{2g}$ induced from the action (2.2.10). It commutes with the action $\bar{\sigma}$ in (2.2.4) and so μ_n^{2g} also acts on $\mathcal{M}'_n$. Note that the map $\mathrm{SL}(n,\mathbb{K}) \times \mathbb{K}^\times \to \mathrm{GL}(n,\mathbb{K})$ given by multiplication is the categorical quotient of the action of the subgroup scheme $\mu_n = \{(\zeta_n^d I_n, \zeta_n^{-d}), d = 1, \ldots, n\} \subset \mathrm{SL}(n,\mathbb{K}) \times \mathbb{K}^\times$ on $\mathrm{SL}(n,\mathbb{K}) \times \mathbb{K}^\times$. Therefore we can identify $\mathcal{U}_n = (\mathcal{U}'_n \times (\mathbb{K}^\times)^{2g})//\mu_n^{2g}$ and taking quotients we have

$$\mathcal{M}_n \cong \left(\mathcal{M}'_n \times (\mathbb{K}^\times)^{2g}\right)//\mu_n^{2g}. \tag{2.2.14}$$

In particular we see that the categorical quotient

$$\tilde{\mathcal{M}}_n = \mathcal{M}_n//(\mathbb{K}^\times)^{2g} \cong \left(\mathcal{M}'_n \times (\mathbb{K}^\times)^{2g}\right)//\left(\mu_n^{2g} \times (\mathbb{K}^\times)^{2g}\right) \cong \mathcal{M}'_n//\mu_n^{2g} \tag{2.2.15}$$

is an orbifold of dimension $(n^2-1)(2g-2)$.

When we take cohomologies in (2.2.14) we get:

$$\begin{aligned} H^*(\mathcal{M}_n/\mathbb{C}) = \left(H^*\left(\mathcal{M}'_n/\mathbb{C} \times \mathcal{M}_1/\mathbb{C}\right)\right)^{\mu_n^{2g}} &= H^*\left(\mathcal{M}'_n/\mathbb{C}\right)^{\mu_n^{2g}} \otimes H^*(\mathcal{M}_1/\mathbb{C}) \\ &= H^*(\tilde{\mathcal{M}}_n/\mathbb{C}) \otimes H^*(\mathcal{M}_1/\mathbb{C}), \end{aligned}$$

by (2.2.15), the Künneth theorem, the fact that μ_n^{2g} acts trivially on $H^*(\mathcal{M}_1/\mathbb{C})$ and the observation of Grothendieck [25] that the rational cohomology of a quotient of a smooth variety by a finite group like (2.2.15) and (2.2.14) is the invariant part of the cohomology of the space. The theorem follows. □

For $g = 1$ we now determine our varieties $\mathcal{M}_n$ and $\tilde{\mathcal{M}}_n$ explicitly.

Lemma 2.2.13. *Let $H \subset \mathrm{GL}(n, \mathbb{K})$ be the subgroup generated by $A, B \in \mathrm{GL}(n, \mathbb{K})$ satisfying*

$$[A, B] = \zeta_n I_n, \tag{2.2.16}$$

where I_n is the identity matrix. Then the corresponding action of H on $\mathbb{K}^n$ is irreducible.

Proof. The proof is the same as in the proof of Lemma 2.2.6. □

Lemma 2.2.14. *With the notation of the previous Lemma 2.2.13 we have*

$$A^n = \alpha I_n, \quad B^n = \beta I_n, \quad \alpha, \beta \in \mathbb{K}^\times. \tag{2.2.17}$$

Proof. From (2.2.16) we easily deduce that

$$A^j B^k = \zeta_n^{jk} B^k A^j, \quad j, k \in \mathbb{Z}. \tag{2.2.18}$$

In particular, A^n and B^n are in the center of H and our claim follows from Schur's lemma and Lemma 2.2.13. □

Lemma 2.2.15. *There exists a unique solution, up to conjugation, to the equations*

$$A^n = B^n = I_n, \quad [A, B] = \zeta_n I_n, \quad A, B \in \mathrm{GL}(n, \mathbb{K}), \tag{2.2.19}$$

where I_n is the identity matrix.

Remark 2.2.16. The group H generated by the matrices in the hypothesis of the lemma is a finite Heisenberg group. The lemma is a version of the Stone–von-Neumann theorem on the uniqueness of the Heisenberg representation.

Proof. Let $v \in \mathbb{K}^n$ be an eigenvector of A, say $Av = \zeta v$ with $\zeta^n = 1$. Then $ABv = \zeta_n BAv = \zeta\zeta_n Bv$ and Bv is also an eigenvector of A. Repeating the process we see that $B^k v$ is an eigenvector of A for all $k \in \mathbb{Z}$.

Since the action of H is irreducible by Lemma 2.2.13 we must have that $v, Bv, \dots, B^{n-1}v$ is a basis of $\mathbb{K}^n$ (their span is clearly stable under H). Replacing v by an appropriate vector $B^k v$ if necessary we may assume that $\zeta = 1$. Hence in this basis A is the diagonal matrix with entries $1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1}$ along the diagonal and B is the permutation matrix corresponding to the n-cycle $(12 \cdots n)$. It is easy to verify that these particular matrices are indeed solutions to (2.2.19) and we have shown all pairs of matrices satisfying (2.2.19) are conjugate to these proving our claim. □

Theorem 2.2.17. *The orbits of the action of* $\mathrm{GL}(n, \mathbb{K})$ *acting on the solutions to*

$$[A, B] = \zeta_n I_n, \quad A, B \in \mathrm{GL}(n, \mathbb{K})$$

by conjugation are in bijection with $\mathbb{K}^\times \times \mathbb{K}^\times$ via $(A, B) \mapsto (\alpha, \beta)$ where $A^n = \alpha I_n$, $B^n = \beta I_n$. Consequently $\mathcal{M}_n \cong \mathbb{K}^\times \times \mathbb{K}^\times$ and $\tilde{\mathcal{M}}_n$ is a point when $g = 1$.

Proof. Consider the action of $(\mathbb{K}^\times)^2$ on $\mathcal{M}_n$ induced by (2.2.10). Let $x_0 \in \mathcal{M}_n$ be the point corresponding to the unique $\mathrm{PGL}(n, \mathbb{K})$ orbit in $\mathcal{U}_n$ of pairs of matrices (A, B) solving (2.2.19). Then for such a pair of matrices $\tau(\lambda_1, \lambda_2)(A, B) = (\lambda_1 A, \lambda_2 B)$ will give a solution of (2.2.16), such that (2.2.17) will hold with $\alpha = \lambda_1^n$ and $\beta = \lambda_2^n$. Because of the uniqueness of x_0, we see that the action of $(\mathbb{K}^\times)^2$ on $\mathcal{M}_n$ is transitive. Therefore $\tilde{\mathcal{M}}_n$ is a point. The stabilizer of x_0 is $\mu_n^2 \subset (\mathbb{K}^\times)^2$. It follows that $\mathcal{M}_n \cong (\mathbb{K}^\times)^2 // \mu_n^2 \cong \mathbb{K}^\times \times \mathbb{K}^\times$, and the isomorphism $\mathcal{M}_n \to \mathbb{K}^\times \times \mathbb{K}^\times$ is given by the map in the theorem. □

2.3 Counting solutions to equations in finite groups. We collect in this section various known results about counting solutions to equations in finite groups that we will need. These and similar results have appeared in the literature in many places see for example [62,17,53]. Interestingly, the first application in Frobenius's [18] of 1896, where he introduced characters of finite groups, were formulas of similar type. (Those that relate to a Riemann sphere with punctures.)

These counting formulas arise naturally, when considering Fourier transform on finite groups. This point of view will be discussed in [34], where it is shown that the counting formulas below and the one in [32] have the same origin.

Let G be a finite group. For a function

$$f : G \longrightarrow \mathbb{C}$$

we define

$$\int_G f(x)dx := \frac{1}{|G|} \sum_{x \in G} f(x)$$

Given a word $w \in F_n$, where $F_n = \langle X_1, \ldots, X_n \rangle$ is the free group in generators $X_1, \ldots, X_n$, and a function f on G as above we define

$$\{f, w\} := \int_{G^n} f(w(x_1, \ldots, x_n))dx_1 \ldots dx_n, \tag{2.3.1}$$

where $w(x_1, \ldots, x_n)$ is a shorthand for $\phi(w) \in G$ with $\phi : F_n \to G$ the homomorphism mapping each X_i to x_i.

Lemma 2.3.1. *With the above notation we have for any $z \in G$ and χ any irreducible character of G*

$$\int_{G^n} \chi(w(x_1, \ldots, x_n)z)dx_1 \ldots dx_n = \{\chi, w\} \frac{\chi(z)}{\chi(1)}. \tag{2.3.2}$$

Proof. Consider the linear endomorphism of the vector space V of a representation ρ of G with character χ

$$W := \frac{1}{|G|^n} \sum_{(x_1,\dots,x_n)\in G^n} \rho(w(x_1, \dots, x_n)).$$

Changing each x_i in the sum defining W by zx_iz^{-1} for some $z \in G$ does not change the sum. On the other hand, $w(zx_1z^{-1}, \dots, zx_nz^{-1}) = zw(x_1, \dots, x_n)z^{-1}$, hence $W = \rho(z)W\rho(z)^{-1}$. In other words, W is G-linear. By Schur's lemma W is a scalar; taking traces we find that

$$W = \frac{\{\chi, w\}}{\chi(1)} \operatorname{id}_V .$$

Multiplying both sides by $\rho(z)$ on the right and taking traces again we obtain (2.3.2). □

Proposition 2.3.2. *With the above notation let $N(z)$ be the number of solutions to*

$$w(x_1 \cdots x_n)z = 1, \quad (x_1, \dots, x_n) \in G^n, \tag{2.3.3}$$

then

$$N(z) = |G|^{n-1} \sum_{\chi} \{\chi, w\}\chi(z) \tag{2.3.4}$$

where the sum is over all irreducible characters of G.

Proof. Write the delta function on G

$$\delta(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

as a linear combination of the irreducible characters of G

$$\delta = \sum_{\chi} c_\chi \chi, \tag{2.3.5}$$

where

$$c_\chi = (\chi, \delta) = \int_G \chi(x)\delta(x)dx = \frac{\chi(1)}{|G|}.$$

On the other hand,

$$N(z) = |G|^n \int_{G^n} \delta(w(x_1, \dots, x_n)z)dx_1 \dots dx_n,$$

which combined with (2.3.5) and (2.3.2) yields our claim. □

Consider now words $w_i \in \langle X_1^{(i)}, \dots, X_{n_i}^{(i)}\rangle$ in disjoint set of variables for $i = 1, \dots, k$ and let $w = w_1 \cdots w_k \in \langle X_1^{(1)}, \dots, X_{n_1}^{(1)}, X_1^{(2)}, \dots, X_{n_2}^{(2)}, \dots\rangle$.

From Lemma 2.3.1 it follows by induction that

$$\{\chi, w\} = \frac{\{\chi, w_1\} \cdots \{\chi, w_k\}}{\chi(1)^{k-1}}. \tag{2.3.6}$$

As an application, consider $w = [x, y] = xyx^{-1}y^{-1}$. It is not hard to verify that

$$\{\chi, w\} = \frac{1}{\chi(1)}.$$

Indeed, consider the linear endomorphism of the vector space V of a representation ρ of G with character χ

$$W := \frac{1}{|G|} \sum_{x \in G} \rho(xyx^{-1}).$$

By changing variables in the sum we see that $W\rho(z) = \rho(z)W$ for all $z \in G$. Hence, by Schur's lemma W is a scalar; taking traces we find that

$$W = \frac{\chi(y)}{\chi(1)} \operatorname{id}_V .$$

Now we note that $\{\chi, w\}$ is the trace of

$$\frac{1}{|G|} \sum_{y \in G} W y^{-1} = \frac{1}{\chi(1)|G|} \sum_{y} \chi(y)\rho(y)^{-1}$$

and our claim follows. We conclude from Proposition 2.3.2 that for any[2] $g \in \mathbb{Z}_{\geq 0}$

$$\#\left\{(x_1, y_1, \ldots, x_g, y_g) \in G^{2g} \middle| [x_1, y_1] \cdots [x_g, y_g]z = 1\right\} = \sum_{\chi} \left(\frac{|G|}{\chi(1)}\right)^{2g-1} \chi(z). \tag{2.3.7}$$

Remark 2.3.3. For $z = 1$ the quantity in (2.3.7) equals $\#\operatorname{Hom}(\Gamma_g, G)$ where Γ_g is the fundamental group of a genus g Riemann surface. Hence we have

$$\#\operatorname{Hom}(\Gamma_g, G) = |G| \sum_{\chi} \left(\frac{|G|}{\chi(1)}\right)^{2g-2}, \tag{2.3.8}$$

which, in particular, implies the remarkable fact that $|G|$ always divides $\#\operatorname{Hom}(\Gamma_g, G)$ for $g > 0$.

2.4 Partitions. We collect in this section some notation and concepts on partitions that we will need later. The main reference is Macdonald's book [50].

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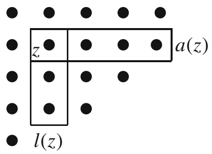
[2] In this paper we avoid the use of the notation $\mathbb{N}$ as the notion of natural numbers is different for the two authors. Instead we use the notation $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ respectively.

Let $\mathcal{P}_m$ be the set of all partitions λ of a non-negative integer $m = |\lambda|$ (where for $m = 0$ we only have the zero partition $\{0\}$) and $\mathcal{P} = \bigcup_m \mathcal{P}_m$. We write a partition of n as $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$, so that $\sum \lambda_i = n$. The *Ferrers diagram* $d(\lambda)$ of λ is the set of lattice points

$$\{(i, j) \in \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0} : j < \lambda_{-i+1}\}. \tag{2.4.1}$$

The *arm length* $a(z)$ and *leg length* $l(z)$ of a point $z \in d(\lambda)$ (sometimes called a *box*) denote the number of points strictly to the right of z and below z, respectively, as indicated in this example:



where $\lambda = (5, 5, 4, 3, 1)$, $z = (-1, 1)$, $a(z) = 3$ and $l(z) = 2$. The *hook length* then is defined as

$$h(z) = l(z) + a(z) + 1. \tag{2.4.2}$$

Given two partitions $\lambda, \mu \in \mathcal{P}$ we define

$$\langle \lambda, \mu \rangle = \sum_{j \geq 1} \lambda'_j \mu'_j, \tag{2.4.3}$$

where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ and $\mu' = (\mu'_1, \mu'_2, \ldots)$ are the dual partitions.

For $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{P}$ we define

$$n(\lambda) := \sum_{i \geq 1} (i - 1)\lambda_i \tag{2.4.4}$$

then

$$\langle \lambda, \lambda \rangle = 2n(\lambda) + |\lambda|. \tag{2.4.5}$$

Let the *hook polynomial* be (see [50, p. 152])

$$\tilde{H}_\lambda(q) = \prod (q^h - 1), \tag{2.4.6}$$

where for the product is taken for the set of boxes $d(\lambda)$ in the Ferrers diagram of λ and we let $h = h(z)$ denote the hook length of a box $z \in d(\lambda)$ as defined in (2.4.2).

It will be convenient for us to work with Laurent polynomials in $q^{\frac{1}{2}}$ and scale the hook polynomial by an appropriate power of q. Concretely, we let

$$\mathcal{H}_\lambda(q) := q^{-\frac{1}{2}\langle \lambda, \lambda \rangle} \prod (1 - q^h). \tag{2.4.7}$$

Hence with this normalization we have

$$\mathcal{H}_\lambda(q^{-1}) = (-1)^{|\lambda|}\mathcal{H}_{\lambda'}(q) \tag{2.4.8}$$

since

$$\sum h = n(\lambda) + n(\lambda') + |\lambda|. \tag{2.4.9}$$

For a non-negative integer g and a partition λ we define

$$\mathcal{H}_\lambda(z, w) := \prod \frac{(z^{2a+1} - w^{2l+1})^{2g}}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})} \tag{2.4.10}$$

a rational function in z, w, where the product runs over the boxes in $d(\lambda)$ with a and l the corresponding arm and leg length. (Typically g will be fixed and hence there is no need to indicate it in the notation. Also the context should make clear whether $\mathcal{H}_\lambda$ represents the one and two variable version.)

We note the following easily checked properties of $\mathcal{H}_\lambda$.

1.

$$\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q}) = \mathcal{H}_\lambda(q)^{2g-2}; \tag{2.4.11}$$

2.

$$\mathcal{H}_\lambda(-z, -w) = \mathcal{H}_\lambda(z, w), \qquad \mathcal{H}_\lambda(w, z) = \mathcal{H}_{\lambda'}(z, w); \tag{2.4.12}$$

3. $\mathcal{H}_\lambda$ has a Laurent series expansion in z and w^{-1}

$$\mathcal{H}_\lambda = \sum_{i \geq i_0, j \geq 0} *_{i,j} z^j w^{-i} \in \mathbb{Z}[[z, w^{-1}]][w] \tag{2.4.13}$$

with $i_0 = -(2g-2)\langle \lambda, \lambda \rangle$.

(To verify the last statement, for example, write

$$\mathcal{H}_\lambda(z, w) = w^{(2g-2)\langle\lambda,\lambda\rangle} \prod \frac{(1 - z^{2a+1}/w^{2l+1})^{2g}}{(1 - z^{2a+2}/w^{2l})(1 - z^{2a}/w^{2l+2})}$$

and expand each factor of the denominator in a geometric series.)

2.5 Formal infinite products. We will need the following formal manipulations of infinite products. For a discussion for general λ-rings see [23] whose notation we will follow.

We first define the crucial maps Exp and Log that we need. Let $K := k(x_1, \ldots, x_N)$ be the field of rational functions in the indeterminates $x_1, \ldots, x_N$ over a ground field k of characteristic zero. In the ring $K[[T]]$ of formal power series in another indeterminate T with coefficients in K we consider the following map

$$\mathrm{Exp} : TK[[T]] \longrightarrow 1 + TK[[T]] \tag{2.5.1}$$

$$V \longmapsto \exp\Big(\sum_{r\geq 1} \frac{1}{r} V\big(x_1^r, \ldots, x_N^r, T^r\big)\Big). \tag{2.5.2}$$

The map Exp has an inverse Log which we now define. Given $F \in 1 + TK[[T]]$ let $U_n \in K$ be the coefficients in the expansion

$$\log(F) =: \sum_{n\geq 1} U_n(x_1, \ldots, x_N)\frac{T^n}{n}.$$

Define

$$V_n(x_1, \ldots, x_N) := \frac{1}{n}\sum_{d|n} \mu(d) U_{n/d}\big(x_1^d, \ldots, x_N^d\big), \tag{2.5.3}$$

where μ is the ordinary Möbius function, and set

$$\mathrm{Log}(F) := \sum_{n\geq 1} V_n(x_1, \ldots, x_N)T^n.$$

We now prove that Exp and Log are indeed inverse maps.

Let $V = \sum_{n\geq 1} V_n(x_1, \ldots, x_N)T^n \in TK[[T]]$ then

$$\begin{aligned}\log(\mathrm{Exp}(V)) &= \sum_{n,r\geq 1} V_n\big(x_1^r, \ldots, x_N^r\big)\frac{T^{nr}}{r} \\ &= \sum_{n\geq 1} \frac{1}{n} \sum_{d|n} dV_d\big(x_1^{n/d}, \ldots, x_N^{n/d}\big)T^n\end{aligned}$$

so that

$$U_n(x_1, \ldots, x_N) = \sum_{d|n} dV_d\big(x_1^{n/d}, \ldots, x_N^{n/d}\big).$$

By Möbius inversion this equality is equivalent to (2.5.3). Therefore $\mathrm{Log} \circ \mathrm{Exp}(V) = V$ and similarly $\mathrm{Exp} \circ \mathrm{Log}(F) = F$ for $F \in 1 + TK[[T]]$.

