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Mixed hp-FEM on anisotropic meshes II: Hanging nodes and tensor products of boundary layer meshes

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Summary. The divergence stability of mixed hp Finite Element Methods for incompressible fluid flow is analyzed. A discrete inf-sup condition is proved for a general class of meshes. The meshes may be refined anisotropically, geometrically and may contain hanging nodes on geometric patches. The inf-sup constant is shown to be independent of the aspect ratio of the anisotropic elements and the dependence on the polynomial degree is analyzed. Numerical estimates of inf-sup constants confirm the theoretical results.

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1 Introduction

Boundary value problems of incompressible fluid dynamics are described by the Navier-Stokes Equations (NSE). Their robust and accurate numerical solution in arbitrary domains can be achieved by the Finite Element Method (FEM). It is well known that the performance of these methods is governed by a) consistency and b) stability.

Consistency is related to the approximation properties of the Finite Element spaces - they should be designed so that for the anticipated solution class of the NSE a high rate of convergence is achievable. For low and moderate Reynolds numbers, solutions of the NSE in polygonal domains exhibit *corner singularities* (see, e.g., the recent monograph [26] and the references

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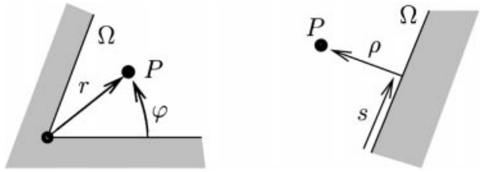

Fig. 1. Polar coordinates (r, φ) near a corner and boundary fitted coordinates (ρ, s) near $\partial\Omega$

there for details). In polar coordinates (r, φ) near a corner they are basically of the form

$$u_c(r, \varphi) = r^{\alpha}\Phi(\varphi) \tag{1.1}$$

for some $|\alpha \geq 0|$ and some analytic function Φ (cf. Fig. 1). At high Reynolds number there arise additionally *boundary layers* due to the singularly perturbed character of the NSE. Boundary layers are solution components which decay rapidly normal to the boundary. A typical example of an *exponential boundary layer* is

$$u_b(\rho, s) = C(s)\exp(-\rho\sqrt{\mathrm{Re}}) \tag{1.2}$$

where (ρ, s) are the usual boundary fitted coordinates in a tubular neighbourhood of the smooth boundary $\partial\Omega$ of the domain with ρ denoting the normal distance to $\partial\Omega$ and s being the arclength on $\partial\Omega$ (cf. Fig. 1). The function $C(s)$ is smooth independently of Re. Since the NSE are nonlinear, the boundary layers do not have necessarily the form (1.2) but rather $u(\rho, s) = C(s)U(\rho, \mathrm{Re})$ where U is analytic and satisfies a certain nonlinear ordinary differential equation (see, e.g., [11, 12]).

The efficient resolution of corner singularity or boundary layer phenomena with the hp version of the FEM requires properly designed meshes: Combining *anisotropic* geometric mesh refinement towards the boundary and the corners with judiciously increased polynomial degrees allows one to approximate corner singularities and boundary layers at an *exponential rate* of convergence (see [9, 10, 19] and the references there for the hp-approximation of corner singularities and [15, 20, 22] for hp-approximation of boundary layers).

Stability problems arise intrinsically in the variational formulations due to the incompressibility constraint $\nabla \cdot \boldsymbol{u} = 0$. It is well known that the velocity and pressure FE spaces can not be chosen independently. Stability is only guaranteed as long as a discrete Babuška-Brezzi condition is satisfied by the velocity and pressure spaces. For many pairs of velocity and pressure spaces this inf-sup condition has been established (see [5, 8, 23] and the references there for h-version FEM and [4, 21, 24, 25, 31] and the references there for p-version/spectral FEM). These stability issues are already present in the much simpler Stokes equations that are obtained by linearization of the NSE. We therefore study here the stability of the FE-spaces in this model problem.

However, the appearance of anisotropic elements of arbitrarily large aspect ratio raises additional stability concerns since almost all the presently available techniques for establishing divergence stability seem to require the shape regularity of the meshes in some sense. This precludes, of course, anisotropic meshes which are mandatory in hp-FEM to resolve boundary layers at exponential rates of convergence. Recently, some attention has been turned to this issue and it has been proved by Becker and Rannacher [2, 3] that a certain nonconforming low order element is indeed stable independently of the element aspect ratio on axiparallel meshes. In [18] the authors proved stability for conforming hp-elements independent of the aspect ratio on anisotropic quadrilateral mesh patches.

