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MIXED HODGE STRUCTURES ON LOG SMOOTH DEGENERATIONS

TARO FUJISAWA

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Abstract. We introduce the notion of a log smooth degeneration, which is a logarithmic analogue of the central fiber of some kind of degenerations of complex manifolds over polydiscs. Under suitable conditions, we construct a natural cohomological mixed Hodge complex on the reduction of a compact log smooth degeneration. In particular, we obtain mixed Hodge structures on the log de Rham cohomologies and E_1-degeneration of the log Hodge to de Rham spectral sequence for a certain kind of compact *reduced* log smooth degenerations.

Introduction.

(0.1) In this article, we introduce the notion of a log smooth degeneration over the $\boldsymbol{N}^k$-log point and of its reduction. For the reduction X of a log smooth degeneration Y over the $\boldsymbol{N}^k$-log point $*^k$, the main result of this article claims that the relative log de Rham cohomology $\mathrm{H}^n(X, \omega^{\cdot}_{X/*^k})$ carries the natural $\boldsymbol{Q}$-mixed Hodge structure for every integer n, if X is compact and if all the irreducible components of X are Kähler complex manifolds. Moreover, we obtain E_1-degeneration of the log Hodge to de Rham spectral sequence for such X. In the special case that Y is *reduced* in addition, we have natural $\boldsymbol{Q}$-mixed Hodge structures on the relative log de Rham cohomology groups and E_1-degeneration of the log Hodge to de Rham spectral sequence for Y itself.

(0.2) Let Δ^k be the k-dimensional polydisc and E_i the i-th coordinate hyperplane in Δ^k for $i = 1, \dots, k$. We set $E = \sum_{i=1}^k E_i$ as a divisor on Δ^k. For a surjective morphism of complex manifolds $f : X \to \Delta^k$, a divisor D on X is defined by $D = f^*E$. Then the central fiber $f^{-1}(0)$ is a log smooth degeneration over the $\boldsymbol{N}^k$-log point if the divisor D_{red} is simple normal crossing and if f is log smooth with respect to the log structures associated to the divisors D and E. This is a typical example of log smooth degenerations. Here we remark that we do not assume the morphism f above to be flat nor equi-dimensional.

(0.3) The case of $k = 1$ in the example above is nothing but the (semi-stable) degeneration over the unit disc Δ in $\boldsymbol{C}$. In [17], Steenbrink constructed a cohomological mixed Hodge complex on the reduction of the central fiber (under suitable conditions), and proved that this cohomological mixed Hodge complex yields the "limiting" mixed Hodge structure for the unipotent monodromy case. (He also treated the general case in [18].)

The theory of log structure in [10] shed a new light on the subject above. Steenbrink introduced the notion of log deformation in [19]. A log deformation is a complex space equipped with a log structure such that it is locally isomorphic, in the sense of log geometry,

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to the central fiber of a semi-stable degeneration over the unit disc in $\boldsymbol{C}$. He constructed a $\boldsymbol{Q}$-cohomological mixed Hodge complex on a proper log deformation in [19] by using Koszul complex obtained from the log structure. Independently, similar result is obtained by Kawamata and Namikawa in [15] under the different definition of log structure. They use the real blow-up for the construction of the $\boldsymbol{Q}$-structure of the $\boldsymbol{Q}$-cohomological mixed Hodge complex in question. The coincidence of the cohomological mixed Hodge complexes by Steenbrink and by Kawamata-Namikawa is proved by Nakayama and the author in [7]. These results are considered as logarithmic analogues of Steenbrink's results in [17].

On the other hand, in [5] some of Steenbrink's results in [17] are generalized by the present author to a certain case over a higher dimensional polydisc. More precisely, he introduces the notion of a morphism of generalized semi-stable type (which is called a morphism of semi-stable type in [6]) and constructs a $\boldsymbol{Q}$-cohomological mixed Hodge complex on its central fiber under a certain condition. A morphism of semi-stable type over Δ^k is the morphism as in (0.2) with the additional condition for D being reduced. Thus the central fiber of a morphism of semi-stable type is a log smooth degeneration over the $\boldsymbol{N}^k$-log point. Here we remark that the additional assumption for D being reduced implies the flatness of the morphism.

In [14] Kawamata presents a new construction of a cohomological mixed Hodge complex on the fiber of a weakly semistable morphism. This gives a new and simple proof of the results obtained in [17] and [5]. He uses the real blow-up, the simplicial method and the weight filtrations on the relative log de Rham complexes. Here we remark again that the weakly semistable morphism is equi-dimensional by definition.

(0.4) These results in (0.3) are concerned with flat or equi-dimensional morphisms. In the context of log geometry, the flatness condition is realized as the exactness (for the definition of exactness see, e.g., [10, Definition (4.6)]). In fact, the central fibers of the morphisms in (0.3), on which $\boldsymbol{Q}$-cohomological mixed Hodge complexes in question were constructed, are exact over the base points as log complex analytic spaces. Moreover, the log deformation is exact over the log point.

However, a log smooth degeneration is not necessarily exact over the log point $*^k$ as suggested in the example (0.2). So the main result of this article is, on the one hand, a logarithmic analogue of results in Steenbrink [17], Fujisawa [5] and Kawamata [14] in the *non-flat* case, on the other hand, a generalization of results in Steenbrink [19], Kawamata-Namikawa [15] and Fujisawa-Nakayama [7] to the *non-exact* case over the $\boldsymbol{N}^k$-log points.

For the case of a proper exact log smooth morphism, the E_1-degeneration of the log Hodge to de Rham spectral sequence is proved by Illusie, Kato and Nakayama in [9] in the algebraic context. (Kato, Matsubara and Nakayama proved similar results in [11] by a different approach.) The E_1-degeneration for a log smooth degeneration over the $\boldsymbol{N}^k$-log point is an analogue of these results for a non-exact case (see [9, Remarks (7.3)]).

(0.5) Now we roughly describe the contents of this article. Section 1 treats preliminary results on Koszul complexes. Taking inductive limits of Koszul complexes is a new ingredient for our construction. In Section 2 we study the relation between Koszul complexes and the log de Rham complexes. In Section 3, for later use, we develop the mixed Hodge theory for a compact Kähler complex manifold equipped with a certain kind of log structures. In Section 4 we introduce the notion of a log smooth degeneration over the $\boldsymbol{N}^k$-log point and of its reduction. Then we construct a simplicial resolution of the relative log de Rham complex for the reduction of a log smooth degeneration over $*^k$ with the nonsingular irreducible components. In Section 5 we treat the reduction X of a log smooth degeneration over $*^k$ satisfying the condition that X is compact and that all the irreducible components of X are nonsingular and Kähler. We construct a weak $\boldsymbol{Q}$-cohomological mixed Hodge complex on X and study the relation between the underlying $\boldsymbol{C}$-structure of the weak $\boldsymbol{Q}$-cohomological mixed Hodge complex above with the Hodge filtration F and the relative log de Rham complex with the stupid filtration F. The construction follows the simplicial method as in [2, 3, 4] and [14]. Here we return to El Zein's method in [3], instead of Kawamata's approach in [14] which uses the weight filtrations on the relative log de Rham complexes. Then we obtain the main results of this article, one of which claims that the relative log de Rham cohomologies carry the natural $\boldsymbol{Q}$-mixed Hodge structures, and the other the E_1-degeneration of the log Hodge to de Rham spectral sequence. In Appendix, we try to relax the finiteness assumption in axioms of cohomological mixed Hodge complexes in [2]. We introduce the notion of a weak $\boldsymbol{Q}$-cohomological mixed Hodge complex. Although the conclusions for weak $\boldsymbol{Q}$-cohomological mixed Hodge complexes are slightly weaker than for $\boldsymbol{Q}$-cohomological mixed Hodge complexes, they are sufficient for our purpose in this article.

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1. Koszul complexes.

(1.1) In this section we collect several results on Koszul complexes. The basic references are [8] and [19]. Moreover, we refer to a previous paper of the present author [5].

(1.2) Let X be a topological space, A a commutative ring and $\varphi : E \to F$ a morphism of A-sheaves on X. For any non-negative integer n, the Koszul complex $\mathrm{Kos}(\varphi; n)$ of φ is defined by

$$\mathrm{Kos}(\varphi; n)^p = \begin{cases} \Gamma_{n-p}(E) \otimes \bigwedge^p F & \text{for } 0 \le p \le n\,, \\ 0 & \text{otherwise} \end{cases}$$

with differential $d : \mathrm{Kos}(\varphi; n)^p \to \mathrm{Kos}(\varphi; n)^{p+1}$ given by

$$d(x_1^{[n_1]} \cdots x_k^{[n_k]} \otimes y) = \sum_{i=1}^{k} x_1^{[n_1]} \cdots x_i^{[n_i-1]} \cdots x_k^{[n_k]} \otimes \varphi(x_i) \wedge y\,,$$

where $x_1, \dots, x_k$ and y are local sections of the sheaves E and $\bigwedge^p F$, respectively, and $n_1, \dots, n_k$ are positive integers with $n_1 + \cdots + n_k = n - p$.

(1.3) The following properties can be easily proved.

(1.3.1) The Koszul complex is compatible with the extension of the base ring.

(1.3.2) The Koszul complex is compatible with the pull-back of the morphism of topological spaces.

(1.3.3) The Koszul complex is functorial in the following sense: for a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \downarrow & & \downarrow \\ E' & \xrightarrow{\varphi'} & F' \end{array}$$

we have a canonical morphism $\mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi'; n)$.

Moreover, we have the following proposition (cf. [5, 8, 19]).

PROPOSITION 1.4. *In the situation above, we have an isomorphism*

$$\Gamma_{n-p}(\mathrm{Ker}\,\varphi) \otimes \bigwedge^{p} \mathrm{Coker}\,\varphi \to \mathrm{H}^p(\mathrm{Kos}(\varphi; n))$$

for every p if E, F and Coker φ *are A-flat.*

(1.5) For any subsheaf G of F and for an integer m with $0 \le m \le p$, we have a morphism

$$\bigwedge^{m} F \otimes \Gamma_{n-p}(E) \otimes \bigwedge^{p-m} G \to \Gamma_{n-p}(E) \otimes \bigwedge^{p} F \tag{1.5.1}$$

by sending $x \otimes y \otimes z$ to $y \otimes x \wedge z$, where x, y and z are local sections of the sheaves $\bigwedge^m F$, $\Gamma_{n-p}(E)$ and $\bigwedge^{p-m} G$, respectively. Then we define a subsheaf $W(G)_m \,\mathrm{Kos}(\varphi; n)^p$ by

$$W(G)_m \,\mathrm{Kos}(\varphi; n)^p = \text{image of the morphism (1.5.1)}\,.$$

If the subsheaf G contains the image of φ, these subsheaves for $0 \le p \le n$ form a subcomplex of $\mathrm{Kos}(\varphi; n)$. Thus we define a finite increasing filtration $W(G)$ on the Koszul complex $\mathrm{Kos}(\varphi; n)$.

PROPOSITION 1.6. *In the situation above, we assume that the subsheaf G contains the image of the morphism φ. The morphism φ induces a morphism $E \to G$, which is denoted by φ_G. Then the morphism* (1.5.1) *induces an isomorphism of complexes*

$$\bigwedge^{m}(F/G) \otimes \mathrm{Kos}(\varphi_G; n-m)[-m] \simeq \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; n)$$

for every integer m with $0 \le m \le n$.

(1.7) Now we fix a global section e of the sheaf $\mathrm{Ker}\,\varphi$. For integers n, m with $n \le m$, we define a morphism

$$\kappa(\varphi; e)_{m,n} : \mathrm{Kos}(\varphi; n)^p = \Gamma_{n-p}(E) \otimes \bigwedge^{p} F \to \Gamma_{m-p}(E) \otimes \bigwedge^{p} F = \mathrm{Kos}(\varphi; m)^p$$

by sending $x \otimes y$ to $(m-n)!e^{[m-n]}x \otimes y$ for every p with $0 \leq p \leq n$. These morphisms form a morphism of complexes

$$\kappa(\varphi; e)_{m,n} : \mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi; m) \tag{1.7.1}$$

by the fact that e is contained in $\mathrm{Ker}\,\varphi$. Moreover, the data above define an inductive system of complexes because of the formula

$$e^{[a]}e^{[b]} = \binom{a+b}{a} e^{[a+b]}$$

in the divided power envelope.

On the other hand, the morphism (1.7.1) preserves the filtration $W(G)$ for any subsheaf G of F containing the image of φ. Therefore the data $W(G)_m \,\mathrm{Kos}(\varphi; n)$ form an inductive system.

DEFINITION 1.8. We define a complex of A-sheaves $\mathrm{Kos}(\varphi; \infty; e)$ by

$$\mathrm{Kos}(\varphi; \infty; e) = \varinjlim_n \mathrm{Kos}(\varphi; n)\,.$$

The canonical morphism $\mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi; \infty; e)$ is denoted by $\kappa(\varphi; e)_n$. We omit the symbol e if there is no danger of confusion. Because the functor of taking inductive limit is an exact functor,

$$W(G)_m \,\mathrm{Kos}(\varphi; \infty; e) = \varinjlim_n W(G)_m \,\mathrm{Kos}(\varphi; n) \tag{1.8.1}$$

is a subcomplex of $\mathrm{Kos}(\varphi; \infty; e)$ for every m. Thus we define an increasing filtration $W(G)$ on $\mathrm{Kos}(\varphi; \infty; e)$.

(1.9) By the properties (1.3.1) through (1.3) we have the similar properties for $\mathrm{Kos}(\varphi; \infty; e)$.