Note that Exp and Log work the same way if we replace $K[[T]]$ by $S[[T]]$ where

$$S := k[[x_1, \ldots, x_N]]\big[x_1^{-1}, \ldots, x_N^{-1}\big]$$

is a Laurent series ring.

The connection with infinite products is the following one. We clearly have that $\mathrm{Exp}(V + W) = \mathrm{Exp}(V)\,\mathrm{Exp}(W)$ and

$$\mathrm{Exp}(x^m T^n) = (1 - x^m T^n)^{-1}, \quad m = (m_1, \ldots, m_N) \in \mathbb{Z}^N, n \in \mathbb{N},$$

where $x^m := x_1^{m_1} \cdots x_N^{m_N}$. Now suppose that the coefficients in $V = \sum_{n \geq 1} V_n(x_1, \ldots, x_N) T^n \in TK[[T]]$ have a Laurent expansion

$$V_n(x_1, \ldots, x_N) = \sum_m a_{m,n} x^m, \quad m = (m_1, \ldots, m_N) \in \mathbb{Z}^N, \quad a_{m,n} \in k \tag{2.5.4}$$

in S. Then formally we may write

$$\mathrm{Exp}(V) = \prod_{m,n} (1 - x^m T^n)^{-a_{m,n}}; \tag{2.5.5}$$

or, in other words, for $F \in 1 + TS[[T]]$ we may think of the coefficients in $\mathrm{Log}(F) = \sum_{m,n} a_{m,n} x^m T^n \in TS[[T]]$ as the exponents of a formal infinite product expansion of F of the form (2.5.5).

In fact, we may actually replace k by $\mathbb{Z}$. Let

$$R := \mathbb{Z}[[x_1, \ldots, x_N]]\big[x_1^{-1}, \ldots, x_N^{-1}\big].$$

Then from (2.5.5) we see that Exp maps $TR[[T]]$ to $1 + TR[[T]]$.

Similarly, Log maps $1 + TR[[T]]$ to $TR[[T]]$. Indeed, we claim that any $F \in 1 + TR[[T]]$ can be written as a formal infinite product

$$F = \prod_{m,n} (1 - x^m T^n)^{-a_{m,n}}, \quad a_{m,n} \in \mathbb{Z}. \tag{2.5.6}$$

We may in fact find the exponents $a_{m,n}$ recursively as follows. Order the m's, say lexicographically. Start with $n = 1$ and let m_0 be the smallest m such that $a_{m,1} \neq 0$. Consider $F(1 - x^{m_0} T)^{a_{m_0,1}} \in 1 + TR[[T]]$; its coefficient of $x^{m_0} T$ is zero by construction. Repeat the process with this series. In the limit we get a series say $F_1 \in 1 + TR[[T]]$ whose coefficient of T is zero. Now set $n = 2$ and start all over again with F_1. In the limit we end up with the constant series 1 from which we obtain an expression of the desired form (2.5.6). Clearly $\mathrm{Log}(F) = \sum_{m,n} a_{m,n} x^m T^n$ hence, in particular, the exponents $a_{m,n}$ are uniquely determined by F.

Usually given $F = \mathrm{Exp}(V)$ with $V \in TK[[T]]$ we have more than one choice for what Laurent series ring to consider for the expansion (2.5.4) of the coefficients of V. This may result in at first puzzlingly different infinite products for the same series F.

A typical example is the following. Let $V = T/(1 - q)$. If we expand it in a Laurent series in q we have

$$V = T \sum_{n \geq 0} q^n \quad \text{in } \mathbb{Z}[[q, T]]$$

and hence

$$F := \mathrm{Exp}(T/(1 - q)) = \prod_{n \geq 0} (1 - q^n T)^{-1}, \quad \text{in } \mathbb{Z}[[q, T]].$$

On the other hand $V = -Tq^{-1}/(1-q^{-1})$ and hence if we expand it in a Laurent series in q^{-1} we find

$$V = -T\sum_{n\geq 1} q^{-n} \quad \text{in } \mathbb{Z}[[q^{-1}, T]]$$

and hence also

$$F = \prod_{n\geq 1}(1-q^{-n}T), \quad \text{in } \mathbb{Z}[[q^{-1}, T]]. \tag{2.5.7}$$

This observation becomes important, for example, when comparing results from different sources.

3 E-polynomial of $\mathcal{M}_n$

3.1 The irreducible characters of the general linear group over a finite field. Throughout this section G_n will denote the group $\mathrm{GL}(n, \mathbb{F}_q)$ for a fixed $n \in \mathbb{Z}_{>0}$ and finite field $\mathbb{F}_q$ of cardinality q. We now recall the description of the irreducible characters of G_n following [50].

Fix an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. For each $r \in \mathbb{Z}_{>0}$ let $\mathbb{F}_{q^r}$ be the unique subfield of $\overline{\mathbb{F}_q}$ of cardinality q^r. Let $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ be the Frobenius automorphism $x \mapsto x^q$. Then $\mathbb{F}_{q^r}$ is the fixed field of Frob_q^r. For $r, s \in \mathbb{Z}_{>0}$ with $r|s$ we have the norm map $\mathbb{N}_{s,r} : \mathbb{F}_{q^s} \to \mathbb{F}_{q^r}$, which is surjective.

Let Γ_r be the character group of $\mathbb{F}_{q^r}^\times$. Composition with $\mathbb{N}_{s,r}$, when $r|s$, gives an injective map $\Gamma_r \to \Gamma_s$. Let

$$\Gamma = \varinjlim \Gamma_r$$

be the direct limit of the Γ_r via these maps. The Frobenius automorphism Frob_q acts on Γ by $\gamma \mapsto \gamma^q$. The fixed group of Frob_q^r is the image of Γ_r in Γ, which, abusing notation, we also denote by Γ_r.

Let $\mathcal{P}_m(\Gamma)$ be the set of all maps $\Lambda : \Gamma \to \mathcal{P}$ which commute with Frob_q and such that

$$|\Lambda| := \sum_{\gamma\in\Gamma} |\Lambda(\gamma)| = m.$$

Set $\mathcal{P}(\Gamma) := \bigcup_m \mathcal{P}_m(\Gamma)$. Given $\Lambda \in \mathcal{P}_m(\Gamma)$ we let $\Lambda' \in \mathcal{P}_m(\Gamma)$ be the function with values $\Lambda'(\gamma) := (\Lambda(\gamma))'$.

For $\gamma \in \Gamma$ we let $\{\gamma\}$ be its orbit in Γ under Frob_q and $d(\gamma)$ be its degree (the size of the orbit). Given $\Lambda \in \mathcal{P}_m(\Gamma)$ we let $m_{d,\lambda}$ be the multiplicity of (d, λ) in Λ, where $d \in \mathbb{Z}_{>0}$ and $0 \neq \lambda \in \mathcal{P}$. I.e.,

$$m_{d,\lambda} := \#\{\{\gamma\} | d(\gamma) = d, \Lambda(\gamma) = \lambda\};$$

for convenience we also set $m_{d,0} = 0$ for all d. We will call the collection of multiplicities $\{m_{d,\lambda}\}$ the *type* of Λ and denote it by $\tau(\Lambda)$. We will write

$$|\tau| := |\Lambda| = \sum_{d,\lambda} m_{d,\lambda} d|\lambda|.$$

There is a canonical bijection $\Lambda \mapsto \chi_\Lambda$ between $\mathcal{P}_n(\Gamma)$ and the irreducible characters of G_n. Under this correspondence, the dimension of the irreducible representation associated to Λ is

$$\chi_\Lambda(1) = \prod_{i=1}^{n}(q^i - 1) / \prod_{\{\gamma\}} q_\gamma^{-n(\Lambda(\gamma)')} \tilde{H}_{\Lambda(\gamma)}(q_\gamma) \tag{3.1.1}$$

where the product is taken over orbits $\{\gamma\}$ of Frob_q *in Γ and $q_\gamma := q^{d(\gamma)}$.*

Moreover, the value of χ_Λ on any central element αI_n with $\alpha \in \mathbb{F}_q^\times$ and $I_n \in G_n$ the identity matrix is given by

$$\chi_\Lambda(\alpha) = \Delta_\Lambda(\alpha)\chi_\Lambda(1), \tag{3.1.2}$$

where

$$\Delta_\Lambda = \prod_{\gamma\in\Gamma} \gamma^{|\Lambda(\gamma)|} \in \Gamma_1. \tag{3.1.3}$$

In particular, note that $\chi_\Lambda(1)$ only depends on the type τ of Λ; we may hence write it as $\chi_\tau(1)$. Let

$$\mathcal{H}_\tau(q) := \prod_{\{\gamma\}} \mathcal{H}_{\Lambda(\gamma)}(q_\gamma) = \prod_{d,\lambda} \mathcal{H}_\lambda(q^d)^{m_{d,\lambda}} \tag{3.1.4}$$

where $\tau = \tau(\Lambda)$. Since

$$|G_n| = q^{\frac{1}{2}n(n-1)} \prod_{i=1}^{n}(q^i - 1)$$

we have

$$\frac{|G_n|}{\chi_\tau(1)} = (-1)^n q^{\frac{1}{2}n^2} \mathcal{H}_{\tau'}(q) \tag{3.1.5}$$

where $\tau' := \tau(\Lambda')$.

Remark 3.1.1. With the above description the Alvis–Curtis duality [1,6] for characters of $GL(n, \mathbb{F}_q)$ is simply given by $\Lambda \mapsto \Lambda'$. In particular, as polynomials in q

$$q^{\frac{n(n-1)}{2}} \chi_\Lambda(1)(q^{-1}) = \chi_{\Lambda'}(1)(q).$$

3.2 Counting solutions on the general linear group. We now apply the results of Subsect. 2.3 to $G_n = \mathrm{GL}(n, \mathbb{F}_q)$ using the results of Subsect. 3.1. We specialize (2.3.7) to this case and where $z = \alpha I_n$ with $\alpha \in \mathbb{F}_q^\times$. In the resulting sum on the right hand side we collect all irreducible characters of the same type τ and obtain

$$|G_n| \left(\frac{|G_n|}{\chi_\tau(1)}\right)^{2g-2} \sum_{\tau(\Lambda)=\tau} \Delta_\Lambda(\alpha). \tag{3.2.1}$$

Our next goal is to compute the sum in the case that α is a primitive n-th root of unity. We will see that a tremendous cancelation takes place and only relatively few Λ's give a non-zero contribution.

Assume then that $\mathbb{F}_q$ contains a primitive n-th root of unity ζ_n and let

$$C_\tau := \sum_{\tau(\Lambda)=\tau} \Delta_\Lambda(\zeta_n). \tag{3.2.2}$$

To simplify the notation let

$$N_n(q) := \#\{x_1, y_1, \ldots, x_g, y_g \in \mathrm{GL}(n, \mathbb{F}_q) | [x_1, y_1] \cdots [x_g, y_g]\zeta_n = 1\}. \tag{3.2.3}$$

At this point we have in combination with (3.1.5)

$$\frac{1}{|G_n|} N_n(q) = \sum_{|\tau|=n} C_\tau \left(q^{\frac{1}{2}n^2} \mathcal{H}_{\tau'}(q)\right)^{2g-2}. \tag{3.2.4}$$

Our next task is to compute C_τ; we will find that C_τ is a constant times $(q-1)$, independent of the choice of ζ_n. In particular, this will show that $N_n(q)/|G_n|$ is a polynomial in q.

3.3 General combinatorial setup. To compute C_τ we will use the inclusion–exclusion principle on a certain partially ordered set. We first describe a slightly more general setup.

Let $I := \{1, 2, \ldots, m\}$ and let $\Pi(I)$ be the poset of partitions of I; it consists of all decompositions π of I into disjoint unions of non-empty subsets $I = \coprod_j I_j$ ordered by refinement, which we denote by $\leq$. Concretely, $\pi \leq \pi'$ in $\Pi(I)$ if every subset in π is a subset of one in π'. We call the I_j's the *blocks* of π.

The group $S(I)$ of permutations of I acts on $\Pi(I)$ in a natural way preserving the ordering $\leq$; for $\rho \in S(I)$ let $\Pi(I)^\rho$ be the subposet of $\Pi(I)$ of elements fixed by ρ.

For $\pi \in \Pi(I)$ let J be the set of its blocks and write $I = \coprod_{j \in J} I_j$. It will be convenient to also think of π as the surjection $\pi : I \to J$ that takes i to j where I_j is the unique block containing i. Then the blocks I_j are just the fibers of this map. For $\pi \in \Pi(I)^\rho$ the blocks of π are permuted by ρ. Denote by ρ_π the induced permutation in $S(J)$.

Fix a variety X defined over $\mathbb{F}_q$ and let $(I, X) := X^I$ be the variety of maps $\xi : I \to X$. We have a natural injection $S(I) \hookrightarrow \mathrm{Aut}((I, X))$. For $\rho \in S(I)$ we let $(I, X)_\rho$ be the twist of (I, X) by ρ. Its $\mathbb{F}_q$-points consists of the maps

$$\xi : I \longrightarrow X(\overline{\mathbb{F}}_q), \quad \xi \circ \rho = \mathrm{Frob}_q \circ \xi.$$

Also let $(I, X)'_\rho \subseteq (I, X)_\rho$ be the open subset of injective maps $\xi : I \to X$.

For $\pi \in \Pi^\rho(I)$ we let $(\pi, X)_\rho \subseteq (I, X)_\rho$ be the closed subset of maps $\xi : I \to X$ which are constant on the blocks of π. (This notation is consistent with our previous one if we think of I as the partition where every block has size 1, the unique minimal element of $\Pi(I)^\rho$.) There is a natural isomorphism

$$\begin{aligned} \iota_\pi : (J, X)_{\rho_\pi} &\longrightarrow (\pi, X)_\rho \\ \xi \quad &\longmapsto \quad \xi \circ \pi \end{aligned}$$

Finally, we let $(\pi, X)'_\rho \subset (\pi, X)_\rho$ be the image of $(J, X)'_{\rho_\pi}$ under ι_π. Concretely, π prescribes some equalities on the values of $\xi : I \to X$ (it is constant on the blocks of I) and $\xi \in (\pi, X)'_\rho$ if and only if these are the *only* equalities among these values. It follows that

$$(I, X)_\rho = \bigsqcup_\pi (\pi, X)'_\rho, \tag{3.3.1}$$

where π runs through the partitions in $\Pi(I)^\rho$. More generally,

$$(\pi^*, X)_\rho = \bigsqcup_{\pi \le \pi^*} (\pi, X)'_\rho, \tag{3.3.2}$$

for any $\pi^* \in \Pi(I)^\rho$.

Now take X to be a commutative algebraic group over $\mathbb{F}_q$. In particular, all the $(\pi, X)_\rho$ are subgroups of $(I, X)_\rho$ and ι_π is a group isomorphism. Fix $n \in \mathbb{Z}_{>0}$. Assume that there exists a character $\varphi : X(\mathbb{F}_q) \to \mu_n$ of exact order n. Let

$$\begin{aligned} \Phi : (I, X)_\rho &\longrightarrow \mu_n \\ \xi &\longmapsto \varphi\Big(\prod_{i\in I} \xi(i)^{\eta(i)}\Big), \end{aligned}$$

where $\eta : I \to \mathbb{Z}_{>0}$ is compatible with ρ, i.e., $\eta \circ \rho = \eta$ (or, equivalently, η is constant on the orbits of ρ) and $\sum_i \eta(i) = n$. Then Φ is a well defined character on $(I, X)_\rho$ (the argument of φ is in $X(\mathbb{F}_q)$ by the compatibility of ξ and η with ρ).

For $\pi \in \Pi(I)^\rho$ and $j \in J$ define $\eta_\pi(j) := \sum_{i\in I_j} \eta(i)$. It is easy to check that η_π is compatible with ρ_π, i.e., $\eta_\pi \circ \rho_\pi = \eta_\pi$ and also $\sum_{j\in J} \eta_\pi(j) = n$. Let Φ_π be the analogue of Φ for $(J, X)_{\rho_\pi}$ constructed using η_π. Then

$$\Phi_\pi = \Phi \circ \iota_\pi. \tag{3.3.3}$$

We will see in the next section that what we need is to compute the following sum

$$S'(I) := \sum_\xi \Phi(\xi), \tag{3.3.4}$$

where ξ runs over $(I, X)'_\rho(\mathbb{F}_q)$. Thanks to (3.3.2) we can calculate $S'(I)$ using the inclusion–exclusion principle on the poset $\Pi(I)^\rho$:

$$S'(I) = \sum_{\pi\in\Pi(I)^\rho} \mu_\rho(\pi) S(\pi), \tag{3.3.5}$$

where

$$S(\pi) := \sum_{\xi} \Phi(\xi), \tag{3.3.6}$$

with ξ running over $(\pi, X)_\rho(\mathbb{F}_q)$, and where μ_ρ is the Möbius function of $\Pi(I)^\rho$.

The advantage of (3.3.6) over (3.3.4) is that it is a complete character sum and, hence, vanishes unless the character is trivial. Using (3.3.3) we get

$$S(\pi) = \sum_{\xi} \Phi_\pi(\xi), \tag{3.3.7}$$

with ξ running over $(J, X)_{\rho_\pi}(\mathbb{F}_q)$. We can now factor $S(\pi)$ as a product over the orbits of ρ_π. Each factor is a complete character sum of the form

$$\sum_{x\in\mathbb{F}_{q^a}^\times} \varphi^b \circ \mathbb{N}_{\mathbb{F}_{q^a}/\mathbb{F}_q}(x),$$

where a and b are, respectively, the size and the common value of η_π of the corresponding orbit of ρ_π. Since φ has exact order n, by assumption, the character $\varphi^b \circ \mathbb{N}_{\mathbb{F}_{q^a}/\mathbb{F}_q}$ is trivial if and only if $n|b$; this can only happen if $|J| = 1$ because $\sum_{j\in J} \eta_\pi(j) = n$ and $\eta_\pi(j) > 0$.

It follows that $S(\pi) = 0$ unless π is the trivial partition $I = I$, the unique maximal element in $\Pi(I)^\rho$; in this case, $S(\pi) = |X(\mathbb{F}_q)|$ since $(\pi, X)_\rho(\mathbb{F}_q) = X(\mathbb{F}_q)$. In order to conclude the calculation we need to know the value of μ_ρ at the maximal element of $\Pi(I)^\rho$. For simplicity denote this by $\bar{\mu}_\rho$. Its value was computed by Hanlon [27]. Abusing notation let ρ also denote the partition of m determined by its cycle structure and write it in multiplicity notation $(1^{m_1}2^{m_2}\cdots)$ where m_d is the number of cycles of size d in ρ. Then we have

$$\bar{\mu}_\rho = \begin{cases} \mu(d)(-d)^{m_d-1}(m_d-1)! & \rho = (d^{m_d}) \\ 0 & \text{otherwise,} \end{cases} \tag{3.3.8}$$

where μ is the ordinary Möbius function. (To be sure, $\rho = (d^{m_d})$ means that ρ consists only of m_d cycles of size d for some d.) Putting this together with (3.3.5) we finally obtain

$$S'(I) = \begin{cases} |X(\mathbb{F}_q)|\mu(d)(-d)^{m_d-1}(m_d-1)! & \rho = (d^{m_d}) \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.9}$$

Note that the value of $S'(I)$ does not actually depend on the actual character φ.

Example 3.3.1. To illustrate the previous calculation consider the simplest case where ρ is the identity, i.e., assume the action of Frobenius is trivial. The situation is the following. Let X be a finite abelian group, $\varphi : X \to \mu_n$ be a character of exact order n and $(n_1, \ldots, n_m)$ be positive integers such that $n_1 + \cdots + n_m = n$. Then (3.3.9) reduces to

$$\sum_{x_i \neq x_j} \varphi\big(x_1^{n_1} \cdots x_m^{n_m}\big) = |X|(-1)^{m-1}(m-1)!,$$

which is not hard to prove directly.