In this paper the earlier work [18] on the divergence stability for the Stokes problem on anisotropic meshes is extended focusing again on the hp version of mixed Finite Element Methods. We prove stability for a family of conforming hp velocity and pressure spaces on *irregular meshes* which may contain anisotropic elements, hanging nodes and on elements with variable polynomial degree, as required in the hp-FEM. In fact, on quadrilaterals, the considered family are the "$\mathcal{P}_N \times \mathcal{P}_{N-2}$" elements already discussed in [4]. The meshes we admit allow for the desired refinement properties, geometrically towards corners and anisotropically towards boundaries, so that singular behaviour as in (1.1) and (1.2) can be resolved. In particular, we prove divergence stability on tensor products of geometrically refined meshes. To do so, we establish first a discrete inf-sup condition for low order elements with hanging nodes with an inf-sup constant depending only on the geometrical grading factor. In this context we introduce an interpolant of Clément type on geometric meshes with hanging nodes which is of independent interest. The corresponding stability results for higher order elements are obtained in a second step with the aid of a macro-element technique and local stability results. The dependence on the polynomial degree k is given explicitly, that is we show that the inf-sup constant is bounded from below by $Ck^{-\frac{1}{2}}$ if the mesh contains no triangles and by the (pessimistic) bound Ck^{-3} otherwise. Numerical estimates of inf-sup constants indicate the sharpness of our results and the dependence on the geometrical grading factor σ. We refer also to [7] where the performance of "$\mathcal{P}_N \times \mathcal{P}_{N-2}$" elements is studied numerically in an L-shaped domain.

The outline of the paper is as follows: In Sect. 2 we formulate the Stokes problem and define the meshes and spaces to be analyzed. In Sect. 3 our main stability result is given and we illustrate the approximation properties of our mesh family in a simple model situation. In Sect. 4 we establish stability results on reference meshes which implies by a macro-element technique

the main result.
The standard notation is used in this paper: For a polygonal domain $D \subseteq \mathbb{R}^2$ or an interval $D = (a, b)$ we denote by $H^k(D)$ the Sobolev spaces of integer orders $k \geq 0$ equipped with the usual norms $\|\cdot\|_{k,D}$ and seminorms $|\cdot|_{k,D}$. We set $H^0(D) = L^2(D)$, $H_0^1(D) = \{u \in H^1(D) : \mathrm{trace(u)} = 0 \text{ on } \partial D\}$ and $L_0^2(D) = \{p \in L^2(D) : (p, 1)_D = 0\}$ where $(\cdot, \cdot)_D$ denotes the $L^2(D)$ inner product. For $s \geq 0$ nonintegral, the Sobolev spaces $H^s(D)$ with norm $\|\cdot\|_{s,D}$ are defined as usually via the K-method of interpolation (see, e.g., [13,29]). The set of all polynomials of total degree $\leq k$ on $D \subseteq \mathbb{R}^2$ is denoted by $\mathcal{P}_k(D)$, the set of all polynomials of degree $\leq k$ in each variable by $\mathcal{Q}_k(D)$. If I is an interval we define $\mathcal{P}_k(I)$ as the set of polynomials on I of degree $\leq k$. In the following we denote by C generic constants not necessarily identical at different places but always independent of the meshwidths and the polynomial degrees.

2 Problem formulation

2.1 Stokes problem

In a bounded, polygonal domain $\Omega \subset \mathbb{R}^2$ we consider the *Stokes* boundary value problem for incompressible fluid flow obtained by linearization of the Navier-Stokes Equations: Find a velocity field $\boldsymbol{u}$ and a pressure p such that

$$-\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega. \tag{2.3}$$

Here, $\nu > 0$ is the kinematic viscosity which is related to the Reynolds number Re of the flow by $\nu = 1/\mathrm{Re}$. The right hand side $\boldsymbol{f}$ is a given body force per unit mass. The usual mixed formulation of (2.1)-(2.3) is the following:

Find $\boldsymbol{u} \in H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$\nu (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_\Omega - (p, \nabla \cdot \boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega, \tag{2.4}$$

$$(q, \nabla \cdot \boldsymbol{u})_\Omega = 0 \tag{2.5}$$

for all $(\boldsymbol{v}, q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$.