PROPOSITION 1.10. *For a subsheaf G of F containing the image of φ we have an isomorphism of complexes*

$$\bigwedge^m (F/G) \otimes \mathrm{Kos}(\varphi_G; \infty; e)[-m] \simeq \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e)$$

for every m.

PROOF. By the exactness of taking inductive limit, we have an isomorphism

$$\mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e) \to \varinjlim_n \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; n)$$

for every m. On the other hand, we have an identification

$$\bigwedge^m (F/G) \otimes \mathrm{Kos}(\varphi_G; n-m)[-m] \simeq \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; n)$$

by Proposition 1.6. It is easy to see that the morphism $\mathrm{Gr}_m^{W(G)} \kappa(\varphi; e)_{n,n'}$ is identified to the one $\mathrm{id} \otimes \kappa(\varphi_G; e)_{n-m,n'-m}$ under the identification above. Thus we obtain the conclusion. □

(1.11) Now we assume that a global section t of the sheaf F is given in addition. We define a morphism of A-sheaves

$$\mathrm{Kos}(\varphi; n)^p = \Gamma_{n-p}(E) \otimes \bigwedge^p F \to \Gamma_{n-p}(E) \otimes \bigwedge^{p+1} F = \mathrm{Kos}(\varphi; n+1)^{p+1}$$

by sending a local section $x \otimes y$ to $x \otimes t \wedge y$. Thus we obtain a morphism of complexes $t\wedge$: $\mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi; n+1)[1]$. By composing this morphism and the canonical morphism $\kappa(\varphi; e)_{n+1}[1] : \mathrm{Kos}(\varphi; n+1)[1] \to \mathrm{Kos}(\varphi; \infty; e)[1]$ we obtain a morphism of complexes $\mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi; \infty; e)[1]$ for every n. Because these morphisms are compatible with the morphisms $\kappa(\varphi; e)_{m,n}$ for all m, n, we obtain a morphism of complexes

$$\mathrm{Kos}(\varphi; \infty; e) \to \mathrm{Kos}(\varphi; \infty; e)[1], \tag{1.11.1}$$

which is denoted by $t\wedge$ again for simplicity. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Kos}(\varphi; n) & \xrightarrow{t\wedge} & \mathrm{Kos}(\varphi; n+1)[1] \\ {\scriptstyle \kappa(\varphi;e)_n}\big\downarrow & & \big\downarrow{\scriptstyle \kappa(\varphi;e)_{n+1}[1]} \\ \mathrm{Kos}(\varphi; \infty; e) & \xrightarrow{t\wedge} & \mathrm{Kos}(\varphi; \infty; e)[1] \end{array}$$

for every n.

(1.12) Let G be a subsheaf of F containing the image of φ. For the case that t is a global section of a subsheaf G, the morphism $t\wedge : \mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi; n+1)[1]$ preserves the filtration $W(G)$. Therefore the induced morphism $t\wedge : \mathrm{Kos}(\varphi; \infty; e) \to \mathrm{Kos}(\varphi; \infty; e)[1]$ preserves the filtration $W(G)$. In this case the morphism

$$\mathrm{Gr}_m^{W(G)}(t\wedge) : \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e) \to \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e)[1]$$

is identified to the morphism $(-1)^m \operatorname{id} \otimes (t\wedge)[-m]$ under the identification

$$\mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e) \simeq \bigwedge^m (F/G) \otimes \mathrm{Kos}(\varphi_G; \infty; e)[-m]$$

$$\mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e)[1] \simeq \bigwedge^m (F/G) \otimes \mathrm{Kos}(\varphi_G; \infty; e)[-m+1]$$

in Proposition 1.10.

Next we consider the case that t is not a global section of the subsheaf G. We denote the image of t by the projection $F \to F/G$ by $\bar{t}$. The morphism $t\wedge : \mathrm{Kos}(\varphi; n) \to \mathrm{Kos}(\varphi; n+1)$ satisfies the condition

$$(t\wedge)(W(G)_m \mathrm{Kos}(\varphi; n)) \subset W(G)_{m+1} \mathrm{Kos}(\varphi; n+1)[1],$$

which induces

$$(t\wedge)(W(G)_m \mathrm{Kos}(\varphi; \infty; e)) \subset W(G)_{m+1} \mathrm{Kos}(\varphi; \infty; e)[1]$$

for every m. The induced morphism

$$\mathrm{Gr}_m^{W(G)}(t\wedge) : \mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e) \to \mathrm{Gr}_{m+1}^{W(G)} \mathrm{Kos}(\varphi; \infty; e)[1]$$

is identified to the morphism $(\bar{t}\wedge) \otimes \mathrm{id}[-m]$ under the identification

$$\mathrm{Gr}_m^{W(G)} \mathrm{Kos}(\varphi; \infty; e) \simeq \bigwedge^m (F/G) \otimes \mathrm{Kos}(\varphi_G; \infty; e)[-m]$$

$$\mathrm{Gr}_{m+1}^{W(G)} \mathrm{Kos}(\varphi; \infty; e)[1] \simeq \bigwedge^{m+1} (F/G) \otimes \mathrm{Kos}(\varphi_G; \infty; e)[-m]$$

in Proposition 1.10, where $\bar{t}\wedge$ denotes the morphism $\bigwedge^m (F/G) \to \bigwedge^{m+1}(F/G)$ sending x to $\bar{t} \wedge x$.

(1.13) Now we treat the case with the following conditions:

(1.13.1) a ring A contains $\boldsymbol{Q}$ as a subring,

(1.13.2) there exists a global section e of the $\mathrm{Ker}\,\varphi$ such that $\mathrm{Ker}\,\varphi = A \cdot e \simeq A$.

Then the morphism

$$\bigwedge^p F \to \mathrm{Kos}(\varphi; n)^p = \Gamma_{n-p}(E) \otimes \bigwedge^p F \tag{1.13.3}$$

sending $x \in \bigwedge^p F$ to $(n-p)! e^{[n-p]} \otimes x$ induces an isomorphism

$$\bigwedge^p \mathrm{Coker}(\varphi) \to \mathrm{H}^p(\mathrm{Kos}(\varphi; n))$$

for every p by Proposition 1.4 because the condition (1.13.2) implies the identification $\Gamma_{n-p}(\mathrm{Ker}\,\varphi) \simeq A$. We can easily see that the morphism (1.13.3) is compatible with the morphism $\kappa(\varphi; e)_{m,n}$ for every m, n. Thus we obtain a morphism

$$\bigwedge^p F \to \mathrm{Kos}(\varphi; \infty; e)^p \tag{1.13.4}$$

for every p. Since the cohomology functor is compatible with inductive limit, we have the following proposition.

PROPOSITION 1.14. *The morphism* (1.13.4) *induces an isomorphism*

$$\bigwedge^p \mathrm{Coker}\,\varphi \to \mathrm{H}^p(\mathrm{Kos}(\varphi; \infty; e))$$

for every p.

COROLLARY 1.15. *In addition to the situation above*, *we assume the condition that the morphism* φ *is surjective. Then we have a quasi-isomorphism*

$$A \to \mathrm{Kos}(\varphi; \infty; e)\,.$$

induced by the morphism $A \to \mathrm{Kos}(\varphi; n)^0 = \Gamma_n(E)$ *which sends* a *to* $n! a e^{[n]}$.

2. Log de Rham complex and Koszul complex for a log complex analytic space.

(2.1) Let X be an fs log complex analytic space, that is, a complex analytic space equipped with a fine saturated log structure $\alpha : M_X \to \mathcal{O}_X$ (cf. [10] and [12]). We denote the sheaf of holomorphic p-forms on X by Ω_X^p, and the sheaf of logarithmic p-forms on X by ω_X^p. Thus the de Rham complex and the log de Rham complex are denoted by $\Omega_X^{\cdot}$ and $\omega_X^{\cdot}$, respectively.

DEFINITION 2.2. We have a morphism

$$\omega_X^m \otimes \Omega_X^{p-m} \to \omega_X^p \,, \tag{2.2.1}$$

which sends $\omega \otimes \eta$ to $\omega \wedge \eta$. We define an $\mathcal{O}_X$-subsheaf $W_m\omega_X^p$ by

$$W_m\omega_X^p = \text{ image of the morphism of (2.2.1)}$$

for every m. Then these sheaves for all p form a subcomplex of $\omega_X^{\cdot}$ because the differential $d : \mathcal{O}_X \to \omega_X^1$ is decomposed into the composite $\mathcal{O}_X \xrightarrow{d} \Omega_X^1 \to \omega_X^1$. It is easy to see that W_m defines a finite increasing filtration on $\omega_X^{\cdot}$.

Moreover, a finite decreasing filtration F on $\omega_X^{\cdot}$ is defined by the same way as the stupid filtration (filtration bête) in [1].

(2.3) We have the exponential sequence

$$0 \to \boldsymbol{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

as in [13, Lemma 54.3], where the morphism $\boldsymbol{Z}_X \to \mathcal{O}_X$ is the canonical inclusion and the morphism $\mathcal{O}_X \to \mathcal{O}_X^*$ sends a local section f of $\mathcal{O}_X$ to $\exp(2\pi\sqrt{-1}f)$. Now M_X^{gp} denotes the abelian sheaf associated to the monoid sheaf M_X. By using the inclusion $\mathcal{O}_X^* \to M_X^{\mathrm{gp}}$, an exact sequence of abelian sheaves

$$0 \to \boldsymbol{Z}_X \to \mathcal{O}_X \to M_X^{\mathrm{gp}} \to M_X^{\mathrm{gp}}/\mathcal{O}_X^* \to 0$$

is obtained. By tensoring $\boldsymbol{Q}$ with the morphism $\mathcal{O}_X \to M_X^{\mathrm{gp}}$ above, we obtain a morphism of $\boldsymbol{Q}$-sheaves $\mathcal{O}_X \to M_X^{\mathrm{gp}} \otimes \boldsymbol{Q}$, which we denote by φ_X. The morphism φ_X fits in the exact sequence

$$0 \to \boldsymbol{Q}_X \to \mathcal{O}_X \xrightarrow{\varphi_X} M_X^{\mathrm{gp}} \otimes \boldsymbol{Q} \to (M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes \boldsymbol{Q} \to 0$$

by definition.

We fix a constant function 1 as a global section of $\operatorname{Ker}\varphi_X$. Then we have complexes of $\boldsymbol{Q}$-sheaves $\mathrm{Kos}(\varphi_X; n)$ and $\mathrm{Kos}(\varphi_X; \infty; 1)$ defined in Section 1, which we denote by $\mathrm{Kos}_X(M_X; n)$ and $\mathrm{Kos}_X(M_X)$, respectively, in this article.

Because the image of the morphism $\varphi_X : \mathcal{O}_X \to M_X^{\mathrm{gp}} \otimes \boldsymbol{Q}$ is the subsheaf $\mathcal{O}_X^* \otimes \boldsymbol{Q}$ of $M_X^{\mathrm{gp}} \otimes \boldsymbol{Q}$, we have an increasing filtration $W(\mathcal{O}_X^* \otimes \boldsymbol{Q})$ on $\mathrm{Kos}_X(M_X)$ as in (1.8.1), which we denote by W for simplicity.

(2.4) The morphism $\mathrm{dlog} : M_X^{\mathrm{gp}} \to \omega_X^1$ defines a morphism of abelian sheaves $\mathrm{dlog} : \bigwedge^p M_X^{\mathrm{gp}} \to \omega_X^p$ for every p. A morphism of $\boldsymbol{Q}$-sheaves

$$\psi_{(X,M_X),n} : \mathrm{Kos}_X(M_X; n)^p \to \omega_X^p$$

is defined by sending a local section $f_1^{[n_1]} \cdots f_k^{[n_k]} \otimes y$ of $\mathrm{Kos}_X(M_X; n)^p = \Gamma_{n-p}(\mathcal{O}_X) \otimes \bigwedge^p M_X^{\mathrm{gp}}$ to a local section $(2\pi\sqrt{-1})^{-p}(n_1! \cdots n_k!)^{-1} f_1^{n_1} \cdots f_k^{n_k}\,\mathrm{dlog}\, y$ of ω_X^p. Because of the equality

$$\mathrm{dlog}(\exp(f)) = df \,, \tag{2.4.1}$$

$\psi_{(X,M_X),n}$ actually defines a morphism of complexes, which is compatible with the morphism

$$(\kappa_X)_{m,n} = \kappa(\varphi_X; 1)_{m,n} : \mathrm{Kos}_X(M_X; n) \to \mathrm{Kos}_X(M_X; m)$$

in (1.7.1) for every n, m with $n \leq m$. Thus a morphism of $\boldsymbol{Q}$-sheaves

$$\psi_{(X,M_X)} : \mathrm{Kos}_X(M_X) \to \omega^{\cdot}_X \tag{2.4.2}$$

is obtained. Sometimes we use the symbol ψ_X simply if there is no danger of confusion.

The morphism ψ_X preserves the weight filtration W on $\mathrm{Kos}_X(M_X)$ and $\omega^{\cdot}_X$ because the equality (2.4.1) implies that the image of the subsheaf $\mathcal{O}^*_X$ by the morphism dlog is contained in the image of the canonical morphism $\Omega^1_X \to \omega^1_X$.

