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Mixed Hodge Structures on Log Deformations.

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ABSTRACT - We study the relationship of constructions of cohomological mixed Hodge complexes and related l-adic constructions by various authors systematically.

Introduction.

J. H. M. Steenbrink showed in [13] that a semistable degeneration over a disk in $\boldsymbol{C}$ yields a cohomological mixed Hodge complex (CMHC) on the central fiber.

He also proved in [14], by means of Koszul complexes, that the central fiber paired with the induced log structure in the sense of Fontaine-Illusie already determines this CMHC and further established that log analytic spaces over the origin locally like the central fibers of semistable degenerations (called log deformations, see 2.15) yield CMHCs even if they do not come from the actual families over disks. These results were independently proved by Y. Kawamata and Y. Namikawa [9] (which were formulated with the log structures in their sense): their CMHC is constructed by means of real blow ups. Both methods are different and we can ask whether both the CMHCs coincide or not for log deformations that do not come from the semistable degenerations.

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The purpose of this article is to study the relationships between the above two CMHCs and their variants systematically. In particular we see that the above two are essentially the same.

On the other hand, M. Rapoport and Th. Zink [12] made a similar construction in the l-adic context and it can be also generalized to the case of (algebraic) log deformations ([11]). Another topic in this article is to compare their construction and the $\boldsymbol{Q}$-structures of the above CMHCs.

In Section 1 we recall double complex constructions due to J. H. M. Steenbrink and S. Zucker in an abstract way, which will appear repeatedly in the sequel. In Section 2 we review some necessary definitions and a few facts in log geometry of Fontaine-Illusie. In Section 3 we prove that a log deformation yields a CMHC. Here we use ringed real blow ups introduced in [8], which are (families of) real blow ups in [9] endowed with the structure rings of log holomorphic functions. In Proposition 3.19, we see that our CMHC coincides with Kawamata-Namikawa's. In Section 4, we consider the case of semistable degeneration as in [13], [15]. In Section 5 we prove that our CMHC coincides also with the one in [14] (Theorem 5.8). In Section 6 we introduce some variants, which intervene between the Hodge construction and the l-adic construction. Using this, we compare in Section 7 the $\boldsymbol{Q}$-structures of the CMHCs with the construction of Rapoport-Zink.

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1. Steenbrink-Zucker construction.

In this section we present a method of constructing complexes which will turn out to be cohomological mixed Hodge complexes. The method is an analogue of the one used in the article [15] by Steenbrink and Zucker. First of all we review the construction in the article [15, § 5] for the sake of convenience.

DEFINITION 1.1. Let X be a complex manifold, and Δ the unit disk in $\boldsymbol{C}$. A morphism of complex manifolds $f: X \rightarrow \Delta$ is said to be a *semistable degeneration* if the fiber $Y := f^{-1}(0)$ is a reduced simple normal crossing divisor and if f is smooth over $\Delta^* := \Delta \backslash \{0\}$.

(1.2) Assume that a semistable degeneration $f: X \to \Delta$ is given. The upper half plane in $\boldsymbol{C}$ is denoted by $\boldsymbol{H}$. The morphism $u \mapsto t = \exp(2\pi\sqrt{-1}u)$ makes $\boldsymbol{H}$ into a universal covering of Δ^*. Let $X^* = f^{-1}(\Delta^*) = X \backslash Y$, $X_\infty = X \times_\Delta \boldsymbol{H} = X^* \times_{\Delta^*} \boldsymbol{H}$, $\pi: X_\infty \to X^*$ be the canonical morphism, $\bar{j}: X^* \to X$ the open immersion, $i: Y \to X$ the closed immersion, and $k = \bar{j}\pi: X_\infty \to X$. These objects fits in the following diagram:

$$\begin{array}{ccccccc}
X_\infty & \xrightarrow{\pi} & X^* & \xrightarrow{\bar{j}} & X & \xleftarrow{i} & Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\boldsymbol{H} & \longrightarrow & \Delta^* & \longrightarrow & \Delta & \longleftarrow & \{0\}
\end{array}$$

in which all the squares are Cartesian.

For any topological space Z, $C^\cdot(Z)$ denotes the complex of sheaves of germs of rational-valued singular cochains on Z. In the situation above, the complex $K^\cdot(X^*)$ and $K^\cdot(X_\infty)$ is defined by

$$K^\cdot(X^*) = i^{-1}\bar{j}_* C^\cdot(X^*), \qquad K^\cdot(X_\infty) = i^{-1}k_* C^\cdot(X_\infty).$$

Then there exists an isomorphism in the derived category $i^{-1}\mathrm{R}k_* \boldsymbol{Q}_{X_\infty} \to K^\cdot(X_\infty)$. On the other hand the complex $K^\cdot(X_\infty)$ carries a monodromy automorphism

$$T: K^\cdot(X_\infty) \to K^\cdot(X_\infty).$$

We remark that the kernel of $T - \mathrm{id}$ coincides with the complex $K^\cdot(X^*)$. For every nonnegative integer m the kernel of the morphism $(T - \mathrm{id})^{m+1}: K^\cdot(X_\infty) \to K^\cdot(X_\infty)$ is denoted by B_m. A subcomplex B of $K^\cdot(X_\infty)$ is defined by $B = \bigcup_{m \geq 0} B_m$. Then B also carries the monodromy automorphism T. By definition of the subcomplex B the logarithm of the automorphism T is well-defined on B.

On a topological space Z an abelian sheaf $\boldsymbol{Z}(r)$ is defined by $\boldsymbol{Z}(r) = (2\pi\sqrt{-1})^r \boldsymbol{Z}_Z \subset \boldsymbol{C}_Z$ for every integer r. For a complex of abelian sheaves K on Z and for an integer r, we define a complex $K(r)$ by $K(r) = K \otimes \boldsymbol{Z}(r)$. Then the morphism

$$\delta = -\frac{1}{2\pi\sqrt{-1}} \log T: B \to B(-1) \tag{1.2.1}$$

is obtained. From the data (B, δ) above Steenbrink and Zucker constructed a complex $\varrho(B)$ by

$$\varrho(B)^p = B^p \oplus B^{p-1}(-1)$$

with the differential

$$d : \varrho(B)^p = B^p \oplus B^{p-1}(-1) \to B^{p+1} \oplus B^p(-1) = \varrho(B)^{p+1}$$

defined by

$$d(x, y) = (dx, -dy + \delta(x)).$$

Then the obvious morphism $K^\cdot(X^*) \to \varrho(B)$ is shown to be a quasi-isomorphism. Moreover the morphism of complexes

$$\theta : \varrho(B) \to \varrho(B)(1)[1] \tag{1.2.2}$$

is defined by $\theta(x, y) = (0, x)$. Here we remark that for a given complex (K, d_K) the differential $d_{K[1]}$ of the complex $K[1]$ is given by $d_{K[1]} = -d_K$ as in [4].

For a complex K the canonical filtration τ is defined by

$$\tau_r K^p = \begin{cases} K^p & \text{if } p < r \\ (\operatorname{Ker} d) \cap K^p & \text{if } p = r \\ 0 & \text{if } p > r \end{cases} \tag{1.2.3}$$

for every p, r.

Finally a double complex $C^{\cdot\cdot}$ is defined by

$$C^{p,q} = \varrho(B)^{p+q+1}(q+1)/\tau_q \varrho(B)^{p+q+1}(q+1)$$

for every p, q with the first differential

$$d' : C^{p,q} \to C^{p+1,q}$$

induced by the differential d of the complex $\varrho(B)$ and the second differential

$$d'' : C^{p,q} \to C^{p,q+1}$$

induced by the morphism of complexes θ in (1.2.2). On the single complex $sC^{\cdot\cdot}$ associated to the double complex $C^{\cdot\cdot}$ a finite increasing filtration L is defined by

$$L_m(sC^{\cdot\cdot})^n = \bigoplus_{p+q=n} \tau_{m+2q+1} \varrho(B)^{p+q+1}(q+1)/\tau_q \varrho(B)^{p+q+1}(q+1).$$

In the article [15] Steenbrink and Zucker proved that the complex

$sC^{\cdot\cdot}$ is isomorphic to the complex $i^{-1}\mathrm{R}k_*\boldsymbol{Q}_{X_\infty}$ in the derived category and that the filtered complex $(sC^{\cdot\cdot}, L)$ underlies a CMHC on Y.

(1.3) Now we come to the point to present an abstract analogue of the Steenbrink-Zucker construction above.

DEFINITION 1.4. Let Y be a topological space. For every non-negative integer n, we are given a complex of abelian sheaves K_n equipped with an increasing filtration W and a morphism of complexes $\theta_n: K_n \to K_{n+1}[1]$ satisfying the conditions

(1.4.1) $K_n^p = 0$ for every $p < 0$
(1.4.2) $W_{-1}K_n = 0$
(1.4.3) $W_m K_n^p = K_n^p$ for every p, m with $m > p$
(1.4.4) $\theta_n(W_m K_n^p) \subset W_{m+1}K_{n+1}^{p+1}$ for every p, m
(1.4.5) $\theta_{n+1}\theta_n = 0$.

Then we define a double complex $D = D(K_n, W, \theta_n)$ by

(1.4.6) $D^{p,q} = K_q^{p+q+1}/W_q$
(1.4.7) $d': D^{p,q} \to D^{p+1,q}$ is induced by the differential of K_q
(1.4.8) $d'': D^{p,q} \to D^{p,q+1}$ is induced by the morphism $\theta_q: K_q \to K_{q+1}[1]$

for every non-negative integers p, q. We denote by $sD = sD(K_n, W, \theta_n)$ the single complex associated to the double complex above. We define an increasing filtration L on the complex sD by

$$L_m sD^n = \bigoplus_{p+q=n} W_{m+2q+1}D^{p,q},$$

where $W_m D^{p,q}$ is the image of $W_m K_q^{p+q+1}$ by the canonical projection $K_q^{p+q+1} \to D^{p,q}$. We call the single complex $sD = sD(K_n, W, \theta_n)$ with the filtration L the *Steenbrink complex* associated to the data $\{(K_n, W), \theta_n\}$.

REMARK 1.5. We can easily see that

$$\mathrm{Gr}_m^L sD = \bigoplus_{\substack{q \geqslant 0 \\ q \geqslant -m}} \mathrm{Gr}_{m+2q+1}^W K_q[1]$$

for every m.

REMARK 1.6. We define a decreasing filtration F on the Steenbrink complex by

$$F^p sD^n = \bigoplus_{\substack{p'+q=n \\ p' \geqslant p}} D^{p',q}$$

for every p and n.

REMARK 1.7. We have the following functoriality for the construction above. Let $\{(K_n, W), \theta_n\}$ and $\{(K'_n, W), \theta'_n\}$ be data satisfying the conditions in Definition 1.4 and $f_n\colon K_n \to K'_n$ a morphism of filtered complexes with the equality $\theta'_n f_n = f_{n+1}[1]\theta_n$:

$$\begin{tikzcd} K_n \arrow[r, "f_n"] \arrow[d, "\theta"'] & K'_n \arrow[d, "\theta'_n"] \\ K_{n+1}[1] \arrow[r, "f_{n+1}[1]"'] & K'_{n+1}[1]. \end{tikzcd}$$

Then we have a morphism of double complexes

$$D(f_n)\colon D(K_n, W, \theta_n) \to D(K'_n, W, \theta'_n)$$

and its associated morphism of single complexes

$$sD(f_n)\colon sD(K_n, W, \theta_n) \to sD(K'_n, W, \theta'_n)$$

preserving the increasing filtration L on both sides.

PROPOSITION 1.8. *In the situation above the morphism of complexes $sD(f_n)$ is a filtered quasi-isomorphism with respect to the filtration L if the morphism f_n is a filtered quasi-isomorphism with respect to the filtration W for every non-negative integer n.*

PROOF. Easy by Remark 1.5. ■

Now we treat the case that we are given a complex K and a morphism $\delta : K \to K(-1)$. This case is an abstract analogue of the complex B with the monodromy logarithm δ (1.2.1). We can construct a double complex (and the single complex associated to it) from the data (K, δ) by the same way as in (1.2).

DEFINITION 1.9. Let K be a complex of abelian sheaves on a topological space Y and $\delta : K \to K(-1)$ a morphism of complexes.

We assume the condition $K^p = 0$ for $p < 0$. Then we define a complex of abelian sheaves $\varrho(K, \delta)$ by

(1.9.1) $\varrho(K, \delta)^p = K^p \oplus K^{p-1}(-1)$ for every p

(1.9.2) $d : \varrho(K, \delta)^p \to \varrho(K, \delta)^{p+1}$ is defined by $d(x, y) = (dx, -dy + \delta(x))$, where $x \in K^p$ and $y \in K^{p-1}(-1)$.

Let (K, δ) be as above. We set $K_0 = \mathrm{Ker}(\delta : K \to K(-1))$. We define a morphism of sheaves $K_0^p \to \varrho(K, \delta)^p$ by

$$K_0^p \ni x \mapsto (x, 0) \in \varrho(K, \delta)^p = K^p \oplus K^{p-1}(-1).$$

These morphisms for all p form a morphism of complexes $K_0 \to \to \varrho(K, \delta)$.

Lemma 1.10. *In the situation above the morphism $K_0 \to \varrho(K, \delta)$ is a quasi-isomorphism if the morphism $\delta : K \to K(-1)$ is surjective.*

Proof. Easy by definition. ■

Definition 1.11. Let (K, δ) be as above. We define a morphism of complexes

$$\theta : \varrho(K, \delta) \to \varrho(K, \delta)(1)[1]$$

by $\theta(x, y) = (0, x)$ for $x \in K^p$, $y \in K^{p-1}(-1)$. We consider a complex of abelian sheaves $K_n = \varrho(K, \delta)(n+1)$ with the canonical filtration $W = \tau$ for every non-negative integer n. Then the morphism θ above defines a morphism

$$\theta_n = \theta(n+1) : K_n = \varrho(K, \delta)(n+1) \to \varrho(K, \delta)(n+2)[1] = K_{n+1}[1]$$

for every n. We can easily see that the data $\{(K_n, W), \theta_n\}$ satisfies the conditions in Definition 1.4. Thus we obtain a double complex $D(K_n, W, \theta)$ and the associated single complex $sD(K_n, W, \theta)$ with the filtration L. The complex $sD(K_n, W, \theta)$ with the filtration L is called the *Steenbrink-Zucker complex* for the data (K, δ) and denoted by $(SZ(K, \delta), L)$.

Remark 1.12. We have

$$\mathrm{Gr}_m^L SZ(K, \delta) = \bigoplus_{\substack{q \geq 0 \\ q \geq -m}} \mathrm{Gr}_{m+2q+1}^{\tau} \varrho(K, \delta)(q+1)[1]$$

by Remark 1.5. Therefore we have a quasi-isomorphism

$$\bigoplus_{\substack{q \geqslant 0 \\ q \geqslant -m}} \mathrm{H}^{m+2q+1}(\varrho(K, \delta))(q+1)[-m-2q] \to \mathrm{Gr}_m^L SZ(K, \delta)$$

for every m.

REMARK 1.13. In the situation above we define a morphism of sheaves $\mu : K^p \to \varrho(K, \delta)(1)^{p+1} = K^{p+1}(1) \oplus K^p$ by

$$\mu(x) = (0, (-1)^p x)$$

for $x \in K^p$. Then the morphism μ above induces a morphism

$$K^p \to D(K_n, W, \theta)^{p,0} = \varrho(K, \delta)(1)^{p+1}/W_0$$

for every p. We can easily see that these morphisms induce a morphism of complexes

$$K \to sD(K_n, W, \theta) = SZ(K, \delta)$$

which is denoted by the same letter μ for simplicity.

LEMMA 1.14. *The morphism $\mu : K \to SZ(K, \delta)$ above is a quasi-isomorphism if the morphism $\delta : K \to K(-1)$ induces a zero map from $\mathrm{H}^p(K)$ to $\mathrm{H}^p(K(-1))$ for every integer p.*

PROOF. We can find the proof in [15, (5.13) Lemma]. ∎

REMARK 1.15. The construction above has the following functoriality. Let (K, δ) and (K', δ') be data as above and $\varphi : K \to K'$ a morphism of complexes such that the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{\delta} & K(-1) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi(-1)} \\ K' & \xrightarrow[\delta']{} & K'(-1). \end{array}$$

Then morphisms of sheaves

$$\varrho(\varphi) : \varrho(K, \delta)^p \to \varrho(K', \delta')^p$$

defined by $\varrho(\varphi)(x, y) = (\varphi(x), \varphi(y))$ form a morphism of complexes

$$\varrho(\varphi) : \varrho(K, \delta) \to \varrho(K', \delta')$$

which has the commutativity $\varrho(\varphi)(1)[1]\theta = \theta\varrho(\varphi)$:

$$\begin{array}{ccc} \varrho(K, \delta) & \xrightarrow{\theta} & \varrho(K, \delta)(1)[1] \\ {\scriptstyle \varrho(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle \varrho(\varphi)(1)[1]} \\ \varrho(K', \delta') & \xrightarrow[\theta]{} & \varrho(K', \delta')(1)[1]. \end{array}$$

Then we easily obtain a morphism of complexes

$$SZ_1(\varphi): SZ(K, \delta) \to SZ(K', \delta')$$

which preserves the filtration L on both sides.