It is well known (see, e.g., [8,19]) that for $\boldsymbol{f} \in L^2(\Omega)^2$ there exists a unique weak solution $(\boldsymbol{u}, p)$ of (2.4)-(2.5) due to the continuous *inf-sup condition*

$$\inf_{0 \neq p \in L_0^2(\Omega)} \sup_{0 \neq \boldsymbol{v} \in H_0^1(\Omega)^2} \frac{(\nabla \cdot \boldsymbol{v}, p)_\Omega}{|\boldsymbol{v}|_{1,\Omega} \|p\|_{0,\Omega}} \geq C(\Omega) > 0. \tag{2.6}$$

A conforming FE-discretization of (2.4)-(2.5) is obtained in the usual way: Given finite dimensional subspaces $\boldsymbol{V}_N \subseteq H_0^1(\Omega)^2$ and $M_N \subseteq L_0^2(\Omega)$, find $(\boldsymbol{u}_N, p_N) \in \boldsymbol{V}_N \times M_N$ such that (2.4)-(2.5) holds for any $(\boldsymbol{v}, q) \in \boldsymbol{V}_N \times M_N$. A family $\{\boldsymbol{V}_N \times M_N\}_N$ is $\gamma(N)$-*stable*, if the following *discrete inf-sup condition* holds

$$\inf_{0 \neq p \in M_N} \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{V}_N} \frac{(\nabla \cdot \boldsymbol{v}, p)_\Omega}{|\boldsymbol{v}|_{1,\Omega} \, \|p\|_{0,\Omega}} \geq \gamma(N) > 0. \tag{2.7}$$

If $\gamma(N)$ in (2.7) does not depend on N, we say that the family $\{\boldsymbol{V}_N \times M_N\}_N$ is *stable*. If a family is $\gamma(N)$-stable, the discrete problem has a unique solution $(\boldsymbol{u}_N, p_N)$ in $\boldsymbol{V}_N \times M_N$ and the rate of convergence of the FE approximations $\{(\boldsymbol{u}_N, p_N)\}_N$ of $(\boldsymbol{u}, p)$ is determined by that of the best approximations of $(\boldsymbol{u}, p)$ in $\{\boldsymbol{V}_N \times M_N\}_N$, i.e. we have the error estimates [5,19]

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_N\|_{1,\Omega} \leq {} & C\gamma^{-1}(N) \inf_{\boldsymbol{v} \in \boldsymbol{V}_N} \|\boldsymbol{u} - \boldsymbol{v}\|_{1,\Omega} \\ & + C\nu^{-1} \inf_{q \in M_N} \|p - q\|_{0,\Omega} , \end{aligned} \tag{2.8}$$

$$\begin{aligned} \|p - p_N\|_{0,\Omega} \leq {} & C\nu\gamma^{-2}(N) \inf_{\boldsymbol{v} \in \boldsymbol{V}_N} \|\boldsymbol{u} - \boldsymbol{v}\|_{1,\Omega} \\ & + C\gamma^{-1}(N) \inf_{q \in M_N} \|p - q\|_{0,\Omega} \end{aligned} \tag{2.9}$$

with $C = C(\Omega)$ independent of N and ν.

2.2 Finite element spaces

We define the velocity-pressure space pairs $\boldsymbol{V}_N \times M_N$ to be analyzed below.

2.2.1 Preliminaries A *mesh* $\mathcal{T}$ on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ is a partition of Ω into disjoint and open quadrilateral and/or triangular elements $\{K\}$ such that $\overline{\Omega} = \cup_{K \in \mathcal{T}} \overline{K}$. We consider only *affine* meshes where each $K \in \mathcal{T}$ is affine equivalent to a reference element $\hat{K}$ which is either the reference triangle $\hat{T} = \{(x, y) : 0 < x < 1, 0 < y < x\}$ or the reference square $\hat{Q} = (0, 1)^2$, i.e. $K = F_K(\hat{K})$ with F_K affine and orientation preserving. The mesh $\mathcal{T}$ is called *regular* if for any two elements $K, K' \in \mathcal{T}$ the intersection $\overline{K} \cap \overline{K'}$ is either empty, a single vertex or an entire side. Otherwise, the mesh $\mathcal{T}$ contains *hanging nodes* and is called *irregular*. For an affine mesh $\mathcal{T}$ and an element $K \in \mathcal{T}$ we denote by h_K the diameter of the element K and by ρ_K the diameter of the largest circle inscribed into K. The *meshwidth* h of $\mathcal{T}$ is given by $h = \max_{K \in \mathcal{T}} h_K$. The