3. Mixed Hodge theory on a log complex manifold.

(3.1) For a reduced effective divisor D on a complex manifold X, we set $M_X(D) = j_*\mathcal{O}^*_{X\setminus D} \cap \mathcal{O}_X$, where $j : X \setminus D \to X$ denotes the inclusion. The inclusion $M_X(D) \to \mathcal{O}_X$ gives us an fs log structure on X, which is called the log structure associated to the divisor D. Moreover, for a given positive integer l, a morphism of monoid sheaves $\alpha : M_X(D) \oplus \boldsymbol{N}^l_X \to \mathcal{O}_X$ is defined by

$$\alpha(f \oplus (n_1, \dots, n_l)) = \begin{cases} 0 & \text{if } (n_1, \dots, n_l) \neq (0, \dots, 0)\,, \\ f & \text{if } (n_1, \dots, n_l) = (0, \dots, 0)\,, \end{cases}$$

where f is a local section of $M_X(D)$ considered as a local section of $\mathcal{O}_X$ by definition. It is easy to see that this actually gives us a log structure on X. We simply denote this log structure by $M_X(D)\oplus\boldsymbol{N}^l_X$. In this situation, we regard $M_X(D)$ and $\boldsymbol{N}^l_X$ as subsheaves of $M_X(D)\oplus\boldsymbol{N}^l_X$.

LEMMA 3.2. *Let X be a complex manifold, D a reduced simple normal crossing divisor on X, $M_X(D)\oplus\boldsymbol{N}^l_X$ the log structure above and $\varphi : M_X(D) \to M_X(D)\oplus\boldsymbol{N}^l_X$ a morphism of log structures. Then φ coincides with the canonical inclusion $M_X(D) \to M_X(D) \oplus \boldsymbol{N}^l_X$.*

PROOF. Take a local section f of $M_X(D)$ and set $\varphi(f) = g \oplus (n_1, \dots, n_l) \in M_X(D) \oplus \boldsymbol{N}^l_X$. Then we have

$$f = \alpha(f) = \alpha\varphi(f) = \alpha(g \oplus (n_1, \dots, n_l)) = \begin{cases} 0 & \text{if } (n_1, \dots, n_l) \neq (0, \dots, 0)\,, \\ g & \text{if } (n_1, \dots, n_l) = (0, \dots, 0)\,, \end{cases}$$

because φ is a morphism of log structures. By the fact that $f \neq 0$, we have $n_1 = \cdots = n_l = 0$ and $g = f$. □

LEMMA 3.3. *Let X, D and $M_X(D) \oplus \boldsymbol{N}^l_X$ be as above. For an isomorphism of log structures $\varphi : M_X(D) \oplus \boldsymbol{N}^l_X \to M_X(D) \oplus \boldsymbol{N}^l_X$, there exists an isomorphism of monoid sheaves $\varphi_1 : \boldsymbol{N}^l_X \to \boldsymbol{N}^l_X$ such that the induced morphism*

$$\bar{\varphi} : (M_X(D)/\mathcal{O}^*_X) \oplus \boldsymbol{N}^l_X \to (M_X(D)/\mathcal{O}^*_X) \oplus \boldsymbol{N}^l_X$$

is given by $\bar{\varphi} = \mathrm{id} \oplus \varphi_1$.

PROOF. The restriction $\varphi|_{M_X(D)}$ gives us a morphism of log structures $M_X(D) \to M_X(D) \oplus \boldsymbol{N}^l_X$. Therefore we have $\varphi(f) = f$ for a local section $f \in M_X(D)$ by the lemma above. We denote the inverse of φ by ψ. For a local section n of $\boldsymbol{N}^l_X$ we set $\varphi(n) = f \oplus n' \in M_X(D) \oplus \boldsymbol{N}^l_X$ and $\psi(n') = g \oplus n''$. Then we have

$$n = \psi\varphi(n) = \psi(f \oplus n') = \psi(f)\psi(n') = f \cdot (g \oplus n'') = fg \oplus n''$$

because $\psi(f) = f$ by the lemma above. Then $fg = 1$ in $M_X(D)$ and $n = n''$. Thus the local section f of $M_X(D)$ is contained in $\mathcal{O}^*_X$. Therefore we have $\bar{\varphi}(n) = n'$. □

(3.4) In the remainder of this section, we study mixed Hodge theory on a compact Kähler complex manifold equipped with a certain kind of log structures.

Let X be a complex manifold equipped with an fs log structure M_X. Here we consider the following condition:

(3.4.1) for any point x of X, there exist an open neighborhood V of x, a reduced simple normal crossing divisor D_V on V and a non-negative integer l_V, such that the log structure $M_X|_V$ is isomorphic to the log structure $M_V(D_V) \oplus \boldsymbol{N}^{l_V}_V$.

LEMMA 3.5. *Let X be an fs log complex manifold satisfying Condition* (3.4) *above. For every point x of X, we have an isomorphism*

$$(M_X/\mathcal{O}^*_X)_x \simeq \boldsymbol{N}^{r(x)}$$

for a non-negative integer $r(x)$. Moreover, the function $r(x)$ is upper semi-continuous.

PROOF. Easy by definition. □

(3.6) Let X be as above. We set $r = \min\{r(x) \mid x \in X\}$ and $U = \{x \in X \mid r(x) = r\}$. Then r is a non-negative integer and U is a non-empty subset of X.

LEMMA 3.7. *In the situation above, we assume that X is connected in addition. Then U is an open dense subset of X. Moreover, the closed subset $D = X \setminus U$ is a reduced normal crossing divisor on X.*

PROOF. By the upper semi-continuity of the function $r(x)$, U is an open subset of X. We denote by $\bar{U}$ the closure of U in X. For any point $x \in \bar{U}$, we take an open neighborhood V of x in X satisfying Condition (3.4). Because $U \cap V \neq \emptyset$, we have $U \cap V = V \setminus D_V$ and the equality $r = l_V$. Then it is easy to see the equality $\bar{U} \cap V = V$, that is, $V \subset \bar{U}$. Therefore $\bar{U}$ is an open subset of X. Because X is connected, we have $\bar{U} = X$. In the situation above, $D|_V = V \setminus U \cap V = D_V$. Thus D is a reduced normal crossing divisor on X. □

(3.8) In the remainder of this section, we fix the notation X, U and D as in Lemma 3.7, that is, assume that X is connected. Then for an open neighborhood V in (3.4) we have $D_V = D|_V$.

COROLLARY 3.9. *Let X be as above. For any point $x \in X$ there exists an open neighborhood V of x such that the log structure $M_X|_V$ on V is isomorphic to the log structure $M_X(D)|_V \oplus \boldsymbol{N}^r_V$.*

LEMMA 3.10. *In the situation above, there exists a unique morphism of log structures* $\iota : M_X(D) \to M_X$. *For an open subset* V *of* X *satisfying* (3.4), *the restriction of this morphism* $\iota|_V : M_X(D)|_V \to M_X|_V$ *is identified with the canonical inclusion* $M_X(D)|_V \to M_X(D)|_V \oplus \boldsymbol{N}^r_V$ *via the identification* $M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$. *Therefore the morphism* ι *is injective.*

PROOF. In the local situation, we may assume the identification $M_X \simeq M_X(D) \oplus \boldsymbol{N}^r_V$. Then Lemma 3.2 implies that the canonical injection $M_X(D) \to M_X(D) \oplus \boldsymbol{N}^r_V$ is the unique morphism of log structures. Thus the uniqueness is proved because it is the question of local nature. Therefore we can obtain the morphism of log structures $M_X(D) \to M_X$ by patching the morphisms of log structures $M_X(D)|_V \to M_X(D)|_V \oplus \boldsymbol{N}^r_V$ on the small open subsets V satisfying the condition (3.4). □

(3.11) From now on, we consider $M_X(D)$ as a subsheaf of M_X by the injection ι. Then we have the inclusions $\mathcal{O}^*_X \subset M_X(D) \subset M_X$.

COROLLARY 3.12. *In the situation above, the sheaf* $M_X/M_X(D)$ *is locally constant. More precisely, we have* $(M_X/M_X(D))|_V \simeq \boldsymbol{N}^r_V$ *for an open subset* V *satisfying* (3.4).

(3.13) The inclusion $\iota : M_X(D) \to M_X$ induces a morphism of monoid sheaves $\bar{\iota}$: $M_X(D)/\mathcal{O}^*_X \to M_X/\mathcal{O}^*_X$.

LEMMA 3.14. *There exists a morphism of monoid sheaves* $\pi : M_X/\mathcal{O}^*_X \to M_X(D)/\mathcal{O}^*_X$ *such that* $\pi \cdot \bar{\iota} = \mathrm{id}$.

PROOF. Once we fix an identification $M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$ on an open subset V satisfying the condition in (3.4), the projection

$$p : (M_X(D)/\mathcal{O}^*_X)|_V \oplus \boldsymbol{N}^r_V \to (M_X(D)/\mathcal{O}^*_X)|_V$$

induces a morphism

$$\pi : (M_X/\mathcal{O}^*_X)|_V \simeq (M_X(D)/\mathcal{O}^*_X)|_V \oplus \boldsymbol{N}^r_V \to (M_X(D)/\mathcal{O}^*_X)|_V,$$

which satisfies the equality $\pi \cdot \bar{\iota} = \mathrm{id}$ on V. Therefore it suffices to prove that the morphism π is independent from the choice of the identification $M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$. We may assume $X = V$ without loss of generality. Let

$$\psi_1, \psi_2 : M_X \to M_X(D)| \oplus \boldsymbol{N}^r_X$$

be two identifications. Then

$$\varphi = \psi_2 \cdot \psi_1^{-1} : M_X(D) \oplus \boldsymbol{N}^r_X \to M_X(D) \oplus \boldsymbol{N}^r_X$$

is an isomorphism of log structures. By Lemma 3.3 the induced morphism

$$\bar{\varphi} : (M_X(D)/\mathcal{O}^*_X)|_V \oplus \boldsymbol{N}^r_V \to (M_X(D)/\mathcal{O}^*_X)|_V \oplus \boldsymbol{N}^r_V$$

is expressed as $\bar{\varphi} = \mathrm{id} \oplus \varphi_1$, where φ_1 is an automorphism of $\boldsymbol{N}^r_X$. Then we have

$$p \cdot \overline{\psi_2} = p \cdot \bar{\varphi} \cdot \overline{\psi_1} = p \cdot (\mathrm{id} \oplus \varphi_1) \cdot \overline{\psi_1} = p \cdot \overline{\psi_1},$$

which shows the independence in question. □

DEFINITION 3.15. We define an abelian sheaf $L_{\boldsymbol{Z}}$ on X by $L_{\boldsymbol{Z}} = M_X^{\mathrm{gp}}/M_X(D)^{\mathrm{gp}}$.

COROLLARY 3.16. *The abelian sheaf $L_{\boldsymbol{Z}}$ is locally constant. More precisely, $L_{\boldsymbol{Z}}|_V \simeq \boldsymbol{Z}^r_V$ for an open subset V satisfying the condition in* (3.4). *Moreover, there exists an isomorphism of abelian sheaves $M_X^{\mathrm{gp}}/\mathcal{O}_X^* \to (M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*) \oplus L_{\boldsymbol{Z}}$ whose restriction on an open subset V as above coincides with the canonical identification $(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)|_V \simeq (M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*)|_V \oplus \boldsymbol{Z}^r_V$ under the identification $L_{\boldsymbol{Z}}|_V \simeq \boldsymbol{Z}^r_V$.*

PROOF. The identification $L_{\boldsymbol{Z}}|_V \simeq \boldsymbol{Z}^r_V$ is easily seen by $M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$. We have an exact sequence

$$0 \to M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^* \to M_X^{\mathrm{gp}}/\mathcal{O}_X^* \to L_{\boldsymbol{Z}} \to 0$$

by definition. Then the morphism $\pi^{\mathrm{gp}} : M_X^{\mathrm{gp}}/\mathcal{O}_X^* \to M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*$ gives the splitting of the exact sequence above. □

LEMMA 3.17. *Let $\varphi : \boldsymbol{N}^n \to \boldsymbol{N}^n$ be an isomorphism of monoids $\boldsymbol{N}^n$. Then there exists an automorphism of the set $\{1, \dots, n\}$ such that we have $\varphi(e_i) = e_{\sigma(i)}$ for every $i = 1, \dots, n$, where e_i is the i-th unit element of $\boldsymbol{N}^n$.*

PROOF. Easy. □

LEMMA 3.18. *The abelian sheaf $L_{\boldsymbol{Z}}$ admits a positive definite bilinear form.*

PROOF. An identification $M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$ induces an identification $L_{\boldsymbol{Z}}|_V \simeq \boldsymbol{Z}^r_V$. Therefore the canonical positive definite bilinear form $\boldsymbol{Z}^r_V \otimes \boldsymbol{Z}^r_V \to \boldsymbol{Z}_V$ gives $L_{\boldsymbol{Z}}|_V$ an positive definite bilinear form. So it is sufficient to prove that this bilinear form on $L_{\boldsymbol{Z}}|_V$ is independent of the choice of the identification $M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$. We may assume $X = V$, take two identifications $\psi_1, \psi_2 : M_X|_V \simeq M_X(D)|_V \oplus \boldsymbol{N}^r_V$ and use the same notation as in the proof of Lemma 3.14. The induced isomorphisms $L_{\boldsymbol{Z}} \to \boldsymbol{Z}^r_X$ are denoted by ζ_1 and ζ_2. Then the isomorphism $\varphi_1 : \boldsymbol{N}^r_X \to \boldsymbol{N}^r_X$ in the proof of Lemma 3.14 satisfies the equality $\zeta_2 = \varphi_1^{\mathrm{gp}} \cdot \zeta_1$. Moreover, we may assume that the isomorphism φ_1 is induced from the automorphism of the monoid $\boldsymbol{N}^r$ by shrinking X sufficiently small. By the lemma above, there exists an automorphism σ of the set $\{1, \dots, r\}$ such that $\varphi_1(e_i) = e_{\sigma(i)}$ for every i. Then we can easily see that the canonical positive definite bilinear form $\boldsymbol{Z}^r_V \otimes \boldsymbol{Z}^r_V \to \boldsymbol{Z}_V$ is independent of the choice of the identification. □

(3.19) In the situation above, we have a sequence of $\mathcal{O}_X$-submodules

$$\Omega^p_X \subset \Omega^p_X(\log D) \subset \omega^p_X$$

for every p. Moreover, we can easily see that the sheaves ω^p_X, $\Omega^p_X(\log D)$ and Ω^p_X are locally free of finite rank.