COROLLARY 1.16. *Let $(K, \delta), (K', \delta')$ and φ be as above. Assume that the morphism φ induces a quasi-isomorphism between the kernels of δ and δ' and that the morphisms $\delta: K \to K(-1)$ and $\delta': K' \to K'(-1)$ are surjective. Then the morphism $\varrho(\varphi): \varrho(K, \delta) \to \varrho(K', \delta')$ is a quasi-isomorphism. In particular the morphism $SZ_1(\varphi): SZ(K, \delta) \to$ $\to SZ(K', \delta')$ is a filtered quasi-isomorphism with respect to the filtration L on both sides.*

PROOF. Easy by Lemma 1.10. ■

REMARK 1.17. The construction of the Steenbrink-Zucker complex has another functoriality which plays an essential role in Section 6.

Let K be a complex of abelian sheaves on a topological space Y with the assumption $K^p = 0$ for every $p < 0$, $\delta, \delta': K \to K(-1)$ morphisms of complexes and $\varphi: K \to K$ a morphism of complexes with the conditions

(1.17.1) $\delta' \varphi = \delta$
(1.17.2) $\varphi(-1) \delta' = \delta' \varphi$.

Notice that the conditions (1.17.1) and (1.17.2) imply the analogue of (1.17.2) for the morphism δ. We define a morphism of sheaves

$$\varrho(\varphi)^p_n: \varrho(K, \delta)(n+1)^p = K^p(n+1) \oplus K^{p-1}(n) \to$$
$$\to \varrho(K, \delta')(n+1)^p = K^p(n+1) \oplus K^{p-1}(n)$$

by

$$\varrho(\varphi)_n^p(x, y) = (\overbrace{\varphi(n+1)\circ\ldots\circ\varphi(n+1)}^{(n+1)\text{ times}}(x), \overbrace{\varphi(n)\circ\ldots\circ\varphi(n)}^{n\text{ times}}(y))$$

for $x \in K^p(n+1)$ and $y \in K^{p-1}(n)$. Then we can see that the morphisms for all p form a morphism of complexes by the conditions (1.17.1) and (1.17.2) for the morphism φ. Moreover we have a commutative diagram

$$\begin{array}{ccc}
\varrho(K, \delta)(n+1) & \xrightarrow{\varrho(\varphi)_n} & \varrho(K, \delta')(n+1) \\
{\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\
\varrho(K, \delta)(n+2)[1] & \xrightarrow[\varrho(\varphi)_{n+1}]{} & \varrho(K, \delta')(n+2)[1]
\end{array}$$

for every n. Thus we obtain a morphism of complexes

$$SZ_2(\varphi) : SZ(K, \delta) \to SZ(K, \delta')$$

which preserves the filtration L on both sides by Remark 1.7.

COROLLARY 1.18. *In the situation above the morphism* $SZ_2(\varphi)$ *is a quasi-isomorphism if the morphism* $\varphi : K \to K$ *is a quasi-isomorphism and if the morphism* $\delta : K \to K(-1)$ *induces a zero map from* $\mathrm{H}^p(K)$ *to* $\mathrm{H}^p(K(-1))$ *for every integer* p.

PROOF. We can easily obtain the conclusion by Lemma 1.14 because the assumptions imply that the morphism $\delta' : K \to K(-1)$ induces zero maps on all cohomologies. ■

2. A review on log geometry.

In this section we review briefly some definitions and facts on log analytic spaces which we shall need later. We do not give the details. See [14] and [8] for them. See [7] and [5] for more on log geometry.

DEFINITION 2.1. ([14] Definition (3.1) and [8] Definition (1.1.1)) Let X be an analytic space. A *pre-log structure* on X is a pair of a sheaf of monoids (= a sheaf of commutative semigroups with unit elements) M on X and a homomorphism $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication on $\mathcal{O}_X$. A pre-log structure (M, α), or simply denoted by M, is said to be a *log structure* if the induced homomorphism $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. A *homomorphism* of (pre-)log structures $(M_1, \alpha_1) \to \to (M_2, \alpha_2)$ is a homomorphism $h : M_1 \to M_2$ of sheaves of monoids satisfying $\alpha_2 \circ h = \alpha_1$.

The inclusion functor from the category of log structures on X to that of pre-log structures on X has the left adjoint $M = (M, \alpha) \mapsto M^a$, which is constructed explicitly as the inductive limit (or push-out) of the diagram $M \leftarrow \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ in the category of sheaves of monoids on X.

Definition 2.2. ([14] Definition (3.4)) Let $f : X \to Y$ be a morphism of analytic spaces, and let M be a log structure on Y. Then we call the log structure $(f^{-1}M \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X)^a$ on X the *pull-back log structure* of M and denote it by f^*M.

Definition 2.3. ([14] Definition (4.1) and [8] Definition (1.1.1)) A *log analytic space* is a pair of an analytic space and a log structure on it. For a log analytic space X, we denote by $\mathring{X}$ the underlying analytic space and by M_X the log structure: $X = (\mathring{X}, M_X)$. A *morphism* of log analytic spaces $f : X \to Y$ is a pair of a morphism of analytic spaces $\mathring{f} : \mathring{X} \to \mathring{Y}$ and a homomorphism of log structures $\mathring{f}^* M_Y \to M_X$ (2.1).

(2.4) Let $\boldsymbol{N}$ be the monoid of nonnegative integers with respect to addition. For a log analytic space X, we consider the following condition: Locally on $\mathring{X}$, there is a homomorphism from the constant sheaf $\boldsymbol{N}_X^r$ for some $r \geqslant 0$ to M_X such that the induced homomorphism of log structures $(\boldsymbol{N}_X^r)^a \to M_X$ is an isomorphism.

In the rest of this article, we consider only the log analytic spaces satisfying the above condition except Remark 2.5 and Section 7.

Remark 2.5. In the above condition, if we allow any fs monoid instead of $\boldsymbol{N}^r$, we get the definition of fs log analytic spaces ([8] Definition (1.1.2)). Therefore a log analytic space satisfying the above condition is an fs log analytic space. Most parts of the rest in this section (including 2.7, 2.11, 2.18 etc.) hold also for fs log analytic spaces.

(2.6) Let X be a log analytic space satisfying the condition in (2.4), and let $f : Y \to X$ be a morphism of analytic spaces. Then $(\mathring{Y}, \mathring{f}^* M_X)$ also satisfies the condition in (2.4). This is deduced from the fact that when $\boldsymbol{N}_X^r \to M_X$ induces an isomorphism of log structures, its pull-back $\boldsymbol{N}_Y^r \to \to \mathring{f}^* M_X$ also induces an isomorphism of log structures.

(2.7) To a log analytic space X satisfying the condition in (2.4), a ringed space $X^{\log} = (X^{\log}, \mathcal{O}_X^{\log})$ and the natural morphism $\tau : X^{\log} \to \overset{\circ}{X}$ of ringed spaces are associated, which we explain in this subsection.

As a set

$$X^{\log} = \left\{ (x, h) \,\middle|\, x \in X, h \in \mathrm{Hom}\,(M_{X,x}, \boldsymbol{S}^1),\ h(f) = \frac{f(x)}{|f(x)|} \text{ for any } f \in \mathcal{O}_{X,x}^* \right\},$$

where $\boldsymbol{S}^1 = \{x \in \boldsymbol{C};\ |x| = 1\}$. The morphism τ sends (x, h) to x. When there is a homomorphism $\beta : \boldsymbol{N}_X^r \to M_X$ as in 2.4 globally on $\overset{\circ}{X}$, we endow $X^{\log}$ with the induced topology from the embedding $X^{\log} \to X \times \times (\boldsymbol{S}^1)^r; (x, h) \mapsto (x, h(e_1), \dots, h(e_r))$, where $(e_i)_i$ is the canonical base of $\boldsymbol{N}^r$. This topology is independent on the choices of r and β so that the topology of $X^{\log}$ is well-defined for the general case.

The sheaf of rings $\mathcal{O}_X^{\log}$ is a $\tau^{-1}\mathcal{O}_X$-algebra generated by «logarithms» of local sections of $\tau^{-1}M_X$. This is defined by

$$\mathcal{O}_X^{\log} = \big(\tau^{-1}(\mathcal{O}_X) \otimes_{\boldsymbol{Z}} \mathrm{Sym}_{\boldsymbol{Z}}(\mathcal{L}_X)\big)/\mathfrak{a}\ ,$$

where $\mathcal{L}_X$ is a sheaf of abelian groups which sits in the commutative diagram

$$(2.7.1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & 2\pi\sqrt{-1}\boldsymbol{Z}_{X^{\log}} & \longrightarrow & \tau^{-1}\mathcal{O}_X & \xrightarrow{\tau^{-1}(\exp)} & \tau^{-1}\mathcal{O}_X^* & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \downarrow & & \\ 0 & \longrightarrow & 2\pi\sqrt{-1}\boldsymbol{Z}_{X^{\log}} & \longrightarrow & \mathcal{L}_X & \xrightarrow{\exp} & \tau^{-1}M_X^{gp} & \longrightarrow & 0 \end{array}$$

with exact rows and $\mathfrak{a}$ is the ideal generated locally by local sections of the form

$$f \otimes \underline{1} - 1 \otimes h(f) \ \text{ for a local section } f \text{ of } \tau^{-1}(\mathcal{O}_X).$$

Here $\underline{1}$ means the $1 \in \boldsymbol{Z} = \mathrm{Sym}^0(\mathcal{L}_X)$, whereas $h(f)$ belongs to $\mathcal{L}_X = \mathrm{Sym}^1(\mathcal{L}_X)$. See [8] or [16] for the precise definitions of $\mathcal{L}_X$ and h.

We describe some properties of $(X^{\log}, \mathcal{O}_X^{\log})$. For an open log subspace $(U, M_X|_U)$ of X, $U^{\log}$ is an open subspace of $X^{\log}$ and $\mathcal{O}_U^{\log} = \mathcal{O}_X^{\log}|_{U^{\log}}$. For each $x \in X$, the fiber $\tau^{-1}(x)$ is homeomorphic to the product of r' copies of $\boldsymbol{S}^1$ and the stalk of $\mathcal{O}_X^{\log}$ at any point y over x is isomorphic as an $\mathcal{O}_{X,x}$-algebra to the polynomial ring of r' indeterminates over $\mathcal{O}_{X,x}$, where r' is the rank of the free monoid $M_{X,x}/\mathcal{O}_{X,x}^*$. This r' also equals to the number of the indexes i such that the image of e_i in $M_{X,x}$ by β does not belong to $\mathcal{O}^*_{X,x}$, where $(e_i)_i$ and $\beta : \boldsymbol{N}^r_X \to M_X$ are as above. See [8] Lemma (1.3)(2) and Lemma (3.3).

We note that $(-)^{\log}$ is a functor from the category of log analytic spaces satisfying the condition in (2.4) to that of ringed spaces and that τ is a natural transformation from $(-)^{\log}$ to $(\overset{\circ}{-})$.

Remark 2.8. It is the ringed space defined above that we called in the introduction a ringed real blow up.

We will use the following two propositions later in Section 3.

Proposition 2.9. *Let $f : X \to Y$ be a proper morphism of log analytic spaces satisfying the condition in* (2.4). *Then $f^{\log}$ is also proper.*

Proof. Since τ in (2.7) is proper, we see that both $X^{\log} \to Y$ and $Y^{\log} \to Y$ are proper. Then $f^{\log}$ is proper. ■

Proposition 2.10. *Let $f : X \to Y$ be a morphism of log analytic spaces satisfying the condition in* (2.4). *Assume that the homomorphism of log structures $\overset{\circ}{f}{}^* M_Y \to M_X$ is an isomorphism. Then the natural homomorphism $\tau^{-1}(\mathcal{O}_X) \otimes_{(f\tau)^{-1}(\mathcal{O}_Y)} f^{\log -1}(\mathcal{O}^{\log}_Y) \to \mathcal{O}^{\log}_X$ is an isomorphism.*

Proof. This is checked at stalks by using the descriptions of the stalks of $\mathcal{O}^{\log}_X$ and $\mathcal{O}^{\log}_Y$ [8] Lemma (3.3). Cf. 2.7 above. ■

The following proposition and its variant 2.17 were proved by T. Matsubara.

Proposition 2.11. *Let X be a log analytic space satisfying the condition in* (2.4). *Let F be a locally free $\mathcal{O}_X$-module of finite rank and $\tau : X^{\log} \to X$ the natural morphism in* 2.7. *Then the natural homomorphism*

$$F \to \mathrm{R}\tau_* \tau^* F$$

is a quasi-isomorphism.

Proof. This is a special case of [10] Proposition 4.6. For reader's convenience, we recall the proof briefly: We may assume that $F = \mathcal{O}_X$. We have to show that for any $x \in X$, $(\mathrm{R}^q \tau_* \mathcal{O}^{\log}_X)_x = 0$ for $q > 0$ (resp. $= \mathcal{O}_{X,x}$ for $q = 0$). Since τ is proper and separated, we can work fiberwise.

As described in 2.7, $\tau^{-1}(x)$ is homeomorphic to $(\boldsymbol{S}^1)^{r'}$ and all the stalks of $\mathcal{O}_X^{\log}|_{\tau^{-1}(x)}$ are isomorphic to $\mathcal{O}_{X,x}[T_1, \dots, T_{r'}]$, where r' is as in 2.7 and T_i's are indeterminates. Further $\mathcal{O}_X^{\log}|_{\tau^{-1}(x)}$ is in fact a locally constant sheaf and the action of $\pi_1(\tau^{-1}(x))$ can be described as $g_i(T_j) = T_j + \delta_{ij} 2\pi\sqrt{-1}$ $(1 \leq i, j \leq r')$ in taking a suitable $(T_j)_j$ and $(g_i)_i$ such that the set $\{g_1, \dots, g_{r'}\}$ generates $\pi_1(\tau^{-1}(x))$. It is enough to show that $\mathrm{H}^q(\tau^{-1}(x), \mathcal{O}_X^{\log}|_{\tau^{-1}(x)}) = 0$ for $q > 0$ (resp. $= \mathcal{O}_{X,x}$ for $q = 0$). The case where $r' = 1$ is deduced from the exactness of $0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}[T_1] \xrightarrow{g_1 - \mathrm{id}} \mathcal{O}_{X,x}[T_1] \to 0$. Here we use the fact that the cohomologies of $\boldsymbol{S}^1$ with a locally constant coefficient sheaf M are calculated by the complex $M_x \xrightarrow{g - \mathrm{id}} M_x$, where $x \in \boldsymbol{S}^1$ and g is the monodromy. The general case is reduced to this case by the Künneth formula. ■

NOTATION 2.12. ([14] (2.6) and [8] (1.2.3)) Let X be a manifold and D a reduced divisor with normal crossings on X. Let $i : D \to X$ be the closed immersion and let $j : U := X \setminus D \hookrightarrow X$ be the open immersion from the complement. In the next definition we denote by $M_{D,X}$ the pull-back log structure $i^*(\mathcal{O}_X \cap j_* \mathcal{O}_U^* \xhookrightarrow{\alpha} \mathcal{O}_X)$ on D.

DEFINITION 2.13. Let $f : X \to \Delta$ be a semistable degeneration and $Y = f^{-1}(0)$ (Definition 1.1). Then f induces a morphism of log analytic spaces $(Y, M_{Y,X}) \to (\{0\}, M_{\{0\},\Delta})$, which we call the *log central fiber* of f. We denote $(\{0\}, M_{\{0\},\Delta})$ simply by 0 and call it the *standard log point.*

EXAMPLE 2.14. Let 0 be the standard log point. Then $0^{\log} = \boldsymbol{S}^1$ and $\mathcal{O}_0^{\log}$ is a locally constant sheaf whose local value is $\boldsymbol{C}[\log t]$, the polynomial ring over $\boldsymbol{C}$, where t is a global section of the log structure M_0 that generates $\Gamma(0, M_0)$ over $\Gamma(0, \mathcal{O}_0^*) = \boldsymbol{C}^*$.

DEFINITION 2.15. ([14] Definition (3.8)) A morphism $Y \to 0$ from a log analytic space to the standard log point is said to be a *log deformation* if locally on $\overset{\circ}{Y}$, Y is isomorphic over 0 to the log central fiber of a semistable degeneration (Definition 2.13) and if each irreducible component of $\overset{\circ}{Y}$ is smooth over $\boldsymbol{C}$.

The log central fiber of a semistable degeneration is clearly a log deformation.