3.4 Calculation of C_τ. We now apply the general setup of the previous section to compute C_τ. We start by describing all $\Lambda \in \mathcal{P}_n(\Gamma)$ with a given type τ.

For each $d \in \mathbb{Z}_{>0}$ and $0 \neq \lambda \in \mathcal{P}$ let $m_{d,\lambda}$ be the multiplicity of (d, λ) in τ. Let

$$m_d := \sum_\lambda m_{d,\lambda}, \quad m := \sum_d m_d,$$

then the *support* of Λ, i.e., those $\gamma \in \Gamma$ with $\Lambda(\gamma) \neq 0$, has size m.

Let $I := \{1, 2, \ldots, m\}$ as in the previous section and fix an element $\rho \in S(I)$ whose cycle type has m_d cycles of length d for each $d \in \mathbb{Z}_{>0}$. Fix also a map

$$\nu : I \longrightarrow \mathcal{P} \setminus \{0\}$$

which is constant on orbits of ρ, i.e., $\nu \circ \rho = \nu$, and such that for any $\lambda \in \mathcal{P} \setminus \{0\}$ and $d \in \mathbb{Z}_{>0}$ there are exactly $m_{d,\lambda}$ orbits of size d.

Given an injective map

$$\xi : I \longrightarrow \Gamma, \quad \xi \circ \rho = \mathrm{Frob}_q \circ \xi \tag{3.4.1}$$

there is a uniquely determined $\Lambda \in \mathcal{P}(\Gamma)$ satisfying

$$\Lambda \circ \xi = \nu.$$

To check that it is indeed in $\mathcal{P}(\Gamma)$ note that

$$(\Lambda \circ \mathrm{Frob}_q) \circ \xi = \Lambda \circ \xi \circ \rho = \nu \circ \rho = \nu = \Lambda \circ \xi$$

hence $\Lambda \circ \mathrm{Frob}_q = \Lambda$. Note also that by construction $\tau(\Lambda) = \tau$ and

$$n := |\Lambda| = \sum_{d,\lambda} dm_{d,\lambda}|\lambda| = \sum_{i \in I} |\nu(i)|.$$

It is clear that every $\Lambda \in \mathcal{P}_n(\Gamma)$ with $\tau(\Lambda) = \tau$ arises in this manner (ξ is just a labelling of the support of Λ and ν fixes its values) but typically in more than one way. More precisely, the assignment $\xi \mapsto \Lambda$ is a z_τ to 1 map, where z_τ is the order of the subgroup consisting of the elements of

$S(I)$ which commute with ρ and preserve ν. It is straightforward to check that

$$z_\tau = \prod_{d,\lambda} d^{m_{d,\lambda}} m_{d,\lambda}! = \prod_d d^{m_d} \prod_\lambda m_{d,\lambda}!$$

Take now $X = \mathbb{G}_m$ in the previous section. We may (non-canonically) identify Γ with $X(\overline{\mathbb{F}}_q)$; then the injective maps ξ of (3.4.1) above correspond to the elements of $(I, X)'_\rho$. Let $\varphi : \mathbb{G}_m(\mathbb{F}_q) \to \mu_n$ correspond to the order n homomorphism $\Gamma_1 \to \mu_n$ given by evaluation at $\zeta_n \in \mathbb{F}_q^\times$. Let $\eta : I \to \mathbb{Z}_{>0}$ be defined by $\eta(i) := |\nu(i)|$. Note that

$$\sum_i \eta(i) = n.$$

Then if ξ corresponds to Λ as above we have

$$\Phi(\xi) = \Delta_\Lambda(\zeta_n)$$

and therefore

$$C_\tau = \frac{1}{z_\tau} S'(I).$$

Hence by (3.3.9)

$$C_\tau = \begin{cases} (-1)^{m_d-1}(q-1)\frac{\mu(d)}{d}\frac{(m_d-1)!}{\prod_\lambda m_{d,\lambda}!} & \rho = (d^{m_d}) \\ 0 & \text{otherwise} \end{cases} \tag{3.4.2}$$

independently of the choice of ζ_n.

3.5 Main formula. Let

$$E_n(q) := \frac{N_n(q)}{|\mathrm{PGL}(n, \mathbb{F}_q)|}. \tag{3.5.1}$$

As we remarked at the end of 3.2 E_n is a polynomial in q. To see this it is enough to plug in (3.4.2) into (3.2.4).

Theorem 3.5.1. *The variety $\mathcal{M}_n/\mathbb{C}$ has polynomial count and its E-polynomial satisfies*

$$E(\mathcal{M}_n/\mathbb{C}; x, y) = E_n(xy).$$

Proof. From the definition (2.2.3) of $\mathcal{U}_n$ it is clear that it can be viewed as a closed subscheme $\mathcal{X}$ of $\mathrm{GL}(n)^{2g}$ over the ring $R := \mathbb{Z}[\zeta_n, \frac{1}{n}]$. Note that we have extended the base ring in Remark 2.2.3. Let $\varphi : R \to \mathbb{C}$ be an embedding, then $\mathcal{X}$ is a spreading out of $\mathcal{U}_n/\mathbb{C}$.

For every homomorphism

$$\phi : R \longrightarrow \mathbb{F}_q \tag{3.5.2}$$

the image $\phi(\zeta_n)$ is a primitive n-th root of unity in $\mathbb{F}_q$, because the identity

$$\prod_{i=1}^{n-1} \left(\zeta_n^i - 1\right) = n$$

guarantees that $1 - \zeta_n^i$ is a unit in R for $i = 1, \dots, n-1$, and therefore cannot be zero in the image. (This is why we have extended our ring R from the one in Remark 2.2.3.) Hence all of our previous considerations apply to compute $\#\mathcal{X}_\phi(\mathbb{F}_q) = N_n(q)$.

On the other hand the group scheme $\mathrm{PGL}(n, R)$ acts on $\mathcal{X}$ by conjugation. We define the affine scheme $\mathcal{Y} = \mathrm{Spec}(R[\mathcal{X}]^{\mathrm{PGL}(n,R)})$ over R. Because $\varphi : R \to \mathbb{C}$ is a flat morphism [63, Lemma 2] implies that $\mathcal{Y}$ is a spreading out of $\mathcal{M}_n/\mathbb{C}$ over R.

Now take an $\mathbb{F}_q$-point of the scheme $\mathcal{Y}_\phi$, obtained from $\mathcal{Y}$ by the extensions of scalars in (3.5.2). By [44, Lemma 3.2] the fiber over it in $\mathcal{X}_\phi(\mathbb{F}_q)$ is non-empty and an orbit of $\mathrm{PGL}(n, \mathbb{F}_q)$. The same argument as in the proof of Theorem 2.2.5 shows that $\mathrm{PGL}(n, \mathbb{F}_q)$ acts freely on $\mathcal{X}_\phi(\mathbb{F}_q)$. Consequently

$$\#\mathcal{Y}_\phi(\mathbb{F}_q) = \frac{\#\mathcal{X}_\phi(\mathbb{F}_q)}{\#\mathrm{PGL}(n, \mathbb{F}_q)} = \frac{N_n(q)}{|\mathrm{PGL}(n, \mathbb{F}_q)|} = E_n(q).$$

Thus $\mathcal{M}_n/\mathbb{C}$ has polynomial count. Now the theorem follows from Theorem 6.1.2.3. □

Let us write

$$E_n(q) := \sum_k e_k^n q^k.$$

We also consider the following normalized version of E_n

$$\overline{E}_n(q) := q^{-\frac{1}{2}d_n} E_n(q) := \sum_k \bar{e}_k^n q^k \tag{3.5.3}$$

a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$.

It will be more convenient to work with the following modified quantity

$$V_n(q) := \frac{q}{(q-1)^2} \overline{E}_n(q) = \frac{q^{-(g-1)n^2}}{(q-1)^2} E_n(q) = q^{-(g-1)n^2} \frac{N_n(q)}{(q-1)|\mathrm{GL}(n, \mathbb{F}_q)|}, \tag{3.5.4}$$

(recall that $d_n = \dim(\mathcal{M}_n) = (2g-2)n^2 + 2$). For $g > 0$ this is a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$; for $g = 0$ we have $N_1(q) = 1$ and $N_n(q) = 0$ for $n > 1$. In this case $V_1 = q/(q-1)^2 = \sum_{n \geq 1} nq^n$ is a power series in $\mathbb{Z}[[q]]$ and $V_n = 0$ for $n > 1$. Also $\overline{E}_1 = E_1 = 1$ and $\overline{E}_n = E_n = 0$ for $n > 1$.

By the formalism of Subsect. 2.5 if we let

$$V := \sum_{n\geq 1} V_n(q)T^n, \qquad V_n(q) =: \sum_{k\in\mathbb{Z}} v_k^n q^k, \quad v_k^n \in \mathbb{Z}$$

then

$$\operatorname{Exp}(V) = \prod_{n\geq 1}\prod_{k\in\mathbb{Z}}(1-q^kT^n)^{-v_k^n}. \tag{3.5.5}$$

Define

$$\overline{E} := \sum_{n\geq 1}\overline{E}_n(q)T^n,$$

and so by (3.5.4)

$$\frac{q}{(q-1)^2}\overline{E} = V. \tag{3.5.6}$$

Taking Exp of (3.5.6) we get

$$\operatorname{Exp}(V) = \prod_{j,n\geq 1}\prod_{k\in\mathbb{Z}}(1-q^{k+j}T^n)^{-j\bar{e}_k^n}. \tag{3.5.7}$$

The main result is the following

Theorem 3.5.2. *For every $g \geq 0$ we have*

$$\sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(q)^{2g-2}T^{|\lambda|} = \prod_{j,n\geq 1}\prod_{k\in\mathbb{Z}}(1-q^{k+j}T^n)^{-j\bar{e}_k^n}. \tag{3.5.8}$$

Proof. Take the logarithm of the left hand side and write the resulting coefficient of T^n as $U_n(q)/n$. Using the multinomial theorem we find that

$$\frac{U_n}{n} = \sum_{m_\lambda}(-1)^{m-1}(m-1)!\prod_\lambda \frac{\mathcal{H}_\lambda^{(2g-2)m_\lambda}}{m_\lambda!}, \quad m = \sum_\lambda m_\lambda \tag{3.5.9}$$

where the sum is over all $m_\lambda \in \mathbb{Z}_{\geq 0}$ satisfying

$$\sum_\lambda m_\lambda|\lambda| = n. \tag{3.5.10}$$

On the other hand comparing (3.5.9) with (3.2.4) after plugging in the value of C_τ from (3.4.2) we obtain

$$V_n(q) := \frac{1}{n}\sum_{d|n} U_{n/d}(q^d)\mu(d). \tag{3.5.11}$$

An application of the (usual) Möbius inversion shows that

$$U_n(q) := \sum_{d|n} dV_{n/d}(q^d).$$

We have then

$$\sum_{n\geq 1} U_n(q)\frac{T^n}{n} = \sum_{n,r\geq 1} \frac{1}{r} V_n(q^r)T^{nr}$$

which together with (3.5.5) and (3.5.7) imply our claim. □

The following is an immediate corollary of this result:

Corollary 3.5.3 (Curious Poincaré duality)**.**

$$\overline{E}_n(q^{-1}) = \overline{E}_n(q) \tag{3.5.12}$$

Proof. Inverting q in (3.5.8) does not change the left hand side by (2.4.8). Hence looking at the right hand side we see that $\bar{e}^n_{-k} = \bar{e}^n_k$, which is equivalent to our claim. □

Remark 3.5.4. We should point out that the above duality satisfied by $\overline{E}_n$ is ultimately a direct consequence of the Alvis–Curtis duality (3.1.1) for characters of $\mathrm{GL}(n, \mathbb{F}_q)$.

Corollary 3.5.5. *The variety $\mathcal{M}_n/\mathbb{C}$ is connected.*

Proof. The statement is clear for $g = 0, 1$ by Example 2.2.1 and Theorem 2.2.17. Therefore we can assume $g > 1$ for the rest of the proof.

Corollary 2.2.9 says that each connected component of $\mathcal{M}_n$ has dimension d_n. Thus the leading coefficient of $E(\mathcal{M}_n/\mathbb{C}; q)$ is the number of components of $\mathcal{M}_n$. By Theorem 3.5.1 $E(\mathcal{M}_n/\mathbb{C}; q) = E_n(q)$, so it is enough to determine the leading coefficient of $E_n(q)$.

For this recall the definition of $U_n(q)$ from (3.5.9). It is a Laurent polynomial in q. In order to determine its lowest degree term, we see that the lowest degree term of the summands in (3.5.9) are

$$n(-1)^{m-1}\frac{(m-1)!}{\prod_\lambda m_\lambda!}q^{-(g-1)\sum_\lambda \langle\lambda,\lambda\rangle m_\lambda}.$$

Lemma 3.5.6. *The maximum of $\sum_\lambda \langle\lambda, \lambda\rangle m_\lambda$ under the constraint* (3.5.10) *occurs only when $m_\lambda = 1$ for $\lambda = (1^n)$ and $m_\lambda = 0$ otherwise.*

Proof. To see this recall (2.4.3) that

$$\langle\lambda, \lambda\rangle = \sum_i \lambda_i'^2.$$

Consider a point in the simplex $\Delta : \sum_i x_i = n, x_i \geq 0$ in $\mathbb{R}^n$ with $\sum_\lambda m_\lambda m_i(\lambda')$ coordinates equal to i. (Here $m_i(\lambda')$ is the multiplicity of i in λ'.) It now suffices to notice that the maximum distance to the origin on Δ occurs at a vertex. □

The lemma implies that the lowest degree term of $U_n(q)$ is $nq^{-(g-1)n^2}$. Formula (3.5.11) now implies that the lowest degree term of $V_n(q)$ is $q^{-(g-1)n^2}$. The definition (3.5.4) gives that the constant term of the polynomial $E_n(q)$ is 1. By (3.5.12) the leading term of $E_n(q)$ is q^{d_n}. The corollary follows. □

3.6 Special cases. We first work out the E-polynomial of $\mathcal{M}_n$ when $n = 1, 2$ from our generating function (3.5.8). Evaluating (3.5.9) we get that $U_1(q) = q^{-(g-1)}(1-q)^{2g-2}$. A short calculation yields $E_1(q) = (1-q)^{2g}$, so by Theorem 3.5.1

$$E(\mathcal{M}_1; x, y) = (1-xy)^{2g},$$

which is consistent with (2.2.6).

For $n = 2$ we again evaluate (3.5.9) to get

$$\frac{U_2(q)}{2} = -\frac{1}{2}\mathcal{H}_{(1)}^{2(2g-2)}(q) + \mathcal{H}_{(11)}^{2g-2}(q) + \mathcal{H}_{(2)}^{2g-2}(q)$$

substituting the hook polynomials (2.4.7) we get

$$\begin{aligned}\frac{U(2)}{2} = \frac{1}{2}&q^{2g-2}(1-q)^{4g-4} + q^{-4g-4}(1-q)^{2g-2}(1-q^2)^{2g-2}\\ &+ q^{-(2g-2)}(1-q)^{2g-2}(1-q^2)^{2g-2}.\end{aligned}$$

Using (3.5.11) combined with (3.5.6), (3.5.3), and Theorem 3.5.1 we get

Corollary 3.6.1. *The E-polynomial of $\mathcal{M}_2/\mathbb{C}$ is*

$$E(\mathcal{M}_2/\mathbb{C}; x, y) = E_2(xy),$$

where

$$\begin{aligned}E_2(q) = &-\frac{1}{2}q^{(2g-2)}(1-q)^{4g-2} + (1-q)^{2g}(1-q^2)^{2g-2}\\ &+ q^{2g-2}(1-q)^{2g}(1-q^2)^{2g-2} - \frac{1}{2}q^{2g-2}(1-q)^2(1-q^2)^{2g-2}.\end{aligned}$$

It is also instructive to consider the special cases $g = 0, 1$ of the theorem in detail. For $g = 0$ the identity (3.5.8) becomes

$$\sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(q)^{-2}T^{|\lambda|} = \prod_{j\geq 1}(1-q^jT)^{-j}. \tag{3.6.1}$$

This formula follows from known combinatorial identities. Indeed

$$s_\lambda = \pm q^{n(\lambda)} H_\lambda(q)^{-1}$$

where s_λ is the Schur function associated to λ evaluated at $x_i = q^{i-1}$ (see [50, I.3 ex. 2]) and

$$H_\lambda(q) = \prod (1 - q^h)$$

is the hook polynomial. Plugging in $x_j = Tq^j$ in the second formula of [50, §I.4 ex. 2] yields (3.6.1). This agrees with our previous calculation: $\bar{e}_k^n = 1$ for $n = 1, k = 0$ and zero otherwise.

For $g = 1$ the identity (3.5.8) becomes

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} = \prod_{n \geq 1} \prod_{r > 0} \prod_{s \geq 0} \frac{(1 - q^{r+s} T^n)^2}{(1 - q^{r+s-1} T^n)(1 - q^{r+s+1} T^n)}, \tag{3.6.2}$$

which by (3.5.5) simplifies to

$$\sum_{n \geq 0} p(n) T^n = \prod_{n \geq 1} (1 - T^n)^{-1}$$

where $p(n)$ is the number of partitions of n (this is an identity of Euler).

Remark 3.6.2. We deduce that $V_n = 1$ for all n, when $g = 1$. Therefore $E(\mathcal{M}_n/\mathbb{C}^n; q) = (q-1)^2$ and $E(\tilde{\mathcal{M}}_n/\mathbb{C}^n; q) = 1$. Because $\tilde{\mathcal{M}}_n$ is zero dimensional by Theorem 2.2.12 it follows that $\tilde{\mathcal{M}}_n$ is a point (cf. Theorem 2.2.17).

3.7 Euler characteristic. We now prove Corollary 1.1.1.

Proof. By (2.2.11), the E-polynomial of $\tilde{\mathcal{M}}_n/\mathbb{C}$ is given by

$$E(\tilde{\mathcal{M}}_n/\mathbb{C}; x, y) = \frac{E(\mathcal{M}_n/\mathbb{C}; x, y)}{(xy-1)^{2g}}.$$

By Remark 2.1.6, Theorem 3.5.1 and Theorem 3.5.2 the Euler characteristic of $\tilde{\mathcal{M}}_n/\mathbb{C}$ equals

$$\left. \frac{N_n(q)}{(q-1)^{2g} |\mathrm{PGL}(n, \mathbb{F}_q)|} \right|_{q=1}. \tag{3.7.1}$$

We should point out that the rational function in q in (3.7.1) is actually a polynomial, the E-polynomial of $\tilde{\mathcal{M}}_n/\mathbb{C}$.

In terms of V_n we get that (3.7.1) equals

$$\left. \frac{V_n(q)}{(q-1)^{2g-2}} \right|_{q=1}. \tag{3.7.2}$$

We certainly have that $(q-1)^n$ divides $\mathcal{H}_\lambda(q)$ for any partition λ of n. Hence, in the notation of the proof of (3.5.8) $(q-1)^{(2g-2)n}$ divides $U_n(q)$ (note that by assumption $2g-2>0$) and it follows from (3.5.11) that $(q-1)^{2g-2}$ divides $V_n(q)$.