(3.20) In order to prove Theorem 3.27 below, we first consider the case where the log structure M_X is the trivial log structure $\mathcal{O}_X^*$. In this case the log de Rham complex $\omega^{\cdot}_X$ is nothing but the usual de Rham complex $\Omega^{\cdot}_X$ and the filtration F defined in Definition 2.2 is nothing but the stupid filtration in [1].

Now Conditions (1.13.1) and (1.13.2) are satisfied for φ_X in (2.3) and for $A = \boldsymbol{Q}$. Moreover, the morphism φ_X is surjective. Therefore we have a quasi-isomorphism

$$\boldsymbol{Q}_X \to \mathrm{Kos}_X(\mathcal{O}_X^*)$$

by Corollary 1.15. For every integer m, we can easily see the commutativity of the diagram

$$\begin{array}{ccccc} \boldsymbol{Q}_X & \longrightarrow & \mathrm{Kos}_X(\mathcal{O}_X^*) & \xrightarrow{(2\pi\sqrt{-1})^m\psi_{(X,\mathcal{O}_X^*)}} & \Omega_X^{\cdot} \\ \downarrow & & & & \| \\ \boldsymbol{Q}(m)_X & \longrightarrow & \boldsymbol{C}_X & \longrightarrow & \Omega_X^{\cdot}, \end{array}$$

where the left vertical arrow is the morphism sending a to $(2\pi\sqrt{-1})^m a$, the morphism $\boldsymbol{Q}(m)_X \to \boldsymbol{C}_X$ is the inclusion, and the morphism $\boldsymbol{C}_X \to \Omega_X^{\cdot}$ is the usual quasi-isomorphism. We denote the composite of the bottom line by $\iota(m)$ for a while. Thus we have the following.

PROPOSITION 3.21. *The data*

$$(\mathrm{Kos}_X(\mathcal{O}_X^*), (\Omega_X^{\cdot}, F), (2\pi\sqrt{-1})^m\psi_{(X,\mathcal{O}_X^*)}) \tag{3.21.1}$$

is identified with the data

$$(\boldsymbol{Q}(m)_X, (\Omega_X^{\cdot}, F), \iota(m))$$

for every integer m. Therefore the data (3.21.1) *is a* $\boldsymbol{Q}$*-cohomological Hodge complex of weight* 0, *if the complex manifold X is compact and Kähler.*

(3.22) Next we treat the general case in (3.4). The morphism dlog : $\bigwedge^m M_X^{\mathrm{gp}} \to \omega_X^m$ induces a morphism of $\mathcal{O}_X$-modules $\bigwedge^m M_X^{\mathrm{gp}} \otimes \Omega_X^{p-m} \to \omega_X^p$ by sending $a \otimes \omega$ to $\mathrm{dlog}(a) \wedge \omega$. By definition the image of this morphism is contained in the subsheaf $W_m\omega_X^p$. Therefore we obtain a morphism

$$\bigwedge^m M_X^{\mathrm{gp}} \otimes \Omega_X^{p-m} \to \mathrm{Gr}_m^W \omega_X^p \tag{3.22.1}$$

for every m.

On the other hand, the canonical projection

$$\bigwedge^m M_X^{\mathrm{gp}} \to \bigwedge^m (M_X^{\mathrm{gp}}/\mathcal{O}_X^*) \simeq \bigwedge^m ((M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*) \oplus L_{\boldsymbol{Z}})$$

defines a surjective morphism

$$\bigwedge^m M_X^{\mathrm{gp}} \otimes \Omega_X^{p-m} \to \bigwedge^m ((M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*) \oplus L_{\boldsymbol{Z}}) \otimes \Omega_X^{p-m}$$

for every m. It is easy to see that the morphism (3.22.1) factors through this surjection. Thus we obtain a morphism of $\mathcal{O}_X$-modules

$$\bigoplus_{l=0}^{m} \bigwedge^{m-l} L_{\boldsymbol{Z}} \otimes \bigwedge^{l} (M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*) \otimes \Omega_X^{p-m} \to \mathrm{Gr}_m^W \omega_X^p \tag{3.22.2}$$

for every m.

LEMMA 3.23. *The morphism* (3.22.2) *induces an isomorphism of complexes*

$$\bigoplus_{l=0}^{m}(i_l)_*\Bigl(i_l^{-1}\Bigl(\bigwedge^{m-l} L_{\mathbf{Z}}\Bigr)\otimes\varepsilon^l\otimes\Omega^{\cdot}_{\tilde{D}^l}[-m]\Bigr)\to \mathrm{Gr}_m^W\,\omega^{\cdot}_X\,, \tag{3.23.1}$$

where $\tilde{D}^l$, i_l *and* ε^l *are defined in Deligne* [1].

PROOF. We can easily see that there exists an isomorphism

$$(i_l)_*\varepsilon^l\simeq\bigwedge^{l}(M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*)\otimes\boldsymbol{C} \tag{3.23.2}$$

for every l. Then easy local computation implies the conclusion. □

COROLLARY 3.24. *Let* F *on* $\omega^{\cdot}_X$ *and on* $\Omega^{\cdot}_X$ *are the decreasing filtrations in Definition* 2.2. *Under the isomorphism* (3.23.1), *the induced filtration* F *on the right hand side coincides with the filtration*

$$\bigoplus_{l=0}^{m}(i_l)_*\Bigl(i_l^{-1}\Bigl(\bigwedge^{m-l} L_{\mathbf{Z}}\Bigr)\otimes\varepsilon^l\otimes F[-m]\Omega^{\cdot}_{\tilde{D}^l}[-m]\Bigr)$$

on the left hand side.

(3.25) On the other hand, we have the following for the filtration W on the complex $\mathrm{Kos}_X(M_X)$.

LEMMA 3.26. *The filtration* W *on* $\mathrm{Kos}_X(M_X)$ *is finite. Moreover*, *we have an isomorphism in the derived category*

$$\bigoplus_{l=0}^{m}(i_l)_*\Bigl(i_l^{-1}\Bigl(\bigwedge^{m-l} L_{\mathbf{Z}}\Bigr)\otimes\varepsilon^l\otimes\mathrm{Kos}_{\tilde{D}^l}(\mathcal{O}^*_{\tilde{D}^l})[-m]\Bigr)\to \mathrm{Gr}_m^W\,\mathrm{Kos}_X(M_X) \tag{3.26.1}$$

for every integer m.

PROOF. Proposition 1.10 for $F=M_X^{\mathrm{gp}}\otimes\boldsymbol{Q}$ and $G=\mathcal{O}_X^*\otimes\boldsymbol{Q}$ implies the isomorphism of the complexes

$$\bigwedge^{m}(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)\otimes\mathrm{Kos}_X(\mathcal{O}_X^*)[-m]\simeq\mathrm{Gr}_m^W\,\mathrm{Kos}_X(M_X)$$

for every m. Because for m big enough $\bigwedge^m(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)=0$, we have $\mathrm{Gr}_m^W\,\mathrm{Kos}_X(M_X)=0$. Then we can easily see the finiteness of the filtration W.

Since it follows from (3.23.2) that

$$\bigwedge^{m}(M_X^{\mathrm{gp}}/\mathcal{O}_X^*)=\bigwedge^{m}((M_X(D)^{\mathrm{gp}}/\mathcal{O}_X^*)\oplus L_{\mathbf{Z}})=\bigoplus_{l=0}^{m}(i_l)_*\Bigl(i_l^{-1}\Bigl(\bigwedge^{m-l} L_{\mathbf{Z}}\Bigr)\otimes\varepsilon^l\Bigr),$$

we obtain an isomorphism of the complexes

$$\bigoplus_{l=0}^{m}(i_l)_*\Bigl(i_l^{-1}\Bigl(\bigwedge^{m-l} L_{\mathbf{Z}}\Bigr)\otimes\varepsilon^l\otimes i_l^{-1}(\mathrm{Kos}_X(\mathcal{O}_X^*))[-m]\Bigr)\to \mathrm{Gr}_m^W\,\mathrm{Kos}_X(M_X)$$

for every m. Now we have a canonical morphism $i_l^{-1}(\mathrm{Kos}_X(\mathcal{O}_X^*)) \to \mathrm{Kos}_{\tilde{D}^l}(\mathcal{O}_{\tilde{D}^l}^*)$ by the functoriality of the Koszul complexes. This canonical morphism is a quasi-isomorphism because we have functorial quasi-isomorphisms $\boldsymbol{Q}_X \to \mathrm{Kos}_X(\mathcal{O}_X^*)$ and $\boldsymbol{Q}_{\tilde{D}^l} \to \mathrm{Kos}_{\tilde{D}^l}(\mathcal{O}_{\tilde{D}^l}^*)$. Thus we complete the proof. □

THEOREM 3.27. *In the situation* (3.4) *the data*

$$((\mathrm{Kos}_X(M_X), W), (\omega_X^{\cdot}, W, F), \psi_{(X,M_X)})$$

is a $\boldsymbol{Q}$-cohomological mixed Hodge complex if X is compact and Kähler.

PROOF. The filtration F on $\omega_X^{\cdot}$ induces the log Hodge to de Rham spectral sequence

$$E_1^{pq} = \mathrm{H}^q(X, \omega_X^p) \Rightarrow E^n = \mathrm{H}^n(X, \omega_X^{\cdot})$$

as usual. Then the E_1-terms are of finite dimension because X is compact. Therefore the cohomology $\mathrm{H}^n(X, \omega_X^{\cdot})$ is finite dimensional because the filtration F is finite. For every integer m we have isomorphisms (3.23.1) and (3.26.1) in the derived category. Corollary 3.24 states that the induced filtration F on $\mathrm{Gr}_m^W \omega_X^{\cdot}$ coincides with the filtration induced by $F[-m]$ on the left hand side in the identification (3.23.1). Moreover, we have a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{l=0}^{m}(i_l)_*\big(i_l^{-1}(\bigwedge^{m-l} L_{\boldsymbol{Z}}) \otimes \varepsilon^l \otimes \mathrm{Kos}_{\tilde{D}^l}(\mathcal{O}_{\tilde{D}^l}^*)[-m]\big) & \longrightarrow & \mathrm{Gr}_m^W \mathrm{Kos}_X(M_X) \\
{\scriptstyle \bigoplus \mathrm{id} \otimes (2\pi\sqrt{-1})^{-m} \psi_{(\tilde{D}^l, \mathcal{O}_{\tilde{D}^l}^*)}} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Gr}_m^W \psi_{(X,M_X)}} \\
\bigoplus_{l=0}^{m}(i_l)_*\big(i_l^{-1}(\bigwedge^{m-l} L_{\boldsymbol{Z}}) \otimes \varepsilon^l \otimes \Omega_{\tilde{D}^l}^{\cdot}[-m]\big) & \longrightarrow & \mathrm{Gr}_m^W \omega_X^{\cdot}
\end{array}$$

for every m. Thus we see that the data

$$(\mathrm{Gr}_m^W \mathrm{Kos}_X(M_X), (\mathrm{Gr}_m^W \omega_X^{\cdot}, F), \mathrm{Gr}_m^W \psi_{(X,M_X)})$$

is a $\boldsymbol{Q}$-cohomological Hodge complex of weight m by Proposition 3.21 and Lemma 3.18 as in Deligne [1]. □

(3.28) Now we assume that we are given global sections $t_1, \ldots, t_k$ of M_X. They give us the global sections $\mathrm{dlog}\, t_1, \ldots, \mathrm{dlog}\, t_k$ of ω_X^1. We have $\boldsymbol{C}$-sheaves $\boldsymbol{C}[u_1, \ldots, u_k] \otimes_{\boldsymbol{C}} \omega_X^p$, where $\boldsymbol{C}[u_1, \ldots, u_k]$ denotes the polynomial ring. A morphism of $\boldsymbol{C}$-sheaves

$$d : \boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^p \to \boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{p+1}$$

is given by the formula

$$d(f \otimes \omega) = f \otimes d\omega + (2\pi\sqrt{-1})^{-1} \sum_{i=1}^{k} \frac{\partial f}{\partial u_i} \otimes \mathrm{dlog}\, t_i \wedge \omega\,,$$

which turns out to satisfy the equality $d^2 = 0$ by easy computation. So we obtain a complex of $\boldsymbol{C}$-sheaves $\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}$ on X.

(3.29) The sections $t_1, \ldots, t_k$ regarded as global sections of M_X^{gp} define morphisms of complexes

$$t_i \wedge : \mathrm{Kos}_X(M_X) \to \mathrm{Kos}_X(M_X)[1]$$

for $i = 1, \ldots, k$ as in (1.11.1). It is easy to see that they satisfy the conditions

$$(t_i\wedge) \cdot (t_i\wedge) = 0\,, \tag{3.29.1}$$

$$(t_i\wedge) \cdot (t_j\wedge) = -(t_j\wedge) \cdot (t_i\wedge)\,. \tag{3.29.2}$$

We have $\boldsymbol{Q}$-sheaves $\boldsymbol{Q}[u_1, \ldots, u_k] \otimes_{\boldsymbol{Q}} \mathrm{Kos}_X(M_X)^p$ and a morphism of $\boldsymbol{Q}$-sheaves

$$d : \boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)^p \to \boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)^{p+1}$$

by

$$d(f \otimes x) = f \otimes dx + \sum_{i=1}^{k} \frac{\partial f}{\partial u_i} \otimes t_i \wedge x\,.$$

The equality $d^2 = 0$ holds by (3.29.1) and (3.29.2). Thus we obtain a complex of $\boldsymbol{Q}$-sheaves $\boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)$ on X.