REMARK 2.16. In 2.12, $(X, \mathcal{O}_X \cap j_* \mathcal{O}_U^*)$ satisfies the condition in (2.4) (cf. [7] Example (2.5)(1)); and by 2.6, $(D, M_{D,X})$ also satisfies the condition in (2.4). Hence any log deformation (and in particular the standard log point also) satisfies the condition in (2.4).

PROPOSITION 2.17. *Let $f: Y \to 0$ be a log deformation. Let Y_1 be the log analytic space $(\overset{\circ}{Y}, \overset{\circ}{f}{}^* M_0)$, which satisfies the condition in* (2.4) *by* (2.6). *Let g be the natural morphism $Y \to Y_1$ induced by f. Let F be a locally free $\mathcal{O}_Y$-module of finite rank and let $\tau: Y^{\log} \to \overset{\circ}{Y}$ and $\tau_1: Y_1^{\log} \to \overset{\circ}{Y}$ be the natural morphisms in* (2.7) *for Y and Y_1 respectively. Then the natural homomorphism*

$$\tau_1^* F \to \mathrm{R}g_*^{\log} \tau^* F$$

is a quasi-isomorphism.

PROOF. This is a special case of [10] Lemma 4.5. The properness of $\overset{\circ}{f}$ is assumed there for another purpose; but as for [10] Lemma 4.5 only, this assumption is not necessary. The proof is similar to the previous Proposition 2.11. We calculate the cohomologies of the fibers of $g^{\log}$, which are again the products of some copies of $\boldsymbol{S}^1$. ■

(2.18) Here we explain log de Rham complexes on a semistable degeneration and on a log deformation.

First, let $f: X \to \Delta$ be a semistable degeneration and $Y = f^{-1}(0)$. We denote by ω_X^1 the sheaf of differential forms with log poles $\Omega_X^1(\log \overset{\circ}{Y})$ in the usual sense ([2]). We have the log de Rham complex $\omega_X^{\cdot}$.

Next let $Y \to 0$ be a log deformation. We denote by ω_Y^1 the sheaf of differential forms with log poles on Y ([8] (3.5)). (In the case that Y is the log central fiber of a semistable degeneration $X \to \Delta$, ω_Y^1 is isomorphic to the pull-back of ω_X^1 to $\overset{\circ}{Y}$ as a coherent sheaf.) We have the log de Rham complex $\omega_Y^{\cdot}$.

Further we consider the $\mathcal{O}_Y^{\log}$-module $\omega_Y^{1,\log} := \tau^* \omega_Y^1$. This module is endowed with the derivation $d: \mathcal{O}_Y^{\log} \to \omega_Y^{1,\log}$, which is compatible with the usual derivation and $d \log: M_Y \to \omega_Y^1$. For the precise definition, see [8] (3.5). Thus we have the complex $\omega_Y^{\cdot,\log}$. We also have the log Poincaré lemma as follows.

PROPOSITION 2.19. *Let $f: Y \to 0$ be a log deformation. Then the natural homomorphism $\boldsymbol{C}_{Y^{\log}} \to \omega_Y^{\cdot,\log}$ is a quasi-isomorphism.*

PROOF. This is a part of [8] Theorem (3.8). We see that the condition in [8] (0.4) is satisfied for a log deformation by taking $P_\lambda = \boldsymbol{N}^r$ for some $r \geq 1$ and $\Sigma_\lambda = \langle (1, \ldots, 1) \rangle$ in the notation there. ■

3. CMHC on a log deformation.

In this section we construct a CMHC on a log deformation. This is an analogue of the Steenbrink's result in [13] in the context of the log geometry. As mentioned in the Introduction, Kawamata-Namikawa [9] and Steenbrink [14] obtained such result independently. Here we present another way to construct CMHC on a log deformation, which is a «logarithmic» analogue of the argument in [15], and prove the coincidence of our CMHC and Kawamata-Namikawa's.

(3.1) Let $f : Y \to 0$ be a log deformation. Then we have a commutative diagram

$$\begin{array}{ccc} Y^{\log} & \xrightarrow{\tau} & Y \\ {\scriptstyle f^{\log}}\downarrow & & \downarrow{\scriptstyle f} \\ \boldsymbol{S}^1 = 0^{\log} & \xrightarrow{\tau} & 0 \end{array}$$

by the functoriality of $(-)^{\log}$. As in the article [16] by S. Usui, we define a topological space Y_∞ and the morphisms $\pi : Y_\infty \to Y^{\log}$, $f_\infty : Y_\infty \to \boldsymbol{R}$ by the cartesian square

$$\begin{array}{ccc} Y_\infty & \xrightarrow{\pi} & Y^{\log} \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f^{\log}} \\ \boldsymbol{R} & \longrightarrow & \boldsymbol{S}^1 = 0^{\log} \end{array} \tag{3.1.1}$$

where the morphism of the bottom line $\mathbf{R} \to \boldsymbol{S}^1 = 0^{\log}$ is the universal covering given by $s \mapsto \exp(2\pi\sqrt{-1}s)$, where s denotes the coordinate function of $\boldsymbol{R}$. The covering transformation of $\boldsymbol{R} \to \boldsymbol{S}^1$ given by $s \mapsto s+1$ gives rise to an automorphism of Y_∞ over $Y^{\log}$. This automorphism induces the monodromy automorphism

$$T : \pi_* \pi^{-1} F \to \pi_* \pi^{-1} F$$

for every abelian sheaf F on $Y^{\log}$. The direct image

$$\tau_* \pi_* \pi^{-1} F \to \tau_* \pi_* \pi^{-1} F$$

of the monodromy automorphism above is called the monodromy automorphism too and denoted by the same letter T by abuse of the language.

LEMMA 3.2. *In the situation above we have an exact sequence*

$$0 \longrightarrow F \longrightarrow \pi_* \pi^{-1} F \xrightarrow{T-\mathrm{id}} \pi_* \pi^{-1} F \to 0 \tag{3.2.1}$$

on $Y^{\log}$. In particular we have an exact sequence

$$0 \longrightarrow \tau_* F \longrightarrow \tau_* \pi_* \pi^{-1} F \xrightarrow{T-\mathrm{id}} \tau_* \pi_* \pi^{-1} F \longrightarrow 0$$

if the abelian sheaf F is τ_*-acyclic.

PROOF. Take an open subset U of $Y^{\log}$ such that the morphism π coincides with the projection $U \times \boldsymbol{Z} \to U$. Then we have

$$\Gamma(U, \pi_* \pi^{-1} F) = \mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F)),$$

where $\mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F))$ denotes the set of all mappings from $\boldsymbol{Z}$ to $\Gamma(U, F)$ as sets. Then the canonical morphism $F \to \pi_* \pi^{-1} F$ induces the diagonal morphism $\Gamma(U, F) \to \mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F))$ sending a of $\Gamma(U, F)$ to the constant map with values a. Moreover the monodromy automorphism T acts on this set by $T(a)(i) = a(i+1)$ for every element a of $\mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F))$. Thus we can see that the diagonal morphism above coincides with the kernel of the morphism $T - \mathrm{id}$. Now we prove the surjectivity of the morphism $T - \mathrm{id}$. Take an element b of $\mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F))$. Define an element a of $\mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F))$ by

$$a(i) = \begin{cases} \sum_{k=0}^{i-1} b(k) & \text{for } i > 0 \\ 0 & \text{for } i = 0 \\ -\sum_{k=i}^{-1} b(k) & \text{for } i < 0 . \end{cases}$$

Then we can easily check the equality $(T - \mathrm{id})(a) = b$. Thus the morphism $T - \mathrm{id}$ on $\Gamma(U, \pi_* \pi^{-1} F) = \mathrm{Map}(\boldsymbol{Z}, \Gamma(U, F))$ is surjective. Thus we obtain the exact sequence (3.2.1) ■

LEMMA 3.3. *In the situation above, let F be a locally free $\mathcal{O}_Y$-module of finite rank. Then there is a quasi-isomorphism*

$$F \otimes_{\boldsymbol{C}} \boldsymbol{C}[u] \to \mathrm{R}(\tau\pi)_* \pi^{-1} \tau^* F$$

that sends the indeterminate u *to* $(2\pi\sqrt{-1})^{-1}\log t$, *where* t *is a generator of the log structure of the standard log point* 0 (*cf. Example* 2.14).

PROOF. Let the notation be as in Proposition 2.17. Define $g_\infty = g^{\log} \times \times_{S^1} \boldsymbol{R} : Y_\infty \to Y_{1,\infty} = \overset{\circ}{Y} \times \boldsymbol{R}$ and denote by π_1 the projection $Y_{1,\infty} = \overset{\circ}{Y} \times \times \boldsymbol{R} \to Y_1^{\log} = \overset{\circ}{Y} \times \boldsymbol{S}^1$. These spaces make a commutative diagram

$$\begin{array}{ccc}
Y_\infty & \xrightarrow{g_\infty} & Y_{1,\infty} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_1} \\
Y^{\log} & \xrightarrow{g^{\log}} & Y_1^{\log} \\
{\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau_1} \\
Y & \xrightarrow{g} & Y_1
\end{array}$$

where the upper square is cartesian. Then $\mathrm{R}(\tau\pi)_* \pi^{-1}\tau^* F = = \mathrm{R}(\tau_1\pi_1)_* \mathrm{R}g_{\infty *}\pi^{-1}\tau^* F$, which equals to $\mathrm{R}(\tau_1\pi_1)_*\pi_1^{-1}\mathrm{R}g_*^{\log}\tau^* F$ by proper base change theorem with respect to the proper map $g^{\log}$ ($g^{\log}$ is proper by Proposition 2.9).

Further by Proposition 2.17, $\pi_1^{-1}\mathrm{R}g_*^{\log}\tau^* F$ is naturally quasi-isomorphic to

$$\pi_1^{-1}\tau_1^* F = \pi_1^{-1}(\tau_1^{-1}F \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_1}^{\log}) \overset{2.10}{=} \pi_1^{-1}(\tau_1^{-1}F \otimes_{\boldsymbol{C}} (f_1^{\log})^{-1}\mathcal{O}_0^{\log}) =$$

$$= (\tau_1\pi_1)^{-1}F \otimes_{\boldsymbol{C}} (\pi_1^{-1}(f_1^{\log})^{-1}\mathcal{O}_0^{\log}),$$

where $f_1 \colon Y_1 \to 0$ is the induced map from f. By Example 2.14, we know that $\mathcal{O}_0^{\log}$ is locally constant and that $\pi_1^{-1}(f_1^{\log})^{-1}\mathcal{O}_0^{\log}$ is constant valued by $\boldsymbol{C}[u]$, $u = (2\pi\sqrt{-1})^{-1}\log t$. Thus we have $\mathrm{R}(\tau\pi)_*\pi^{-1}\tau^* F = = \mathrm{R}(\tau_1\pi_1)_*(\tau_1\pi_1)^{-1}(F \otimes_{\boldsymbol{C}} \boldsymbol{C}[u])$, which is seen to be quasi-isomorphic to $F \otimes_{\boldsymbol{C}} \boldsymbol{C}[u]$ by applying [6] Proposition 2.7.8, taking Y_n there to be $\overset{\circ}{Y}_1 \times \times [-n, n]$ for each n. ■

(3.4) Assume that we are given an injective resolution $\boldsymbol{Q}_{Y^{\log}} \to I$. Then we have a quasi-isomorphism

$$\boldsymbol{Q}_{Y_\infty} = \pi^{-1}\boldsymbol{Q}_{Y^{\log}} \to \pi^{-1}I.$$

The $\boldsymbol{Q}$-sheaf $\pi^{-1}I^p$ is an injective $\boldsymbol{Q}$-sheaf for every p because I^p is injective and because the injectivity of the sheaf is a local property (see [6]

Proposition 2.4.10). Therefore we obtain an isomorphism

$$\mathrm{R}(\tau\pi)_* \boldsymbol{Q}_{Y_\infty} \to \tau_* \pi_* \pi^{-1} I$$

in the derived category. Moreover we have an exact sequence

$$0 \longrightarrow \tau_* I \longrightarrow \tau_* \pi_* \pi^{-1} I \xrightarrow{T - \mathrm{id}} \tau_* \pi_* \pi^{-1} I \longrightarrow 0$$

by Lemma 3.2 because of the τ_*-acyclicity of the sheaf I^p for every p.

We define subcomplexes $B(I)_m$ and $B(I)$ of $\tau_* \pi_* \pi^{-1} I$ by

$$B(I)_m = \mathrm{Ker}\,(T - \mathrm{id})^{m+1} \subset \tau_* \pi_* \pi^{-1} I \tag{3.4.1}$$

for non-negative integer m and by

$$B(I) = \bigcup_{m \geqslant 0} B(I)_m. \tag{3.4.2}$$

The subcomplex $B(I)_0$ is identified with the complex $\tau_* I$ via the canonical morphism $\tau_* I \to \tau_* \pi_* \pi^{-1} I$.

The morphism $\log T$ is well-defined on the subcomplex $B(I)$ by definition. We define an automorphism $U : B(I) \to B(I)$ by

$$U = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (T - \mathrm{id})^k$$

which satisfies the following conditions:

(3.4.3) $U \cdot (T - \mathrm{id}) = (T - \mathrm{id}) \cdot U$
(3.4.4) $U \cdot \log T = \log T \cdot U$
(3.4.5) $U|_{B(I)_0} = \mathrm{id}$
(3.4.6) $(T - \mathrm{id}) \cdot U = \log T$.

Then we see that the morphism $\log T : B(I) \to B(I)$ is surjective because the morphism $T - \mathrm{id}\colon B(I) \to B(I)$ is surjective by Lemma 3.2. Moreover we have $\mathrm{Ker}\,(\log T : B(I) \to B(I)) = \mathrm{Ker}\,(T - \mathrm{id}) = B(I)_0 \simeq \simeq \tau_* I$.

LEMMA 3.5. *For an injective resolution $\boldsymbol{Q}_{Y^{\log}} \to I$ the inclusion $B(I) \to \tau_* \pi_* \pi^{-1} I$ is a quasi-isomorphism.*

PROOF. It is sufficient to prove that the morphism

$$B(I)_{\boldsymbol{C}} = B(I) \otimes \boldsymbol{C} \to \tau_* \pi_* \pi^{-1} I_{\boldsymbol{C}} = \tau_* \pi_* \pi^{-1} I \otimes \boldsymbol{C}$$

obtained by tensoring $\boldsymbol{C}$ is a quasi-isomorphism. By Proposition 2.19 in the last section the canonical morphism $\boldsymbol{C}_{Y^{\log}} \to \omega_Y^{\cdot,\log}$ is a quasi-isomorphism. Then there exists a quasi-isomorphism $\omega_Y^{\cdot,\log} \to I_{\boldsymbol{C}}$ which sits in the commutative diagram

$$
\begin{tikzcd}
\boldsymbol{C}_{Y^{\log}} \arrow[r] \arrow[d, equal] & \omega_Y^{\cdot,\log} \arrow[d] \\
\boldsymbol{C}_{Y^{\log}} \arrow[r] & I_{\boldsymbol{C}}
\end{tikzcd}
\tag{3.5.1}
$$

because $\boldsymbol{C}_{Y^{\log}} \to I_{\boldsymbol{C}}$ is an injective resolution of $\boldsymbol{C}_{Y^{\log}}$. Then we have a quasi-isomorphism

$$\omega_Y^{\cdot}[u] \simeq \mathrm{R}(\tau\pi)_* \pi^{-1} \omega_Y^{\cdot,\log} \to \mathrm{R}(\tau\pi)_* \pi^{-1} I_{\boldsymbol{C}} \simeq \tau_* \pi_* \pi^{-1} I_{\boldsymbol{C}}$$

by using Lemma 3.3. We remark that the monodromy automorphism T on the right hand side corresponds to the automorphism on $\omega_Y^{\cdot}[u]$ induced by the homomorphism of algebras $\boldsymbol{C}[u] \to \boldsymbol{C}[u]$ sending u to $u+1$ via the quasi-isomorphism above. Thus we see that the quasi-isomorphism above factors through the subcomplex $B(I)_{\boldsymbol{C}} = B(I) \otimes \boldsymbol{C}$. Therefore it is sufficient to prove that the morphism $\omega_Y^{\cdot}[u] \to B(I)_{\boldsymbol{C}}$ is a quasi-isomorphism.