We now see that the only contribution in (3.5.11) to (3.7.2) can come from the term $d=n$ and

$$\left.\frac{V_n(q)}{(q-1)^{2g-2}}\right|_{q=1} = \left.\frac{\mu(n)U_1(q^n)}{n(q-1)^{2g-2}}\right|_{q=1}.$$

But $U_1(q)=(q-1)^{2g-2}$ since $\mathcal{H}_{(1)}(q)=q-1$ for the unique partition (1) of 1. We conclude that the Euler characteristic of $\tilde{\mathcal{M}}_n/\mathbb{C}$ is $\mu(n)n^{2g-3}$ finishing the proof. □

3.8 The untwisted case. From the above calculation we may now actually deduce the number of solutions to the untwisted equation (see Remark 2.3.3)

$$\begin{aligned}&\#\operatorname{Hom}(\Gamma_g,\mathrm{GL}(n,\mathbb{F}_q))\\ &\qquad = \#\{x_1,y_1,\ldots,x_g,y_g\in\mathrm{GL}(n,\mathbb{F}_q) \mid [x_1,y_1]\cdots[x_g,y_g]=1\}.\end{aligned}$$

Assume $g>0$ since otherwise Γ_g is trivial. We prove

Theorem 3.8.1. *If $\Gamma_g=\pi_1(\Sigma)$ is the fundamental group of a closed Riemann surface of genus $g>0$, then, using the formalism of Subsect. 2.5, we have*

$$\sum_{n\geq 0}\frac{\#\operatorname{Hom}(\Gamma_g,\mathrm{GL}(n,\mathbb{F}_q))}{q^{(g-1)n^2}|\mathrm{GL}(n,\mathbb{F}_q)|}T^n = \operatorname{Exp}\left((q-1)\operatorname{Log}\left(\sum_{\lambda\in\mathcal{P}}\mathcal{H}_\lambda(q)^{2g-2}T^{|\lambda|}\right)\right). \tag{3.8.1}$$

Remark 3.8.2. This gives an explicit formula for the number of representations of $\pi_1(\Sigma)$ to $\mathrm{GL}(n,\mathbb{F}_q)$. The asymptotics for these numbers as n tends to infinity has been studied in [49].

Proof. One way to express the main formula (3.5.8) is in terms of zeta functions of colorings as in [61], whose notation we will follow. We consider colorings on $X=\mathbb{G}_m$ with values on partitions and weight function

$$W(\lambda) := \mathcal{H}_\lambda^{2g-2}(q)\in\mathbb{Z}[q,q^{-1}].$$

We recognize the left hand side of (3.5.8) as $Z_C(\bullet,q,T)$ with this setup. Hence the main formula (3.5.8) can be written (in the notation of Subsect. 2.5) as

$$\operatorname{Log}\left(Z_C(\bullet,q,T)\right) = V = \sum_{n\geq 1}V_n(q)T^n. \tag{3.8.2}$$

Similarly, by (2.3.8) and (3.1.5) we find that

$$Z_C(\mathbb{G}_m, q, T) = \sum \frac{\#\operatorname{Hom}(\Gamma_g, \operatorname{GL}(n, \mathbb{F}_q))}{q^{(g-1)n^2}|\operatorname{GL}(n, \mathbb{F}_q)|} T^n. \tag{3.8.3}$$

In particular this implies that

$$\#\operatorname{Hom}(\Gamma_g, \operatorname{GL}(n, \mathbb{F}_q))/|\operatorname{GL}(n, \mathbb{F}_q)| \in \mathbb{Z}[q] \ \text{ if } g > 0$$

since $W(\lambda)$ is a Laurent polynomial. This is consistent with the observation at the end of Remark 2.3.3 that for $g > 0$ $|G|$ always divides $\#\operatorname{Hom}(\Gamma_g, G)$.

Using formula [61, (24)] we deduce from (3.8.2) that

$$\operatorname{Log}(Z_C(\mathbb{G}_m, q, T)) = (q-1) \sum_{n \ge 1} V_n(q) T^n.$$

If we take Exp of both sides we get

$$Z_C(\mathbb{G}_m, q, T) = \operatorname{Exp}((q-1)V).$$

This proves (3.8.1) assuming that $\mathbb{F}_q$ contains a primitive n-th root of unity. However, as we pointed out, the coefficients on the left hand side of (3.8.1) are Laurent polynomials in q so the statement is true for all q. The theorem follows. □

4 Mixed Hodge polynomial of $\mathcal{M}_n$

4.1 Cohomology of $\mathcal{M}_n$. In this section we take $\mathbb{K} = \mathbb{C}$. According to Theorem 2.2.12 $H^*(\mathcal{M}_n) = H^*(\tilde{\mathcal{M}}_n) \otimes H^*(\mathcal{M}_1)$, where $\mathcal{M}_1 \cong (\mathbb{C}^\times)^{2g}$, and the factor $H^*(\mathcal{M}_1)$ is generated by $2g$ degree 1 classes $\epsilon_j \in H^1((\mathbb{C}^\times)^{2g})$ for $j = 1, \ldots, 2g$. By slight abuse of notation we use the same notation for the corresponding classes in $\epsilon_j \in H^1(\mathcal{M}_n)$ for $j = 1, \ldots, 2g$.

To get more interesting cohomology classes on $\mathcal{M}_n$, we construct cohomology classes in $H^*(\tilde{\mathcal{M}}_n) \cong H^*(\mathcal{M}'_n)^{\mu_n^{2g}}$. We construct a differentiable principal bundle over $\mathcal{M}'_n \times \Sigma$ by following [43]. Let $\bar{G} = \operatorname{PGL}(n, \mathbb{C})$. Any $\rho \in \mathcal{U}'_n$ induces a well-defined homomorphism $\pi_1(\Sigma) \to \bar{G}$. Let $\tilde{\Sigma}$ be the universal cover of Σ, which is acted on by $\pi_1(\Sigma)$ via deck transformations. There is then a free action of $\pi_1(\Sigma) \times \operatorname{GL}(n, \mathbb{C})$ on $\bar{G} \times \mathcal{U}'_n \times \tilde{\Sigma}$ given by

$$(p, g) \cdot (h, \rho, x) = (\bar{g}\rho(p)h, \bar{g}\rho\bar{g}^{-1}, p \cdot x),$$

where $\bar{g}$ denotes the image of g in $\bar{G}$. This action commutes with the action of μ^{2g} on $\mathcal{U}'_n$. The quotient is the desired (μ^{2g}-equivariant) principal $\bar{G}$-bundle on $\mathcal{M}'_n$, which we denote by $\mathbb{U}$. Like any principal $\bar{G}$-bundle, it has characteristic classes $\bar{c}_2(\mathbb{U}), \ldots, \bar{c}_r(\mathbb{U})$, where $\bar{c}_i(\mathbb{U}) \in H^{2i}(\mathcal{M}'_n \times \Sigma)^{\mu_n^{2g}}$. In terms of formal Chern roots ξ_k, $\bar{c}_i$ can be described as the i-th elementary symmetric polynomial in the $\xi_k - \zeta$, where ζ is the average of the ξ_k. In particular $\bar{c}_1 = 0$.

Now let $\sigma \in H^2(\Sigma)$ be the fundamental cohomology class, and let $e_1, \ldots, e_{2g}$ be a standard symplectic basis of $H^1(\Sigma)$. In terms of these, each of the characteristic classes has a Künneth decomposition

$$\bar{c}_i(\mathbb{U}) = \alpha_i \sigma + \beta_i + \sum_{j=1}^{2g} \psi_{i,j} e_j, \tag{4.1.1}$$

defining classes $\alpha_i \in H^{2i-2}(\mathcal{M}'_n)^{\mu_n^{2g}} \subset H^{2i-2}(\mathcal{M}_n)$, $\beta_i \in H^{2i}(\mathcal{M}'_n)^{\mu_n^{2g}} \subset H^{2i}(\mathcal{M}_n)$, and $\psi_{i,j} \in H^{2i-1}(\mathcal{M}'_n)^{\mu_n^{2g}} \subset H^{2i-1}(\mathcal{M}_n)$ for $i = 2, \ldots, n$. In [51] Markman proves that

Theorem 4.1.1. *The classes ϵ_j; α_i, $\psi_{i,j}$ and β_i generate $H^*(\mathcal{M}_n)$.*

Construction 4.1.2. For what follows we need the following construction. Let $f : Y \to X$, $x \in X$ and $F = f^{-1}(x)$. Then we have the following commutative diagram

$$\begin{tikzcd}
H^i(Y) \arrow[r, "i_F^*"] & H^i(F) \arrow[r, "d"] & H^{i+1}(Y, F) \arrow[r, "i_Y^*"] & H^{i+1}(Y) \\
 & & H^{i+1}(X, x) \arrow[u, "f^*"] \arrow[r, phantom, "\overset{i_X^*}{\cong}"] & H^{i+1}(X). \arrow[u, "f^*"'] \arrow[ul, "q^*"']
\end{tikzcd}$$

Here the first row is the cohomology long exact sequence of the pair (Y, F). The second row is the cohomology long exact sequence of the pair (X, x), the map i_X^* is an isomorphism for the map $H^i(X) \to H^i(x)$ is always surjective. Finally $q^* = f^*(i_X^*)^{-1} : H^{i+1}(X) \to H^{i+1}(Y, F)$. By the commutativity of the diagram q^* induces a map $\ker(f^*) \to \ker(i_Y^*) \cong \operatorname{im}(d) \cong \operatorname{coker}(i_F^*)$. We denote the resulting map

$$\sigma^f : \ker(f^*) \longrightarrow \operatorname{coker}\left(i_F^*\right).$$

We can also give the map σ in terms of cochains. Let $x \in C^{i+1}(X)$ be a cocycle. Then $f^*(x)$ will be a cocyle in $C^{i+1}(Y)$ vanishing on F. If $[x] \in \ker(f^*)$ then $f^*(x)$ is exact. Let $\tilde{y} \in C^i(Y)$ be a cochain such that $d\tilde{y} = f^*(x)$. Let $y = i_F^*(\tilde{y})$. Then $dy = i_F^*(f^*(x)) = 0$, so y is a cocycle. We can define $\sigma([x]) = [y]$.

Example 4.1.3. When f is a fibration the map σ is called the *suspension map* (see [52, §8.2.2 p. 298]). When the fibration is the path fibration of the space X then $Y = PX$ is the based path space and so contractible, while F can homotopically be identified with the based loop space ΩX. In this case the suspension map $\sigma : H^{i+1}(X) \to H^i(\Omega X)$ can be identified with the map $\sigma = p_* ev^*$, where $ev : S^1 \times \Omega X \to X$ is the evaluation map and $p : S^1 \times \Omega X \to \Omega X$ is the projection. A particular case of the path fibration is the universal bundle: $\pi : EG \xrightarrow{G} BG$ for a connected G complex linear group G. Here $G \sim \Omega BG$ and $EG \sim PBG$. The suspension map then is a map $\sigma^\pi : H^{i+1}(BG) \to H^i(G)$.

Remark 4.1.4. We will also need an equivariant version of this construction. If we assume that G is a topological group, which acts on (X, x) and Y in a way so that f is equivariant, then we have the same diagram and construction above in equivariant cohomology. This way we get the equivariant map

$$\sigma_G^f : \ker_G(\pi^*) \longrightarrow \operatorname{coker}_G\left(i_F^*\right).$$

In particular, G acts on itself by conjugation and consequently on BG and EG making the fibration π equivariant. We will then have the equivariant suspension map

$$\sigma_G^\pi : H_G^{i+1}(BG) \longrightarrow H_G^i(G). \tag{4.1.2}$$

Lemma 4.1.5. *When X, Y are complex algebraic varieties and f is algebraic then* $\ker(f^*)$, $\operatorname{coker}(i_F^*)$ *have natural mixed Hodge structures, and σ preserves it. Additionally if a complex linear group G acts on (X, x) and Y so that f is equivariant then* $\ker_G(f^*)$, $\operatorname{coker}_G(i_F^*)$ *have natural mixed Hodge structures, and σ_G preserves it.*

Proof. Deligne in [9, Example 8.3.8] constructs a mixed Hodge structure on relative cohomology, and shows in [9, Proposition 8.3.9] that all maps in the cohomology long exact sequence of a pair preserve mixed Hodge structure. The first statement follows.

For the second statement Deligne constructs in [9, Theorem 9.1.1] a mixed Hodge structure on $H^*(BG)$, by considering a model for BG as a simplicial scheme. Similarly one can construct the Borel construction $X \times_G EG$ as a simplicial scheme, which will give a mixed Hodge structure on $H_G^*(X) = H^*(X \times_G EG)$ see e.g. [16]. Then we see that all maps in the equivariant cohomology sequence of a pair preserve mixed Hodge structures. In turn we get that σ_G too preserves mixed Hodge structures. $\square$

Definition 4.1.6. We say that a cohomology class $\gamma \in H^i(X) = H^i(X, \mathbb{Q})$ or $\gamma \in H^i(X; \mathbb{R})$ has *homogenous weight k* if its complexification satisfies $\gamma^{\mathbb{C}} = \gamma \otimes 1 \in W_{2k} H^i(X)^{\mathbb{C}} \cap F^k H^i(X; \mathbb{C})$.

Remark 4.1.7. Note that if $\gamma \in H^i(X)$ and $\gamma^{\mathbb{C}} \in W_l^{\mathbb{C}} \cap F^m$ with $2m > l$ then $Gr_l^{W^{\mathbb{C}}} \cap F^m \cap \overline{F^m} = 0$ by (2.1.1) and $\overline{\gamma^{\mathbb{C}}} = \gamma^{\mathbb{C}}$ imply that $\gamma^{\mathbb{C}} \in W_{l-1}^{\mathbb{C}}$. By induction we have $\gamma = 0$. Thus we get that

$$\begin{gathered}\text{if } \gamma \in H^i(X) \text{ has homogenous weight } k \text{ and} \\ \gamma^{\mathbb{C}} \in F^{k+1} \text{ or } \gamma \in W_{2k-1} \text{ then } \gamma = 0.\end{gathered} \tag{4.1.3}$$

In particular, a non-zero cohomology class cannot have different homogenous weights. Moreover as the cup-product preserves mixed Hodge structures by Theorem 2.1.2.3 we have that if γ_1 has homogenous weight l_1 and γ_2 has homogenous weight l_2 then $\gamma_1 \cup \gamma_2$ has homogenous weight $l_1 + l_2$. In particular we see that if the cohomology $H^*(X)$ of an algebraic variety is generated by classes with homogenous weight then the MHS on $H^*(X)$ will be of type (p, p) i.e.

$$W_l H^*(X)^{\mathbb{C}} \cap F^m H^*(X; \mathbb{C}) = 0, \text{ when } 2m > l. \tag{4.1.4}$$

Namely if $0 \neq x = \sum_k a_k + ib_k \in W_l H^*(X)^{\mathbb{C}} \cap F^m H^*(X; \mathbb{C})$ with $a_k, b_k \in H^*(X; \mathbb{R})$ homogenous of weight k then we can consider $k_{min} := \min_k\{a_k + ib_k \neq 0\}$ and $k_{max} = \max_k\{a_k + ib_k \neq 0\}$ and get $m \leq k_{min} \leq k_{max} \leq l/2$ from (4.1.3).

Finally for a complex algebraic map $f : X \to Y$ the map $f^* : H^*(Y) \to H^*(X)$ preserves mixed Hodge structures, we have that if $\alpha \in H^*(Y)$ has homogenous weight l so does $f^*(\alpha)$.

Now we determine the weights of the universal generators. First we know from (2.2.6) that the homogenous weight of ϵ_j is 1. To determine the weight of the rest of the universal classes we will use Jeffrey's [43] group cohomology description of them as interpreted in [4,54,59].

We note that [43,4,54,59] work with the compact groups $\mathrm{SU}(n)$, however the arguments are correct with complex groups too. Another way to see that Jeffrey's formulas (4.1.5), (4.1.6) and (4.1.7) for the universal classes are valid for $G := \mathrm{SL}(n, \mathbb{C})$ is to note that Lemma 4.1.12 below implies that the natural inclusion map of the twisted $\mathrm{SU}(n)$-character variety into the twisted $\mathrm{SL}(n, \mathbb{C})$-character variety $\mathcal{M}'_n$ induces an isomorphism on (μ_n^{2g}-invariant) cohomology below degree $2(g-1)(n-1)+2$. Now $2(g-1)(n-1)+2$ is larger than the degree of any universal class, except possibly of β_n (which has degree $2n$), when $g = 2$. However Jeffrey's formula for β_n is trivially correct for the complex character varieties as we will see below. Another difference in our application of [43,4,54,59] is that we work on the level of cohomology instead of differential forms or cochains, but our cohomological interpretation of [43,4,54,59] is straightforward using the last paragraph in Construction 4.1.2.

The easiest is to determine the weight of the β_k. By their construction $\beta_k = c_k(\mathbb{U}|_{\mathcal{M}'_n \times \{p\}})$ are the Chern classes of the differentiable $\mathrm{PGL}(n, \mathbb{C})$-bundle $\mathbb{U}$ constructed above, restricted to $\mathcal{M}'_n \times \{p\}$, where p is a point on Σ. It is straightforward to identify $\mathbb{U}|_{\mathcal{M}'_n \times \{p\}}$ with the $\bar{G}$-bundle $\pi'_n : \mathcal{U}'_n \to \mathcal{M}'_n$ in (2.2.13), thus

$$\beta_k = c_k\big(\mathcal{U}'_n\big). \tag{4.1.5}$$

Now π'_n is an algebraic principal bundle, therefore its Chern classes are pulled back from $H^*(B\bar{G})$ by a complex algebraic map. It now follows from [9, Theorem 9.1.1] that the homogenous weight of β_k is indeed k.

We next determine the weight of the $\psi_{k,j}$. Let $\bar{c}_k \in H_G^{2k}(B\bar{G})$ be the k-th equivariant Chern class of the $\bar{G}$-equivariant bundle $\pi : E\bar{G} \to B\bar{G}$. Clearly $H_G^*(B\bar{G}) \cong H^*(B(G \ltimes_\phi \bar{G}))$, where $\phi : G \to \mathrm{Aut}(\bar{G})$ is given by conjugation. By [9, Theorem 9.1.1] $\bar{c}_k$ has homogeneous weight k. Using the map (4.1.2) we construct the class $\eta_G^k = \sigma_G(\bar{c}_k) \in H_G^{2k-1}(G)$. It follows that η_G^k has homogenous weight k. Let $p_j : G^{2g} \to G$ be the projection to the j-th factor, which is equivariant with respect to the conjugation action of G. Thus $p_j^*(\eta_G^k) \in H_G^{2k-1}(G^{2g})$ has homogenous weight k. If i denotes the G-equivariant embedding of $\mathcal{U}'_n$ into G^{2g}, then we have that $i^*p_j^*(\eta_G^k) \in H_G^{2k-1}(p)$ has homogenous weight k. Now [59, Theorem 3.2] implies that

$$\psi_{k,j} = i^*p_j^*\left(\eta_G^k\right) \in H_G^{2k-1}\left(\mathcal{U}'_n\right) \cong H^{2k-1}\left(\mathcal{M}'_n\right). \tag{4.1.6}$$

Thus $\psi_{k,j}$ has homogenous weight k as claimed.

To calculate the weight of α_k we recall Construction 4.1.2 for the case $Y = G^{2g}$, $X = G$, $\pi = \mu'$ and $x = \zeta_n I_n \in G$. Then $F = \mathcal{U}'_n$. This gives a map $\sigma_G : \ker_G(\pi^*) \to \mathrm{coker}_G(i_F^*)$. Now it follows from [59, Lemma 2.4] that

$$\eta_G^k \in \ker_G(\pi^*) \subset H_G^{2k-1}(G)$$

and also by [59, Theorem 3.2] that

$$\sigma_G^\pi\left(\eta_G^k\right) = p(\alpha_k), \tag{4.1.7}$$

where

$$p : H_G^{2k-2}\left(\mathcal{U}'_n\right) \longrightarrow \mathrm{coker}_G^{2k-2}\left(i_F^*\right)$$

denotes the projection. We know that the homogenous weight of η_G^k is k and so $p(\alpha_k)$ has homogenous weight k. By the previous paragraph $\mathrm{im}(i_F^*) \subset H_G^*(\mathcal{U}'_n) \cong H^*(\mathcal{M}'_n)$ is exactly the subring generated by the $\psi_{k,j}$ and β_k for $k = 2, \dots, n$ and $j = 1, \dots, 2g$. This shows in particular that p is an isomorphism when $k = 2$. Thus the homogenous weight of $\alpha_2 \in H^2(\mathcal{M}'_n)$ is 2.