(3.30) The morphism ψ_X in (2.4.2) induces a morphism of $\boldsymbol{Q}$-sheaves

$$\mathrm{id} \otimes \psi_X : \boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)^p \to \boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^p$$

for every p. Easy computation shows that the morphism $\mathrm{id} \otimes \psi_X$ is a morphism of complexes.

(3.31) On the complexes $\boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)$ and $\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}$ we define increasing filtrations L by

$$L_m(\boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)) = \bigoplus_{q \in \boldsymbol{N}^k} u^q \otimes W_{m-2|q|} \mathrm{Kos}_X(M_X)\,,$$

$$L_m(\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}) = \bigoplus_{q \in \boldsymbol{N}^k} u^q \otimes W_{m-2|q|}\omega_X^{\cdot}$$

for every m., where $u^q = u_1^{q_1} \cdots u_k^{q_k}$ and $|q| = q_1 + \cdots + q_k$ for $q = (q_1, \ldots, q_k) \in \boldsymbol{N}^k$. The facts

$$(t_i\wedge)(W_m \mathrm{Kos}_X(M_X)) \subset W_{m+1} \mathrm{Kos}_X(M_X)[1]\,,$$

$$\mathrm{dlog}\, t_i \wedge W_m\omega_X^{\cdot} \subset W_{m+1}\omega_X^{\cdot}[1]$$

for $i = 1, \ldots, k$ imply that they are subcomplexes of $\boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_X(M_X)$ and of $\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}$.

On $\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}$ we define a decreasing filtration F by

$$F^n(\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}) = \bigoplus_{q \in \boldsymbol{N}^k} u^q \otimes F^{n-|q|}\omega_X^{\cdot}$$

for every n. This defines a subcomplex of $\boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega_X^{\cdot}$ by the facts

$$\mathrm{dlog}\, t_i \wedge F^n\omega_X^{\cdot} \subset F^{n+1}\omega_X^{\cdot}[1]$$

for $i = 1, \ldots, k$.

REMARK 3.32. We have the equalities

$$L_{-1}(\boldsymbol{Q}[u_1,\dots,u_k]\otimes \mathrm{Kos}_X(M_X))=0\,,$$
$$L_{-1}(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot})=0$$

trivially. We remark that L is not a finite filtration. However, it satisfies the conditions

$$\bigcup_m L_m(\boldsymbol{Q}[u_1,\dots,u_k]\otimes \mathrm{Kos}_X(M_X))=\boldsymbol{Q}[u_1,\dots,u_k]\otimes \mathrm{Kos}_X(M_X)$$

and

$$\bigcup_m L_m(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot})=\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot}\,.$$

Similarly, we have

$$F^0(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot})=\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot}\,,$$

but F is not a finite filtration. We have the equality

$$\bigcap_p F^p(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot})=0\,.$$

Since the morphism ψ_X preserves the weight filtrations W on $\mathrm{Kos}_X(M_X)$ and on $\omega_X^{\cdot}$, the morphism $\mathrm{id}\otimes\psi_X$ preserves the filtrations L on $\boldsymbol{Q}[u_1,\dots,u_k]\otimes \mathrm{Kos}_X(M_X)$ and on $\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot}$.

THEOREM 3.33. *In the situation above*, *the data*

$$((\boldsymbol{Q}[u_1,\dots,u_k]\otimes \mathrm{Kos}_X(M_X),L),(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot},L,F),\mathrm{id}\otimes\psi_X)$$

is a weak $\boldsymbol{Q}$*-cohomological mixed Hodge complex on* X, *if* X *is compact and Kähler* (*for the definition of a weak* $\boldsymbol{Q}$*-cohomological mixed Hodge complex*, *see Definition* A.4 *in Appendix*).

PROOF. We have equalities

$$\mathrm{Gr}^L_m\,\boldsymbol{Q}[u_1,\dots,u_k]\otimes \mathrm{Kos}_X(M_X)=\bigoplus_{q\in\boldsymbol{N}^k}\mathrm{Gr}^W_{m-2|q|}\,\mathrm{Kos}_X(M_X)\,,$$
$$\mathrm{Gr}^L_m(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot})=\bigoplus_{q\in\boldsymbol{N}^k}\mathrm{Gr}^W_{m-2|q|}\,\omega_X^{\cdot}\,,$$
$$F\,\mathrm{Gr}^L_m(\boldsymbol{C}[u_1,\dots,u_k]\otimes \omega_X^{\cdot})=\bigoplus_{q\in\boldsymbol{N}^k}F[-|q|]\,\mathrm{Gr}^W_{m-2|q|}\,\omega_X^{\cdot}\,,$$
$$\mathrm{Gr}^L_m(\mathrm{id}\otimes\psi_X)=\bigoplus_{q\in\boldsymbol{N}^k}\mathrm{Gr}^W_{m-2|q|}\,\psi_X$$

by easy computation. From Theorem 3.27 the data

$$(\mathrm{Gr}^W_{m-2|q|}\,\mathrm{Kos}_X(M_X),(\mathrm{Gr}^W_{m-2|q|}\,\omega_X^{\cdot},F[-|q|]\,\mathrm{Gr}^W_{m-2|q|}\,\omega_X^{\cdot}),\mathrm{Gr}^W_{m-2|q|}\,\psi_X)$$

is a $\boldsymbol{Q}$-cohomological Hodge complex of weight $m-2|q|-2(-|q|)=m$ on X. □

4. Log smooth degeneration and its reduction.

(4.1) For a positive integer k, a pre-log structure $\boldsymbol{N}^k \to \boldsymbol{C}$ over the point $(\operatorname{Spec} \boldsymbol{C})_{\mathrm{an}}$ is given by sending $0 \in \boldsymbol{N}^k$ to $1 \in \boldsymbol{C}$ and sending $x \in \boldsymbol{N}^k$, $x \neq 0$, to $0 \in \boldsymbol{C}$. The log structure induced from the pre-log structure above is nothing but $((\operatorname{Spec} \boldsymbol{C})_{\mathrm{an}}, \boldsymbol{C}^* \oplus \boldsymbol{N}^k)$. The point with this log structure is called the $\boldsymbol{N}^k$-log point and denoted simply by $(*, \boldsymbol{N}^k)$ or $*^k$. The $\boldsymbol{N}$-log point $(*, \boldsymbol{N})$ is called the standard log point in Steenbrink [19]. For the module of "log differential forms" $\omega^1_{*^k}$ on $*^k$, we have $\omega^1_{*^k} = \boldsymbol{C}^k$.

Let E be the divisor on $\boldsymbol{C}^k$ which is the sum of all the coordinate hyperplanes. Moreover, we denote by $(\boldsymbol{C}^k, E)$ the complex manifold $\boldsymbol{C}^k$ equipped with the log structure associated to E. Then the point $0 \in \boldsymbol{C}^k$ equipped with the pull-back of the log structure above is nothing but the $\boldsymbol{N}^k$-log point. Thus we have a canonical strict closed immersion

$$(*, \boldsymbol{N}^k) \to (\boldsymbol{C}^k, E) \tag{4.1.1}$$

by definition.

DEFINITION 4.2. Let $f : U \to \boldsymbol{C}^k$ be a morphism of complex manifolds. We assume that the effective divisor $D = f^*E$ on U is well-defined. Then we have a log structure M_U associated to the effective divisor D, and obtain a morphism of log complex analytic spaces $f : (U, M_U) \to (\boldsymbol{C}^k, E)$. In this article, the morphism of log complex analytic spaces f is called a *log degeneration* over $\boldsymbol{C}^k$, if the divisor D_{red} is a simple normal crossing divisor on U. If the morphism f is log smooth, in addition, it is called a *log smooth degeneration* over $\boldsymbol{C}^k$.

DEFINITION 4.3. Let Y be an fs log complex analytic space, and $g : Y \to *^k$ a morphism of log complex analytic spaces. The morphism g (or simply Y) is said to be a log degeneration over the $\boldsymbol{N}^k$-log point $*^k$ if for any point $x \in Y$ there exist an open neighborhood V of x in Y, a log degeneration $f : (U, M_U) \to (\boldsymbol{C}^k, E)$ and a morphism of log complex analytic spaces $V \to U$ such that the diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ {\scriptstyle g|_V}\downarrow & & \downarrow{\scriptstyle f} \\ *^k & \longrightarrow & \boldsymbol{C}^k, \end{array}$$

is Cartesian in the category of log complex analytic spaces, where the bottom arrow is the canonical morphism (4.1.1). If we can take a log smooth degeneration $f : (U, M_U) \to (\boldsymbol{C}^k, E)$ in the definition above, Y is called a log smooth degeneration over $*^k$. If Y is compact, in addition, we call Y a proper log (smooth) degeneration over $*^k$. Moreover, the complex analytic space $X = Y_{\mathrm{red}}$ with the pull-back log structure from Y is called the reduction of the log (smooth) degeneration Y.

REMARK 4.4. If Y is a log smooth degeneration over $*^k$, then Y is log smooth over $*^k$ by definition.

EXAMPLE 4.5. Let $f : \mathfrak{X} \to \Delta$ be a surjective morphism from a complex manifold $\mathfrak{X}$ to the unit disc Δ, such that f is smooth outside the origin $\{0\}$ in Δ and that the divisor $f^{-1}(0)$ is a normal crossing divisor. Then the fiber $Y = f^{-1}(0)$ as a complex space is a log smooth degeneration over the $\boldsymbol{N}$-log point $* = \{0\}$.

EXAMPLE 4.6. A log deformation defined by Steenbrink [19] is a log smooth degeneration over the $\boldsymbol{N}$-log point.

EXAMPLE 4.7. Let $f : \mathfrak{X} \to \Delta^k$ be a surjective morphism from a complex manifold $\mathfrak{X}$ to the k-dimensional polydisc Δ^k. We use the same symbols E_i and E as in (4.1) for the coordinate hyperplanes and the divisor which is the sum of the coordinate hyperplanes. Then the fiber $Y = f^{-1}(0)$ is a log degeneration over the $\boldsymbol{N}^k$-log point if the divisor $(f^*E)_{\mathrm{red}}$ is a normal crossing divisor on $\mathfrak{X}$. If f is log smooth in addition, then Y is a log smooth degeneration. For the case where the morphism f is of generalized semi-stable type in [5], Y is a *reduced* log smooth degeneration.

EXAMPLE 4.8. Let $f : \mathfrak{X} \to S$ be a morphism of complex manifolds. If the morphism f is a weak semistable reduction in the sense of Kawamata [14], every fiber of this morphism turns out to be a log smooth degeneration under suitable log structures on the fiber and the base point.

EXAMPLE 4.9. We give a simple and explicit example which shows that a log degeneration is not necessarily equi-dimensional. We consider $\boldsymbol{C}^3$ with the coordinate (x, y, z) and $\boldsymbol{C}^2$ with the coordinate (s, t). We consider the log structures on $\boldsymbol{C}^3$ and $\boldsymbol{C}^2$ associated to the simple normal crossing divisors $xyz = 0$ and $st = 0$, respectively. The morphism $f : \boldsymbol{C}^3 \to \boldsymbol{C}^2$ given by $f(x, y, z) = (xy, xz)$ is a log smooth degeneration over $\boldsymbol{C}^2$. Then the fiber $f^{-1}(0)$ with the induced log structure is a log smooth degeneration over $*^2$, which is not equi-dimensional.

EXAMPLE 4.10. Let $f : U \to \boldsymbol{C}^k$ be a log degeneration over $\boldsymbol{C}^k$ as in Definition 4.2 and $\pi : \tilde{U} \to U$ a log blow-up (for the definition see [9, Definition (6.1.1)]) whose underlying morphism is a blow-up along an intersection of some irreducible components of $f^{-1}(0)$. Then we can see that the composite $\pi f : \tilde{U} \to \boldsymbol{C}^k$ is a log degeneration again. If we assume that f is a log smooth degeneration, then πf is also a log smooth degeneration. Therefore the fiber $(\pi f)^{-1}(0)$ turns out to be a log (smooth) degeneration over the $\boldsymbol{N}^k$-log point. Here we remark that $(\pi f)^{-1}(0)$ is not equi-dimensional for $k \geq 2$ because it contains the exceptional divisor.

DEFINITION 4.11. Let X be the reduction of a log degeneration over $*^k$. We denote the irreducible components of the reduced complex analytic space X by $\{X_\lambda\}_{\lambda \in \Lambda}$. We set

$$X[\Gamma] = \bigcap_{\lambda \in \Gamma} X_\lambda$$

for any subset Γ of Λ (for $\Gamma = \emptyset$, $X[\emptyset] = X$). We give an fs log structure $M_{X[\Gamma]}$ on $X[\Gamma]$ by pulling back the log structure of X.

REMARK 4.12. For a non-empty subset Γ, $X[\Gamma]$ above is a complex manifold satisfying the condition (3.4), if the irreducible component X_λ is nonsingular for every λ.