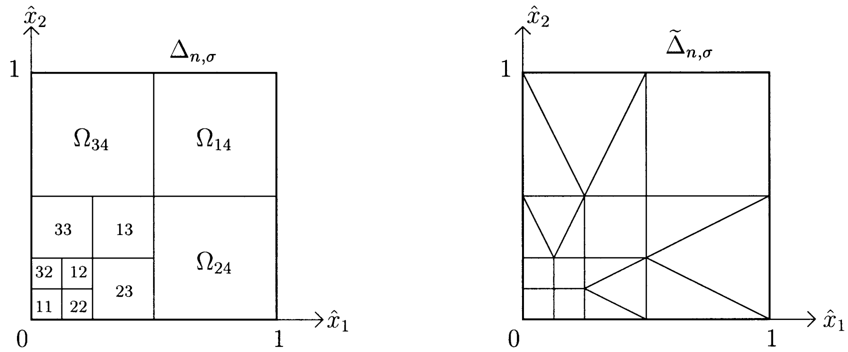

Fig. 2. The geometric meshes $\Delta_{n,\sigma}$ and $\widetilde{\Delta}_{n,\sigma}$ with $n = 3$ and $\sigma = 0.5$

fraction $\sigma_K := \frac{h_K}{\rho_K}$ is the *aspect ratio* of the cell K. A (regular or irregular) affine mesh $\mathcal{T}$ is called κ*-uniform* if there exists $\kappa > 0$ such that

(2.10) $$\max_{K\in\mathcal{T}} \sigma_K \leq \kappa < \infty.$$

2.2.2 Reference meshes Our hp-FEM will be based on certain two-level families of meshes: A *macroscopic* κ-uniform mesh denoted $\mathcal{T}_{\mathrm{m}}$ which will be *locally* refined either towards corners or towards the boundary. To this end, we introduce now some meshes on the reference elements $\hat{Q}$ and $\hat{T}$ (which are the reference elements for $\mathcal{T}_{\mathrm{m}}$). Most of these reference meshes are irregular or contain anisotropic elements.

Definition 2.1 Let $n \in \mathbb{N}_0$ and $\sigma \in (0, 1)$. On $\hat{Q}$, the *(irregular) geometric mesh* $\Delta_{n,\sigma}$ with $n + 1$ *layers* and *grading factor* σ is created recursively as follows: If $n = 0$, $\Delta_{0,\sigma} = \{\hat{Q}\}$. Given $\Delta_{n,\sigma}$ for $n \geq 0$, $\Delta_{n+1,\sigma}$ is generated by subdividing that square $K \in \Delta_{n,\sigma}$ with $0 \in \overline{K}$ into four smaller rectangles by dividing the sides of K in a $\sigma : (1 - \sigma)$ ratio.
The *(regular) geometric mesh* $\widetilde{\Delta}_{n,\sigma}$ is obtained from $\Delta_{n,\sigma}$ by removing the hanging nodes as indicated in Fig. 2.

In Fig. 2 the geometric mesh is shown for $n = 3$ and $\sigma = 0.5$. Clearly, $\Delta_{n,\sigma}$ is an irregular affine mesh, it contains *hanging nodes*. The elements of the geometric mesh $\Delta_{n,\sigma}$ are numbered as in Fig. 2, i.e.

(2.11) $$\Delta_{n,\sigma} = \{\Omega_{11}\} \cup \{\Omega_{ij} : 1 \leq i \leq 3, 2 \leq j \leq n + 1\}.$$

The elements Ω_{1j}, Ω_{2j} and Ω_{3j} constitute the *layer* j.