We summarize our findings in the following

Proposition 4.1.8. *The cohomology classes ϵ_j have homogenous weight* 1*, while $\psi_{k,j}$, β_k have homogenous weight k. Finally α_2 has homogenous weight* 2 *and $p(\alpha_k) \in \mathrm{coker}_G^{2k-2}(i_F^*)$ have homogenous weight k.*

Remark 4.1.9. It is most probably true that α_k has homogenous weight k even for $k > 2$, the result for $p(\alpha_k)$ however will suffice for our purposes. Here we show that $p(\alpha_k) \neq 0$. By the previous paragraph $\mathrm{im}(i_F^*) \subset H_G^*(\mathcal{U}'_n) \cong H^*(\mathcal{M}'_n)$ is exactly the subring generated by the $\psi_{k,j}$ and β_k for $k = 2, \dots, n$ and $j = 1, \dots, 2g$. Because the degree of α_k is $2k - 2 \leq 2n - 2 \leq 2(g - 1)(n - 1)$ Lemma 4.1.12 below implies that $\alpha_k \notin \mathrm{im}(i_F^*)$ i.e. $p(\alpha_k) \neq 0$.

Corollary 4.1.10. *The pure part* $PH^*(\mathcal{M}_n) = \bigoplus_k W_k H^k(\mathcal{M}_n)$ *is generated by the classes* $\beta_i \in H^{2i}(\mathcal{M}_n)$ *for* $i = 2, \ldots, n$.

Proof. The previous proposition shows that among the $\psi_{i;j}$ and β_i only the classes $\beta_i \in H^{2i}(\mathcal{M}_n)$ are pure classes, i.e., have pure homogenous weight i. This shows that the pure part of the subring $\operatorname{im}(i^*F) \subset H^*_G(\mathcal{U}'_n) \cong H^*(\mathcal{M}'_n)$ they generate is generated by the β_i classes. Moreover the μ_n^{2g}-invariant part of $\operatorname{coker}_G(i^*_F)$ is generated by the classes $p(\alpha_i)$ none of which has pure homogenous weight. Thus the pure part of $H^*(\mathcal{M}'_n)^{\mu_n^{2g}} \cong H^*(\tilde{\mathcal{M}}_n)$ is generated by the classes β_i. By Theorem 2.2.12 the result follows. □

Corollary 4.1.11. *The cohomology of* $\mathcal{M}_n$ *is of type* (p, p)*, i.e.,* $h^{p,q;j}(\mathcal{M}_n) = 0$ *unless* $p = q$*. In particular* $H(\mathcal{M}_n; x, y, t)$ *is a polynomial in* xy *and* t*. In the notation of* (2.1.3)

$$H(\mathcal{M}_n; x, y, t) = H(\mathcal{M}_n; xy, t).$$

Proof. By Remark 4.1.7 and Proposition 4.1.8 we know that both the μ_n^{2g}-invariant part of $\operatorname{coker}_G(i^*_F)$, which is generated by the classes $p(\alpha_i)$ and the subring $\operatorname{im}(i^*_F) \subset H^*_G(\mathcal{U}'_n)^{\mu_n^{2g}} \cong H^*(\mathcal{M}'_n)^{\mu_n^{2g}}$ generated by the β_i and $\psi_{i;j}$ have MHS of type (p, p) in other words (4.1.4) holds. Thus $H^*(\mathcal{M}'_n)^{\mu_n^{2g}} \cong H^*(\tilde{\mathcal{M}}_n)$ has MHS of type (p, p). By Theorem 2.2.12 so does $H^*(\mathcal{M})$. □

Lemma 4.1.12. *There are no relations among the universal generators in the cohomology of* $H^*(\mathcal{M}'_n)$ *until degree* $2(g-1)(n-1)+2$.

Proof. This follows from the same statement for the twisted $\mathrm{SU}(n)$ character variety, which in turn follows from [2, (7.16)]. □

4.2 Main conjecture. Recall the definition of the $\mathcal{H}_\lambda$ from (2.4.10) and its properties thereafter.

Let $U_n(z, w)$ be defined by

$$\log\Big(\sum_\lambda \mathcal{H}_\lambda(z, w) T^{|\lambda|}\Big) = \sum_{n \geq 0} U_n(z, w) \frac{T^n}{n}.$$

As in (3.5.9) we find that

$$\frac{U_n(z, w)}{n} = \sum_{m_\lambda} (-1)^{m-1}(m-1)! \prod_\lambda \frac{\mathcal{H}_\lambda(z, w)^{m_\lambda}}{m_\lambda!}, \quad m = \sum_\lambda m_\lambda \tag{4.2.1}$$

where the sum is over all $m_\lambda \in \mathbb{Z}_{\geq 0}$ satisfying

$$\sum_\lambda m_\lambda |\lambda| = n. \tag{4.2.2}$$

Expanding $U_n(z,w)$ in Laurent series in z, w^{-1} as in (2.4.13) we see that the leading term in w^{-1} of the summand is

$$n(-1)^{m-1}\frac{(m-1)!}{\prod_\lambda m_\lambda!}w^{(2g-2)\sum_\lambda\langle\lambda,\lambda\rangle m_\lambda}.$$

From Lemma 3.5.6 it follows that the leading term of U_n in w^{-1} is $nw^{(2g-2)n^2}$.

Let

$$V_n(z,w) := \frac{1}{n}\sum_{d|n}\mu(d)U_{n/d}(z^d,w^d). \tag{4.2.3}$$

By the formalism explained in Subsect. 2.5 we know that

$$\sum_\lambda \mathcal{H}_\lambda(z,w)T^{|\lambda|} = \exp\left(\sum_{k,n\geq 1} V_n(z^k,w^k)\frac{T^{nk}}{k}\right). \tag{4.2.4}$$

From our previous calculation we deduce that the leading term in w^{-1} of V_n is $w^{(2g-2)n^2}$.

Let also

$$\overline{H}_n(z,w) := (z^2-1)(1-w^2)V_n(z,w). \tag{4.2.5}$$

Both V_n and $\overline{H}_n$ are rational functions of z and w. We should remark that by (2.4.11) we have

$$V_n(\sqrt{q},1/\sqrt{q}) = V_n(q)$$

and therefore

$$\overline{H}_n(\sqrt{q},1/\sqrt{q}) = \overline{E}_n(q). \tag{4.2.6}$$

From (2.4.12) we deduce that

$$V_n(w,z) = V_n(z,w), \quad V_n(-z,-w) = V_n(z,w)$$

and

$$\overline{H}_n(w,z) = \overline{H}_n(z,w), \quad \overline{H}_n(-z,-w) = \overline{H}_n(z,w). \tag{4.2.7}$$

We expand V_n and $\overline{H}$ as Laurent series in $z, 1/w$

$$V_n(z,w) = \sum_{i\geq i_0, j\geq 0} v^n_{i,j} z^j w^{-i}, \quad \overline{H}_n(z,w) = \sum_{i\geq -d_n, j\geq 0} \overline{h}^n_{i,j} z^j w^{-i}$$

where $i_0 = -(2g-2)n^2$ and hence $i_0 - 2 = -d_n = -\dim(\mathcal{M}_n)$. Our calculation of leading terms implies that $\overline{h}^n_{-d_n,j} = 1$ for $j = 0$ and is 0 otherwise.

In terms of these coefficients we can write our generating function as the infinite products

$$\sum_{\lambda} \mathcal{H}_{\lambda}(z,w)T^{|\lambda|} = \prod_{n\geq 1}\prod_{i\geq i_0,\, j\geq 0}(1-z^j w^{-i}T^n)^{-v^n_{i,j}}$$
$$= \prod_{n\geq 1}\prod_{r>0, s\geq 0}\prod_{i\geq -d_n,\, j\geq 0}(1-z^{2s+j}w^{-(2r+i)}T^n)^{-\overline{h}^n_{i,j}}. \tag{4.2.8}$$

Our main conjecture is the following

Conjecture 4.2.1

$$H(\mathcal{M}_n;q,t) = (t\sqrt{q})^{d_n}\,\overline{H}_n\left(\sqrt{q}, \frac{-1}{t\sqrt{q}}\right). \tag{4.2.9}$$

Remark 4.2.2. In view of (4.2.6) and (4.2.7) Conjecture 4.2.1 is true specialized to $t=-1$ as it reduces to (3.5.8).

Because of the second identity in (4.2.7) and because d_n is even by Corollary 2.2.9, we have that the RHS of (4.2.9) is actually a rational function in q. The geometric Conjecture 4.2.1 implies the following combinatorial conjectures

Conjecture 4.2.3. 1. $\overline{H}_n(z,w)$ is a polynomial in z, w.

2. The coefficients $(-1)^j\overline{h}^n_{i,j}$ of $\overline{H}_n(z,-w)$ are non-negative integers.

In light of (4.2.7), our main Conjecture 4.2.1 implies the following.

Conjecture 4.2.4 (Curious Poincaré duality).

$$H\left(\mathcal{M}_n;\frac{1}{qt^2},t\right) = (qt)^{-d_n}H(\mathcal{M}_n;q,t)$$

Remark 4.2.5. When $t=-1$, this formula specializes to the known Corollary 3.5.12.

Remark 4.2.6. On the level of mixed Hodge numbers this conjecture is equivalent to

$$h^{p,p;k}(\mathcal{M}_n) = h^{d_n-p,d_n-p;d_n+k-2p}(\mathcal{M}_n). \tag{4.2.10}$$

Because $\mathcal{M}_n$ is non-singular, $h^{p,p;k}(\mathcal{M}_n)=0$ for $2p<k$. Dually (4.2.10) implies that $h^{p,p;k}(\mathcal{M}_n)=0$ for $k>d_n$. The vanishing of the cohomology of $\mathcal{M}_n$ above middle dimension can be deduced from the fact that $\mathcal{M}_n$ is diffeomorphic to the space of twisted flat $\mathrm{GL}(n,\mathbb{C})$-connections on the Riemann surface Σ, which is a Stein manifold with its natural hyperkähler metric [40].

In particular (4.2.10) implies that the pure mixed Hodge numbers $h^{p,p;2p}(\mathcal{M}_n)$ should be curious Poincaré dual to $h^{d_n-p,d_n-p;d_n}(\mathcal{M}_n)$, i.e., to the mixed Hodge numbers of the middle (top non-vanishing) cohomology of $\mathcal{M}_n$.

Finally we have a geometric conjecture which would imply the above curious Poincaré duality. Define the Lefschetz map

$$L : H^i(\tilde{\mathcal{M}}_n) \longrightarrow H^{i+2}(\tilde{\mathcal{M}}_n)$$

by $x \mapsto \alpha \cup x$, where $\alpha = \alpha_2$ is the universal class in $H^2(\tilde{\mathcal{M}}_n)$ defined in (4.1.1). As it respects mixed Hodge structures by Theorem 2.1.2.3 and α has homogenous weight 2 by Proposition 4.1.8 it defines a map on the graded pieces of the homogenous weight filtration $L : Gr_l^W H^i(\tilde{\mathcal{M}}_n) \to Gr_{l+4}^W H^{i+2}(\tilde{\mathcal{M}}_n)$.

Conjecture 4.2.7 (Curious hard Lefschetz). Recall that $\tilde{d}_n = \dim(\tilde{\mathcal{M}}_n) = (n^2-1)(2g-2)$. Then

$$L^l : Gr^W_{\tilde{d}_n-2l} H^{i-l}(\tilde{\mathcal{M}}_n) \longrightarrow Gr^W_{\tilde{d}_n+2l} H^{i+l}(\tilde{\mathcal{M}}_n)$$

is an isomorphism.

Remark 4.2.8. Here we prove a consequence of this conjecture. As $\tilde{\mathcal{M}}_n$ is an orbifold and the non-trivial weights in the weight filtration on $H^*(\tilde{\mathcal{M}}_n)$ are even by Proposition 4.1.8, we have that for $0 < k \le \tilde{d}_n/2$

$$Gr^W H^{\tilde{d}_n/2-k}(\tilde{\mathcal{M}}_n) = \bigoplus_{j=0}^{[\tilde{d}_n/4-k/2]} Gr^W_{d_n-2k-2j} H^{\tilde{d}_n/2-k}(\tilde{\mathcal{M}}_n).$$

Conjecture 4.2.7 says that the map

$$L^{k+j} : Gr^W_{d_n-2k-2j} H^{\tilde{d}_n/2-k}(\tilde{\mathcal{M}}_n) \longrightarrow Gr^W_{d_n+2k+2j} H^{\tilde{d}_n/2+k+j}(\tilde{\mathcal{M}}_n)$$

is an isomorphism. This implies that

$$L^k : Gr^W_{d_n-2k-2j} H^{\tilde{d}_n/2-k}(\tilde{\mathcal{M}}_n) \longrightarrow Gr^W_{d_n+2k-2j} H^{\tilde{d}_n/2+k}(\tilde{\mathcal{M}}_n)$$

is injective. Thus Conjecture 4.2.7 implies that the map

$$L^k : H^{\tilde{d}_n/2-k}(\tilde{\mathcal{M}}_n) \longrightarrow H^{\tilde{d}_n/2+k}(\tilde{\mathcal{M}}_n)$$

is an injection. This statement follows from [30, Corollary 4.3] (cf. also [30, Remark 4.4]) when applied to the moduli space of Higgs bundles diffeomorphic to $\tilde{\mathcal{M}}_n$.

4.3 Special cases of the main conjecture. First we verify the cases of $n = 1, 2$ of Conjecture 4.2.1. From (4.2.3) and (4.2.1)

$$V_1(z, w) = U_1(z, w) = \frac{(z-w)^{2g}}{(z^2-1)(1-w^2)}.$$

By (4.2.5)

$$\overline{H}_1(z,w) = (z-w)^{2g}.$$

Thus Conjecture 4.2.1 implies

$$H_1(\mathcal{M}_1, q, t) = (t\sqrt{q})^{2g}\left(\sqrt{q} + \frac{1}{t\sqrt{q}}\right)^{2g} = (1+tq)^{2g},$$

which checks with (2.2.6).

From (4.2.1) we have

$$\frac{U_2(z,w)}{2} = -\frac{1}{2}\mathcal{H}^{2(2g-2)}_{(1)}(z,w) + \mathcal{H}^{2g-2}_{(11)}(z,w) + \mathcal{H}^{2g-2}_{(2)}(z,w).$$

Combining (4.2.3), (2.4.10) and (4.2.5)

$$\begin{aligned}\overline{H}_2(z,w) = &-\frac{1}{2}\frac{(z-w)^{4g}}{(z^2-1)(1-w^2)} + \frac{(z^3-w)^{2g}(z-w)^{2g}}{(z^4-1)(z^2-w^2)}\\ &+ \frac{(z-w^3)^{2g}(z-w)^{2g}}{(z^2-w^2)(1-w^4)} - \frac{1}{2}\frac{(z^2-w^2)^{2g}}{(z^2+1)(1+w^2)}.\end{aligned}$$

Substituting $z = \sqrt{q}$ and $w = \frac{-1}{t\sqrt{q}}$ we see that Theorem 1.1.3, proved in Subsect. 5.2, is equivalent to Conjecture 4.2.1 for $n = 2$.

Next we consider the special cases of $g = 0, 1$. For $g = 0$ we have $\mathcal{M}_n$ is a point for $n = 1$ and is empty otherwise. Hence

$$H(\mathcal{M}_n; q, t) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and according to the conjecture (4.2.1) we find

$$\overline{h}^n_{i,j} = \begin{cases} 1 & n = 1, i = j = 0 \\ 0 & \text{otherwise} \end{cases}$$

hence, after replacing z^2 by z and w^2 by w, we should have

$$\sum_\lambda \frac{1}{\prod(z^{a+1} - w^l)(z^a - w^{l+1})} T^{|\lambda|} = \prod_{r>0,s\geq 0}(1 - z^s w^{-r} T)^{-1}. \qquad (4.3.1)$$

In fact we can prove this identity.

Theorem 4.3.1. *The identity* (4.3.1) *is true.*

Proof. We know from [20, Thm. 3.10 (f)] that

$$\sum_{|\lambda|=n} \frac{1}{\prod (w^l - z^{a+1})(z^a - w^{l+1})} = \sum_{|\lambda|=n} \frac{z^{n(\lambda')} w^{n(\lambda)}}{\prod (1-z^h)(1-w^h)}, \qquad (4.3.2)$$

where $h = a + l + 1$ is the hook length. On the other hand we know [50, I.3 ex. 2] that

$$s_\lambda(1, x, x^2, \dots) = \frac{x^{n(\lambda)}}{\prod (1-x^h)}$$

where s_λ is the Schur function and hence

$$s_\lambda(1, 1/x, 1/x^2, \dots) = \frac{(-x)^{|\lambda|} x^{n(\lambda')}}{\prod (1-x^h)}.$$

Summing over all n we then find

$$\sum_\lambda \frac{1}{\prod (w^l - z^{a+1})(z^a - w^{l+1})} T^{|\lambda|} = \sum_\lambda s_\lambda(1, z, z^2, \dots) s_\lambda(T/w, T/w^2, T/w^3, \dots)$$

and by Cauchy's formula [50, I (4.3)] this equals the right hand side of (4.3.1). □

Now let us consider the case $g = 1$. We have that $\mathcal{M}_n \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ for all n (see Theorem 2.2.17). Hence

$$H(\mathcal{M}_n; q, t) = (1 + qt)^2$$

and according to Conjecture 4.2.1 we should have

$$\overline{H}_n(z, w) = (z - w)^2, \quad n \in \mathbb{Z}_{>0}.$$

Consequently Conjecture 4.2.1 implies

Conjecture 4.3.2. The following identity holds

$$\sum_\lambda \prod \frac{(z^{2a+1} - w^{2l+1})^2}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})} T^{|\lambda|} = \prod_{n \geq 1} \prod_{r > 0} \prod_{s \geq 0} \frac{(1 - z^{2s+1} w^{-2r+1} T^n)^2}{(1 - z^{2s} w^{-2r+2} T^n)(1 - z^{2s+2} w^{-2r} T^n)}. \qquad (4.3.3)$$

Remark 4.3.3. The conjecture is a purely combinatorial one. The specialization $z = \sqrt{q}$, $w = 1/\sqrt{q}$ is essentially Euler's identity which we already encountered in (3.6.2). We also prove below in Remark 4.4.3 the specialization $z = 0$, $w = \sqrt{q}$.

We have checked (4.3.3) numerically up to the T^6 terms. For this it is more convenient to write in its additive form (4.2.4)

$$\sum_{\lambda} \prod \frac{(z^{2a+1} - w^{2l+1})^2}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})} T^{|\lambda|}$$
$$= \exp\left(\sum_{k \geq 1} \frac{(z^k - w^k)^2}{(z^{2k} - 1)(1 - w^{2k})(1 - T^k)} \frac{T^k}{k} \right)$$

and check that the coefficient of T^n on both sides (as rational functions in z, w) agree.

4.4 Purity conjecture

Theorem 4.4.1. *Let $A_n(q)$ be the number of absolutely indecomposable g-tuples of n by n matrices over the finite field $\mathbb{F}_q$ modulo conjugation. Then*

$$\overline{H}_n(0, \sqrt{q}) = A_n(q). \tag{4.4.1}$$

Proof. It is immediate to verify that

$$\mathcal{H}_\lambda(0, \sqrt{q}) = \frac{q^{(g-1)\langle\lambda,\lambda\rangle}}{b_\lambda(1/q)}, \tag{4.4.2}$$

where $b_\lambda(q) = \prod_{i \geq 1} (1 - q) \cdots (1 - q^{m_i})$ with m_i is the multiplicity of i in λ.