LEMMA 4.13. *For the reduction X of a log degeneration Y over $*^k$, $\omega^p_{X[\Gamma]}$ is locally free of finite rank for every p and for every $\Gamma \subset \Lambda$. Moreover, $\omega^p_{X[\Gamma]/*^k}$ is also locally free, if Y is log smooth degeneration over $*^k$.*

PROOF. By Kato and Nakayama [12, Lemma (3.6)(2)], the canonical morphism $\mathcal{O}_{X[\Gamma]} \otimes \omega^1_Y \to \omega^1_{X[\Gamma]}$ turns out to be an isomorphism. Also we have the canonical isomorphism $\mathcal{O}_{X[\Gamma]} \otimes \omega^1_{Y/*^k} \xrightarrow{\simeq} \omega^1_{X[\Gamma]/*^k}$ by similar arguments. Then we have isomorphisms

$$\mathcal{O}_{X[\Gamma]} \otimes \omega^p_Y \xrightarrow{\simeq} \omega^p_{X[\Gamma]}, \quad \mathcal{O}_{X[\Gamma]} \otimes \omega^p_{Y/*^k} \xrightarrow{\simeq} \omega^p_{X[\Gamma]/*^k} \tag{4.13.1}$$

for every p. Thus we can easily see the conclusions. □

(4.14) For the reduction X of a log degeneration Y, we set

$$X_n = \coprod_{|\Gamma|=n+1} X[\Gamma] \tag{4.14.1}$$

for any non-negative integer n. Once we fix a total order on the finite set Λ in addition, we obtain a simplicial "resolution"

$$\cdots X_n \;\begin{smallmatrix}\to\\ \to\\ \cdots\\ \to\\ \to\end{smallmatrix}\; X_{n-1} \;\begin{smallmatrix}\to\\ \to\\ \cdots\\ \to\end{smallmatrix}\; \cdots \;\begin{smallmatrix}\to\\ \to\\ \to\end{smallmatrix}\; X_1 \;\begin{smallmatrix}\to\\ \to\end{smallmatrix}\; X_0 \to X \tag{4.14.2}$$

as usual. We denote by a_n the natural morphism $X_n \to X$.

(4.15) From the simplicial resolution above, we have a sequence of the morphisms of sheaves

$$0 \to \mathcal{O}_X \xrightarrow{\delta} (a_0)_*\mathcal{O}_{X_0} \xrightarrow{\delta} (a_1)_*\mathcal{O}_{X_1} \xrightarrow{\delta} (a_2)_*\mathcal{O}_{X_2} \xrightarrow{\delta} \cdots \tag{4.15.1}$$

by using the "Čech type" morphism δ as usual.

PROPOSITION 4.16. *In the situation above, we assume that the irreducible component X_λ of X is nonsingular for every λ. Then the sequence* (4.15.1) *is an exact sequence.*

PROOF. We may check the exactness locally because of the smoothness condition on the irreducible components of X. So, we may assume that Y is given by a Cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & U \\ \downarrow & & \downarrow f \\ *^k & \longrightarrow & \boldsymbol{C}^k, \end{array}$$

where $f : U \longrightarrow \boldsymbol{C}^k$ is a log degeneration over $\boldsymbol{C}^k$. Moreover, we may assume that U is an open neighborhood of the origin 0 in $\boldsymbol{C}^n$ with the coordinate functions $x_1, \ldots, x_n$ and that the morphism f is given by

$$f^*t_j = g_j \prod_{i=1}^{n} x_i^{a_{ij}} \quad j = 1, \dots, k, \tag{4.16.1}$$

where g_j's are invertible holomorphic functions over U and a_{ij}'s are non-negative integers. In this situation, we look at the stalk at the origin of the sequence (4.15.1). We set $A = \boldsymbol{C}\{x_1, \dots, x_n\}$ and denote by $\mathcal{I}$ the defining ideal of X in A. Then we can easily see that the irreducible decomposition of X at the origin is given by

$$\mathcal{I} = \mathcal{I}_1 \cap \dots \cap \mathcal{I}_N, \tag{4.16.2}$$

where $\mathcal{I}_\alpha$ is generated by some of coordinate functions x_λ in the ring $\boldsymbol{C}\{x_1, \dots, x_n\}$. So, what we have to prove is the exactness of the sequence

$$0 \to \mathcal{I} \to A \xrightarrow{\delta} \textstyle\bigoplus_{\alpha=1}^{N} A/\mathcal{I}_\alpha \xrightarrow{\delta} \bigoplus_{1 \le \alpha_0 < \alpha_1 \le N} A/(\mathcal{I}_{\alpha_0} + \mathcal{I}_{\alpha_1}) \xrightarrow{\delta} \cdots,$$

where the morphism

$$\delta : \bigoplus_{1 \le \alpha_0 < \dots < \alpha_j \le N} A/(\mathcal{I}_{\alpha_0} + \dots + \mathcal{I}_{\alpha_j}) \to \bigoplus_{1 \le \alpha_0 < \dots < \alpha_{j+1} \le N} A/(\mathcal{I}_{\alpha_0} + \dots + \mathcal{I}_{\alpha_{j+1}})$$

sends an element $f = (f_{\alpha_0 \cdots \alpha_j})$ to the element $g = (g_{\alpha_0 \cdots \alpha_{j+1}})$ with

$$g_{\alpha_0 \cdots \alpha_{j+1}} = \sum_{l=0}^{j+1} (-1)^{l+1} f_{\alpha_0 \cdots \hat{\alpha}_l \cdots \alpha_{j+1}}$$

as usual. Since the exactness of the sequence

$$0 \to \mathcal{I} \xrightarrow{\delta} A \xrightarrow{\delta} \bigoplus_{\alpha=1}^{N} A/\mathcal{I}_\alpha$$

is trivial by (4.16.2), we have to prove the exactness of the sequence

$$A \xrightarrow{\delta} \bigoplus_{\alpha=1}^{N} A/\mathcal{I}_\alpha \xrightarrow{\delta} \bigoplus_{1 \le \alpha_1 < \alpha_2 \le N} A/(\mathcal{I}_{\alpha_1} + \mathcal{I}_{\alpha_2}) \xrightarrow{\delta} \cdots. \tag{4.16.3}$$

The completion $\hat{A} = \boldsymbol{C}[[x_1, \dots, x_n]]$ is faithfully flat over $A = \boldsymbol{C}\{x_1, \dots, x_n\}$ (see, for instance, Matsumura [16, Theorem 8.14]). Therefore we can check the exactness after tensoring $\hat{A}$. Since every ideal $\mathcal{I}_\alpha$ and the differential d are homogeneous with respect to the $\boldsymbol{N}^n$ grading, we can check the exactness degree by degree. So we fix a degree $q = (q_1, \dots, q_n) \in \boldsymbol{N}^n$. If $x_1^{q_1} \cdots x_n^{q_n}$ is contained in the ideal $\mathcal{I}_{\alpha_0} + \dots + \mathcal{I}_{\alpha_j}$, then $x_1^{q_1} \cdots x_n^{q_n}$ is an element of $\mathcal{I}_{\alpha_l}$ for some l, because every $\mathcal{I}_\alpha$ is generated by some of x_i's. Therefore we may exclude all the ideals $\mathcal{I}_{\alpha_l}$ containing $x_1^{q_1} \cdots x_n^{q_n}$ at first, and identify the degree q part of the sequence (4.16.3) with the sequence of $\boldsymbol{C}$-vector spaces

$$\boldsymbol{C} \xrightarrow{\delta} \boldsymbol{C}^{N'} \xrightarrow{\delta} \bigwedge^2 (\boldsymbol{C}^{N'}) \xrightarrow{\delta} \bigwedge^3 (\boldsymbol{C}^{N'}) \cdots$$

for some non-negative integer N'. This complex is nothing but the complex $\mathrm{Kos}(\Delta; N')$ for the diagonal morphism $\Delta : \boldsymbol{C} \to \boldsymbol{C}^{N'}$. Then we obtain the exactness of the sequence above by using Proposition 1.4. □

COROLLARY 4.17. *If X is the reduction of a log smooth degeneration with the smooth irreducible components, we have exact sequences*

$$0 \to \omega^p_{X/*^k} \xrightarrow{\delta} (a_0)_*\omega^p_{X_0/*^k} \xrightarrow{\delta} (a_1)_*\omega^p_{X_1/*^k} \xrightarrow{\delta} (a_2)_*\omega^p_{X_2/*^k} \xrightarrow{\delta} \cdots$$

for all p by the "Čech type" morphisms δ.

PROOF. The isomorphisms in (4.13.1) and Proposition 4.16 imply the conclusion. □

(4.18) In the situation above, the single complex associated to the double complex

$$0 \to (a_0)_*\omega^{\cdot}_{X_0/*^k} \xrightarrow{\delta} (a_1)_*\omega^{\cdot}_{X_1/*^k} \xrightarrow{\delta} (a_2)_*\omega^{\cdot}_{X_2/*^k} \xrightarrow{\delta} \cdots$$

is denoted by $(a_{\cdot})_*\omega^{\cdot}_{X_{\cdot}/*^k}$. The stupid filtrations (filtrations bêtes) in Deligne [1] on $\omega^{\cdot}_{X/*^k}$ and $\omega^{\cdot}_{X_n/*^k}$ are denoted by F as usual. Moreover, subcomplexes $F^p(a_{\cdot})_*\omega^{\cdot}_{X_{\cdot}/*^k}$ which are the associated single complexes to

$$0 \to (a_0)_*F^p\omega^{\cdot}_{X_0/*^k} \xrightarrow{\delta} (a_1)_*F^p\omega^{\cdot}_{X_1/*^k} \xrightarrow{\delta} (a_2)_*F^p\omega^{\cdot}_{X_2/*^k} \xrightarrow{\delta} \cdots$$

define a decreasing filtration F on $(a_{\cdot})_*\omega^{\cdot}_{X_{\cdot}/*^k}$.

PROPOSITION 4.19. *For the reduction X of a log smooth degeneration over* $*^k$ *the "Čech type" morphism*

$$\omega^{\cdot}_{X/*^k} \to (a_{\cdot})_*\omega^{\cdot}_{X_{\cdot}/*^k} \tag{4.19.1}$$

is a filtered quasi-isomorphism with respect to the filtration F, if all the irreducible components of X are nonsingular.

PROOF. We have $\mathrm{Gr}^p_F\,\omega^{\cdot}_{X/*^k} = \omega^p_{X/*^k}[-p]$. On the other hand, $\mathrm{Gr}^p_F(a_{\cdot})_*\omega^{\cdot}_{X_{\cdot}/*^k}$ is the complex obtained by shifting the complex

$$0 \to (a_0)_*\omega^p_{X_0/*^k} \xrightarrow{\delta} (a_1)_*\omega^p_{X_1/*^k} \xrightarrow{\delta} (a_2)_*\omega^p_{X_2/*^k} \xrightarrow{\delta} \cdots$$

by p to the right. Thus we obtain the conclusion by Corollary 4.17. □

5. Mixed Hodge structures on the relative log de Rham cohomologies.

(5.1) Let X be the reduction of a *proper* log smooth degeneration over $*^k$. Moreover, we assume that all the irreducible components of X are nonsingular and Kähler in addition. In the last section we obtain a resolution of $\omega^{\cdot}_{X/*^k}$ (4.19.1). In this section we construct mixed Hodge structures on the cohomology groups $\mathrm{H}^n(X, \omega^{\cdot}_{X/*^k})$ by using the resolution above.

By the additional assumption, X_n in (4.14.1) is a compact Kähler log complex manifold satisfying Condition (3.4). We denote by $f_n : X_n \to *^k$ the composite of the morphisms

$a_n : X_n \to X$ and $X \to *^k$, which yields global sections $t_1, \dots, t_k$ of M_{X_n}. Therefore we have a weak $\boldsymbol{Q}$-cohomological mixed Hodge complex

$$((\boldsymbol{Q}[u_1, \dots, u_k] \otimes \mathrm{Kos}_{X_n}(M_{X_n}), L), (\boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}, L, F), \mathrm{id} \otimes \psi_{X_n})$$

on X_n by Theorem 3.33.

(5.2) A morphism of $\boldsymbol{C}$-sheaves $\boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^p_{X_n} \to \omega^p_{X_n}$ sending $f \otimes \omega$ to $f(0)\omega$ induces a morphism of complexes

$$\theta_{X_n/*^k} : \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n} \to \omega^{\cdot}_{X_n/*^k}$$

because $\mathrm{dlog}\, t_i = 0$ in $\omega^1_{X_n/*^k}$ for every i. We can easily see that this morphism preserves the filtrations F on both sides.

PROPOSITION 5.3. *In the situation above, the morphism $\theta_{X_n/*^k}$ induces a quasi-isomorphism*

$$\mathrm{Gr}^p_F\, \theta_{X_n/*^k} : \mathrm{Gr}^p_F\, \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n} \to \mathrm{Gr}^p_F\, \omega^{\cdot}_{X_n/*^k}$$

for every integer p.