An increasing filtration fil on $\omega_Y^{\cdot}[u]$ is defined by the degree of the indeterminate u. On the other hand the subcomplex $B(I)_m$ of $B(I)$ defines a filtration on $B(I)$. It is easy to see that the morphism above sends $\mathrm{fil}_m \omega_Y^{\cdot}[u]$ to $B(I)_{m,\boldsymbol{C}}$. Therefore it suffices to prove that the induced morphism

$$\mathrm{Gr}_m^{\mathrm{fil}} \omega_Y^{\cdot}[u] \to B(I)_{m,\boldsymbol{C}} / B(I)_{m-1,\boldsymbol{C}}$$

is a quasi-isomorphism. Trivially we have an isomorphism $\omega_Y^{\cdot} \to \mathrm{Gr}_m^{\mathrm{fil}} \omega_Y^{\cdot}[u]$ by sending x to xu^m. On the other hand the morphism $(T-\mathrm{id})^m$ induces an isomorphism $B(I)_{m,\boldsymbol{C}}/B(I)_{m-1,\boldsymbol{C}} \to B(I)_{0,\boldsymbol{C}}$. Moreover these isomorphisms make the following diagram commutative

$$
\begin{tikzcd}
\omega_Y^{\cdot} \arrow[r] \arrow[d] & \mathrm{Gr}_m^{\mathrm{fil}} \omega_Y^{\cdot}[u] \arrow[d] \\
B(I)_{0,\boldsymbol{C}} & B(I)_{m,\boldsymbol{C}}/B(I)_{m-1,\boldsymbol{C}}, \arrow[l]
\end{tikzcd}
$$

where the left vertical arrow stands for the composite of the quasi-isomorphism

$$\omega^{\cdot}_{Y} \to \mathrm{R}\tau_{*}\omega^{\cdot,\log}_{Y}$$

in Proposition 2.11 and the quasi-isomorphism

$$\mathrm{R}\tau_{*}\omega^{\cdot,\log}_{Y} \to \mathrm{R}\tau_{*}I_{\boldsymbol{C}} \simeq B(I)_{0,\boldsymbol{C}}$$

induced by the quasi-isomorphism $\omega^{\cdot,\log}_{Y} \to I_{\boldsymbol{C}}$ before. Thus we complete the proof. ■

(3.6) Now we fix an injective resolution $\boldsymbol{Q}_{Y^{\log}} \to I$ and a quasi-isomorphism $\omega^{\cdot,\log}_{Y} \to I_{\boldsymbol{C}}$ as above which we call a reference morphism. Then the reference morphism induces a quasi-isomorphism $\varphi : \omega^{\cdot}_{Y}[u] \to B(I)_{\boldsymbol{C}}$ as in the proof of the last lemma.

We denote the morphism of complexes

$$-\frac{1}{2\pi\sqrt{-1}}\log T : B(I) \to B(I)(-1)$$

by δ. Then we obtain the Steenbrink-Zucker complex $(SZ(B(I), \delta), L)$ which is a complex of $\boldsymbol{Q}$-sheaves on Y.

On the other hand a morphism of complexes

$$\delta = -\frac{1}{2\pi\sqrt{-1}}\frac{d}{du} : \omega^{\cdot}_{Y}[u] \to \omega^{\cdot}_{Y}[u]$$

gives us the Steenbrink-Zucker complex $(SZ(\omega^{\cdot}_{Y}[u], \delta), L)$.

We can easily see that the morphism φ fits in the commutative diagram

$$\begin{array}{ccc}
\omega^{\cdot}_{Y}[u] & \xrightarrow{\varphi} & B(I)_{\boldsymbol{C}} \\
{\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} \\
w^{\cdot}_{Y}[u] & \xrightarrow{\varphi} & B(I)_{\boldsymbol{C}}.
\end{array}$$

Therefore we have a morphism of the Steenbrink-Zucker complexes

$$\text{(3.6.1)} \qquad SZ_{1}(\varphi) : SZ(\omega^{\cdot}_{Y}[u], \delta) \to SZ(B(I)_{\boldsymbol{C}}, \delta) = SZ(B(I), \delta)_{\boldsymbol{C}}$$

preserving the filtration L on both sides by the functoriality in Remark 1.15.

LEMMA 3.7. *The morphism $SZ_1(\varphi)$ is a filtered quasi-isomorphism with respect to the filtration L on both sides.*

PROOF. We have

$$\mathrm{Ker}\,(\delta : \omega_Y^{\cdot}[u] \to \omega_Y^{\cdot}[u]) = \omega_Y^{\cdot}$$

and

$$\mathrm{Ker}\,(\delta : B(I)_{\mathbf{C}} \to B(I)_{\mathbf{C}}) = B(I)_{0,\mathbf{C}} \simeq \tau_* I_{\mathbf{C}}.$$

The morphism φ induces a quasi-isomorphism from $\omega_Y^{\cdot}$ to $B(I)_{0,\mathbf{C}}$ as in the proof of Lemma 3.5. Moreover the endomorphisms δ on $\omega_Y^{\cdot}[u]$ and $B(I)_{\mathbf{C}}$ are surjective. Thus we obtain the conclusion by Corollary 1.16. ■

(3.8) On the complex $\omega_Y^{\cdot}$ Steenbrink defines a finite increasing filtration W in [14, § 4] which is an analogue of the usual weight filtration.

A morphism of complexes

$$\theta : \omega_Y^{\cdot} \to \omega_Y^{\cdot}[1] \tag{3.8.1}$$

is given by

$$\theta(x) = d \log t \wedge x$$

for a local section x of ω_Y^p, where t is the generator of the log structure of the standard log point 0 (cf. Example 2.14).

Then we define a data $\{(K_n, W), \theta_n\}$ by

(3.8.2) $K_n = \omega_Y^{\cdot}$ with the increasing filtration W for every n
(3.8.3) $\theta_n = \theta$ for every n.

We can easily see that this data satisfies the conditions (1.4.1)-(1.4.5) by definition. Thus we obtain the Steenbrink complex of the data $\{(K_n, W), \theta_n\}$ which is denoted by $(A_{\mathbf{C}}, L)$ as in [13] and in [14].

(3.9) Now we will establish the relation between the complexes $SZ(\omega_Y^{\cdot}[u], \delta)$ and $A_{\mathbf{C}}$ constructed above.

We define a morphism

$$\psi : \varrho(\omega_Y^{\cdot}[u], \delta)^p = \omega_Y^p[u] \oplus \omega_Y^{p-1}[u] \to \omega_Y^p$$

by

$$\psi(x, y) = x_0 + d\log t \wedge y_0, \tag{3.9.1}$$

where $x = \sum_{j=0}^{r} x_j(u^j/j!)$ and $y = \sum_{j=0}^{r} y_j(u^j/j!)$ are elements of $\omega_Y^p[u]$ and $\omega_Y^{p-1}[u]$. We can easily see that the morphisms ψ for all p form a morphism of complexes $\psi : \varrho(\omega_Y^{\cdot}[u], \delta) \to \omega_Y^{\cdot}$.

LEMMA 3.10. *The morphism ψ above is a quasi-isomorphism.*

PROOF. The inclusion $\omega_Y^{\cdot} \to \omega_Y^{\cdot}[u]$ induces a quasi-isomorphism $\iota : \omega_Y^{\cdot} \to \varrho(\omega_Y^{\cdot}[u], \delta)$ by Lemma 1.10 because the inclusion above coincides with the kernel of the morphism δ and because the morphism δ is surjective. We can easily see that the composite of the morphisms $\psi\iota : \omega_Y^{\cdot} \to \omega_Y^{\cdot}$ is nothing but the identity. Thus we obtain the conclusion.

By definition the identity on $\omega_Y^{\cdot}$ induces a morphism of filtered complexes

$$(\omega_Y^{\cdot}, \tau) \to (\omega_Y^{\cdot}, W),$$

where τ denotes the canonical filtration. In [14] this is proved to be a filtered quasi-isomorphism. Thus the morphism of complexes ψ induces a filtered quasi-isomorphism

$$(\varrho(\omega_Y^{\cdot}[u], \delta), \tau) \to (\omega_Y^{\cdot}, W)$$

which we denote by the same letter ψ.

The morphism $\theta : \varrho(\omega_Y^{\cdot}[u], \delta) \to \varrho(\omega_Y^{\cdot}[u], \delta)[1]$, the morphism $\theta : \omega_Y^{\cdot} \to \omega_Y^{\cdot}[1]$ and the morphism $\psi : \varrho(\omega_Y^{\cdot}[u], \delta) \to \omega_Y^{\cdot}$ satisfies the equality $\theta\psi = \psi\theta$. By setting

$$\psi_n = \psi : (\varrho(\omega_Y^{\cdot}[u], \delta), \tau) \to (\omega_Y^{\cdot}, W)$$

for every n, we obtain a morphism of the Steenbrink complexes

$$sD(\psi) : SZ(\omega_Y^{\cdot}[u], \delta) = sD(\varrho(\omega_Y^{\cdot}[u], \delta), \tau, \theta) \to A_C$$

by the functoriality.

LEMMA 3.11. *The morphism*

$$sD(\psi) : SZ(\omega_Y^{\cdot}[u], \delta) \to A_C$$

is a filtered quasi-isomorphism with respect to the filtration L on both sides.

PROOF. Easy by Proposition 1.8 and by the fact that $\psi : (\varrho(\omega^{\cdot}_Y[u], \delta), \tau) \to (\omega^{\cdot}_Y, W)$ is a filtered quasi-isomorphism. ■

The filtered quasi-isomorphisms $sD(\psi) : SZ(\omega^{\cdot}_Y[u], \delta) \to A_{\boldsymbol{C}}$ and $SZ_1(\varphi) : SZ(\omega^{\cdot}_Y[u], \delta) \to SZ(B(I), \delta)_{\boldsymbol{C}}$ in (3.6.1) define an isomorphism

$$\alpha : SZ(B(I), \delta)_{\boldsymbol{C}} \to A_{\boldsymbol{C}}$$

in the derived category.

On the other hand we have a finite decreasing filtration F on $A_{\boldsymbol{C}} = sD(\omega^{\cdot}_Y, W, \theta)$ defined in Remark 1.6.

THEOREM 3.12. *Assume that the analytic space Y is compact and that the irreducible components of Y are Kähler. Then the data*

$$((SZ(B(I), \delta), L), (A_{\boldsymbol{C}}, L, F), \alpha) \tag{3.12.1}$$

is a cohomological mixed Hodge complex on Y.

PROOF. Let $Y_1, Y_2, \ldots, Y_l$ be the irreducible components of the analytic space Y. We set

$$Y[m] = \coprod_{1 \le i_1 < i_2 < \ldots < i_m \le l} Y_{i_1} \cap Y_{i_2} \cap \ldots \cap Y_{i_m}$$

for every m. Then every $Y[m]$ is nonsingular because Y is the underlying analytic space of a log deformation. Thus every $Y[m]$ is a compact Kähler complex manifold by the assumption. In [14] Steenbrink constructed an isomorphism of complexes

$$R_m : \mathrm{Gr}^W_m \omega^{\cdot}_Y \to \Omega^{\cdot}_{Y[m]}[-m]$$

for every m which is the analogue of the Poincaré residue isomorphism. By using this isomorphism we obtain the conclusion as in [13], [15]. ■

REMARK 3.13. We can easily see that the same conclusion holds for the case that Y is algebraic.

REMARK 3.14. The $\boldsymbol{C}$-structure $A_{\boldsymbol{C}}$ of the CMHC above is the same as the ones in [9] and [14]. But the method to construct $\boldsymbol{Q}$-structure is different from the methods used in [9] and [14]. We will see later that the resulting $\boldsymbol{Q}$-structures are essentially the same. (cf. Proposition 3.19 and Theorem 5.8 below.)

REMARK 3.15. We can see that the CMHC in the theorem above is independent of the choice of the reference morphism $\omega_Y^{\cdot,\log} \to I_{\boldsymbol{C}}$ as follows. Let us assume that we have two reference morphisms $\nu_1, \nu_2 \colon \omega_Y^{\cdot,\log} \to I_{\boldsymbol{C}}$. These morphisms induce morphisms $\varphi_1, \varphi_2 \colon \omega_Y^{\cdot}[u] \to B(I)_{\boldsymbol{C}}$ as before. There exist a complex J consisting of injective $\boldsymbol{C}$-sheaves bounded below and a quasi-isomorphism $\nu : I_{\boldsymbol{C}} \to J$ such that the morphisms $\nu\nu_1, \nu\nu_2 \colon \omega_Y^{\cdot,\log} \to J$ are homotopic because of the commutativity of the diagram (3.5.1). The morphism $\nu : I_{\boldsymbol{C}} \to J$ induces a morphism $\varphi : B(I)_{\boldsymbol{C}} \to B(J)$. Then the homotopy between $\nu\nu_1$ and $\nu\nu_2$ induces the homotopy between the morphisms

$$SZ_1(\varphi)SZ_1(\varphi_1),\ SZ_1(\varphi)\,SZ_1(\varphi_2) : SZ(\omega_Y^{\cdot}[u],\ \delta) \to SZ(B(J),\ \delta)$$

which preserves the filtration. Therefore the morphisms $SZ_1(\varphi_1)$ and $SZ_1(\varphi_2)$ from $SZ(\omega_Y^{\cdot}[u], \delta)$ to $SZ(B(I), \delta)_{\boldsymbol{C}}$ coincide in the filtered derived category. Hence the morphisms ν_1 and ν_2 induce the same morphism from $SZ(B(I), \delta)_{\boldsymbol{C}}$ to $A_{\boldsymbol{C}}$ in the derived category.

REMARK 3.16. We see that our CMHC is independent of the choice of an injective resolution $\boldsymbol{Q}_{Y^{\log}} \to I$ as follows. Let I and I' be injective resolutions of $\boldsymbol{Q}_{Y^{\log}}$. Then we have a morphism of complexes $\nu : I' \to I$ such that the diagram

$$\begin{array}{ccc} \boldsymbol{Q}_{Y^{\log}} & \longrightarrow & I' \\ \| & & \downarrow{\scriptstyle \nu} \\ \boldsymbol{Q}_{Y^{\log}} & \longrightarrow & I \end{array}$$

is commutative. By the remark above we may assume that the reference morphisms make the following commutative square

$$\begin{array}{ccc} \omega_Y^{\cdot,\log} & \longrightarrow & I'_{\boldsymbol{C}} \\ \| & & \downarrow{\scriptstyle \nu} \\ \omega_Y^{\cdot,\log} & \longrightarrow & I_{\boldsymbol{C}}. \end{array}$$

Then we have a commutative diagram

$$\begin{array}{ccccc} \omega^{\cdot}_{Y}[u] & \longrightarrow & B(I') & \longrightarrow & \tau_* \pi_* \pi^{-1} I'_{C} \\ \| & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ \omega^{\cdot}_{Y}[u] & \longrightarrow & B(I) & \longrightarrow & \tau_* \pi_* \pi^{-1} I_{C} \end{array}$$

from which we obtain an isomorphism

$$((SZ(B(I'), \delta), L), (A_C, L, F), \alpha') \simeq ((SZ(B(I), \delta), L), (A_C, L, F), \alpha)$$

of CMHC on Y. In addition the isomorphism above depends only on the homotpy class of the morphism $\nu : I' \to I$ as in the last remark.

(3.17) The morphism of complexes $\mu : B(I) \to SZ(B(I), \delta)$ is defined in Remark 1.13. For this morphism to be a quasi-isomorphism it is sufficient that the morphism δ induces a zero map from $\mathrm{H}^p(B(I))$ to $\mathrm{H}^p(B(I)(-1))$ by Lemma 1.14.

This is checked stalkwise as follows. Take a point y of Y and put $r :=$ $= \operatorname{rank}(M_Y^{gp}/\mathcal{O}_Y^*)_y$. Consider y as an fs log analytic space endowed with the pull-back log structure of M_Y. Let $y_\infty \xrightarrow{\pi_y} y^{\log} \xrightarrow{\tau_y} y$ be the base change of $Y_\infty \xrightarrow{\pi} Y^{\log} \xrightarrow{\tau} Y$ with respect to the closed immersion $y \to Y$. Since $Y^{\log}$ is locally contractible and τ is proper, we have $(\mathrm{R}^q(\tau\pi)_* \boldsymbol{Q}_{Y_\infty})_y \cong \mathrm{H}^q(y_\infty, \boldsymbol{Q})$ by base change. The space y_∞ is homeomorphic to $(\boldsymbol{S}^1)^{r-1} \times \boldsymbol{R}$, and the monodromy acts on the cohomology via the transformation $\boldsymbol{R} \to \boldsymbol{R}$; $u \mapsto u+1$. Since $\mathrm{H}^q(y_\infty, \boldsymbol{Q}) \cong \mathrm{H}^q((\boldsymbol{S}^1)^{r-1}, \boldsymbol{Q})$, the action is trivial.

Thus we obtain isomorphisms

$$\mathrm{R}(\tau\pi)_* \boldsymbol{Q}_{Y_\infty} \simeq \mathrm{R}(\tau\pi)_* \pi^{-1} \boldsymbol{Q}_{Y^{\log}} \simeq \tau_* \pi_* \pi^{-1} I \leftarrow B(I) \to SZ(B(I), \delta)$$

in the derived category. Therefore we obtain the following by Theorem 3.12.