It follows that the left hand side of (4.2.8) for $z = 0,\ w = \sqrt{q}$ equals the left hand side of Hua's formula [41, Theorem 4.9] for the S_g quiver. On the right hand side we get

$$\prod_{n \geq 1} \prod_{r > 0, i \geq -d_n} (1 - q^{-(r+i)} T^n)^{-\overline{h}^n_{2i,0}}$$

(note that $\overline{h}_{i,0} = 0$ for i odd thanks to (2.4.12)). By the formalism of (2.5.7) we may rewrite this as

$$\prod_{n \geq 1} \prod_{r \geq 0, i \leq d_n} (1 - q^{r+i} T^n)^{\overline{h}^n_{-2i,0}}.$$

Comparing with the right hand side of Hua's formula we deduce that $\overline{h}_{-2i,0} = t^n_i$ proving our claim. □

Remark 4.4.2. Combining Theorem 4.4.1 and Conjecture 4.2.1 is what we call the *purity conjecture*: the pure part of the mixed Hodge polynomial of the character variety $\mathcal{M}_n$ is the reverse of the A-polynomial of the quiver S_g (a vertex with g loops) with dimension vector n. By a result of Kac [45] $A_n(q)$, and therefore also $\overline{H}_n(0, \sqrt{q})$, is a polynomial in q, which is implied by Part (1) of Conjecture 4.2.3. Then Part (2) of Conjecture 4.2.3 implies non-negativity of the coefficients of $A_n(q)$, which is Conjecture 2 of Kac [45] for the S_g quiver with dimension n at the vertex. Since S_g with this dimension vector is divisible (for $n > 1$) the conjecture is still open (the indivisible case was proved in [5]). To summarize this discussion we can claim: the purity conjecture implies Kac's [45, Conjecture 2] for the quiver S_g. In [34] a detailed discussion, motivation and the origin for this and more general purity conjectures will be given.

Remark 4.4.3. For $g = 0$ Theorem 4.4.1 implies that Hua's formula [41, (5.1)] is the specialization of (4.3.1) at $z = 0$. On the other hand, for $g = 1$ the theorem shows that Hua's formula [41, (5.2)] is the specialization $z = 0$, $w = \sqrt{q}$ of our conjecture (4.3.3).

Proposition 4.4.4. *For all $n, g > 0$ we have that $q^{(g-1)n+1}$ divides $A_n(q)$ and*

$$\left.\frac{A_n(q)}{q^{(g-1)n+1}}\right|_{q=0} = 1.$$

Proof. This is a consequence of Hua's formula but for convenience we will express the result in our notation using (4.4.1). For a partition λ of $n > 0$ we have from (4.4.2)

$$\mathcal{H}_\lambda(0, \sqrt{q}) = \frac{q^{(g-1)\langle\lambda,\lambda\rangle+m}}{\prod_{i\geq 1}(q-1)(q^2-1)\cdots(q^{m_i}-1)}$$

where

$$m := \sum_{i\geq 1}\binom{m_i+1}{2}$$

with $m_i = m_i(\lambda)$ the multiplicity of i in λ. Among all partitions λ of n the exponent $(g-1)\langle\lambda,\lambda\rangle + m$ of q takes its minimum value $(g-1)n+1$ only for $\lambda = (n)$. In particular, $q^{(g-1)n+1}$ divides $\mathcal{H}_\lambda(0, \sqrt{q})$ and, moreover,

$$\left.\frac{\mathcal{H}_{(n)}(0, \sqrt{q})}{q^{(g-1)n+1}}\right|_{q=0} = -1.$$

After some calculation we find that

$$\left.\frac{V_n(0, \sqrt{q})}{q^{(g-1)n+1}}\right|_{q=0} = -1,$$

which combined with (4.2.5) proves our claim. $\square$

Remark 4.4.5. One consequence of Proposition 4.4.4 is that the purity conjecture or more generally our main Conjecture 4.2.1 implies, that the largest non-trivial degree of $PH^*(\mathcal{M}_n)$ is $2(g-1)n(n-1)$. Interestingly, [12, Theorem 7 and Proposition 9] proves the same about the "pure ring" of the twisted $U(n)$-character variety $\mathcal{N}_n^d$, i.e., the subring generated by the classes β_k. This and the known situation for $n=2$ (see the next section) indicates that the "pure ring" of $\mathcal{N}_n^d$ maybe isomorphic with $PH^*(\mathcal{M}_n)$. An interesting consequence of this would be that the "pure ring" of $\mathcal{N}_n^d$ is independent of d, unlike the whole cohomology $H^*(\mathcal{N}_n^d)$, which does depend on d. Finally, combining the reasoning above with the purity conjecture suggests that the "pure ring" of $\mathcal{N}_n^d$ could also be used for a cohomological interpretation of the A-polynomials $A_n(q)$, implying [45, Conjecture 2] for the S_g quiver.

Remark 4.4.6. If we combine the purity conjecture with Remark 4.2.6, we get that the middle cohomology of $\mathcal{M}_n$ should have dimension $A_n(1)$. We list below the formulas for the value of $A_n(1)$ for $n=2,3$ and 4 as a polynomial in $\chi=2g-2$ obtained by computer calculations:

$$\begin{aligned} A_2(1) &= \frac{1}{2}\chi+1 \\ A_3(1) &= \frac{1}{2}\chi^2+\frac{3}{2}\chi+1 \\ A_4(1) &= \frac{2}{3}\chi^3+\frac{5}{2}\chi^2+\frac{17}{6}\chi+1. \end{aligned}$$

It is known that the middle cohomology of $\mathcal{M}_2$ has dimension $g=\frac{1}{2}(2g-2)+1$ by [40] and that the middle Betti number of $\mathcal{M}_3$ is $2g^2-g=\frac{1}{2}(2g-2)^2+\frac{3}{2}(2g-2)+1$ dimensional [24]. For $n\geq 4$ the middle Betti number of $\mathcal{M}_n$ is not known. However one can say something about the leading coefficient of $A_n(1)$ as a polynomial in χ. In the above formulas it is $\frac{n^{n-3}}{(n-1)!}$ (a proof of this fact will appear elsewhere). One can also guess the leading coefficient of $\dim H^{d_n}(\mathcal{M}_n)$ as a function of $\chi=2g-2$. The dimension $\dim H^{d_n}(\mathcal{M}_n)$ is exactly the number of fixed point components of the natural circle-action on the corresponding moduli space of Higgs bundles. These fixed point components are not well understood in general, but one class of fixed point components, the so-called type $(1,1,\ldots,1)$ is well understood (see the proof of [38, Proposition 10.1]). Their number turns out to be a degree $n-1$ polynomial in χ with leading coefficient $\frac{n^{n-3}}{(n-1)!}$, which is the volume of a certain skew hypercube, given by inequalities dictated by the stability condition for Higgs bundles of type $(1,1,\ldots,1)$, which appear in the proof of [38, Proposition 10.1]. As the rest of the fixed point components are expected to be counted by a polynomial in χ of degree less then $n-1$, the quantity $\frac{n^{n-3}}{(n-1)!}$ should be the leading coefficient of $\dim H^{d_n}(\mathcal{M}_n)$, in agreement with the prediction coming from the above conjecture.

4.5 Intersection form. Another consequence of Conjecture 4.2.1 and Proposition 4.4.4 is that the $\tilde{d}_n$ (=middle) dimensional cohomology of $\tilde{\mathcal{M}}_n$ does not have pure part. This implies the following

Corollary 4.5.1. *Conjecture 3.5.2 implies that the intersection form on $H_c^{\tilde{d}_n}(\tilde{\mathcal{M}}_n)$ is trivial. Equivalently the forgetful map $H_c^*(\tilde{\mathcal{M}}_n) \to H^*(\tilde{\mathcal{M}}_n)$ is* 0.

Proof. Conjecture 3.5.2 and Proposition 4.4.4 implies that there is no pure part in $H^{\tilde{d}_n}(\tilde{\mathcal{M}}_n)$, consequently all the non-trivial weights in the weight filtration are $> \tilde{d}_n$. Now Poincaré duality (2.1.2) implies that $H_c^{\tilde{d}_n}(\tilde{\mathcal{M}}_n)$ has no pure part either; consequently all the non-trivial weights in the weight filtration $< \tilde{d}_n$. However Theorem 2.1.3.1 shows that the map $H^{\tilde{d}_n}{}_c(\tilde{\mathcal{M}}_n) \to H^{\tilde{d}_n}(\tilde{\mathcal{M}})_n$ preserves the weight filtration. This proves that the map has to be 0. □

5 Mixed Hodge polynomial of $\mathcal{M}_2$

5.1 Cohomology ring of $\mathcal{M}_2$. Here we compute $H(\mathcal{M}_2; q, t)$ by using the explicit description of the ring $H^*(\mathcal{M}_2)$ given

in [37,39]. According to [37] the cohomology ring $H^*(\mathcal{M}_2)$ is generated by classes $\epsilon_i \in H^1(\mathcal{M}_2)$, $\psi_i \in H^3(\mathcal{M}_2)$ for $i = 1, \ldots, 2g$ and $\alpha \in H^2(\mathcal{M}_2)$ and $\beta \in H^4(\mathcal{M}_2)$. In the notation of (4.1.1) $\alpha = \alpha_2$, $\psi_j = \psi_{2,j}$ and $\beta = \beta_2$. The paper [39] then proceeds by determining the relations in these universal generators. The result is as follows.

Let Γ be the group $\mathrm{Sp}(2g, \mathbb{Z})$. Let $\Lambda^k(\psi)$ be the kth exterior power of the standard representation of Γ, with basis $\psi_1, \ldots, \psi_{2g}$. Define the *primitive part* $\Lambda_0^k(\psi)$ to be the kernel of the natural map $\Lambda^k(\psi) \to \Lambda^{2g+2-k}(\psi)$ given by the wedge product with γ^{g+1-k}, where $\gamma = 2\sum_{i=1}^g \psi_i\psi_{i+g}$. The primitive part is complementary to $\gamma\Lambda^{k-2}(\psi) \subset \Lambda^k(\psi)$, and is an irreducible representation of Γ: this is well-known for $\mathrm{Sp}(2g, \mathbb{C})$, and so remains true for the Zariski dense subgroup Γ. Consequently,

$$\dim\left(\Lambda_0^k(\psi)\right) = \binom{2g}{k} - \binom{2g}{k-2}. \tag{5.1.1}$$

For any $g, n \geq 0$, let I_n^g be the ideal within the polynomial ring $\mathbb{Q}[\alpha, \beta, \gamma]$ generated by γ^{g+1} and the polynomials

$$\rho_{r,s,t}^{n,g} = \sum_{i=0}^{\min(r,s,g-t)} (c-i)! \frac{\alpha^{r-i}}{(r-i)!} \frac{\beta^{s-i}}{(s-i)!} \frac{(2\gamma)^{t+i}}{i!}, \tag{5.1.2}$$

where $c = r + 3s + 2t - 2g + 2 - n$, for all $r, s, t \geq 0$ such that

$$t \leq g, \quad r + 3s + 3t > 3g - 3 + n \quad \text{and} \quad r + 2s + 2t \geq 2g - 2 + n. \tag{5.1.3}$$

The following is then the main result of [39].

Theorem 5.1.1. *As a* Γ*-algebra,*

$$H^*(\mathcal{M}_2) = \Lambda(\epsilon) \otimes \Big(\bigoplus_{k=0}^{g} \Lambda_0^k(\psi) \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_k^{g-k} \Big).$$

Proof. There is a slight difference in the classes $\rho_{r,s,t}^{n,g}$ in (5.1.2) and the classes $\rho_{r,s,t}^{c}$ in [39]. In [39] the sum for i is between 0 and $\min(r, s, c)$. First of all c is unnecessary in the min because $s \le c$ by the third inequality in (5.1.3). Second difference is that in (5.1.2) we have the sum going from 0 to $\min(r, s, g - t)$. So the relations are slightly different, here any monomial which is divisible by γ^{g+1} is left out. However as $\gamma^{g+1} \in I_n^g$ the two sets of polynomials generate the same ideal. □

Remark 5.1.2. We note that the I_k^{g-k} has the following additive basis: take all classes $\rho_{r,s,t}^{n,g}$ satisfying (5.1.3) and monomials of the form $\alpha^r \beta^s \gamma^t$ with $t > g$. It is an additive basis because their leading terms in the lexicographical ordering additively generate an ideal.

For the calculation of the mixed Hodge polynomial of $\mathcal{M}_2$ we only need to know that a monomial basis for $\mathbb{Q}[\alpha, \beta, \gamma]/I_k^{g-k}$ is given by $\alpha^r \beta^s \gamma^t$, for

$$\begin{gathered} 0 \le r, 0 \le s, 0 \le t \le g' \ \text{ and} \\ (r + 3s + 3t \le 3g' - 3 + k \ \text{ or } \ r + 2s + 2t < 2g' - 2 + k), \end{gathered} \tag{5.1.4}$$

where $g' = g - k$.

5.2 Calculation of the mixed Hodge polynomial. We introduce the notation $S_k^{g'}$ for the set of triples (r, s, t) of non-negative integers satisfying (5.1.4). To simplify notation we will use g instead of g' below.

Lemma 5.2.1

$$\begin{aligned} \sum_{(r,s,t)\in S_k^g} a^r b^s c^t = {} & \frac{1 - c^{g+1}}{(1-a)(1-b)(1-c)} - \frac{a^{k-2} b^g \Big(1 - \frac{c^{g+1}}{b^{g+1}}\Big)}{(1-a)\big(1-\frac{c}{b}\big)\big(1-\frac{b}{a^2}\big)} \\ & - \frac{(b^{g+[(k+1)/2]-1} + ab^{g+[k/2]-1})\Big(1 - \frac{c^{g+1}}{b^{g+1}}\Big)}{(1-b)\big(1-\frac{c}{b}\big)\Big(1-\frac{a^2}{b}\Big)} \\ & - \frac{a^{3g+k-2}\big(1 - \frac{c^g}{a^{3g}}\big)}{(1-a)\big(1-\frac{c}{a^3}\big)\big(1-\frac{b}{a^3}\big)} + \frac{a^{k-2} b^g \big(1 - \frac{c^g}{b^g}\big)}{(1-a)\big(1-\frac{c}{b}\big)\big(1-\frac{b}{a^3}\big)}. \end{aligned} \tag{5.2.1}$$

Proof. Fix g. It is clear that $S_k^g \subset S_{k+1}^g$. Furthermore we can separate $S_{k+1}^g \setminus S_k^g = R_1^k \amalg R_2^k$ into the following two disjoint sets:

$$R_1^k := \{(r,s,t) \in \mathbb{Z}_{\geq 0}^3 \mid r+3s+3t = 3g-3+k+1 \text{ and } r+2s+2t > 2g-2+k \text{ and } t \leq g\}$$

$$R_2^k := \{(r,s,t) \in \mathbb{Z}_{\geq 0}^3 \mid r+3s+3t \geq 3g-3+k+1 \text{ and } r+2s+2t = 2g-2+k \text{ and } t \leq g\}.$$

We can calculate

$$\begin{aligned}\sum_{(r,s,t)\in R_1^k} a^r b^s c^t &= \sum_{t=0}^{g-1} \sum_{s=0}^{g-1-t} a^{3g-3+k+1}(b/a^3)^s (c/a^3)^t \\ &= \sum_{t=0}^{g-1} a^{3g-3+k+1} \frac{1-(b/a^3)^{g-t}}{1-b/a^3} (c/a^3)^t \\ &= \frac{a^{3g-3+k+1}(1-(c/a^3)^g)}{(1-c/a^3)(1-b/a^3)} - \frac{a^{k-2}b^g(1-(c/b)^g)}{(1-b/a^3)(1-c/b)} \qquad (5.2.2)\end{aligned}$$

and

$$\begin{aligned}\sum_{(r,s,t)\in R_2^k} a^r b^s c^t &= \sum_{t=0}^{g} \sum_{s=g-t}^{g-1+[k/2]-t} a^{2g-2+k}(b/a^2)^s (c/a^2)^t \\ &= \sum_{t=0}^{g} a^{2g-2+k} (c/a^2)^t (b/a^2)^{g-t} \frac{1-(b/a^2)^{[k/2]}}{1-(b/a^2)} \\ &= \frac{a^{k-2}b^g(1-(c/b)^{g+1})(1-(b/a^2)^{[k/2]})}{(1-c/b)(1-b/a^2)}, \qquad (5.2.3)\end{aligned}$$

thus

$$\sum_{(r,s,t)\in S_{k+1}^g\setminus S_k^g} a^r b^s c^t = \sum_{(r,s,t)\in R_1^k} a^r b^s c^t + \sum_{(r,s,t)\in R_2^k} a^r b^s c^t. \qquad (5.2.4)$$

As

$$\bigcup_{k'=k}^{\infty} S_{k'}^g = \{(r,s,t) \in \mathbb{Z}_{\geq 0}^3 \mid t \leq g\}$$

we can deduce that

$$\begin{aligned}\frac{1-c^{g+1}}{(1-a)(1-b)(1-c)} &= \sum_{(r,s,t)\in \cup_{k'=k}^{\infty} S_{k'}^g} a^r b^s c^t \\ &= \sum_{(r,s,t)\in S_k^g} a^r b^s c^t + \sum_{k'=k}^{\infty} \sum_{(r,s,t)\in S_{k'+1}^g \setminus S_{k'}^g} a^r b^s c^t.\end{aligned}$$

Using (5.2.4) and summing up (5.2.2) and (5.2.3) proves the lemma. □

We can now prove Theorem 1.1.3.

Proof. By Proposition 4.1.8 we know that the classes α, ψ_k and β have homogenous weight 2. Thus γ has homogenous weight 4. As the cup product is compatible with mixed Hodge structures by Theorem 2.1.2.4 the homogenous weights of a monomial in the universal generators will be the sum of the homogenous weights of the factors (see Remark 4.1.7). Thus a method to calculate the mixed Hodge polynomial of $\mathcal{M}_2$ is to take the monomial basis of $\mathbb{Q}[\alpha, \beta, \gamma]/I_k^{g-k}$ given in (5.1.4) evaluate the homogenous weights of the individual monomials and sum this up over all monomials.

First we have

Lemma 5.2.2

$$\sum_{(r,s,t)\in S_k^g} (q^2t^2)^r(q^2t^4)^s(q^4t^6)^t$$
$$= \frac{q^{2g-2}t^{4g-4-2k}(1-q^{4g-4k+4}t^{2g-2k+2})}{(1-q^4t^2)(q^2-1)(q^2t^2-1)} + \frac{1-q^{4g-4k+4}t^{6g-6k+6}}{(1-q^4t^6)(q^2t^2-1)(q^2t^4-1)} - \frac{1}{2}\frac{q^{2g-2-k}t^{4g-4-2k}(1-q^{2g-2k+2}t^{2g-2k+2})}{(1-q^2t^2)(q-1)(qt^2-1)} - \frac{1}{2}\frac{(-q)^{2g-2-k}t^{4g-4-2k}(1-q^{2g-2k+2}t^{2g-2k+2})}{(1-q^2t^2)(q+1)(qt^2+1)}.$$

Proof. Substitute $a = q^2t^2$, $b = q^2t^4$, $c = q^4t^6$ in (5.2.1). To prove that the resulting rational function is the same as the one above, one can multiply over with the denominators and get an identical expression. $\square$

We can now use the description of the cohomology ring of $\mathcal{M}_2$ in Theorem 5.1.1 to get the mixed Hodge polynomial $H(\mathcal{M}_2; q, t)$. We have

$$\frac{H(\mathcal{M}_2; q, t)}{(1+qt)^{2g}} = \sum_{k=0}^{g} \left(\binom{2g}{k} - \binom{2g}{k-2} \right) (q^2t^3)^k \times \sum_{(r,s,t)\in S_k^{g-k}} (q^2t^2)^r(q^2t^4)^s(q^4t^6)^t.$$

Writing in Lemma 5.2.2 and summing it up with k yields exactly Theorem 1.1.3. $\square$

Remark 5.2.3. The above proof of Theorem 1.1.3 follows closely the geometry behind the proof of Theorem 5.1.1 in [39]. There certain spaces $\mathcal{H}_k$ of rank 2 Higgs bundles with a pole of order at most k are introduced. It is shown there that $\mathcal{H}_0 \cong \mathcal{M}_2$ are diffeomorphic, they form a tower: $\mathcal{H}_k \subset \mathcal{H}_{k+1}$ and the direct limit $\mathcal{H}_\infty := \bigcup_{k=0} \mathcal{H}_k$ is homotopically equivalent with the classifying space of a certain gauge group. The cohomology ring $H^*(\mathcal{H}_k)$ is also generated by the same classes $\epsilon_i, \alpha, \psi_i, \beta$. One can show by the description of their cohomology ring in [39] that there exists an abstract weight filtration on $H^*(\mathcal{H}_k)$ by setting the weight of the universal generators α, ψ_i, β be 4, and the weight of ϵ_i to be 2. Lemma 5.2.1 then can be considered as calculating the natural two-variable polynomial associated to this abstract weight filtration on the Γ-invariant part of $H^*(\mathcal{H}_k)$. Similarly to the calculation above, one can obtain the following formula for the two-variable polynomial associated to this filtration on the whole cohomology $H^*(\mathcal{H}_k)$:

$$\frac{(q^2t^3+1)^{2g}(qt+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4+2k}(q^2t+1)^{2g}(qt+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \frac{1}{2}\frac{q^{2g-2+k}t^{4g-4+2k}(qt+1)^{2g}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{(-q)^{2g-2+k}t^{4g-4+2k}(qt-1)^{2g}(qt+1)^{2g}}{(q+1)(qt^2+1)}.$$

This polynomial has some remarkable properties. First we see that in the $k \to \infty$ limit only the first term survives, which gives the two-variable rational function associated to this abstract filtration on the cohomology $H^*(\mathcal{H}_\infty)$ of the classifying space of the gauge group, which is known to be a free anticommutative algebra on the universal generators $\epsilon_i, \alpha, \psi_i, \beta$.