Definition 2.2 Let $\mathcal{T}_x$ be an arbitrary mesh on $I = (0, 1)$, given by a partition of I into subintervals $\{K_x\}$. On $\hat{Q}$, the *boundary layer mesh* $\Delta_{\mathcal{T}_x}$ is the product mesh

$$\Delta_{\mathcal{T}_x} = \{K : K = K_x \times I, K_x \in \mathcal{T}_x\}.$$

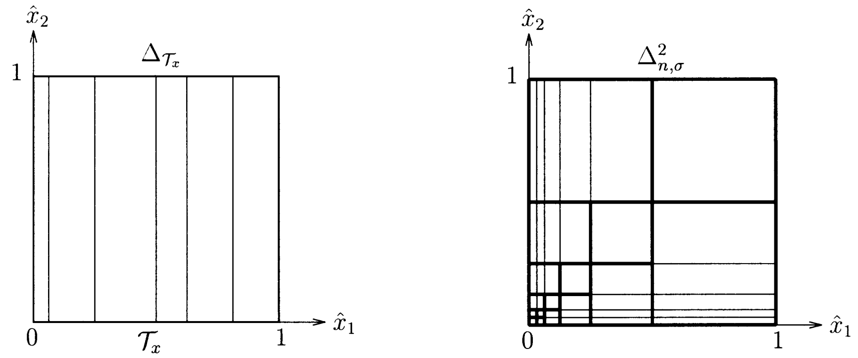

Fig. 3. Boundary layer mesh and geometric tensor product mesh on $\hat{Q}$

Figure 3 shows a typical boundary layer mesh. We emphasize that any $\mathcal{T}_x$ is allowed. In particular, rectangles of arbitrarily high aspect ratio can be used such that boundary layer meshes $\varDelta_{\mathcal{T}_x}$ are not κ-uniform.

Definition 2.3 Let $n \in \mathbb{N}_0$ and $\sigma \in (0,1)$. On $I = (0,1)$, let $\mathcal{T}_{n,\sigma}$ be the one dimensional geometric mesh refined towards 0 given by a partition of I into subintervals $\{I_j\}_{j=1}^{n+1}$ where

$$I_j = (x_{j-1}, x_j) \quad \text{with } x_0 = 0 \text{ and } x_j = \sigma^{n+1-j},\ j = 1, \ldots, n+1.$$

On $\hat{Q}$, the *geometric tensor product mesh* $\varDelta^2_{n,\sigma}$ is then given by $\mathcal{T}_{n,\sigma} \otimes \mathcal{T}_{n,\sigma}$, i.e.

$$\varDelta^2_{n,\sigma} = \{I_j \times I_k : I_j \in \mathcal{T}_{n,\sigma}, I_k \in \mathcal{T}_{n,\sigma}\}\,.$$

The tensor product mesh $\varDelta^2_{n,\sigma}$ contains anisotropic rectangles with arbitrarily large aspect ratio (see Fig. 3). For the proof of the inf-sup conditions ahead, it is important to notice that $\varDelta^2_{n,\sigma}$ can be understood as the geometric mesh $\varDelta_{n,\sigma}$ into which appropriately scaled versions of boundary layer meshes $\varDelta_{\mathcal{T}_x}$ are inserted to remove the hanging nodes. A geometric tensor product mesh is shown in Fig. 3 with $n = 5$ and $\sigma = 0.5$. The underlying geometric mesh $\varDelta_{n,\sigma}$ is indicated by bold lines.

Remark 2.4 The geometric meshes $\varDelta_{n,\sigma}$, $\widetilde{\varDelta}_{n,\sigma}$ and the tensor product mesh $\varDelta^2_{n,\sigma}$ can also be defined on the reference triangle $\hat{T}$. This is shown in Fig. 4. On the reference square $\hat{Q}$ we can even admit mixtures of geometric tensor product meshes and geometric meshes as illustrated in Fig. 5. Of course, other combinations are imaginable.