Corollary 3.18. *The cohomology group* $\mathrm{H}^p(Y_\infty, \boldsymbol{Q})$ *carries a mixed Hodge structure.*

In [9] Y. Kawamata and Y. Namikawa define their CMHC for a compact Kähler normal crossing variety with a log structure in their sense, which corresponds to our log deformation Y in a natural way. Then the real blow up $\widetilde{Y}$ in their sense is nothing but the fiber of the morphism $Y^{\log} \to 0^{\log} = \boldsymbol{S}^1$ over the point $1 \in \boldsymbol{S}^1$, and the $\boldsymbol{Q}$-structure of their CMHC is $\mathrm{R}(\tau|_{\widetilde{Y}})_* \boldsymbol{Q}_{\widetilde{Y}}$.

PROPOSITION 3.19. *Our CMHC* (3.12.1) *coincides with Kawamata-Namikawa's CMHC in* [9].

PROOF. It is sufficient to check the coincidence of the $\boldsymbol{Q}$-structures because of Remark 3.14. Because $\tilde{Y}$ is isomorphic to the fiber $Y_{\infty,0}$ of the morphism $Y_\infty \to \mathbf{R}$ over the point 0, the sheaf $\mathrm{R}(\tau|_{\tilde{Y}})_* \boldsymbol{Q}_{\tilde{Y}}$ is identified with the sheaf $\mathrm{R}(\tau\pi)_* i_* \boldsymbol{Q}_{Y_{\infty,0}}$, where i denotes the inclusion $Y_{\infty,0} \to Y_\infty$. Then it is sufficient to prove that the canonical morphism $\mathrm{R}(\tau\pi)_* \boldsymbol{Q}_{Y_\infty} \to$ $\to \mathrm{R}(\tau\pi)_* i_* \boldsymbol{Q}_{Y_{\infty,0}}$ is a quasi-isomorphism. We consider the stalk of both sides over a point y of Y. Let $y_{\infty,0} \xrightarrow{i_y} y_\infty \xrightarrow{\pi_y} y^{\log} \xrightarrow{\tau_y} y$ be the base change of $Y_{\infty,0} \xrightarrow{i} Y_\infty \xrightarrow{\pi} Y^{\log} \xrightarrow{\tau} Y$ as in (3.17). We remarked in (3.17) that the stalk $(\mathrm{R}^q(\tau\pi)_* \boldsymbol{Q}_{Y_\infty})_y$ is identified with the cohomology group $\mathrm{H}^q(y_\infty, \boldsymbol{Q})$. Moreover the stalk $(\mathrm{R}^q(\tau\pi)_* i_* \boldsymbol{Q}_{Y_{\infty,0}})_y$ is identified with $\mathrm{H}^q(y_{\infty,0}, \boldsymbol{Q})$ by base change because $Y_{\infty,0} \to Y$ is proper. The space y_∞ is homeomorphic to $(\boldsymbol{S}^1)^{r-1} \times \boldsymbol{R}$ as in (3.17) and the space $y_{\infty,0}$ is homeomorphic to the closed subspace $(\boldsymbol{S}^1)^{r-1} = (\boldsymbol{S}^1)^{r-1} \times \{0\}$ of $(\boldsymbol{S}^1)^{r-1} \times \boldsymbol{R}$. Therefore the canonical morphism $\mathrm{H}^q(y_\infty, \boldsymbol{Q}) \to \mathrm{H}^q(y_{\infty,0}, \boldsymbol{Q})$ is an isomorphism. Thus we complete the proof. ∎

4. CMHC for a semistable degeneration.

(4.1) In this section we consider the case that the log deformation Y comes from a semistable degeneration as in [13], [15].

Let $f : X \to \varDelta$ be a semistable degeneration as in Definition 1.1. We denote by Y the central fiber $f^{-1}(0)$ which is a reduced simple normal crossing divisor on X and the inclusion $Y \to X$ by i. The coordinate function of $\varDelta$ is denoted by t and the pull back $f^* t$ on X is denoted by t again by abuse of the language if there is no danger of confusion. Then the morphism $f : X \to \varDelta$ turns out to be a morphism of log analytic spaces and defines a log deformation $Y \to 0$ as in Notation 2.12 and in Definition 2.13. We denote the morphism $Y \to 0$ above by f again by abuse of the language.

Then we obtain a cartesian square

$$\begin{array}{ccc} Y^{\log} & \xrightarrow{i^{\log}} & X^{\log} \\ {\scriptstyle f^{\log}}\downarrow & & \downarrow{\scriptstyle f^{\log}} \\ \boldsymbol{S}^1 = 0^{\log} & \longrightarrow & \varDelta^{\log}. \end{array}$$

The open immersion $\bar{j} : X^* \to X$ induces an open immersion $\bar{j}^{\log} : X^* \to$

$\to X^{\log}$. The closed subsets $0^{\log}$ of $\Delta^{\log}$ and $Y^{\log}$ of $X^{\log}$ are the complements of the open subsets Δ^* and X^* respectively. The open immersions $\Delta^* \to \to \Delta$ and $\bar{j}^{\log}: X^* \to X$ coincide with the restriction of the morphisms $\tau: \Delta^{\log} \to \Delta$ and $\tau: X^{\log} \to X$ respectively.

We denote the universal covering of the topological space $\Delta^{\log}$ by $\overline{\boldsymbol{H}}$. Then the upper half plane $\boldsymbol{H}$ is an open subset of $\overline{\boldsymbol{H}}$ and the complement $\overline{\boldsymbol{H}} \setminus \boldsymbol{H}$ is the universal covering $\boldsymbol{R}$ of the closed subset $0^{\log} = \boldsymbol{S}^1$ of $\Delta^{\log}$.

We define a topological space $\overline{X}_\infty$ by the cartesian square

$$\begin{array}{ccc} \overline{X}_\infty & \longrightarrow & X^{\log} \\ \downarrow & & \downarrow \\ \overline{\boldsymbol{H}} & \longrightarrow & \Delta^{\log}. \end{array}$$

We denote the morphism in the top line by $\pi: \overline{X}_\infty \to X^{\log}$. Then the complex manifold X_∞ defined in (1.2) is an open subset of $\overline{X}_\infty$ and the complement $\overline{X}_\infty \setminus X_\infty$ coincides with Y_∞. We denote the open immersion $X_\infty \to \to \overline{X}_\infty$ by $\bar{j}_\infty$ and the closed immersion $Y_\infty \to \overline{X}_\infty$ by i_∞. The morphisms $\pi: X_\infty \to X^*$ and $\pi: Y_\infty \to Y^{\log}$ coincide with the restriction of the morphism $\pi: \overline{X}_\infty \to X^{\log}$ to the open subset X_∞ and to the closed subset Y_∞ respectively. Moreover the monodromy action on $\overline{X}_\infty$, X_∞ and Y_∞ is compatible with the inclusions $\bar{j}_\infty: X_\infty \to \overline{X}_\infty$ and $i_\infty: Y_\infty \to \overline{X}_\infty$. We summarize the situation in the diagram

$$\begin{array}{ccccc} X_\infty & \xrightarrow{\bar{j}_\infty} & \overline{X}_\infty & \xleftarrow{i_\infty} & Y_\infty \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X^* & \xrightarrow{\bar{j}^{\log}} & X^{\log} & \xleftarrow{i^{\log}} & Y^{\log} \\ \| & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ X^* & \xrightarrow{\bar{j}} & X & \xleftarrow{i} & Y \end{array}$$

in which the four squares are cartesian.

(4.2) Let $C^\cdot(X_\infty)$, $K^\cdot(X_\infty)$, T, B and δ be as in (1.2). As we reviewed in (1.2) the $\boldsymbol{Q}$-structure of the CMHC in [15] is the Steenbrink-Zucker complex $SZ(B, \delta)$.

The log de Rham complex $\omega^\cdot_X$ in (2.18) carries the weight filtration W

as usual. A morphism of complexes $\theta : \omega^{\cdot}_X \to \omega^{\cdot}_X[1]$ is defined by

$$\theta(x) = \frac{dt}{t} \wedge x$$

for a local section x of ω^p_X. Then the $\boldsymbol{C}$-structure of the CMHC in [13], [15] is the Steenbrink complex $sD(\omega^{\cdot}_X, W, \theta)$.

The morphism

$$(\bar{j}\pi)^{-1}\omega^{\cdot}_X = \Omega^{\cdot}_{X_\infty} \to C^{\cdot}(X_\infty)_{\boldsymbol{C}}$$

defined by integrating differential forms on singular chains induces a morphism of complexes

$$i^{-1}\omega^{\cdot}_X[u] \to K^{\cdot}(X_\infty)_{\boldsymbol{C}}$$

which factors through the subcomplex $B_{\boldsymbol{C}}$. The morphism $\delta : B_{\boldsymbol{C}} \to B_{\boldsymbol{C}}$ corresponds to the morphism

$$\delta = -\frac{1}{2\pi\sqrt{-1}} \frac{d}{du} : i^{-1}\omega^{\cdot}_X[u] \to i^{-1}\omega^{\cdot}_X[u]$$

as before. Thus a morphism between Steenbrink-Zucker complex

$$SZ(i^{-1}\omega^{\cdot}_X[u], \delta) \to SZ(B, \delta)_{\boldsymbol{C}}$$

is obtained. Moreover a morphism $\varrho(i^{-1}\omega^{\cdot}_X[u], \delta)^p \to i^{-1}\omega^p_X$ is defined by the same formula as (3.9.1). Hence a morphism of complexes

$$SZ(i^{-1}\omega^{\cdot}_X[u], \delta) \to sD(\omega^{\cdot}_X, W, \theta) \tag{4.2.1}$$

is obtained by the same way as in Lemma 3.11. Steenbrink-Zucker [15] shows that these morphisms define an isomorphism $\alpha : SZ(B, \delta)_{\boldsymbol{C}} \to \to sD(\omega^{\cdot}_X, W, \theta)$ in the derived category and that the data

$$((SZ(B, \delta), L), (sD(\omega^{\cdot}_X, W, \theta), L, F), \alpha) \tag{4.2.2}$$

is a CMHC on Y.

Now we fix an injective resolution $\boldsymbol{Q}_{X^{\log}} \to J$ and a quasi-isomorphism

$$\omega^{\cdot, \log}_X \to J_{\boldsymbol{C}}$$

which is compatible with the morphisms $\boldsymbol{C}_{X^{\log}} \to J_{\boldsymbol{C}}$ and $\boldsymbol{C}_{X^{\log}} \to \omega^{\cdot, \log}_X$ as before. Then the complex $i^{-1}\tau_*\pi_*\pi^{-1}J$ carries the monodromy automorphism T and a subcomplex $B(J)$ is defined by the same formula as

(3.4.1)-(3.4.2). We can define the morphism $\log T$ on $B(J)$ and denote the morphism

$$-\frac{1}{2\pi\sqrt{-1}}\log T : B(J)\to B(J)(-1)$$

by δ as before. Then we obtain the Steenbrink-Zucker complex $SZ(B(J), \delta)$ on Y. On the other hand the morphism $\omega_X^{\cdot\,\log}\to J_C$ induces a morphism $i^{-1}\omega_X^{\cdot}[u]\to i^{-1}\tau_*\pi_*\pi^{-1}J_C$ which factors through the subcomplex $B(J)_C$ as before. Moreover the morphism δ on $B(J)_C$ corresponds to the morphism

$$-\frac{1}{2\pi\sqrt{-1}}\frac{d}{du} : i^{-1}\omega_X^{\cdot}[u]\to i^{-1}\omega_X^{\cdot}[u]$$

which we denote by δ again. Thus we obtain a morphism of complexes $SZ(i^{-1}\omega_X^{\cdot}[u], \delta)\to SZ(B(J)_C, \delta)$ which turns out to be a quasi-isomorphism as before. Combining with the morphism (4.2.1) we obtain an isomorphism $\alpha : SZ(B(J), \delta)_C\to sD(\omega_X^{\cdot}, W, \theta)$ in the derived category. By the similar way as before we can prove that the data

$$((SZ(B(J), \delta), L), (sD(\omega_X^{\cdot}, W, \theta), L, F), \alpha) \tag{4.2.3}$$

is a CMHC on Y. This CMHC is independent of the choice of the injective resolution $\boldsymbol{Q}_{X^{\log}}\to J$ and of the quasi-isomorphism $\omega_X^{\cdot\,\log}\to J_C$ as before.

(4.3) At first we will prove that these two CMHC's are isomorphic.

By pulling back the injective resolution $\boldsymbol{Q}_{X^{\log}}\to J$ by $\pi : \overline{X}_\infty\to X^{\log}$ we obtain an injective resolution $\boldsymbol{Q}_{\overline{X}_\infty}\to\pi^{-1}J$. Restricting to the open subset X_∞ we have an injective resolution $\boldsymbol{Q}_{X_\infty}\to\bar{j}_\infty^{-1}\pi^{-1}J$. Then we can find a quasi-isomorphism of complexes

$$C^{\cdot}(X_\infty)\to\bar{j}_\infty^{-1}\pi^{-1}J$$

which is compatible with the morphisms $\boldsymbol{Q}_{X_\infty}\to C^{\cdot}(X_\infty)$ and $\boldsymbol{Q}_{X_\infty}\to$ $\to\bar{j}_\infty^{-1}\pi^{-1}J$. Then we have a quasi-isomorphism of complexes

$$\begin{aligned} K^{\cdot}(X_\infty) &\to i^{-1}\bar{j}_*\pi_*\bar{j}_\infty^{-1}\pi^{-1}J \\ &= i^{-1}\tau_*\bar{j}_*^{\log}\pi_*\bar{j}_\infty^{-1}\pi^{-1}J \\ &= i^{-1}\tau_*\pi_*(\bar{j}_\infty)_*\bar{j}_\infty^{-1}\pi^{-1}J \end{aligned}$$

which is compatible with the monodromy automorphism. In the local situation such as X is a polydisc, the topological space $\overline{X}_\infty$ is identified with the space $([0,1)\times \boldsymbol{S}^1)^k \times \Delta^l$ and the space X_∞ with $((0,1)\times \boldsymbol{S}^1)^k \times \times \Delta^l$ for some k, l. Therefore the canonical morphism of complexes

$$\pi^{-1}J \to (\bar{j}_\infty)_* \bar{j}_\infty^{-1}\pi^{-1}J$$

is a quasi-isomorphism. Then we obtain a quasi-isomorphism of complexes

$$i^{-1}\tau_*\pi_*\pi^{-1}J \to i^{-1}\tau_*\pi_*(\bar{j}_\infty)_*\bar{j}_\infty^{-1}\pi^{-1}J$$

which is also compatible with the monodromy automorphism. Therefore we obtain a morphism in the derived category

$$B \to B(J) \tag{4.3.1}$$

which turns out to be a quasi-isomorphism because both sides are quasi-isomorphic to the complexes $K^\cdot(X_\infty)$ and $i^{-1}\tau_*\pi_*\pi^{-1}J$. Then the quasi-isomorphism (4.3.1) induces an isomorphism of complexes $SZ(B, \delta) \to SZ(B(J), \delta)$ in the derived category. We can easily see that this isomorphism induces an isomorphism of CMHC's (4.2.2) and (4.2.3) in the derived category.

Next we will prove that the CMHC (4.2.3) is isomorphic to our CMHC (3.12.1). The pull back $\boldsymbol{Q}_{Y^{\log}} \to (i^{\log})^{-1}J$ of the injective resolution $\boldsymbol{Q}_{X^{\log}} \to \to J$ is a quasi-isomorphism. Then we have a quasi-isomorphism of complexes $(i^{\log})^{-1}J \to I$ which is compatible with the morphisms $\boldsymbol{Q}_{Y^{\log}} \to I$ and $\boldsymbol{Q}_{Y^{\log}} \to (i^{\log})^{-1}J$, where $\boldsymbol{Q}_{Y^{\log}} \to I$ is the injective resolution fixed in (3.6). Then we have a morphism of complexes

$$\begin{aligned} i^{-1}\tau_*\pi_*\pi^{-1}J &\to \tau_*\pi_* i_\infty^{-1}\pi^{-1}J \\ &= \tau_*\pi_*\pi^{-1}(i^{\log})^{-1}J \\ &\to \tau_*\pi_*\pi^{-1}I, \end{aligned}$$

where the morphism in the top line is the canonical base change morphism. By definition the left hand side is identified with $i^{-1}\mathrm{R}(\tau\pi)_*\boldsymbol{Q}_{\overline{X}_\infty}$ and the right with $\mathrm{R}(\tau\pi)_*\boldsymbol{Q}_{Y_\infty} = \mathrm{R}(\tau\pi)_* i_\infty^{-1}\boldsymbol{Q}_{\overline{X}_\infty}$ and the morphism above is also identified with the canonical base change morphism. Thus we can see that the morphism above is a quasi-isomorphism by using the base change theorem as in (3.17). Because the morphism above is compatible with the monodromy automorphism, we obtain a quasi-isomorphism of complexes $B(J)\to B(I)$. This morphism induces a quasi-isomorphism

$$SZ(B(J),\,\delta)\to SZ(B(I),\,\delta)$$

by functoriality. This gives us an isomorphism between the $\boldsymbol{Q}$-structures of CMHC (4.2.3) and of our CMHC (3.12.1). The $\boldsymbol{C}$-structure of these CMHC's are isomorphic because we have $\omega_X^{p+q+1}/W_q\simeq i_*(\omega_Y^{p+q+1}/W_q)$ for every $p,q\geqslant 0$. These data gives us an isomorphism between the CMHC (4.2.3) and our CMHC (3.12.1).