PROOF. From the equality $\omega^1_{*^k} = \boldsymbol{C}^k$, we obtain the morphism of $\mathcal{O}_{X_n}$-sheaves

$$f_n^* \omega^1_{*^k} = \mathcal{O}^{\oplus k}_{X_n} \to \omega^1_{X_n},$$

which is denoted by f_n^* for a while. By Lemma 4.13 this morphism fits in an exact sequence

$$0 \to \mathcal{O}^{\oplus k}_{X_n} \xrightarrow{f_n^*} \omega^1_{X_n} \to \omega^1_{X_n/*^k} \to 0$$

and the cokernel $\omega^1_{X_n/*^k}$ is locally free $\mathcal{O}_{X_n}$-module. We have

$$\mathrm{Gr}^p_F\, \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^l_{X_n} = \bigoplus_{q \in \boldsymbol{N}^k} \mathrm{Gr}^{p-|q|}_F\, \omega^l_{X_n} = \bigoplus_{\substack{q \in \boldsymbol{N}^k \\ |q| = p-l}} \omega^l_{X_n} = \mathrm{Sym}^{p-l}(\mathcal{O}^{\oplus k}_{X_n}) \otimes \bigwedge^l \omega^1_{X_n}$$

for every l and p. By the equality above together with easy computation on the differentials the complex $\mathrm{Gr}^p_F\, \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}$ can be identified with the Koszul complex $\mathrm{Kos}(f_n^*; p)$. On the other hand, the equality

$$\mathrm{Gr}^p_F\, \omega^{\cdot}_{X_n/*^k} = \omega^p_{X_n/*^k}[-p] = \bigwedge^p \omega^1_{X_n/*^k}[-p]$$

holds trivially. Then we have the conclusion by Proposition 1.4. □

(5.4) Since the simplicial resolution (4.14.2) is compatible with the morphisms $f_n : X_n \to *^k$ for all n, the "Čech type" morphisms

$$\delta : (a_n)_* \boldsymbol{Q}[u_1, \dots, u_k] \otimes \mathrm{Kos}_{X_n}(M_{X_n}) \to (a_{n+1})_* \boldsymbol{Q}[u_1, \dots, u_k] \otimes \mathrm{Kos}_{X_{n+1}}(M_{X_{n+1}}),$$
$$\delta : (a_n)_* \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n} \to (a_{n+1})_* \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_{n+1}}$$

are morphisms of complexes which preserves the filtrations L and F involved. Thus we obtain double complexes

$$((a_n)_* \boldsymbol{Q}[u_1, \dots, u_k] \otimes \mathrm{Kos}_{X_n}(M_{X_n}), \delta), \quad ((a_n)_* \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}, \delta)$$

and the associated single complexes which we denote by $K_{\boldsymbol{Q}}$ and $K_{\boldsymbol{C}}$. We define subcomplexes $W_m K_{\boldsymbol{Q}}$ and $W_m K_{\boldsymbol{C}}$ by the associated single complexes to the double complexes

$$((a_n)_* L_{m+n}(\boldsymbol{Q}[u_1, \dots, u_k] \otimes \mathrm{Kos}_{X_n}(M_{X_n})), \delta),$$
$$((a_n)_* L_{m+n}(\boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}), \delta)$$

for every m. Moreover, the associated single complex to the double complex

$$((a_n)_* F^p(\boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}), \delta)$$

defines a subcomplex $F^p K_{\boldsymbol{C}}$ for every p. Thus we obtain filtered complex of $\boldsymbol{Q}$-sheaves $(K_{\boldsymbol{Q}}, W)$ and bifiltered complex of $\boldsymbol{C}$-sheaves $(K_{\boldsymbol{C}}, W, F)$ on X.

Here we remark that the filtrations above are not finite. However, we can easily see the following conditions:

(5.4.1) For sufficiently small m, $W_m K_{\boldsymbol{Q}} = 0$ and $W_m K_{\boldsymbol{C}} = 0$.

(5.4.2) The equalities $\bigcup_m W_m K_{\boldsymbol{Q}} = K_{\boldsymbol{Q}}$ and $\bigcup_m W_m K_{\boldsymbol{C}} = K_{\boldsymbol{C}}$ hold.

(5.4.3) We have $F^0 K_{\boldsymbol{C}} = K_{\boldsymbol{C}}$ and $\bigcap_p F^p K_{\boldsymbol{C}} = 0$.

(5.5) Trivially the morphisms of complexes

$$\mathrm{id} \otimes \psi_{X_n} : \boldsymbol{Q}[u_1, \dots, u_k] \otimes \mathrm{Kos}_{X_n}(M_{X_n}) \to \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}$$

and

$$\theta_{X_n/*^k} : \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n} \to \omega^{\cdot}_{X_n/*^k}$$

are compatible with the morphism δ for every n. Thus we obtain morphisms of complexes preserving weights and Hodge filtartions, which are denoted by

$$\psi : K_{\boldsymbol{Q}} \to K_{\boldsymbol{C}},$$
$$\theta_{X_{\cdot}/*^k} : K_{\boldsymbol{C}} \to (a_{\cdot})_* \omega^{\cdot}_{X_{\cdot}/*^k},$$

respectively.

LEMMA 5.6. *The morphism $\theta_{X_{\cdot}/*^k}$ induces a quasi-isomorphism*

$$\mathrm{Gr}_F^p(\theta_{X_{\cdot}/*^k}) : \mathrm{Gr}_F^p K_{\boldsymbol{C}} \to \mathrm{Gr}_F^p(a_{\cdot})_* \omega^{\cdot}_{X_{\cdot}/*^k}$$

for every p.

PROOF. The complexes $\mathrm{Gr}_F^p K_{\boldsymbol{C}}$ and $\mathrm{Gr}_F^p(a_{\cdot})_* \omega^{\cdot}_{X_{\cdot}/*^k}$ are the associated single complexes to $((a_n)_* \mathrm{Gr}_F^p \boldsymbol{C}[u_1, \dots, u_k] \otimes \omega^{\cdot}_{X_n}, \delta)$ and $((a_n)_* \mathrm{Gr}_F^p \omega^{\cdot}_{X_n/*^k}, \delta)$, respectively. Thus we obtain the conclusion by Proposition 5.3. □

COROLLARY 5.7. *In the derived category we have a morphism*

$$K_{\boldsymbol{C}} \to \omega^{\cdot}_{X/*^k}$$

preserving the filtrations F on both sides, which induces an isomorphism

$$\mathrm{Gr}_F^p K_{\boldsymbol{C}} \to \mathrm{Gr}_F^p \omega^{\cdot}_{X/*^k} = \omega^p_{X/*^k}$$

in the derived category for every p.

PROOF. By the lemma above together with Proposition 4.19. □

PROPOSITION 5.8. *In the situation above, the data $((K_{\boldsymbol{Q}}, W), (K_{\boldsymbol{C}}, W, F), \psi)$ is a weak $\boldsymbol{Q}$-cohomological mixed Hodge complex on X (see Definition A.4 in Appendix).*

PROOF. For every m, we have

$$\mathrm{Gr}_m^W K_{\boldsymbol{Q}} = \bigoplus_{n \geq 0} \mathrm{Gr}_{m+n}^L \boldsymbol{Q}[u_1, \ldots, u_k] \otimes \mathrm{Kos}_{X_n}(M_{X_n})[-n]\,,$$

$$\mathrm{Gr}_m^W K_{\boldsymbol{C}} = \bigoplus_{n \geq 0} \mathrm{Gr}_{m+n}^L \boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega^{\cdot}_{X_n}[-n]\,,$$

$$\mathrm{Gr}_m^W \psi = \bigoplus_{n \geq 0} \mathrm{Gr}_{m+n}^L (\mathrm{id} \otimes \psi_{X_n})[-n]$$

by easy computation. Moreover, the filtration F induces

$$F\,\mathrm{Gr}_m^W K_{\boldsymbol{C}} = \bigoplus_{n \geq 0} F(\mathrm{Gr}_{m+n}^L \boldsymbol{C}[u_1, \ldots, u_k] \otimes \omega^{\cdot}_{X_n})[-n]$$

for every m. Therefore the data $(\mathrm{Gr}_m^W K_{\boldsymbol{Q}}, (\mathrm{Gr}_m^W K_{\boldsymbol{C}}, F), \mathrm{Gr}_m^W \psi)$ is a $\boldsymbol{Q}$-cohomological Hodge complex of weight $m + n + (-n) = m$ by Theorem 3.33. □

THEOREM 5.9. *Let X be the reduction of a proper log smooth degeneration over the $\boldsymbol{N}^k$-log point $*^k$. Assume that all the irreducible components of X are Kähler complex manifolds in addition. Then we have the following:*

(5.9.1) *For every integer n, the cohomology group $\mathrm{H}^n(X, \omega^{\cdot}_{X/*^k})$ underlies a $\boldsymbol{Q}$-mixed Hodge structure, whose Hodge filtration is induced by the stupid filtration on $\omega^{\cdot}_{X/*^k}$.*

(5.9.2) *The log Hodge to de Rham spectral sequence*

$$E_1^{pq} = \mathrm{H}^q(X, \omega^p_{X/*^k}) \Rightarrow E^n = \mathrm{H}^n(X, \omega^{\cdot}_{X/*^k}) \tag{5.9.3}$$

degenerate at E_1-terms.

PROOF. Corollary 5.7 tells us that there exists a positive integer N such that $\mathrm{Gr}_F^p K_{\boldsymbol{C}}$ is acyclic for all $p \geq N$. The exact sequence in (A.6.4) implies that $F^p\mathrm{H}^n(X, K_{\boldsymbol{C}}) = F^N\mathrm{H}^n(X, K_{\boldsymbol{C}})$ for $p \geq N$. Since the filtration F on $\mathrm{H}^n(X, K_{\boldsymbol{C}})$ induces a finite filtration on $\mathrm{Gr}_m^W \mathrm{H}^n(X, K_{\boldsymbol{C}})$, we have $F^N \mathrm{Gr}_m^W \mathrm{H}^n(X, K_{\boldsymbol{C}}) = 0$ for every integer m. Then we have the inclusions

$$F^N \cap W_m \subset F^N \cap W_{m-1} \subset \cdots$$

on $\mathrm{H}^n(X, K_{\boldsymbol{C}})$. Then we have the equality $F^N \cap W_m = 0$ because $W_l\mathrm{H}^n(X, K_{\boldsymbol{C}}) = 0$ for sufficiently small l. Therefore we have $F^N = 0$ by using the condition $\bigcup_m W_m\mathrm{H}^n(X, K_{\boldsymbol{C}}) =$

$\mathrm{H}^n(X, K_{\boldsymbol{C}})$. Thus the filtration F on $\mathrm{H}^n(X, K_{\boldsymbol{C}})$ is finite. Moreover, the cohomology $\mathrm{H}^n(X, K_{\boldsymbol{C}})$ is of finite dimension, because we have

$$\mathrm{Gr}_F^p \mathrm{H}^n(X, K_{\boldsymbol{C}}) \simeq \mathrm{H}^n(X, \mathrm{Gr}_F^p K_{\boldsymbol{C}}) \simeq \mathrm{H}^{n-p}(X, \omega^p_{X/*^k})$$

for every p. Then the filtration W is finite and the data

$$((\mathrm{H}^n(X, K_{\boldsymbol{Q}}), W[n]), \quad (\mathrm{H}^n(X, K_{\boldsymbol{C}}), W[n], F)) \tag{5.9.4}$$

turns out to be a $\boldsymbol{Q}$-mixed Hodge structure by (A.6.1) in Lemma A.6. Now we have the identification

$$\mathrm{H}^n(X, K_{\boldsymbol{C}}) = \mathrm{H}^n(X, K_{\boldsymbol{C}})/F^N\mathrm{H}^n(X, K_{\boldsymbol{C}}) \simeq \mathrm{H}^n(X, K_{\boldsymbol{C}}/F^N K_{\boldsymbol{C}}) \simeq \mathrm{H}^n(X, \omega^{\cdot}_{X/*^k})$$

by $F^N\mathrm{H}^n(X, K_{\boldsymbol{C}}) = 0$ as well as by the fact that the morphism $K_{\boldsymbol{C}} \to \omega^{\cdot}_{X/*^k}$ induces an isomorphism $K_{\boldsymbol{C}}/F^N K_{\boldsymbol{C}} \xrightarrow{\simeq} \omega^{\cdot}_{X/*^k}$ in the derived category. Thus the cohomology group $\mathrm{H}^n(X, \omega^{\cdot}_{X/*^k})$ underlies the mixed Hodge structure (5.9.4). Then E_1-degeneration (A.6.3) for the filtered complex $(K_{\boldsymbol{C}}, F)$ implies the E_1-degeneration of the log Hodge to de Rham spectral sequence (5.9.3). Moreover, we can easily see the coincidence of the filtrations F on $\mathrm{H}^n(X, K_{\boldsymbol{C}})$ and on $\mathrm{H}^n(X, \omega^{\cdot}_{X/*^k})$ by Corollary 5.7. □

Appendix. In this appendix, we try to relax the finiteness assumption in axioms of (cohomological) mixed Hodge complexes in [2].

DEFINITION A.1. A weak $\boldsymbol{Q}$-mixed Hodge complex consists of the following data:

(A.1.1) a bounded below complex of $\boldsymbol{Q}$-vector spaces $K_{\boldsymbol{Q}}$,

(A.1.2) an increasing filtration W on $K_{\boldsymbol{Q}}$ such that $\bigcup_m W_m K_{\boldsymbol{Q}} = K_{\boldsymbol{Q}}$ and that for every n there exists an integer m with $W_m K^n_{\boldsymbol{Q}} = 0$,

(A.1.3) a bounded below complex of $\boldsymbol{C}$-vector spaces $K_{\boldsymbol{C}}$,

(A.1.4) an increasing filtration W on $K_{\boldsymbol{C}}$ satisfying the same conditions as for W on $K_{\boldsymbol{Q}}$ above,

(A.1.5) a decreasing filtration F on $K_{\boldsymbol{C}}$,

(A.1.6) a morphism $\alpha : K_{\boldsymbol{Q}} \otimes \boldsymbol{C} \to K_{\boldsymbol{C}}$ preserving the filtration W on both sides such that the morphism $\mathrm{Gr}_m^W \alpha : \mathrm{Gr}_m^W K_{\boldsymbol{Q}} \otimes \boldsymbol{C} \to \mathrm{Gr}_m^W K_{\boldsymbol{C}}$ is a quasi-isomorphism for every integer m,

satisfying the condition

(wMHC) the data $(\mathrm{Gr}_m^W K_{\boldsymbol{Q}}, (\mathrm{Gr}_m^W K_{\boldsymbol{C}}, F), \mathrm{Gr}_m^W \alpha)$ is a $\boldsymbol{Q}$-Hodge complex of weight m for every integer m.