2.2.3 Geometric boundary layer meshes We define:

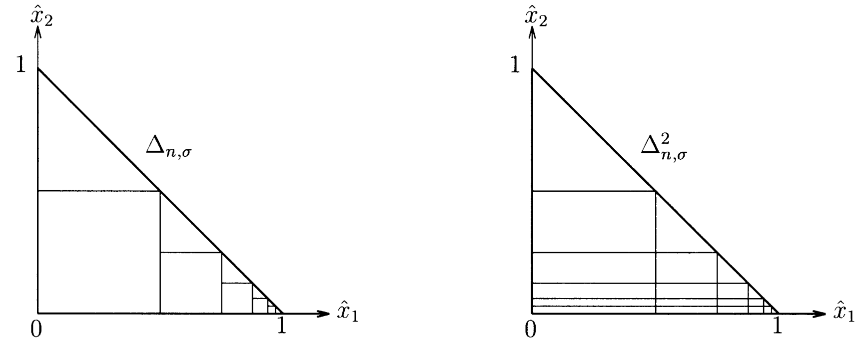

Fig. 4. The meshes $\Delta_{n,\sigma}$ and $\Delta^2_{n,\sigma}$ on the reference triangle $\hat{T}$

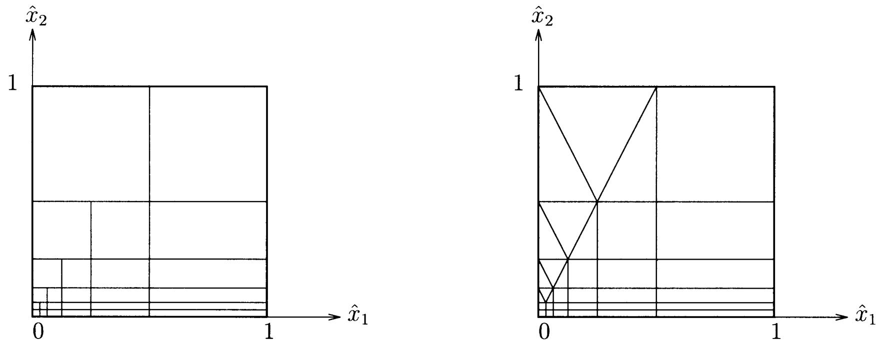

Fig. 5. Further reference meshes on $\hat{Q}$

Definition 2.5 Consider a (coarse) κ-uniform affine mesh $\mathcal{T}_{\rm m}$ on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$. An affine mesh $\mathcal{T}$ on Ω is called *geometric boundary layer mesh* with *macro-element mesh* $\mathcal{T}_{\rm m}$ if $\mathcal{T}$ is obtained from $\mathcal{T}_{\rm m}$ in the following way: Some elements $K \in \mathcal{T}_{\rm m}$ are further partitioned into $F_K(\hat{\mathcal{T}})$ where $\hat{\mathcal{T}}$ is any of the possibly irregular affine reference meshes on $\hat{K}$ introduced in the previous subsection (Definitions 2.1, 2.2, 2.3 and Remark 2.4) and F_K is the affine mapping between $\hat{K}$ and K.

The elements of $\mathcal{T}_{\rm m}$ are called macro-elements. If no macro-element in $\mathcal{T}_{\rm m}$ is further refined, the notion "geometric boundary layer mesh" reduces to the already introduced notion of "κ-uniform affine meshes" (such meshes can of course also contain geometric refinements but they are not allowed to have anisotropic elements) and the notion of "macro-elements" becomes in that case unnecessary. "Geometric boundary layer meshes" are a very general class of possibly highly irregular and anisotropic meshes. We will show below that they are well suited for the effective resolution of boundary layer and corner singularity phenomena, i.e. the hp-FEM based on such meshes can resolve boundary layers and corner singularities at an exponential rate. Typically, mesh-patches from $\mathcal{T}_{\rm m}$ near the boundary of the domain are par-