5. Koszul complex construction.

In this section we will prove that our CMHC (3.12.1) constructed in last the section 3 is isomorphic to the one constructed by J. Steenbrink in [14]. In this section we use the same terminology as in [14, §1] for divided power envelopes and for Koszul complexes.

(5.1) At first we briefly recall the Koszul complex construction by Steenbrink in [14].

Let $Y\to 0$ be a log deformation. We denote the log structure on Y by M_Y. We denote by M_Y^{gp} the abelian sheaf associated to the monoid sheaf M_Y. A morphism $\boldsymbol{e}:\mathcal{O}_Y\to M_Y^{gp}$ is defined by $\boldsymbol{e}(f)=\exp(2\pi\sqrt{-1}f)$ for a local section f of $\mathcal{O}_Y$. We fix the positive integer $N=\dim Y+1$.

In [14] Steenbrink constructed torsion free abelian sheaves L^0 and L^1 such that the following conditions are satisfied:

(5.1.1) the sheaves L^0 and L^1 fit in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}(\varepsilon) & \longrightarrow & L^0 & \xrightarrow{\varepsilon} & L^1 & \longrightarrow & \operatorname{Coker}(\varepsilon) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v_1} & & \downarrow & & \\
0 & \longrightarrow & \boldsymbol{Z}_Y & \longrightarrow & \mathcal{O}_Y & \xrightarrow{\boldsymbol{e}} & M_Y^{gp} & \longrightarrow & \operatorname{Coker}(\boldsymbol{e}) & \longrightarrow & 0\,,
\end{array}$$

where the morphisms $\operatorname{Ker}(\varepsilon)\to\boldsymbol{Z}_Y$ and $\operatorname{Coker}(\varepsilon)\to\operatorname{Coker}(\boldsymbol{e})$ are isomorphisms.

(5.1.2) there exists a global section of L^1 whose image by v_1 is the global section t of M_Y^{gp}.

We denote by t the global section of L^1 mentioned in (5.1.2) by abuse of the language.

Now the Koszul complex $\operatorname{Kos}^n(\varepsilon)$ is considered for every non-negative integer n. An increasing filtration W is defined on $\mathrm{Kos}^n(\varepsilon)$ in [14] which is an analogue of the weight filtration on $\omega^{\cdot}_Y$. Moreover a morphism of complexes

$$\theta : \mathrm{Kos}^n(\varepsilon)^p \to \mathrm{Kos}^{n+1}(\varepsilon)(1)^{p+1}$$

is defined by $\theta(x \otimes y) = (2\pi\sqrt{-1})\, x \otimes t \wedge y$ as in [14] for a local section x and y of $\Gamma_{n-p}(L^0)$ and $\bigwedge^p L^1$ respectively. It is easy to check that the morphism θ defines a morphism of complexes

$$\theta : \mathrm{Kos}^n(\varepsilon) \to \mathrm{Kos}^{n+1}(\varepsilon)(1)[1].$$

A morphism of abelian sheaves

$$\Gamma(v_0) : \Gamma(L^0) \to \mathcal{O}_Y$$

is defined by

$$\Gamma(v_0)(x_1^{[a_1]} x_2^{[a_2]} \dots x_k^{[a_k]}) = \frac{1}{a_1!\, a_2! \dots a_k!} v_0(x_1)^{a_1} v_0(x_2)^{a_2} \dots v_0(x_k)^{a_k}, \tag{5.1.3}$$

where $x_1, x_2, \dots, x_k$ are local sections of L^0 and $a_1, a_2, \dots, a_k$ are non-negative integers. We can easily check that this is a morphism of sheaves of rings. On the other hand the morphism of abelian sheaves

$$d \log \cdot v_1 : L^1 \to \omega^1_Y$$

induces a morphism

$$\wedge^p (d \log \cdot v_1) : \bigwedge^p L^1 \to \omega^p_Y$$

for every p, where the symbol $\cdot$ stands for the composition of the morphisms. Then the morphism of abelian sheaves $\phi : \mathrm{Kos}^n(\varepsilon)^p \to \omega^p_Y$ is defined by

$$\phi(x \otimes y) = (2\pi\sqrt{-1})^{-p}\, \Gamma(v_0)(x) \bigwedge^p (d \log \cdot v_1)(y) \tag{5.1.4}$$

for local sections x of $\Gamma_{n-p}(L^0)$ and y of $\bigwedge^p L^1$. It is easy to see that this morphism gives us a morphism of complexes $\phi : \mathrm{Kos}^n(\varepsilon) \to \omega^{\cdot}_Y$ preserving the filtration W on both sides.

The data consisting of the complexes $K_n = \mathrm{Kos}^{N+n+1}(\varepsilon)(n+1)$, the filtration W and the morphism of complexes $\theta : K_n = \mathrm{Kos}^{N+n+1}(\varepsilon)(n+1) \to \to K_{n+1}[1] = \mathrm{Kos}^{N+n+2}(\varepsilon)(n+2)[1]$ satisfies the conditions (1.4.1)-(1.4.5). Thus the Steenbrink complex $sD(K_n, W, \theta)$ is obtained as in Definition 1.4.

The morphism of complexes $\phi : \mathrm{Kos}^{N+n+1}(\varepsilon) \to \omega^{\cdot}_Y$ induces a morphism of complexes $\phi : K_n = \mathrm{Kos}^{N+n+1}(\varepsilon)(n+1) \to \omega^{\cdot}_Y$ in the trivial way. Then the morphisms $\theta : \omega^{\cdot}_Y \to \omega^{\cdot}_Y[1]$ in (3.8.1) and $\theta : K_n \to K_{n+1}[1]$ are compatible via the morphism ϕ. Thus the morphism of complexes

$$sD(\phi) : sD(K_n, W, \theta)_{\boldsymbol{C}} \to sD(\omega^{\cdot}_Y, W, \theta) = A_{\boldsymbol{C}}$$

is obtained by the functoriality of the Steenbrink complex. Then the data

(5.1.5) $$((sD(K_n, W, \theta), L), (A_{\boldsymbol{C}}, L, F), sD(\phi))$$

is proved to be a CMHC on Y in [14].

(5.2) We define an abelian sheaf $\tilde{L}^0$ by $\tilde{L}^0 = L^0 \oplus \boldsymbol{Z}_Y$. The projection $\mathrm{pr} : \tilde{L}^0 = L^0 \oplus \boldsymbol{Z}_Y \to L^0$ induces a morphism $\mathrm{pr} : \Gamma_n(\tilde{L}^0) \to \Gamma_n(L^0)$ for every non-negative integer n.

Now l denotes the global section $0 \oplus 1$ of the abelian sheaf $L^0 \oplus \boldsymbol{Z}_Y = \tilde{L}^0$. Then we have

(5.2.1) $$\Gamma_n(\tilde{L}^0) = \bigoplus_{k=0}^{n} \Gamma_{n-k}(L^0)\, l^{[k]}$$

for every n. The morphism $\mathrm{pr} : \Gamma_n(\tilde{L}^0) \to \Gamma_n(L^0)$ is given by

$$\mathrm{pr}\left(\sum_{k=0}^{n} x_k\, l^{[k]}\right) = x_0,$$

where x_k is a local section of $\Gamma_{n-k}(L^0)$ and then $\sum_{k=0}^{n} x_k\, l^{[k]}$ is a local section of $\Gamma_n(\tilde{L}^0)$. We define a morphism of abelian sheaves $\delta' : \Gamma_n(\tilde{L}^0) \to \Gamma_{n-1}(\tilde{L}^0)$ for every positive integer n by

$$\delta'\left(\sum_{k=0}^{n} x_k\, l^{[k]}\right) = -\sum_{k=0}^{n-1} x_{k+1}\, l^{[k]}$$

by using the direct sum decomposition (5.2.1).

We denote by e the image of the global section 1 of $\boldsymbol{Z}_Y$ by the canonical injection $\boldsymbol{Z}_Y = \mathrm{Ker}(\varepsilon) \to L^0$. Then the multiplication with e in $\Gamma(L^0)$ gives us a morphism of abelian sheaves

$$m : \Gamma_n(L^0) \to \Gamma_{n+1}(L^0)$$

for every n. Moreover e can be regarded as a global section of $\tilde{L}^0$ by the inclusion $L^0 \to \tilde{L}^0$. Then the multiplication with e in $\Gamma(\tilde{L}^0)$ defines a morphism of abelian sheaves

$$\widetilde{m} : \Gamma_n(\widetilde{L}^0) \to \Gamma_{n+1}(\widetilde{L}^0)$$

for every n.

On the other hand we define an automorphism $T : \Gamma_n(\widetilde{L}^0) \to \Gamma_n(\widetilde{L}^0)$ by

$$T\left(\sum_{k=0}^{n} x_k l^{[k]}\right) = \sum_{k=0}^{n} x_k (l+e)^{[k]}$$

by using the direct sum decomposition (5.2.1). Then we can see that the logarithm of the automorphism T coincides with the morphism $-\widetilde{m}\delta'$.

(5.3) The global section t of L^1 gives us a morphism of abelian sheaves $\boldsymbol{Z}_Y \to L^1$ which is denoted by ε' here. We define a morphism of sheaves $\tilde{\varepsilon} : \widetilde{L}^0 = L^0 \oplus \boldsymbol{Z}_Y \to L^1$ by $\tilde{\varepsilon} = \varepsilon + \varepsilon'$. Then we have $\operatorname{Coker}(\tilde{\varepsilon}) \simeq \simeq M_Y^{gp} / \mathcal{O}_Y^* \cdot t^{\boldsymbol{Z}}$ and $\operatorname{Ker}(\tilde{\varepsilon}) = \operatorname{Ker}(\varepsilon) \simeq \boldsymbol{Z}_Y$.

Now we have a complex of abelian sheaves $\operatorname{Kos}^n(\tilde{\varepsilon})$ for every non-negative integer n. The morphisms above induce morphisms of sheaves

(5.3.1) $$\mathrm{pr} : \operatorname{Kos}^n(\tilde{\varepsilon})^p \to \operatorname{Kos}^n(\varepsilon)^p$$

(5.3.2) $$\delta' : \operatorname{Kos}^n(\tilde{\varepsilon})^p \to \operatorname{Kos}^{n-1}(\tilde{\varepsilon})^p$$

(5.3.3) $$m : \operatorname{Kos}^n(\varepsilon)^p \to \operatorname{Kos}^{n+1}(\varepsilon)^p$$

(5.3.4) $$\widetilde{m} : \operatorname{Kos}^n(\tilde{\varepsilon})^p \to \operatorname{Kos}^{n+1}(\tilde{\varepsilon})^p$$

(5.3.5) $$T : \operatorname{Kos}^n(\tilde{\varepsilon})^p \to \operatorname{Kos}^n(\tilde{\varepsilon})^p$$

by tensoring with the identity. We can easily see that the morphisms δ', m, $\widetilde{m}$ and T define morphisms of complexes by the fact that the global section e is contained in $\operatorname{Ker}(\varepsilon) \simeq \operatorname{Ker}(\tilde{\varepsilon})$. As for the morphism pr, we have the equality

(5.3.6) $$\mathrm{pr} \cdot d = d \cdot \mathrm{pr} - (2\pi\sqrt{-1})^{-1} \theta \cdot \mathrm{pr} \cdot \delta' : \operatorname{Kos}^n(\tilde{\varepsilon})^p \to \operatorname{Kos}^n(\varepsilon)^{p+1}.$$

On the other hand we have the equality $-\log T = \widetilde{m}\delta'$. We define a morphism of complexes $\delta : \operatorname{Kos}^n(\tilde{\varepsilon}) \to \operatorname{Kos}^n(\tilde{\varepsilon})(-1)$ by $\delta = -(2\pi\sqrt{-1})^{-1} \log T = = (2\pi\sqrt{-1})^{-1} \widetilde{m}\delta'$.

Now we have a morphism of complexes $\delta : \operatorname{Kos}^N(\tilde{\varepsilon}) \to \operatorname{Kos}^N(\tilde{\varepsilon})(-1)$.

Thus we obtain the Steenbrink-Zucker complex

$$SZ(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta) = sD(\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1), \tau, \theta).$$

We define a morphism

$$\psi : \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1)^p =$$
$$= \mathrm{Kos}^N(\tilde{\varepsilon})(n+1)^p \oplus \mathrm{Kos}^N(\tilde{\varepsilon})(n)^{p-1} \to K_n^p = \mathrm{Kos}^{N+n+1}(\varepsilon)(n+1)^p$$

by

$$\psi(x, y) = m^{n+1} \cdot \mathrm{pr}(x) + m^n \cdot \theta \cdot \mathrm{pr}(y),$$

where x and y are local sections of $\mathrm{Kos}^N(\tilde{\varepsilon})^p$ and $\mathrm{Kos}^N(\tilde{\varepsilon})^{p-1}$. Then the morphisms ψ for all p define a morphism of complexes

$$\psi : \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1) \to K_n$$

because of the equalities (5.3.6) and $\theta^2 = 0$.

LEMMA 5.4. *The morphism*

$$\psi_{\boldsymbol{Q}} : \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1)_{\boldsymbol{Q}} \to K_{n,\boldsymbol{Q}}$$

is a quasi-isomorphism, where the subscript stands for the tensor product with $\boldsymbol{Q}$.

PROOF. The inclusion $i : L^0 \to \widetilde{L}^0 = L^0 \oplus \boldsymbol{Z}_Y$ induces a morphism of complexes $i : \mathrm{Kos}^N(\varepsilon) \to \mathrm{Kos}^N(\tilde{\varepsilon})$ because of the equality $\varepsilon = \tilde{\varepsilon} i$. We can easily see that the morphism $i : \mathrm{Kos}^N(\varepsilon)^p \to \mathrm{Kos}^N(\tilde{\varepsilon})^p$ factors through the kernel of the morphism δ. Therefore the composite of the morphism i above and the canonical morphism $\mathrm{Kos}^N(\tilde{\varepsilon})^p \to \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)^p = \mathrm{Kos}^N(\tilde{\varepsilon})^p \oplus \mathrm{Kos}^N(\tilde{\varepsilon})(-1)^{p-1}$ for every p defines a morphism of complexes $\iota : \mathrm{Kos}^N(\varepsilon) \to \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)$. Then the composite $\psi\iota : \mathrm{Kos}^N(\varepsilon)(n+1) \to K_n = \mathrm{Kos}^{N+n+1}(\varepsilon)(n+1)$ coincides with the morphism m^{n+1}. Thus it suffices to prove that the morphism $\iota_{\boldsymbol{Q}}$ is a quasi-isomorphism because the morphism $m_{\boldsymbol{Q}} : \mathrm{Kos}^k(\varepsilon)_{\boldsymbol{Q}} \to \mathrm{Kos}^{k+1}(\varepsilon)_{\boldsymbol{Q}}$ is a quasi-isomorphism for every $k \geqslant N$ by Lemma (1.4) in [14].

We have an exact sequence of complexes

$$0 \to \mathrm{Kos}^N(\tilde{\varepsilon})(-1)[-1] \to \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta) \to \mathrm{Kos}^N(\tilde{\varepsilon}) \to 0$$

by definition. Then we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{p-1}(\mathrm{Kos}^N(\tilde{\varepsilon})(-1)) \longrightarrow \mathrm{H}^p(\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)) \longrightarrow \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon}))$$
$$\longrightarrow \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})(-1)) \longrightarrow \cdots$$

for every p. We can easily see that the morphism $\mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})) \to \to \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})(-1))$ is induced from the morphism δ. On the other hand we have a long exact sequence

$$\cdots \longrightarrow \mathrm{H}^p(\mathrm{Kos}^N(\varepsilon)) \longrightarrow \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})) \longrightarrow \mathrm{H}^p(\mathrm{Kos}^{N-1}(\tilde{\varepsilon})) \longrightarrow \cdots$$

from an exact sequence

$$0 \longrightarrow \mathrm{Kos}^N(\varepsilon) \longrightarrow \mathrm{Kos}^N(\tilde{\varepsilon}) \xrightarrow{\delta'} \mathrm{Kos}^{N-1}(\tilde{\varepsilon}) \longrightarrow 0 .$$

Then we can easily see that these two long exact sequences fit in the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{H}^p(\mathrm{Kos}^N(\varepsilon)) & \longrightarrow & \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})) & \longrightarrow & \mathrm{H}^p(\mathrm{Kos}^{N-1}(\tilde{\varepsilon})) & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (2\pi\sqrt{-1})^{-1}\widetilde{m}} & \\
\cdots \longrightarrow & \mathrm{H}^p(\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)) & \longrightarrow & \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})) & \longrightarrow & \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})(-1)) & \longrightarrow \cdots
\end{array}$$

because the morphism $\mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})) \to \mathrm{H}^p(\mathrm{Kos}^N(\tilde{\varepsilon})(-1))$ in the bottom line is induced from the morphism δ. Thus the morphism $\iota_{\boldsymbol{Q}}$ is a quasi-isomorphism because the morphism $\widetilde{m}_{\boldsymbol{Q}}\colon \mathrm{Kos}^{N-1}(\tilde{\varepsilon})_{\boldsymbol{Q}} \to \mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}}$ is a quasi-isomorphism by Lemma (1.4) in [14].