Second, the polynomial satisfies a curious Poincaré duality, when replacing q by $1/qt^2$. Thus when we set $t=-1$ in the above polynomial we have a palindromic polynomial in q. It has degree $8g-6+2k$. We may therefore expect that there is a character variety version $\mathcal{M}_2^k$ of the Higgs moduli spaces $\mathcal{H}_k$ so that the abstract weight filtration we put on $H^*(\mathcal{H}_k)$ is the actual weight filtration coming from the mixed Hodge structure on $H^*(\mathcal{M}_2^k)$. However if this was the case the E-polynomial of $\mathcal{M}_2^k$ would be of degree $8g-6+2k$, and therefore $\mathcal{M}_2^k$ would have dimension $8g-6+2k$. The dimension of $\mathcal{H}_k$ is $8g-6+3k$. Therefore what we could expect is perhaps a deformation retract of $\mathcal{H}_k$ being diffeomorphic to a certain character variety $\mathcal{M}_2^k$ of dimension $8g-6+2k$ with the above mixed Hodge polynomial.

Remark 5.2.4. We can now deduce Corollary 1.1.4 by combining Theorem 1.1.3 and Remark 4.2.

5.3 Curious hard Lefschetz. Define the Lefschetz map $L: H^i(\tilde{\mathcal{M}}_2) \to H^{i+2}(\tilde{\mathcal{M}}_2)$ by $x \mapsto \alpha \cup x$, where $\alpha=\alpha_2$ is the universal class in $H^2(\tilde{\mathcal{M}}_2)$ defined in (4.1.1). As it respects mixed Hodge structures and α has homogenous weight 2 by Proposition 4.1.8 it defines a map on the graded pieces of the weight filtration $L: Gr_l^W H^i(\tilde{\mathcal{M}}_2) \to Gr_{l+4}^W H^{i+2}(\tilde{\mathcal{M}}_2)$. We now prove Theorem 1.1.5.

Proof. We start with a few lemmas. Let us call the last (in the ordering of the sum in (5.1.2)) monomial $\alpha^{r_0}\beta^{s_0}\gamma^{t_0}$ appearing in $\rho^{n,g}_{r,s,t}$ its *tail*. Clearly $r_0 = r - \min(r, s, g - t)$, $s_0 = s - \min(r, s, g - t)$ and $t_0 = t - \min(r, s, g - t)$. Thus if a monomial $\alpha^{r_0}\beta^{s_0}\gamma^{t_0}$ is the tail of the polynomial $\rho^{n,g}_{r,s,t}$ then

$$r_0 = 0 \text{ or } s_0 = 0 \text{ or } t_0 = g. \tag{5.3.1}$$

Let us denote by T^g_n the set of triples $(r, s, t) \in \mathbb{Z}^3_{\geq 0}$ which satisfy (5.1.3).

Lemma 5.3.1. *Let $r_0, s_0, t_0 \in \mathbb{Z}_{\geq 0}$, $t_0 \leq g$ and satisfying* (5.3.1). *Denote by d the number of polynomials $\rho^{n,g}_{r,s,t}$ with $(r, s, t) \in T^g_n$ and tail $\alpha^{r_0}\beta^{s_0}\gamma^{t_0}$. It satisfies*

$$\begin{aligned} d = \min[&t_0 + 1, \max(r_0 + 3s_0 + 4t_0 - (3g - 3 + n), 0),\\ &\max(r_0 + 2s_0 + 3t_0 - (2g - 3 + n), 0)]. \end{aligned} \tag{5.3.2}$$

The $d \times d$ matrix $A = (a_{ij})^{d-1}_{i,j=0}$ is non-singular, where

$$a_{ij} = \begin{cases} 0 & i + j > t_0 \\ \frac{(r_0+3s_0+3t_0-2g+2-n+i-j)!2^{t_0-i}}{(r_0+i)!(s_0+i)!(t_0-i-j)!} & i + j \leq t_0, \end{cases}$$

which is the coefficient of $\alpha^{r_0+i}\beta^{s_0+i}\gamma^{t_0-i}$ in $\rho^{n,g}_{r_0+t_0-j,s_0+t_0-j,j}$.

Proof. To prove the first statement we need to count the number of $0 \leq i \leq t_0$ such that $(r_0 + i, s_0 + i, t_0 - i) \in T^n_g$ consequently they satisfy

$$r_0 + 3s_0 + 3t_0 + i > 3g - 3 + n,$$

thus

$$t_0 \geq i > 3g - 3 + n - (r_0 + 3s_0 + 3t_0).$$

Similarly, we have

$$r_0 + 2s_0 + 2t_0 + i > 2g - 3 + n,$$

which yields

$$t_0 \geq i > 2g - 3 + n - (r_0 + 2s_0 + 2t_0).$$

This proves (5.3.2).

For the second statement consider the matrix $B = (b_{ij})^{d-1}_{i,j=0}$ with $b_{ij} = \frac{(r_0+3s_0+3t_0-2g+2-n+i-j)!}{(t_0-i-j)!}$, if $t_0 - i - j \geq 0$, and $b_{ij} = 0$ otherwise. As the matrix B is obtained from A by multiplying the rows and columns by non-zero constants it is enough to show that B is non-singular. Introduce the notation $(a)_j = a(a + 1)\ldots(a + j - 1)$ and $(a)_0 = 1$. Now we can write

$$\begin{aligned} b_{ij} &= \frac{(r_0 + 3s_0 + 3t_0 - 2g + 2 - n + i)!(t_0 - i - j + 1)_j}{(r_0 + 3s_0 + 3t_0 - 2g + 2 - n + i - j + 1)_j(t_0 - i)!}\\ &= \frac{(r_0 + 3s_0 + 3t_0 - 2g + 2 - n + i)!(t_0 - i - j + 1)_j}{(-1)^j(-r_0 - 3s_0 - 3t_0 + 2g - 2 + n - i)_j(t_0 - i)!}, \end{aligned}$$

which is valid for any i and j as $(t_0 - i - j + 1)_j = 0$ if and only if $t_0 - i - j < 0$ (note that

$$t_0 - i \geq t_0 - d + 1 \geq 0$$

and

$$\begin{aligned}r_0 + 3s_0 + 3t_0 - 2g + 2 - n + i - j + 1 \\ \geq r_0 + 3s_0 + 3t_0 - 2g + 2 - n + i - d + 2 > 0\end{aligned}$$

by (5.3.2)). Now multiplying the rows and columns of B by non-zero constants we get the matrix $C = (c_{ij})_{i,j=0}^{d-1}$ with

$$c_{ij} = \frac{(\alpha_i - \beta_j)_j}{(\alpha_i)_j},$$

where $\alpha_i = -r_0 - 3s_0 - 3t_0 + 2g - 2 + n - i$ and $\beta_j = -r_0 - 3s_0 - 4t_0 + 2g - 3 + n + j$. The determinant of a matrix like C was calculated in [21, Lemma 19]. Their formula gives

$$|C| = \prod_{i=0}^{d-1} \frac{(\beta_i)_i}{(\alpha_i)_{d-1}} \prod_{0 \leq i < j < d} (i - j).$$

Because

$$\begin{aligned}\beta_i + i - 1 &\leq -r_0 - 3s_0 - 4t_0 + 2g - 3 + n + 2d - 2 - 1 \\ &\leq -r_0 - 3s_0 - 4t_0 + 3g - 5 + n + d < 0\end{aligned}$$

by (5.3.2) we get $|C| \neq 0$ and consequently $|A| \neq 0$. This completes the proof. □

Lemma 5.3.2. *Let* (r_0, s_0, t_0) *satisfy* (5.3.1) *and let*

$$w = 6g - 6 + 2n - (2r_0 + 2s_0 + 4t_0). \tag{5.3.3}$$

The number of monomials of the form $\alpha^{r_0+i}\beta^{s_0+i}\gamma^{t_0-i}$ *for which* $0 \leq r_0+i < 3g - 3 + n - w$ *and* $0 \leq i \leq t_0$ *is at least* d.

Proof. We distinguish three cases depending on which of the cases of (5.3.1) is satisfied.

First case is when $r_0 = 0$. The number of monomials of the form $\alpha^i\beta^{s_0+i}\gamma^{t_0-i}$ for which $0 \leq i < 3g - 3 - n - w$ and $0 \leq i \leq t_0$ is clearly

$$\begin{aligned}\min(t_0 + 1, 3g - 3 + n - w) &= \min(t_0 + 1, 3g - 3 + n - w) \\ &= \min(t_0 + 1, 2r_0 + 2s_0 + 4t_0 - (3g - 3 + n)) \\ &\geq \min(t_0 + 1, r_0 + 2s_0 + 3t_0 - (2g - 3 + n)) \\ &\geq d\end{aligned}$$

because of (5.3.3), $0 \leq r_0, t_0 \leq g$ and (5.3.2).

Second case is when $s_0 = 0$. The number of monomials of the form $\alpha^{r_0+i}\beta^i\gamma^{t_0-i}$ for which $r_0 + i < 3g - 3 - n - w$ and $0 \leq i \leq t_0$ is clearly

$$\begin{aligned}\min(t_0 + 1, 3g - 3 + n - w - r_0)\\ &= \min(t_0 + 1, 3g - 3 + n - w - (r_0 - s_0))\\ &= \min(t_0 + 1, 2r_0 + 2s_0 + 4t_0 - (3g - 3 + n) - (r_0 - s_0))\\ &= \min(t_0 + 1, r + 3s_0 + 4t_0 - (3g - 3 + n))\\ &\geq d\end{aligned}$$

because of (5.3.3) and (5.3.2).

Finally the third case is when $t_0 = g$. Now the number of monomials of the form $\alpha^{r_0+i}\beta^{s_0+i}\gamma^{g-i}$ for which $r_0 + i < 3g - 3 - n - w$ and $0 \leq i \leq t_0$ is clearly

$$\begin{aligned}\min(t_0{+}1, 3g - 3 + n - w - r_0)\\ &= \min(t_0 + 1, 3g - 3 + n - w - (r_0 + t_0 - g))\\ &= \min(t_0 + 1, 2r_0 + 2s_0 + 4t_0 - (3g - 3 + n) - (r_0 + t_0 - g))\\ &= \min(t_0 + 1, r_0 + 2s_0 + 3t_0 - (2g - 3 + n))\\ &\geq d.\end{aligned}$$

□

We say that $x \in \mathbb{Q}[\alpha, \beta, \gamma]$ has homogenous weight $w = w(x)$ if it is homogeneous of degree w when α, β, γ are assigned degrees 2, 2 and 4 respectively. Note that all the classes $\rho^{n,g}_{r,s,t}$ have homogenous weight $2r + 2s + 4t$.

Lemma 5.3.3. *Let $x \in \mathbb{Q}[\alpha, \beta, \gamma]$ have homogenous weight $w < 3g-3+n$. Then $x\alpha^{3g-3+n-w} \in I^g_n$ implies $x \in I^g_n$.*

Proof. By Remark 5.1.2 we can write

$$x\alpha^{3g-3+n-w(x)} = \gamma^{g+1}y + \sum_{(r,s,t)\in T^g_n(w)} \lambda_{r,s,t}\rho^{n,g}_{r,s,t}, \tag{5.3.4}$$

where $y \in \mathbb{Q}[\alpha, \beta, \gamma]$, $\lambda_{r,s,t} \in \mathbb{Q}$ and $T^g_n(w)$ is the set of non-negative triples (r, s, t) satisfying (5.1.3) and $w(\rho^{n,g}_{r,s,t}) = 2r + 2s + 4t = 6g - 6 + 2n - w = w(x\alpha^{3g-3+n-w})$.

We show that all $\lambda_{r,s,t} = 0$. Take $(r, s, t) \in T^g_n(w)$. Let $\alpha^{r_0}\beta^{s_0}\gamma^{t_0}$ be the tail of $\rho^{n,g}_{r,s,t}$. In particular $w(\rho^{n,g}_{r,s,t}) = 2r_0 + 2s_0 + 4t_0$. According to Lemma 5.3.1 the number of relations of $(r', s', t') \in T^g_n(w)$ with tail $\alpha^{r_0}\beta^{s_0}\gamma^{t_0}$ is d given by (5.3.2). On the other hand Lemma 5.3.2 implies that the number of monomials $\alpha^{r_0+i}\beta^{s_0+i}\gamma^{t_0-i}$ such that $0 \leq r_0 + i < 3g - 3 + n - w$ and $0 \leq i \leq t_0$ is at least d. But these monomials do not appear on the LHS of (5.3.4) because all the terms there are divisible by $\alpha^{3g-3+n-w}$. They are only contained in relations with tail $\alpha^{r_0}\beta^{s_0}\gamma^{t_0}$ therefore Lemma 5.3.1 implies $\lambda_{r,s,t} = 0$.

Consequently $x\alpha^{3g-3+n-w(x)} = \gamma^{g+1}y$, which implies x is divisible by γ^{g+1}, thus $x \in I_n^g$. Lemma 5.3.3 follows. $\square$

We can now prove Theorem 1.1.5. By Corollary 1.1.4 (proved in Remark 5.2.4) we know that

$$\begin{aligned}\dim\left(Gr^W_{6g-6-2l}H^{i-l}(\tilde{\mathcal{M}}_2)\right) &= h^{3g-3-l,3g-3-l;i-l}(\tilde{\mathcal{M}}_2)\\ &= h^{3g-3+l,3g-3+l;i+l}(\tilde{\mathcal{M}}_2)\\ &= \dim\left(Gr^W_{6g-6+2l}H^{i+l}(\tilde{\mathcal{M}}_2)\right)\end{aligned}$$

thus it is enough to show that L^l in the theorem above is an injection.

Now by Theorem 5.1.1 any element $z \in Gr^W_{6g-6-2l}H^{i-l}(\tilde{\mathcal{M}}_2)$ can be represented as $z = \sum_{k=0}^g y_k[x_k]$, where $y_k \in \Lambda_0^k(\psi)$ and $[x_k] \in \mathbb{Q}[\alpha, \beta, \gamma]/I_k^{g-k}$, with a representative $x_k \in \mathbb{Q}[\alpha, \beta, \gamma]$ of homogenous weight. As $w(y_k) = 2k$ we have $w(x_k) = 3g-3-l-2k = 3(g-k)-3+k-l$ or equivalently $l = 3(g-k)-3+k-w(x_k)$. Assume now that $z\alpha^l = 0$. This would imply that $x_k\alpha^l \in I_k^{g-k}$ for all k. By the previous Lemma 5.2.1 $x_k \in I_k^{g-k}$ and so $z = 0$. Theorem 1.1.5 follows. $\square$

Corollary 4.1.10 implies that the pure part of $H^*(\tilde{\mathcal{M}}_2)$ is g dimensional, due to the Newstead relation $\beta^g = 0$ proved in [39]. This combined with Theorem 1.1.3 and Theorem 4.4.1 proves Theorem 1.1.6.

We know from [40] that the middle cohomology $H^{6g-6}(\tilde{\mathcal{M}}_2)$ is also g-dimensional. The curious hard Lefschetz map then gives a natural isomorphism between the associated graded of the weight filtration on the vector spaces $PH^*(\tilde{\mathcal{M}}_2)$ and $H^{6g-6}(\tilde{\mathcal{M}}_2)$ (cf. Remark 4.2.6 and Remark 4.4.6).

5.4 Intersection form. Theorem 1.1.3 and Proposition 4.4.4 implies that the middle cohomology of $\tilde{\mathcal{M}}_2$ does not have pure part and as explained in Corollary 4.5.1 we have

Corollary 5.4.1. *The intersection form on $H_c^{6g-6}(\tilde{\mathcal{M}}_2)$ is trivial, i.e., there are no "topological L^2 harmonic forms" on $\tilde{\mathcal{M}}_2$.*

6 Appendix by Nicholas M. Katz: E-polynomials, zeta-equivalence, and polynomial-count varieties

Given a noetherian ring R, we denote by (Sch/R) the category of separated R-schemes of finite type, morphisms being the R-morphisms. We denote by $K_0(\mathrm{Sch}/R)$ its Grothendieck group. By definition, $K_0(\mathrm{Sch}/R)$ is the quotient of the free abelian group on elements $[X]$, one for each separated R-scheme of finite type, by the subgroup generated by all the relation elements

$$[X] - [Y], \text{ whenever } X^{red} \cong Y^{red},$$

and

$$[X] - [X \setminus Z] - [Z], \text{ whenever } Z \subset X \text{ is a closed subscheme.}$$

It follows easily that if X is a finite union of locally closed subschemes Z_i, then in $K_0(\mathrm{Sch}/R)$ we have the inclusion–exclusion relation

$$[X] = \sum_i [Z_i] - \sum_{i<j} [Z_i \cap Z_j] + \dots.$$

For any ring homomorphism $R \to R'$ of noetherian rings, the "extension of scalars" morphism from (Sch/R) to (Sch/R') which sends X/R to $X \otimes_R R'/R'$, extends to a group homomorphism from $K_0(\mathrm{Sch}/R)$ to $K_0(\mathrm{Sch}/R')$.

Suppose A is an abelian group, and ρ is an "additive function" from (Sch/R) to A, i.e., a rule which assigns to each $X \in (\mathrm{Sch}/R)$ an element $\rho(X) \in A$, such that $\rho(X)$ depends only on the isomorphism class of X^{red}, and such that whenever $Z \subset X$ is a closed subscheme, we have

$$\rho(X) = \rho(X - Z) + \rho(Z).$$

Then ρ extends uniquely to a group homomorphism from $K_0(\mathrm{Sch}/R)$ to A, by defining $\rho(\sum_i [X_i]) = \sum_i \rho(X_i)$.

When $R = \mathbb{C}$, we have the following simple lemma, which we record now for later use.