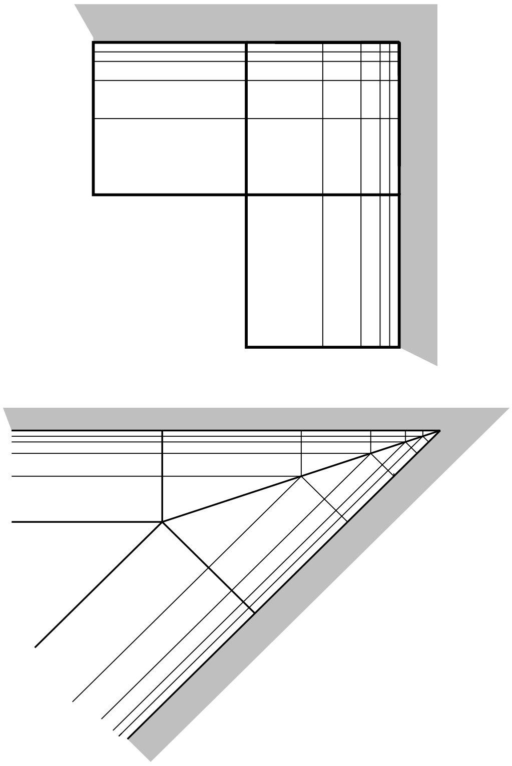

Fig. 6. Geometric boundary layer meshes near convex corners

titioned anisotropically using $\Delta_{\mathcal{T}_x}$-meshes to approximate boundary layers. Patches near corners are geometrically refined towards the corners with the meshes $\Delta_{n,\sigma}$ or $\Delta^2_{n,\sigma}$. This takes into account boundary layers as well as the singular behaviour of the solution near a corner. In the interior of the domain a simple κ-uniform mesh can be used. Some examples of geometric boundary layer meshes are shown in Figs. 6 and 7.

Remark 2.6 Of course, other reference meshes are imaginable for the further local refinement in the macro-elements. As long as these reference meshes are divergence stable (cf. the macro-element technique in Proposition 4.11) they can be added to the "family of local refinement strategies". Further, we remark that no restriction on the regularity of the mesh between two adjacent macro-elements is imposed (even if one demands the macro-element mesh to be regular). For example, a mesh as in Fig. 8 is admissible.

2.2.4 hp-FEM spaces We introduce the hp-FEM spaces to be investigated later on. Therefore, let $\mathcal{T}$ be an affine mesh on Ω. With each element $K \in \mathcal{T}$ we associate a polynomial degree k_K. All degrees are combined into a degree

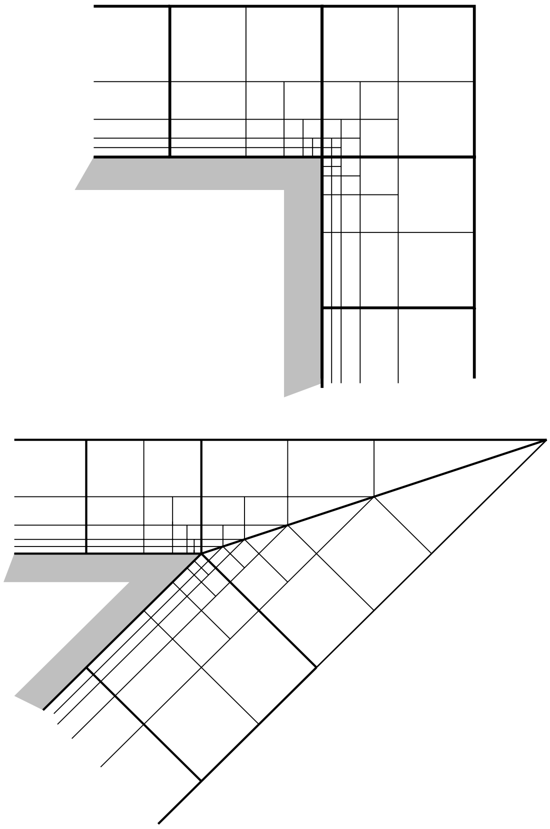

Fig. 7. Geometric boundary layer meshes near reentrant corners

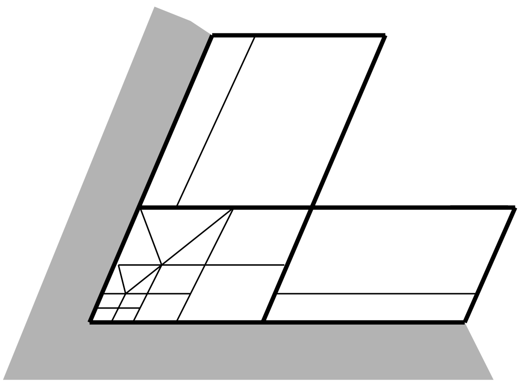

Fig. 8. The macro-elements are irregularly connected in this mesh

vector

$$\underline{k} = \{k_K : K \in \mathcal{T}\} \tag{2.12}$$

and we set $|\underline{k}| = \max\{k_K : K \in \mathcal{T}\}$.
We define the velocity and pressure spaces