In [14] it is proved that the identity on K_n induces a filtered quasi-isomorphism

$$(K_n, \tau) \to (K_n, W),$$

where τ denotes the canonical filtration. Therefore the morphism of complexes ψ induces a morphism of filtered complexes

$$\psi : (\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1), \tau) \to (K_n, W)$$

which is a quasi-isomorphism after tensoring with $\boldsymbol{Q}$. Then the morphism of filtered complexes

$$\psi : (\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1), \tau) \to (K_n, W)$$

is compatible with the morphisms $\theta : \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1) \to \to \varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+2)[1]$ and $\theta : K_n \to K_{n+1}[1]$. Therefore we obtain a morphism of the Steenbrink complexes

$$sD(\psi): SZ(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta) = sD(\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta)(n+1), \tau, \theta) \to sD(K_n, W, \theta)$$

preserving the filtration L on both sides.

PROPOSITION 5.5. *The morphism $sD(\psi)_{\boldsymbol{Q}}$ is a filtered quasi-isomorphism with respect to the filtration L. Therefore the data*

$$(5.5.1) \qquad (SZ(\mathrm{Kos}^N(\tilde{\varepsilon}), \delta), (A_{\boldsymbol{C}}, L, F), sD(\phi)\, sD(\psi))$$

give us a CMHC on Y which is isomorphic to the Steenbrink's CMHC (5.1.5) *after tensoring with $\boldsymbol{Q}$.*

PROOF. Easy by Lemma 5.4. ■

(5.6) Now we use the same notation such as $Y^{\log}$, Y_∞, $0^{\log}$ as in the previous sections. We fix an injective resolution $\boldsymbol{Q}_{Y^{\log}} \to I$ and a reference quasi-isomorphism $\omega_Y^{\cdot, \log} \to I_{\boldsymbol{C}}$.

From the diagram (2.7.1) we obtain a commutative diagram

$$(5.6.1) \qquad \begin{array}{ccccccc} \tau^{-1}L^0 & \longrightarrow & \tau^{-1}\mathcal{O}_Y & \xrightarrow{(2\pi\sqrt{-1})\mathrm{id}} & \tau^{-1}\mathcal{O}_Y & \xrightarrow{h} & \mathcal{L}_Y \\ \downarrow{\scriptstyle \tau^{-1}\varepsilon} & & \downarrow{\scriptstyle \tau^{-1}\boldsymbol{e}} & & \downarrow{\scriptstyle \tau^{-1}\exp} & & \downarrow{\scriptstyle \exp} \\ \tau^{-1}L^1 & \xrightarrow{\tau^{-1}v_1} & \tau^{-1}M_Y^{gp} & = & \tau^{-1}M_Y^{gp} & = & \tau^{-1}M_Y^{gp}. \end{array}$$

For the later use we denote the morphism $\exp: \mathcal{L}_Y \to \tau^{-1}M_Y^{gp}$ by $\exp_{\mathcal{L}_Y}$. Now we consider the Koszul complex $\mathrm{Kos}^N(\exp_{\mathcal{L}_Y})$. Because of the exactness of the bottom line in the diagram (2.7.1) the morphism $2\pi\sqrt{-1}\boldsymbol{Z}_{Y^{\log}} \to \mathcal{L}_Y$ induces a quasi-isomorphism

$$(5.6.2) \qquad (2\pi\sqrt{-1})^N \boldsymbol{Z}_{Y^{\log}} = \Gamma_N((2\pi\sqrt{-1})\,\boldsymbol{Z}_{Y^{\log}}) \to \mathrm{Kos}^N(\exp_{\mathcal{L}_Y})$$

by Lemma (1.4) in [14]. Therefore we have a quasi-isomorphism

$$\nu: \mathrm{Kos}^N(\exp_{\mathcal{L}_Y})_{\boldsymbol{Q}} \to I$$

which fits in the commutative diagram

$$(5.6.3) \qquad \begin{array}{ccc} (2\pi\sqrt{-1})^N \boldsymbol{Q}_{Y^{\log}} & \longrightarrow & \mathrm{Kos}^N(\exp_{\mathcal{L}_Y})_{\boldsymbol{Q}} \\ \downarrow & & \downarrow{\scriptstyle \nu} \\ \boldsymbol{Q}_{Y^{\log}} & \longrightarrow & I\,, \end{array}$$

where the left vertical arrow stands for the morphism given by sending $(2\pi\sqrt{-1})^N a$ to $a/N!$.

On the topological space Y_∞ we have a commutative diagram

$$\begin{tikzcd}
\pi^{-1}\tau^{-1}L^0 \arrow[r, "\pi^{-1}\tau^{-1}\varepsilon"] \arrow[d] & \pi^{-1}\tau^{-1}L^1 \arrow[d, "\pi^{-1}\tau^{-1}v_1"] \\
\pi^{-1}\mathcal{L}_Y \arrow[r, "\pi^{-1}\exp_{\mathcal{L}_Y}"] & \pi^{-1}\tau^{-1}M_Y^{gp}
\end{tikzcd}$$

by pulling back the diagram (5.6.1). On the other hand there exists a global section $\log t$ of $\pi^{-1}\mathcal{L}_Y$ such that its image by $\pi^{-1}\exp_{\mathcal{L}_Y}$ coincides with the global section t of $\pi^{-1}\tau^{-1}M_Y^{gp}$. This global section defines a morphism of abelian sheaves

$$\boldsymbol{Z}_{Y_\infty} \to \pi^{-1}\mathcal{L}_Y$$

and this morphism fits in the commutative diagram

$$\begin{tikzcd}
\boldsymbol{Z}_{Y_\infty} \arrow[r, "\pi^{-1}\tau^{-1}\varepsilon'"] \arrow[d] & \pi^{-1}\tau^{-1}L^1 \arrow[d, "\pi^{-1}\tau^{-1}v_1"] \\
\pi^{-1}\mathcal{L}_Y \arrow[r, "\pi^{-1}\exp_{\mathcal{L}_Y}"] & \pi^{-1}\tau^{-1}M_Y^{gp}
\end{tikzcd}$$

by definition. Therefore we obtain a commutative diagram

$$\begin{tikzcd}
\pi^{-1}\tau^{-1}\widetilde{L}^0 = \pi^{-1}\tau^{-1}L^0 \oplus \boldsymbol{Z}_{Y_\infty} \arrow[r, "\pi^{-1}\tau^{-1}\tilde{\varepsilon}"] \arrow[d] & \pi^{-1}\tau^{-1}L^1 \arrow[d, "\pi^{-1}\tau^{-1}v_1"] \\
\pi^{-1}\mathcal{L}_Y \arrow[r, "\pi^{-1}\exp_{\mathcal{L}_Y}"] & \pi^{-1}\tau^{-1}M_Y^{gp}
\end{tikzcd}$$

which induces a morphism of complexes

$$\begin{aligned}
\pi^{-1}\tau^{-1}\mathrm{Kos}^N(\tilde{\varepsilon}) = \mathrm{Kos}^N(\pi^{-1}\tau^{-1}\tilde{\varepsilon}) \to \\
\to \mathrm{Kos}^N(\pi^{-1}\exp_{\mathcal{L}_Y}) = \pi^{-1}\mathrm{Kos}^N(\exp_{\mathcal{L}_Y}).
\end{aligned} \tag{5.6.4}$$

By composing this morphism and the morphism $\pi^{-1}\nu$ we obtain a morphism of complexes

$$\kappa : \pi^{-1}\tau^{-1}\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}} \to \pi^{-1}I \tag{5.6.5}$$

which fits in the commutative diagram

$$\begin{array}{ccc} \boldsymbol{Q}_{Y_\infty} & \longrightarrow & \pi^{-1}\tau^{-1}\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}} \\ \downarrow & & \downarrow{\scriptstyle \kappa} \\ \boldsymbol{Q}_{Y_\infty} & \longrightarrow & \pi^{-1}I\,, \end{array}$$

where the left vertical arrow stands for the morphism given by sending a to $a/N!$. Hence we obtain a morphism

$$\tau_*\pi_*\kappa : \mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}} \to \tau_*\pi_*\pi^{-1}I$$

by taking adjoint.

The automorphism T (5.3.5) on $\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}}$ corresponds to the monodromy automorphism T on $\tau_*\pi_*\pi^{-1}I$ via the morphism $\tau_*\pi_*\kappa$. Then the morphism $\tau_*\pi_*\kappa$ factors through the subcomplex $B(I)$ of $\tau_*\pi_*\pi^{-1}I$ because the morphism $T-\mathrm{id}$ on $\mathrm{Kos}^N(\tilde{\varepsilon})$ is nilpotent. Hence we obtain a morphism of complexes $\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}} \to B(I)$ which is denoted by $\tau_*\pi_*\kappa$ by abuse of the language. The morphisms $\delta : B(I) \to B(I)(-1)$ and $\delta : \mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}} \to \mathrm{Kos}^N(\tilde{\varepsilon})(-1)_{\boldsymbol{Q}}$ are compatible via the morphism $\tau_*\pi_*\kappa$. Therefore we obtain a morphism of the Steenbrink-Zucker complexes

$$SZ_1(\tau_*\pi_*\kappa) : SZ(\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{Q}},\, \delta) \to SZ(B(I),\, \delta) \tag{5.6.6}$$

by the functoriality.

Proposition 5.7. *The morphism $SZ_1(\tau_*\pi_*\kappa)$ above induces an isomorphism from the CMHC* (5.5.1) *to the CMHC* (3.12.1).

Proof. It is sufficient to prove that the diagram in the filtered derived category

$$\begin{array}{ccc} SZ(\mathrm{Kos}^N(\tilde{\varepsilon}),\, \delta)_{\boldsymbol{C}} & \xrightarrow{sD(\psi)} & sD(K_n,\, W,\, \theta) \\ {\scriptstyle SZ_1(\tau_*\pi_*\kappa)}\downarrow & & \downarrow{\scriptstyle sD(\phi)} \\ SZ(B(I),\, \delta)_{\boldsymbol{C}} & \xrightarrow{\alpha} & A_{\boldsymbol{C}} \end{array}$$

is commutative. Because the morphism α in the derived category is represented by the morphisms

$$SZ(B(I),\, \delta)_{\boldsymbol{C}} \xleftarrow{SZ_1(\varphi)} SZ(\omega^{\cdot}_Y[u],\, \delta) \xrightarrow{sD(\psi)} A_{\boldsymbol{C}},$$

it is sufficient to find a morphism of complexes

$$\widetilde{\phi}:\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{C}}\to\omega^{\cdot}_Y[u]$$

such that we have the equalities $SZ_1(\varphi)SZ_1(\widetilde{\phi})=SZ_1(\tau_*\pi_*\kappa)$ and $sD(\psi)\,SZ_1(\widetilde{\phi})=sD(\phi)\,sD(\psi)$ in the filtered derived category.

Now we define a morphism $\widetilde{\phi}:\mathrm{Kos}^N(\tilde{\varepsilon})^p\to\omega^p_Y[u]$ for every p by

$$\widetilde{\phi}\left(\sum_{k=0}^{n-p}x_k\,l^{[k]}\otimes y\right)=\sum_{k=0}^{n-p}\frac{1}{(2\pi\sqrt{-1})^{k+p}\,k!}\,\Gamma(v_0)(x_k)\overset{p}{\bigwedge}(d\log\cdot v_1)(y)\,u^k,$$

where x_k is a local section of $\Gamma_{n-p-k}(L^0)$ for every k. We can easily check that these morphisms form a morphism of complexes which we denote by $\widetilde{\phi}:\mathrm{Kos}^N(\tilde{\varepsilon})\to\omega^{\cdot}_Y[u]$ again. By tensoring $\boldsymbol{C}$ we obtain a morphism $\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{C}}\to\omega^{\cdot}_Y[u]$ which we denote by $\widetilde{\phi}$ too. Then we can easily see that the diagram

$$\begin{array}{ccc}
\varrho(\mathrm{Kos}^N(\tilde{\varepsilon}),\,\delta)(n+1) & \xrightarrow{\psi} & K_n \\
{\scriptstyle \varrho(\widetilde{\phi})}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\
\varrho(\omega^{\cdot}_Y[u],\,\delta) & \xrightarrow{\psi} & \omega^{\cdot}_Y
\end{array}$$

is commutative. Then we have the equality $sD(\psi)\,SZ_1(\widetilde{\phi})=$ $=sD(\phi)\,sD(\psi)$.

On the topological space $Y^{\log}$ we have a morphism of abelian sheaves

$$v_0(\mathcal{L}_Y):\mathcal{L}_Y\to\mathcal{O}^{\log}_Y$$

by the definition of $\mathcal{O}^{\log}_Y$. Then we obtain a morphism of sheaves of rings $\Gamma(v_0(\mathcal{L}_Y)):\Gamma\mathcal{L}_Y\to\mathcal{O}^{\log}_Y$ by the same formula as (5.1.3). On the other hand the morphism $d\log:M^{gp}_Y\to\omega^1_Y$ is extended to the morphism

$$\tau^{-1}M^{gp}_Y\to\omega^{1,\log}_Y$$

on $Y^{\log}$ which we denote by $d\log$ again. Thus we obtain a morphism of abelian sheaves

$$\phi^{\log}:\mathrm{Kos}^N(\exp_{\mathcal{L}_Y})^p=\Gamma_{N-p}\mathcal{L}_Y\otimes\overset{p}{\bigwedge}\tau^{-1}M^{gp}_Y\to\omega^{p,\log}_Y$$

by

$$\phi^{\log}(x\otimes y)=(2\pi\sqrt{-1})^{-N}\,\Gamma(v_0(\mathcal{L}_Y))(x)\overset{p}{\bigwedge}(d\log)(y)$$

for every p as in (5.1.4). These morphisms for all p define a morphism of complexes

$$\phi^{\log}\colon \mathrm{Kos}^N(\exp_{\mathcal{L}_Y}) \to \omega_Y^{\cdot,\log}$$

because we have the equality $d\cdot v_0(\mathcal{L}_Y) = d\log\cdot\exp_{\mathcal{L}_Y}\colon \mathcal{L}_Y \to \omega_Y^{1,\log}$, where d on the left hand side denotes the differential on the complex $\omega_Y^{\cdot,\log}$. Then we have a commutative diagram

$$\begin{array}{ccc}
(2\pi\sqrt{-1})^N \boldsymbol{Z}_{Y^{\log}} & \longrightarrow & \mathrm{Kos}^N(\exp_{\mathcal{L}_Y}) \\
\big\downarrow & & \big\downarrow{\scriptstyle \phi^{\log}} \\
\boldsymbol{C}_{Y^{\log}} & \longrightarrow & \omega_Y^{\cdot,\log},
\end{array}$$

where the top horizontal arrow denotes the morphism (5.6.2) and the bottom the canonical one and the left vertical arrow is given by sending $(2\pi\sqrt{-1})^N a$ to $a/N!$. Therefore the diagram

$$\begin{array}{ccc}
\mathrm{Kos}^N(\exp_{\mathcal{L}_Y})_{\boldsymbol{C}} & \xrightarrow{\phi^{\log}} & \omega_Y^{\cdot,\log} \\
{\scriptstyle \nu}\big\downarrow & & \big\downarrow \\
I_{\boldsymbol{C}} & = & I_{\boldsymbol{C}}
\end{array}$$

commutes in the derived category because of the commutative diagram (5.6.3).