REMARK A.2. A weak $\boldsymbol{Q}$-mixed Hodge complex $((K_{\boldsymbol{Q}}, W), (K_{\boldsymbol{C}}, W, F), \alpha)$ becomes a $\boldsymbol{Q}$-mixed Hodge complex, if the cohomology $\mathrm{H}^n(K_{\boldsymbol{C}}) \simeq \mathrm{H}^n(K_{\boldsymbol{Q}}) \otimes \boldsymbol{C}$ is of finite dimension for every n.

LEMMA A.3. *Let* $((K_{\boldsymbol{Q}}, W), (K_{\boldsymbol{C}}, W, F), \alpha)$ *be a weak* $\boldsymbol{Q}$*-mixed Hodge complex. Then we have the followings*:

(A.3.1) $(\mathrm{Gr}_m^W \mathrm{H}^n(K_{\boldsymbol{Q}}), (\mathrm{Gr}_m^W \mathrm{H}^n(K_{\boldsymbol{C}}), F))$ *is a* $\boldsymbol{Q}$*-Hodge structure of weight* $m+n$ *for every* n *and* m.

(A.3.2) *The spectral sequence associated to the filtration* W *degenerates at* E_2*-terms.*

(A.3.3) *The spectral sequence associated to the filtration* F *degenerates at* E_1*-terms.*

(A.3.4) *We have the natural exact sequences*

$$0 \to \mathrm{H}^i(F^n K_{\boldsymbol{C}}) \to \mathrm{H}^i(K_{\boldsymbol{C}}) \to \mathrm{H}^i(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}) \to 0,$$
$$0 \to \mathrm{H}^i(F^{n+1} K_{\boldsymbol{C}}) \to \mathrm{H}^i(F^n K_{\boldsymbol{C}}) \to \mathrm{H}^i(\mathrm{Gr}_F^n K_{\boldsymbol{C}}) \to 0$$

for every i *and* n.

PROOF. By the morphism α we have isomorphisms $E_r^{p,q}(K_{\boldsymbol{Q}}, W) \otimes \boldsymbol{C} \simeq E_r^{p,q}(K_{\boldsymbol{C}}, W)$ etc. Moreover, the morphisms $d : E_r^{p,q}(K_{\boldsymbol{C}}, W) \to E_r^{p+r,q-r+1}(K_{\boldsymbol{C}}, W)$ etc. are defined over $\boldsymbol{Q}$.

First we prove the following by induction on r.

(A.3.5) We have an exact sequence

$$0 \to E_r^{p,q}(F^n K_{\boldsymbol{C}}, W) \to E_r^{p,q}(K_{\boldsymbol{C}}, W) \to E_r^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W) \to 0$$

for every p, q, r and n. Therefore the three filtrations F_d, F_{d^*} and F_{rec} coincide. Moreover, $E_r^{p,q}(K_{\boldsymbol{C}}, W)$ with the filtration $F_d = F_{d^*} = F_{\mathrm{rec}}$ is a Hodge structure of weight q for every p, q and $r \geq 1$.

For the case of $r = 0$, it is easy to see that the sequence

$$0 \to E_0^{p,q}(F^n K_{\boldsymbol{C}}, W) \to E_0^{p,q}(K_{\boldsymbol{C}}, W) \to E_0^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W) \to 0$$

is exact. So we proved (A.3.5) for $r = 0$. Moreover, the filtration $F_d = F_{d^*} = F_{\mathrm{rec}}$ is nothing but the induced filtration F on $\mathrm{Gr}_{-p}^W K_{\boldsymbol{C}}^{p+q} = E_0^{p,q}(K_{\boldsymbol{C}}, W)$. Then the morphism

$$d : E_0^{p,q}(K_{\boldsymbol{C}}, W) = \mathrm{Gr}_{-p}^W K_{\boldsymbol{C}}^{p+q} \to \mathrm{Gr}_{-p}^W K_{\boldsymbol{C}}^{p+q+1} = E_0^{p,q+1}(K_{\boldsymbol{C}}, W)$$

is strictly compatible with respect to the filtration $F_d = F_{d^*} = F_{\mathrm{rec}}$ by the assumption (wMHC) and the definition of Hodge complex (see [2]). Therefore we obtain an exact sequence

$$0 \to E_1^{p,q}(F^n K_{\boldsymbol{C}}, W) \to E_1^{p,q}(K_{\boldsymbol{C}}, W) \to E_1^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W) \to 0$$

by the "Lemma on two filtrations" in [2]. This shows that $F_d = F_{d^*} = F_{\mathrm{rec}}$ on $E_1^{p,q}(K_{\boldsymbol{C}}, W)$. Since we have $F_{\mathrm{rec}} = F$ under the identification $E_1^{p,q}(K_{\boldsymbol{C}}, W) \simeq \mathrm{H}^{p+q}(\mathrm{Gr}_{-p}^W K_{\boldsymbol{C}})$ by definition, $E_1^{p,q}(K_{\boldsymbol{C}}, W)$ with the filtration $F_d = F_{d^*} = F_{\mathrm{rec}}$ is a Hodge structure of weight $p + q + (-p) = q$ by the assumption (wMHC). Thus we proved (A.3.5) for the case of $r = 1$. Now we proceed the induction process. We assume Condition (A.3.5) for $r \geq 1$. Since the morphism $d : E_r^{p,q}(K_{\boldsymbol{C}}, W) \to E_r^{p+r,q-r+1}(K_{\boldsymbol{C}}, W)$ preserves the filtration F_d, it is a morphism of Hodge structures. Then d is strictly compatible with the Hodge filtrations $F_d = F_{d^*} = F_{\mathrm{rec}}$. The "Lemma on two filtrations" tells us that we have an exact sequence

$$0 \to E_{r+1}^{p,q}(F^n K_{\boldsymbol{C}}, W) \to E_{r+1}^{p,q}(K_{\boldsymbol{C}}, W) \to E_{r+1}^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W) \to 0$$

for every p, q and n. Therefore $F_d = F_{d^*} = F_{\mathrm{rec}}$ on $E_{r+1}^{p,q}(K_{\boldsymbol{C}}, W)$. Moreover, $E_{r+1}^{p,q}(K_{\boldsymbol{C}}, W)$ with the filtration $F_{\mathrm{rec}} = F_d = F_{d^*}$ is a Hodge structure of weight q, because the morphism d is a morphism of Hodge structures. Thus we proved (A.3.5) by induction.

The weights of the Hodge structures $E_r^{p,q}(K_{\boldsymbol{C}}, W)$ and $E_r^{p+r,q-r+1}(K_{\boldsymbol{C}}, W)$ are q and $q-r+1$, respectively. Therefore the morphism of Hodge structures $d : E_r^{p,q}(K_{\boldsymbol{C}}, W) \to E_r^{p+r,q-r+1}(K_{\boldsymbol{C}}, W)$ is the zero map for $r \geq 2$ because of the inequality $q > q-r+1$. This fact combined with Assumption (A.1.4) in Definition A.1 implies that the spectral sequence associated to the filtration W on $K_{\boldsymbol{C}}$ degenerates at E_2-terms. Thus we obtain the conclusion (A.3.2).

By the E_2-degeneration above, we have the equalities

$$E_\infty^{p,q}(K_{\boldsymbol{C}}, W) \simeq E_2^{p,q}(K_{\boldsymbol{C}}, W),$$
$$E_\infty^{p,q}(F^n K_{\boldsymbol{C}}, W) \simeq E_2^{p,q}(F^n K_{\boldsymbol{C}}, W),$$
$$E_\infty^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W) \simeq E_2^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W),$$

and the exact sequence

(A.3.6) $$0 \to E_\infty^{p,q}(F^n K_{\boldsymbol{C}}, W) \to E_\infty^{p,q}(K_{\boldsymbol{C}}, W) \to E_\infty^{p,q}(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}, W) \to 0$$

for every p, q and n. Therefore we have $F_d = F_{d^*} = F$ on $E_\infty^{p,q}(K_{\boldsymbol{C}}, W)$. Moreover, $E_\infty^{p,q}(K_{\boldsymbol{C}}, W)$ with the filtration $F_d = F_{d^*} = F$ is a Hodge structure of weight q. Therefore

$$\mathrm{Gr}_m^W \mathrm{H}^n(K_{\boldsymbol{C}}) \simeq E_\infty^{-m,m+n}(K_{\boldsymbol{C}}, W)$$

with the filtration F is a Hodge structure of weight $m+n$. So we obtain the conclusion (A.3.1).

From the exact sequence (A.3.6) we obtain the exact sequence

$$0 \to \mathrm{Gr}_m^W \mathrm{H}^i(F^n K_{\boldsymbol{C}}) \to \mathrm{Gr}_m^W \mathrm{H}^i(K_{\boldsymbol{C}}) \to \mathrm{Gr}_m^W \mathrm{H}^i(K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}) \to 0$$

for every i, m and n. Thus we obtain the first exact sequence in (A.3.4) by using Assumption (A.1.4). Injectivity of the morphism $\mathrm{H}^i(F^{n+1} K_{\boldsymbol{C}}) \to \mathrm{H}^i(K_{\boldsymbol{C}})$ implies the injectivity of the morphism $\mathrm{H}^i(F^{n+1} K_{\boldsymbol{C}}) \to \mathrm{H}^i(F^n K_{\boldsymbol{C}})$ for every i and n. Then we can easily see the second exact sequence in (A.3.4), which implies the isomorphism

$$\mathrm{Gr}_F^n \mathrm{H}^i(K_{\boldsymbol{C}}) \simeq \mathrm{H}^i(\mathrm{Gr}_F^n K_{\boldsymbol{C}})$$

for every i and n. Thus we obtain the conclusion (A.3.3). □

DEFINITION A.4. Let X be a topological space. A weak $\boldsymbol{Q}$-cohomological mixed Hodge complex consists of the following data:

(A.4.1) a bounded below complex of $\boldsymbol{Q}$-sheaves $K_{\boldsymbol{Q}}$,

(A.4.2) an increasing filtration W on $K_{\boldsymbol{Q}}$ such that $\bigcup_m W_m K_{\boldsymbol{Q}} = K_{\boldsymbol{Q}}$ and that for every n there exists an integer m with $W_m K_{\boldsymbol{Q}}^n = 0$,

(A.4.3) a bounded below complex of $\boldsymbol{C}$-sheaves $K_{\boldsymbol{C}}$,

(A.4.4) an increasing filtration W on $K_{\boldsymbol{C}}$ satisfying the same conditions as for W on $K_{\boldsymbol{Q}}$ above,

(A.4.5) a decreasing filtration F on $K_{\boldsymbol{C}}$, (A.4.6) a morphism $\alpha : K_{\boldsymbol{Q}} \otimes \boldsymbol{C} \to K_{\boldsymbol{C}}$ preserving the filtration W on both sides such that the morphism $\mathrm{Gr}_m^W \alpha : \mathrm{Gr}_m^W K_{\boldsymbol{Q}} \otimes \boldsymbol{C} \to \mathrm{Gr}_m^W K_{\boldsymbol{C}}$ is a quasi-isomorphism for every integer m,

satisfying the condition

(wCMHC) the data $(\mathrm{Gr}_m^W K_{\boldsymbol{Q}}, (\mathrm{Gr}_m^W K_{\boldsymbol{C}}, F), \mathrm{Gr}_m^W \alpha)$ is a $\boldsymbol{Q}$-cohomological Hodge complex of weight m for every integer m.

(A.5) For a weak $\boldsymbol{Q}$-cohomological mixed Hodge complex $((K_{\boldsymbol{Q}}, W), (K_{\boldsymbol{C}}, W, F), \alpha)$ on a topological space X, we can see that the data

$$((\mathrm{R}\Gamma(X, K_{\boldsymbol{Q}}), W), (\mathrm{R}\Gamma(X, K_{\boldsymbol{C}}), W, F), \mathrm{R}\Gamma(\alpha))$$

becomes a weak $\boldsymbol{Q}$-mixed Hodge complex by using the Godement resolution. So we have the following.

LEMMA A.6. *Let X be a topological space and $((K_{\boldsymbol{Q}}, W), (K_{\boldsymbol{C}}, W, F), \alpha)$ a weak $\boldsymbol{Q}$-cohomological mixed Hodge complex on X. Then we have the followings*:

(A.6.1) $(\mathrm{Gr}_m^W \mathrm{H}^n(X, K_{\boldsymbol{Q}}), (\mathrm{Gr}_m^W \mathrm{H}^n(X, K_{\boldsymbol{C}}), F))$ *is a $\boldsymbol{Q}$-Hodge structure of weight $m + n$ for every n and m.*

(A.6.2) *The spectral sequence associated to the filtration W degenerates at E_2-terms.*

(A.6.3) *The spectral sequence associated to the filtration F degenerates at E_1-terms.*

(A.6.4) *We have the natural exact sequences*

$$0 \to \mathrm{H}^i(X, F^n K_{\boldsymbol{C}}) \to \mathrm{H}^i(X, K_{\boldsymbol{C}}) \to \mathrm{H}^i(X, K_{\boldsymbol{C}}/F^n K_{\boldsymbol{C}}) \to 0\,,$$
$$0 \to \mathrm{H}^i(X, F^{n+1} K_{\boldsymbol{C}}) \to \mathrm{H}^i(X, F^n K_{\boldsymbol{C}}) \to \mathrm{H}^i(X, \mathrm{Gr}_F^n K_{\boldsymbol{C}}) \to 0$$

for every i and n.

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NAGANO NATIONAL COLLEGE OF TECHNOLOGY
716 TOKUMA, NAGANO 381
JAPAN

E-mail address: fujisawa@ge.nagano-nct.ac.jp