Lemma 6.1.1. *Every element of* $K_0(\mathrm{Sch}/\mathbb{C})$ *is of the form* $[S] - [T]$, *with* S *and* T *both projective smooth (but not necessarily connected)* $\mathbb{C}$*-schemes.*

Proof. To show this, we argue as follows. It is enough to show that for any separated $\mathbb{C}$-scheme of finite type X, $[X]$ is of this type. For then $-[X] = [T] - [S]$, and

$$[S_1] - [T_1] + [S_2] - [T_2] = [S_1 \sqcup S_2] - [T_1 \sqcup T_2],$$

and the disjoint union of two projective smooth schemes is again one. (Indeed, if we embed each in a large projective space, say $S_i \subset \mathbb{P}^{N_i}$ and pick a point $a_i \in \mathbb{P}^{N_i} \setminus S_i$, then $S_1 \times a_2$ and $a_1 \times S_2$ are disjoint in $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$.)

We first remark that for any X as above, $[X]$ is of the form $[V] - [W]$ with V and W affine. This follows from inclusion–exclusion by taking a finite covering of X by affine open sets, and noting that the disjoint union of two affine schemes of finite type is again an affine scheme of finite type. So it suffices to prove our claim for affine X. Embedding X as a closed subscheme of some affine space $\mathbb{A}^N$ and using the relation

$$[X] = [\mathbb{A}^N] - [\mathbb{A}^N \setminus X],$$

it now suffices to prove our claim for smooth quasiaffine X. By resolution, we can find a projective smooth compactification Z of X, such that $Z \setminus X$ is a union of smooth divisors D_i in Z with normal crossings. Then by inclusion–exclusion we have

$$[X] = [Z] - \sum_i [D_i] + \sum_{i,j} [D_i \cap D_j] + \dots.$$

In this expression, each summand on the right hand side is projective and smooth. Taking for S the disjoint union of the summands with a plus sign and for T the disjoint union of the summands with a minus sign, we get the desired expression of our $[X]$ as $[S] - [T]$, with S and T both projective and smooth. □

Now take for R a finite field $\mathbb{F}_q$. For each integer $n \geq 1$, the function on $(\mathrm{Sch}/\mathbb{F}_q)$ given by $X \mapsto \#X(\mathbb{F}_{q^n})$ is visibly an additive function from $(\mathrm{Sch}/\mathbb{F}_q)$ to $\mathbb{Z}$. Its extension to $K_0(\mathrm{Sch}/\mathbb{F}_q)$ will be denoted

$$\gamma \longmapsto \#\gamma(\mathbb{F}_{q^n}).$$

We can also put all these functions together, to form the zeta function. Recall that the zeta function $Z(X/\mathbb{F}_q, t)$ of $X/\mathbb{F}_q$ is the power series (in fact it is a rational function) defined by

$$Z(X/\mathbb{F}_q, t) = \exp\Big(\sum_{n\geq 1} \#X(\mathbb{F}_{q^n}) t^n/n\Big).$$

Then $X \mapsto Z(X/\mathbb{F}_q, t)$ is an additive function with values in the multiplicative group $\mathbb{Q}(t)^\times$. We denote by

$$\gamma \longmapsto \mathrm{Zeta}(\gamma/\mathbb{F}_q, t)$$

its extension to $K_0(\mathrm{Sch}/\mathbb{F}_q)$. We say that an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{F}_q)$ is *zeta-trivial* if $\mathrm{Zeta}(\gamma/\mathbb{F}_q, t) = 1$, i.e., if $\#\gamma(\mathbb{F}_{q^n}) = 0$ for all $n \geq 1$. We say that two elements of $K_0(\mathrm{Sch}/\mathbb{F}_q)$ are *zeta-equivalent* if they have the same zeta functions, i.e., if their difference is zeta-trivial.

We say that an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{F}_q)$ is *polynomial-count* (or has *polynomial count*) if there exists a (necessarily unique) polynomial $P_{\gamma/\mathbb{F}_q}(t) = \sum_i a_i t^i \in \mathbb{C}[t]$ such that for every finite extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, we have

$$\#\gamma(\mathbb{F}_{q^n}) = P_{\gamma/\mathbb{F}_q}(q^n).$$

If $\gamma/\mathbb{F}_q$ has polynomial count, its counting polynomial $P_{\gamma/\mathbb{F}_q}(t)$ lies in $\mathbb{Z}[t]$. (To see this, we argue as follows. On the one hand, from the series definition of the zeta function, and the polynomial formula for the number of rational points, we have

$$(td/dt)\log(Z(\gamma/\mathbb{F}_q,t)) = \sum_i a_i q^i t/(1-q^i t).$$

As the zeta function is a rational function, say $\prod_i(1-\alpha_i t)/\prod_j(1-\beta_j t)$ in lowest terms, we first see by comparing logarithmic derivatives that each of its zeroes and poles is a non-negative power of $1/q$. Thus for some integers b_n, the zeta function is of the form $\prod_{n\geq 0}(1-q^n t)^{-b_n}$. Again comparing logarithmic derivatives, we see that we have $a_n = b_n$ for each n.)

Equivalently, an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{F}_q)$ is polynomial-count if it is zeta-equivalent to a $\mathbb{Z}$-linear combination of classes of affine spaces $[\mathbb{A}^i]$, or, equivalently, to a $\mathbb{Z}$-linear combination of classes of projective spaces $[\mathbb{P}^i]$ (since $[\mathbb{A}^i] = [\mathbb{P}^i] - [\mathbb{P}^{i-1}]$, with the convention that $\mathbb{P}^{-1}$ is the empty scheme). If $\gamma/\mathbb{F}_q$ is polynomial-count, then so is its extension of scalars from $\mathbb{F}_q$ to any finite extension field, with the *same* counting polynomial. (But an element $\gamma/\mathbb{F}_q$ which is not polynomial-count can become polynomial-count after extension of scalars, e.g., a nonsplit torus over $\mathbb{F}_q$, or, even more simply, the zero locus of a square-free polynomial $f(z) \in \mathbb{F}_q[z]$ which does not factor completely over $\mathbb{F}_q$.)

Now let R be a ring which is finitely generated as a $\mathbb{Z}$-algebra. We say that an element $\gamma \in K_0(\mathrm{Sch}/R)$ is zeta-trivial if, for every finite field k, and for every ring homomorphism $\phi : R \to k$, the element $\gamma_{\phi,k}/k$ in $K_0(\mathrm{Sch}/\mathbb{F}_q)$ deduced from γ by extension of scalars is zeta-trivial. And we say that two elements are *zeta-equivalent* if their difference is zeta-trivial.

We say that an element $\gamma \in K_0(\mathrm{Sch}/R)$ is *strongly polynomial-count* with (necessarily unique) counting polynomial $P_{\gamma/R}(t) \in \mathbb{Z}[t]$ if, for every finite field k, and for every ring homomorphism $\phi : R \to k$, the element $\gamma_{\phi,k}/k$ in $K_0(\mathrm{Sch}/\mathbb{F}_q)$ deduced from γ by extension of scalars is polynomial-count with counting polynomial $P_{\gamma/R}(t)$.

We say that an element $\gamma \in K_0(\mathrm{Sch}/R)$ is *fibrewise polynomial-count* if, for every ring homomorphism $\phi : R \to k$, the element $\gamma_{\phi,k}/k$ in $K_0(\mathrm{Sch}/\mathbb{F}_q)$ deduced from γ by extension of scalars is polynomial-count (but we allow its counting polynomial to vary with the choice of (k,ϕ)).

All of these notions, zeta-triviality, zeta equivalence, being strongly or fibrewise polynomial-count, are stable by extension of scalars of finitely generated rings.

We now pass to the complex numbers $\mathbb{C}$. Given an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$, by a “spreading out” of $\gamma/\mathbb{C}$, we mean an element $\gamma_R \in K_0(\mathrm{Sch}/R)$, R a subring of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra, which gives back $\gamma/\mathbb{C}$ after extension of scalars from R to $\mathbb{C}$. It is standard that such spreadings out exist, and that given two spreadings out $\gamma_R \in K_0(\mathrm{Sch}/R)$ and $\gamma_{R'} \in K_0(\mathrm{Sch}/R')$, then over some larger finitely generated ring R'' containing both R and R', the two spreadings out will agree in $K_0(\mathrm{Sch}/R'')$.

We say that an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$ is zeta-trivial if it admits a spreading out $\gamma_R \in K_0(\mathrm{Sch}/R)$ which is zeta-trivial. One sees easily, by taking spreadings out to a common R, that the zeta-trivial elements form a subgroup of $K_0(\mathrm{Sch}/\mathbb{C})$.

We say that two elements are zeta-equivalent if their difference is zeta-trivial. We say that an element is strongly polynomial-count, with counting polynomial $P_\gamma(t) \in \mathbb{Z}[t]$, (respectively fibrewise polynomial-count) if it admits a spreading out which has this property.

Given $X/\mathbb{C}$ a separated scheme of finite type, its *E-polynomial* $E(X; x, y) \in \mathbb{Z}[x, y]$ is defined as follows. The compact cohomology groups $H_c^i(X^{an}, \mathbb{Q})$ carry Deligne's mixed Hodge structure, cf. [8] and [9, 8.3.8], and one defines

$$E(X; x, y) = \sum_{p,q} e_{p,q} x^p y^q,$$

where the coefficients $e_{p,q}$ are the virtual Hodge numbers, defined in terms of the pure Hodge structures which are the associated gradeds for the weight filtration on the compact cohomology as follows:

$$e_{p,q} := \sum_i (-1)^i h^{p,q}\big(gr_W^{p+q}\big(H_c^i(X^{an}, \mathbb{C})\big)\big).$$

Notice that the value of $E(X; x, y)$ at the point $(1, 1)$ is just the (compact, or ordinary, they are equal, by [48]) Euler characteristic of X. One knows that the formation of the E-polynomial is additive (because the excision long exact sequence is an exact sequence in the abelian category of mixed Hodge structures, cf. [9, 8.3.9]). So we can speak of the E-polynomial $E(\gamma; x, y)$ attached to an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$.

Theorem 6.1.2. *We have the following results.*

(1) If $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$ *is zeta-trivial, then*

$$E(\gamma; x, y) = 0.$$

(2) If $\gamma_1 \in K_0(\mathrm{Sch}/\mathbb{C})$ *and* $\gamma_2 \in K_0(\mathrm{Sch}/\mathbb{C})$ *are zeta-equivalent, then*

$$E(\gamma_1; x, y) = E(\gamma_2; x, y).$$

In particular, if X and Y in $(\mathrm{Sch}/\mathbb{C})$ *are zeta-equivalent, then*

$$E(X; x, y) = E(Y; x, y).$$

(3) If $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$ *is strongly polynomial-count, with counting polynomial* $P_\gamma(t) \in \mathbb{Z}[t]$*, then*

$$E(\gamma; x, y) = P_\gamma(xy).$$

In particular, if $X \in (\mathrm{Sch}/\mathbb{C})$ is strongly polynomial-count, with counting polynomial $P_X(t) \in \mathbb{Z}[t]$, then

$$E(X; x, y) = P_X(xy).$$

Proof. Assertion (2) is an immediate consequence of (1), by the additivity of the E-polynomial. Statement (3) results from (2) as follows. If $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$ is strongly polynomial-count, with counting polynomial $P_\gamma(t) = \sum_i a_i t^i \in \mathbb{Z}[t]$, then by definition γ is zeta-equivalent to $\sum_i a_i[\mathbb{A}^i] \in K_0(\mathrm{Sch}/\mathbb{C})$. So we are reduced to noting that $E(\mathbb{A}^i; x, y) = x^i y^i$, which one sees by writing $[\mathbb{A}^i] = [\mathbb{P}^i] - [\mathbb{P}^{i-1}]$ and using the basic standard fact that $E(\mathbb{P}^i; x, y) = \sum_{0 \le j \le i} x^j y^j$. So it remains only to prove Assertion (1) of the theorem. By Lemma 6.1.1, every element $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$ is of the form $[X] - [Y]$, with X and Y are projective smooth $\mathbb{C}$-schemes. So Assertion (1) results from the following theorem, which is proven, but not quite stated, in [66][3]. (What Wang proves is that "K-equivalent" projective smooth connected $\mathbb{C}$-schemes have the same Hodge numbers, through the intermediary of using motivic integration to show that K-equivalent projective smooth connected $\mathbb{C}$-schemes are zeta-equivalent.) □

Theorem 6.1.3. *Suppose X and Y are projective smooth $\mathbb{C}$-schemes which are zeta-equivalent. Then*

$$E(X; x, y) = E(Y; x, y).$$

Proof. Pick spreadings out $\mathcal{X}/R$ and $\mathcal{Y}/R$ over a common R which are zeta-equivalent. At the expense of inverting some nonzero element in R, we may further assume that both $\mathcal{X}/R$ and $\mathcal{Y}/R$ are projective and smooth, and that R is smooth over $\mathbb{Z}$. We denote the structural morphisms of $\mathcal{X}/R$ and $\mathcal{Y}/R$ by

$$f : \mathcal{X} \longrightarrow \mathrm{Spec}(R), \quad g : \mathcal{Y} \longrightarrow \mathrm{Spec}(R).$$

One knows [47, 5.9.3] that, for any finitely generated subring $R \subset \mathbb{C}$, there exists an integer $N \ge 1$ such that for all primes ℓ which are prime to N, there exists a finite extension $E/\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}$ and an injective ring homomorphism from R to $\mathcal{O}$. Fix one such prime number ℓ, which we choose larger than both $\dim(X)$ and $\dim(Y)$, and one such inclusion of R into $\mathcal{O}$.

Over $\mathrm{Spec}(R[1/\ell])$, the $\mathbb{Q}_\ell$-sheaves $R^i f_* \mathbb{Q}_\ell$ and $R^i g_* \mathbb{Q}_\ell$ are lisse, and pure of weight i [11, 3.3.9]. By the Lefschetz Trace Formula and proper base change, for each finite field k, and for each k-valued point ϕ of $\mathrm{Spec}(R[1/\ell])$, we have

$$\mathrm{Zeta}(\mathcal{X}_{k,\phi}/k, t) = \prod_i \det\left(1 - t\,\mathrm{Frob}_{k,\phi} \,\middle|\, R^i f_* \mathbb{Q}_\ell\right)^{(-1)^{i+1}}$$

[3] It was later stated explicitly by Ito [42, Corollary 6.2].

and

$$\mathrm{Zeta}(\mathcal{Y}_{k,\phi}/k,t)=\prod_i \det\left(1-t\,\mathrm{Frob}_{k,\phi}\,\middle|R^ig_*\mathbb{Q}_\ell\right)^{(-1)^{i+1}}.$$

By the assumed zeta-equivalence, we have, for each finite field k, and for each k-valued point ϕ of $\mathrm{Spec}(R[1/\ell])$, the equality of rational functions

$$\mathrm{Zeta}(\mathcal{X}_{k,\phi}/k,t)=\mathrm{Zeta}(\mathcal{Y}_{k,\phi}/k,t).$$

Separating the reciprocal zeroes and poles by absolute value, we infer by purity that for every i, we have

$$\det\left(1-t\,\mathrm{Frob}_{k,\phi}\,\middle|R^if_*\mathbb{Q}_\ell\right)=\det\left(1-t\,\mathrm{Frob}_{k,\phi}\,\middle|R^ig_*\mathbb{Q}_\ell\right).$$

Therefore by Chebotarev the virtual semisimple representations of $\pi_1(\mathrm{Spec}(R[1/\ell]))$ given by $(R^if_*\mathbb{Q}_\ell)^{ss}$ and $(R^ig_*\mathbb{Q}_\ell)^{ss}$ are equal:

$$\left(R^if_*\mathbb{Q}_\ell\right)^{ss}\cong\left(R^ig_*\mathbb{Q}_\ell\right)^{ss}.$$

Now make use of the inclusion of R into $\mathcal{O}$, which maps $R[1/\ell]$ to E. The pullbacks $X_{\mathcal{O}}$ and $\mathcal{Y}_{\mathcal{O}}$ of $\mathcal{X}/R$ and $\mathcal{Y}/R$ to $\mathcal{O}$ are proper and smooth over $\mathcal{O}$. Thus their generic fibres, X_E and $\mathcal{Y}_E$ are projective and smooth over E, of dimension strictly less than ℓ, and they have *good reduction*. Via the chosen map from $\mathrm{Spec}(E)$ to $\mathrm{Spec}(R[1/\ell])$, we may pull back the representations $R^if_*\mathbb{Q}_\ell$ and $R^ig_*\mathbb{Q}_\ell$ of $\pi_1(\mathrm{Spec}(R[1/\ell]))$ to $\pi_1(\mathrm{Spec}(E))$, the galois group $Gal_E := Gal(E^{sep}/E)$. Their pullbacks are the etale cohomology groups $H^i(X_{E^{sep}},\mathbb{Q}_\ell)$ and $H^i(\mathcal{Y}_{E^{sep}},\mathbb{Q}_\ell)$ respectively, viewed as representations of Gal_E. These representations of Gal_E need not be semisimple, but their semisimplifications are isomorphic:

$$H^i(X_{E^{sep}},\mathbb{Q}_\ell)^{ss}\cong H^i(\mathcal{Y}_{E^{sep}},\mathbb{Q}_\ell)^{ss}.$$

By a fundamental result of Fontaine–Messing [15, Theorems A and B] (which applies in the case of good reduction, $E/\mathbb{Q}_\ell$ unramified, and dimension less than ℓ) and Faltings [14, 4.1] (which treats the general case, of a projective smooth generic fibre), we know that $H^i(X_{E^{sep}},\mathbb{Q}_\ell)$ and $H^i(\mathcal{Y}_{E^{sep}},\mathbb{Q}_\ell)$ are Hodge–Tate representations of Gal_E, with Hodge–Tate numbers exactly the Hodge numbers of the complex projective smooth varieties X and Y respectively (i.e., the dimension of the Gal_E-invariants in $H^a(X_{E^{sep}},\mathbb{Q}_\ell)(b)\otimes\mathbb{C}_\ell$ under the semilinear action of Gal_E is the Hodge number $H^{b,a-b}(X)$, and similarly for Y). By an elementary argument of Wang [66, 5.1], the semisimplification of a Hodge–Tate representation is also Hodge–Tate, with the same Hodge–Tate numbers. So the theorem of Fontaine–Messing and Faltings tells us that for all i, $H^i(X)$ and $H^i(Y)$ have the same Hodge numbers. This is precisely the required statement, that $E(X;x,y)=E(Y;x,y)$. □

The reader may wonder why we introduced the notion of being fibrewise polynomial-count, for an element $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$. In fact, this notion is entirely superfluous, as shown by the following Theorem.

Theorem 6.1.4. *Suppose $\gamma \in K_0(\mathrm{Sch}/\mathbb{C})$ is fibrewise polynomial-count. Then it is strongly polynomial-count.*

Proof. Write γ as $[X]-[Y]$, with X and Y projective smooth $\mathbb{C}$-schemes. Repeat the first paragraph of the proof of the previous theorem. Extending R if necessary, we may assume that the element $[\mathcal{X}/R]-[\mathcal{Y}/R] \in K_0(\mathrm{Sch}/R)$ is fibrewise polynomial-count. So for each finite field k and each ring homomorphism $\phi : R \to k$, there exists a polynomial $P_{k,\phi} = \sum_n a_{n,k,\phi} t^n \in \mathbb{Z}[t]$ such that

$$\mathrm{Zeta}(\mathcal{X}_{k,\phi}/k, t)/\mathrm{Zeta}(\mathcal{Y}_{k,\phi}/k, t) = \prod_n (1-(\#k)^n t)^{-a_{n,k,\phi}}.$$

Writing the cohomological expressions of the zeta functions and using purity, we see that the coefficient $a_{n,k,\phi}$ is just the difference of the $2n$'th ℓ-adic Betti numbers of $\mathcal{X}_{k,\phi}\otimes\overline{k}$ and $\mathcal{Y}_{k,\phi}\otimes\overline{k}$, which is in turn the difference of the ranks of the two lisse sheaves $R^{2n}f_*\mathbb{Q}_\ell$ and $R^{2n}g_*\mathbb{Q}_\ell$. This last difference is independent of the particular choice of (k,ϕ). □

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