On the other hand we can easily see that the diagram

$$\begin{array}{ccc}
\pi^{-1}\tau^{-1}\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{C}} & \xrightarrow{\pi^{-1}\tau^{-1}\tilde{\phi}} & \pi^{-1}\tau^{-1}\omega_Y^{\cdot}[u] \\
\big\downarrow & & \big\downarrow \\
\pi^{-1}\mathrm{Kos}^N(\exp_{\mathcal{L}_Y})_{\boldsymbol{C}} & \xrightarrow{\pi^{-1}\phi^{\log}} & \pi^{-1}\omega_Y^{\cdot,\log}
\end{array}$$

is commutative, where the left vertical arrow is the morphism (5.6.4) and the right is the morphism defined by sending the indeterminate u to the global section $\log t$ of $\mathcal{O}_Y^{\log}$. Thus we have the diagram

$$\begin{array}{ccc}
\pi^{-1}\tau^{-1}\mathrm{Kos}^N(\tilde{\varepsilon})_{\boldsymbol{C}} & \xrightarrow{\pi^{-1}\tau^{-1}\tilde{\phi}} & \pi^{-1}\tau^{-1}\omega_Y^{\cdot}[u] \\
{\scriptstyle \kappa}\big\downarrow & & \big\downarrow \\
\pi^{-1}I_{\boldsymbol{C}} & = & \pi^{-1}I_{\boldsymbol{C}}
\end{array}$$

which commutes in the derived category. Because the canonical isomorphism $\omega_Y^{\cdot}[u] \to \tau_*\pi_*\pi^{-1}I_{\boldsymbol{C}}$ is the adjoint of the right vertical arrow, we obtain the commutative diagram

$$\begin{array}{ccc} SZ(\mathrm{Kos}^N(\tilde{\varepsilon}),\,\delta)_C & \xrightarrow{SZ_1(\tilde{\varphi})} & SZ(\omega^{\cdot}_Y[u],\,\delta) \\ {\scriptstyle SZ_1(\tau_*\pi_*\kappa)}\downarrow & & \downarrow{\scriptstyle SZ_1(\varphi)} \\ B(I)_C & = & B(I)_C \end{array}$$

and complete the proof. ■

Now we obtain the following:

THEOREM 5.8. *Steenbrink's CMHC* (5.1.5) *is isomorphic to ours* (3.12.1).

6. An analogue of Rapoport-Zink construction.

In this section we present another construction of the $\boldsymbol{Q}$-structure of our CMHC (3.12.1) which is an analogue of Rapoport-Zink construction in [12].

(6.1) Let $Y \to 0$ be a log deformation. We use the same notation such as $Y^{\log}$, Y_∞, $0^{\log}$ as before. We fix an injective resolution $\boldsymbol{Q}_{Y^{\log}} \to I$ and a reference morphism $\omega^{\cdot\,\log}_Y \to I_C$ as in (3.6). Then we obtain the complex $\tau_*\pi_*\pi^{-1}I$ on Y with the monodromy automorphism T. We denote the morphism

$$\frac{1}{2\pi\sqrt{-1}}(T-\mathrm{id}): \tau_*\pi_*\pi^{-1}I \to \tau_*\pi_*\pi^{-1}I(-1)$$

by δ'. Now we obtain the Steenbrink-Zucker complex

$$SZ(\tau_*\pi_*\pi^{-1}I,\,\delta')$$

with the increasing filtration L. This is an analogue of the complex constructed by Rapoport-Zink in [12].

(6.2) We will prove that the complex $SZ(\tau_*\pi_*\pi^{-1}I,\,\delta')$ is filtered quasi-isomorphic to the $\boldsymbol{Q}$-structure of our CMHC (3.12.1) $SZ(B(I),\,\delta)$ with respect to the filtration L.

We denote the restriction of δ' on the subcomplex $B(I)$ by the same letter δ'. Thus we obtain a morphism of complexes

$$\delta' : B(I) \to B(I)(-1)$$

from which we obtain the Steenbrink-Zucker complex $SZ(B(I), \delta')$ with the filtration L.

Because the inclusion $\iota : B(I) \to \tau_* \pi_* \pi^{-1} I$ is compatible with the morphism δ', it defines a morphism of complexes

$$SZ_1(\iota) : SZ(B(I), \delta') \to SZ(\tau_* \pi_* \pi^{-1} I, \delta')$$

preserving the filtration L on both sides by the functoriality in Remark 1.15.

LEMMA 6.3. *The morphism $SZ_1(\iota)$ above is a filtered quasi-isomorphism with respect to the filtration L on both sides.*

PROOF. We have

$$B(I)_0 = \mathrm{Ker}\,(\delta' : B(I) \to B(I)(-1)) =$$

$$= \mathrm{Ker}\,(\delta' : \tau_* \pi_* \pi^{-1} I \to \tau_* \pi_* \pi^{-1} I(-1)).$$

Then we obtain the conclusion by Corollary 1.16 because the morphisms $\delta' : \tau_* \pi_* \pi^{-1} I \to \tau_* \pi_* \pi^{-1} I(-1)$ and $\delta' : B(I) \to B(I)(-1)$ are surjective by Lemma 3.2. ■

On the other hand there exists a morphism of complexes

$$U : B(I) \to B(I)$$

such that the conditions (3.4.3)-(3.4.6) are satisfied. Therefore the morphism $-U$ satisfies the conditions (1.17.1) and (1.17.2). Thus we obtain a morphism of complexes

$$SZ_2(-U) : SZ(B(I), \delta) \to SZ(B(I), \delta')$$

preserving the filtration L by the functoriality in Remark 1.17.

LEMMA 6.4. *The morphism $SZ_2(-U)$ is a filtered quasi-isomorphism with respect to the filtration L.*

PROOF. Notice that the morphism U is an isomorphism of complexes. By Corollary 1.18 it is sufficient to prove that the morphisms δ and δ' induce zero maps from $\mathrm{H}^p(B(I))$ to $\mathrm{H}^p(B(I)(-1))$ for every p. This is proved in (3.17). Thus we complete the proof. ■

Composing these filtered quasi-isomorphisms we obtain the following:

PROPOSITION 6.5. *There exists a filtered quasi-isomorphism*

$$SZ(B(I), \delta) \to SZ(\tau_* \pi_* \pi^{-1} I, \delta')$$

with respect to the filtration L on both sides.

7. Degeneracy of l-adic weight spectral sequences.

In this section we compare our construction and the construction in [11].

THEOREM 7.1. *Let l be a prime number and T a topological generator of $\boldsymbol{Z}_l(1)$. Let $f: Y \to$* (Spec $\boldsymbol{C}$, $\boldsymbol{N}$-constant log structure) *be a proper semistable morphism of fs log schemes* (*loc. cit. Definition* (1.2)). *Assume that each irreducible component of $\overset{\circ}{Y}$ is smooth over* Spec $\boldsymbol{C}$. *Then the Steenbrink-Rapoport-Zink l-adic spectral sequence*

$$\mathrm{E}_1^{-r, q+r} = \bigoplus_{\substack{k \geqslant 0 \\ k \geqslant -r}} \bigoplus_{\substack{E \subset \{1, \dots, m\} \\ \operatorname{Card} E = r+2k+1}} \mathrm{H}^{q-r-2k}(Y_E, \boldsymbol{Q}_l(-r-k)) \Rightarrow \mathrm{H}^q(Y_{\mathrm{t}l}, \boldsymbol{Q}_l)$$

(*see loc. cit. Proposition* (1.8.3) *for the notation*) *is isomorphic to the $\boldsymbol{Q}_l$-tensored spectral sequence associated to the $\boldsymbol{Q}$-structure of CMHC in Theorem* 3.12 *for Y_{an}* (*cf. Remark* 3.13).

For the proof, we need a lemma:

LEMMA 7.2. *Let $Y \to 0$ be a log deformation, and $0' \to 0$ the endomorphism of the standard log point induced by multiplication by n on the monoid $\boldsymbol{N}$, $n \geqslant 1$.* Let $Y' := Y \times_0 0'$ the fiber product in the category of fs log analytic spaces. Then $Y'^{\log}$ *is the fiber product of $Y^{\log}$ and $0'^{\log}$ over $0^{\log}$ in the category of topological spaces.*

PROOF. Since the problem is local, it is enough to prove that $(\operatorname{Spec} \boldsymbol{C}[P])^{\log}_{\mathrm{an}} \to \varprojlim (\operatorname{Spec}(\boldsymbol{C}[D]))^{\log}_{\mathrm{an}}$ is a homeomorphism, where P is the push-out of the diagram $D: \boldsymbol{N} \overset{n}{\leftarrow} \boldsymbol{N} \overset{\text{diag.}}{\to} \boldsymbol{N}^r$ $(r \geqslant 1)$ in the category of fs monoids. In general $(\operatorname{Spec} \boldsymbol{C}[Q])^{\log}_{\mathrm{an}} = \operatorname{Hom}(Q, \boldsymbol{R}^{\mathrm{mult}}_{\geqslant 0}) \times \operatorname{Hom}(Q, \boldsymbol{S}^1)$ as topological spaces for any fs monoid Q. Further $\operatorname{Hom}(-, \boldsymbol{S}^1)$ sends push-outs to fiber products. Finally $\operatorname{Hom}(P, \boldsymbol{R}^{\mathrm{mult}}_{\geqslant 0}) = \varprojlim \operatorname{Hom}(D, \boldsymbol{R}^{\mathrm{mult}}_{\geqslant 0})$ since P is actually the push-out of D in the category of monoids. ■

PROOF OF 7.1. First we modify the construction in the proof of [11] Proposition (1.4) slightly in the following three points. Let the notation be as in there.

1. Replacing the quasi-isomorphism $K \leftarrow s(B^{\cdot\cdot}) \rightarrow s(A^{\cdot\cdot})$ there with the quasi-isomorphism $K \stackrel{\mathrm{id}\otimes T}{\rightarrow} s(A^{\cdot\cdot})$ as in [15, (5.13) Lemma], we have a spectral sequence which is isomorphic to the one in [11] Proposition (1.8.3).

2. Replacing $L = s(K \stackrel{T-1}{\rightarrow} K)$ there with $L' := s(K \stackrel{(T-1)\otimes \check{T}}{\rightarrow} K(-1))$, we have an isomorphic spectral sequence, which is associated to the double complex $A'^{\cdot\cdot}$ that is isomorphic to $A^{\cdot\cdot}$, where $\check{T}$ is the generator of $\boldsymbol{Z}_l(-1)$ such that $T \otimes \check{T} = 1$.

3. Replacing T with another T', we have an isomorphic spectral sequence by [11] Lemma (1.3.1) and (the algebraic version of) the functoriality in 1.17.

Thus in the following we may assume that T is the image of $(s \mapsto s + 1) \in \mathrm{Aut}_{S^1}(\boldsymbol{R}) = \pi_1(0^{\log})$ by the natural homomorphism $\pi_1(0^{\log}) \rightarrow \pi_{1,\,\mathrm{loget}}(0) \cong \widehat{\boldsymbol{Z}}(1) \rightarrow \boldsymbol{Z}_l(1)$, and compare the spectral sequence associated to $A'^{\cdot\cdot}$ for this T with the Hodge-theoretic one.

In the following we denote by e_X the natural morphism of topoi $(X_{\mathrm{an}}^{\log})^{\sim} \rightarrow (X_{\text{ét}}^{\log})^{\sim}$ in [8] (2.1) for an fs log scheme X that is locally of finite type over $\boldsymbol{C}$. Let $\Lambda \rightarrow I$ be a resolution as in [11] (1.3), and $e_Y^{-1} I \rightarrow J$ an injective resolution on $Y_{\mathrm{an}}^{\log}$. Here Λ is the constant sheaf $\boldsymbol{Z}/l^h\boldsymbol{Z}$ for an integer $h \geqslant 1$. We claim that then there is a natural homomorphism $e_{\mathring{Y}}^{-1} \varepsilon_{Y*} \pi_* \pi^* I \rightarrow \tau_* \pi_* \pi^{-1} J$, where ε_Y is the forgetting log morphism. In fact, first, there is a natural homomorphism

$$e_{\mathring{Y}}^{-1} \varepsilon_{Y*} \pi_* \pi^* I = e_{\mathring{Y}}^{-1} \varepsilon_{Y*} \varinjlim_n (\pi_{1/l^n *} \pi_{1/l^n}^* I) =$$

$$= \varinjlim_n e_{\mathring{Y}}^{-1} \varepsilon_{Y*} \pi_{1/l^n *} \pi_{1/l^n}^* I \rightarrow \varinjlim_n \tau_* \pi'_{n*} \pi_n'^{-1} e_Y^{-1} I.$$

Here the second equality comes from the fact that ε_Y is quasi-compact and quasi-separated. See [11] (1.3) for π_{1/l^n}; π'_n is $(\pi_{1/l^n})_{\mathrm{an}}^{\log}: (Y_{1/l^n})_{\mathrm{an}}^{\log} \rightarrow Y_{\mathrm{an}}^{\log}$. Note that the fs log analytic space $(Y_{1/l^n})_{\mathrm{an}}$ no longer satisfies the condition in (2.4). Second, Lemma 7.2 implies that $(Y_{\mathrm{an}})_\infty \rightarrow Y_{\mathrm{an}}^{\log}$ uniquely factors through π'_n. Hence we have $\pi'_{n*} \pi_n'^{-1} e_Y^{-1} I \rightarrow \pi_* \pi^{-1} e_Y^{-1} I \rightarrow \pi_* \pi^{-1} J$. Thus we have $e_{\mathring{Y}}^{-1} K \rightarrow \tau_* \pi_* \pi^{-1} J$, which is compatible with

$(T-1)\otimes\check{T}$ and $\delta'=\frac{1}{2\pi\sqrt{-1}}(T-1)$, and we have $e_{\overset{\circ}{Y}}^{-1}(A'^{\cdot\cdot})\to\to SZ(\tau_*\pi_*\pi^{-1}J,\delta')$.

We consider the associated spectral sequences. By [1] XVI Théorème 4.1, we see that the associated spectral sequence of $e_{\overset{\circ}{Y}}^{-1}(A'^{\cdot\cdot})$ is isomorphic to the one associated to $A'^{\cdot\cdot}$. Hence we have a homomorphism ι of spectral sequences from the one associated to $A'^{\cdot\cdot}$ to the one associated to $SZ(\tau_*\pi_*\pi^{-1}J,\delta')$. Now the calculation of E_1-terms of the latter is the same as in [11] Proposition (1.8.3):

$$\mathrm{E}_1^{-r,q+r}=\bigoplus_{\substack{k\geqslant 0\\ k\geqslant -r}}\ \bigoplus_{\substack{E\subset\{1,\dots,m\}\\ \mathrm{Card}\,E=r+2k+1}}\mathrm{H}^{q-r-2k}((Y_E)_{\mathrm{an}},\Lambda(-r-k))$$

in virtue of [8] Lemma (1.5) and [14] (3.9), which are analytic analogues of [8] Theorem (2.4) and [11] Lemma (1.8.1) respectively. Again by [1] XVI Théorème 4.1, we see that ι is an isomorphism.

In the following we denote by s.s.(Λ), where Λ is the constant sheaf $\boldsymbol{Z}/l^h\boldsymbol{Z}$ $(h\geqslant 1)$, $\boldsymbol{Z}_l$, $\boldsymbol{Q}_l$, or $\boldsymbol{Q}$ on Y_{an}, the spectral sequence associated to $SZ(\tau_*\pi_*\pi^{-1}J,\delta')$ for an injective resolution $\Lambda\to J$. By taking $\varprojlim_h$ of the above ι, we see that the l-adic one with $\boldsymbol{Z}_l$-coefficient is isomorphic to $\varprojlim_h$ s.s.$(\boldsymbol{Z}/l^h\boldsymbol{Z})$. The rest is to show that $\varprojlim_h$ s.s.$(\boldsymbol{Z}/l^h\boldsymbol{Z})\overset{\cong}{\leftarrow}$ s.s.$(\boldsymbol{Z}_l)$, s.s.$(\boldsymbol{Z}_l)\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l\overset{\cong}{\to}$ s.s.$(\boldsymbol{Q}_l)$ and s.s.$(\boldsymbol{Q}_l)\overset{\cong}{\leftarrow}$ s.s.$(\boldsymbol{Q})\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$ because Proposition 6.5 says that s.s.$(\boldsymbol{Q})$ is isomorphic to the spectral sequence associated to $SZ(B(I),\delta)$. It is enough to show that there are suitable maps as above; for once maps exist, they are isomorphisms because E_1-terms of various s.s.(Λ) are all calculated similarly as before and the claims are reduced to the universal coefficients theorem and the finiteness of cohomologies of compact manifolds. Only the existence of the first map may not be trivial. But it comes from the fact that the map s.s.$(\boldsymbol{Z}_l)\to$ s.s.$(\boldsymbol{Z}/l^h\boldsymbol{Z})$ is independent of the choices of compatible resolutions of $\boldsymbol{Z}_l$ and $\boldsymbol{Z}/l^h\boldsymbol{Z}$ (cf. the last sentence of Remark 3.16). ■

REMARK 7.3. By [3] Scholie (8.1.9)(iv), we know that the Hodge theoretic spectral sequence in Theorem 7.1 degenerates in E_2-terms. Theorem 7.1 and [11] Theorem (2.1) give an algebraic proof for this fact. Conversely, Theorem 7.1 and [3] Scholie (8.1.9)(iv) give an alternative proof of [11] Theorem (2.1) under the assumption where the characteristic is zero. In fact, by [11] Lemma (2.2) and a standard argument in [11] (2.3), we reduce the above case of [11] (2.1) to the case where $F=\boldsymbol{C}$.

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