$$S^{\underline{k},1}(\Omega,\mathcal{T}) := \Big\{u \in H^1(\Omega) : u|_K \circ F_K \in \begin{cases} \mathcal{Q}_{k_K}(\hat{Q}) \text{ if } K \text{ is a quadrilateral} \\ \mathcal{P}_{k_K}(\hat{T}) \text{ if } K \text{ is a triangle} \end{cases} \forall K \in \mathcal{T}\Big\} \tag{2.13}$$

and

$$S^{\underline{k},0}(\Omega,\mathcal{T}) := \Big\{p \in L^2(\Omega) : p|_K \circ F_K \in \begin{cases} \mathcal{Q}_{k_K}(\hat{Q}) \text{ if } K \text{ is a quadrilateral} \\ \mathcal{P}_{k_K}(\hat{T}) \text{ if } K \text{ is a triangle} \end{cases} \forall K \in \mathcal{T}\Big\}. \tag{2.14}$$

Implementationally, some care is required to ensure interelement continuity in (2.13) if k_K is variable. In some elements the external (or side) modes in the polynomial spaces must be reduced whereas the internal (or bubble) modes are of full degree k_K. This can be achieved by introducing edge-degrees as in [19].
We set further

$$S_0^{\underline{k},1}(\Omega,\mathcal{T}) = S^{\underline{k},1}(\Omega,\mathcal{T}) \cap H_0^1(\Omega), \quad S_0^{\underline{k},0}(\Omega,\mathcal{T}) = S^{\underline{k},0}(\Omega,\mathcal{T}) \cap L_0^2(\Omega).$$

If the polynomial degree is constant throughout the mesh $\mathcal{T}$ (i.e. $k_K = k \; \forall K \in \mathcal{T}$), we use the shorthand notations $S^{k,1}(\Omega,\mathcal{T})$ and $S^{k,0}(\Omega,\mathcal{T})$.

3 Main results

3.1 Stability

In this section our main result on the divergence stability of $S^{\underline{k},1}(\Omega,\mathcal{T})^2 \times S^{\underline{k}-2,0}(\Omega,\mathcal{T})$ on a geometric boundary layer mesh $\mathcal{T}$ with underlying macro-element mesh $\mathcal{T}_{\mathrm{m}}$ is stated. Let $K \in \mathcal{T}_{\mathrm{m}}$ be a macro-element and $\mathcal{T}_K$ the restriction of $\mathcal{T}$ to K. We permit general polynomial degree distributions $\underline{k}$ as in (2.12) on $\mathcal{T}$ which satisfy

(i) If $\mathcal{T}_K = F_K(\Delta_{\mathcal{T}_x})$, then $\underline{k}$ is constant on $\mathcal{T}_K$.
(ii) If $\mathcal{T}_K = F_K(\Delta)$, where the reference mesh Δ on $\hat{K}$ contains anisotropic elements and has an underlying geometric mesh $\Delta_{n,\sigma}$ (e.g. $\Delta = \Delta^2_{n,\sigma}$), then $\underline{k}$ is constant on $F_K(\Delta_{n,\sigma})$.

Theorem 3.1 *Let $\mathcal{T}$ be a geometric boundary layer mesh on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ such that the underlying macro-element mesh $\mathcal{T}_{\mathrm{m}}$ is regular and κ-uniform for $\kappa > 0$. Assume that all the geometric refinements in $\mathcal{T}$ are obtained with a fixed grading factor $\sigma \in (0,1)$. Let $\underline{k}$ be a polynomial degree distribution on $\mathcal{T}$ which satisfies (i) and (ii) above and let $|\underline{k}| = \max\{k_K : K \in \mathcal{T}\}$. Then there exists a constant $C > 0$ (depending only on κ, σ and Ω) such that the spaces*

$$\boldsymbol{V}_N = S_0^{\underline{k},1}(\Omega,\mathcal{T})^2, \quad M_N = S_0^{\underline{k}-2,0}(\Omega,\mathcal{T})$$

satisfy the inf-sup condition (2.7) with $\gamma(N) \geq C\,|\underline{k}|^{-\alpha}$ where $\alpha = \frac{1}{2}$ if $\mathcal{T}$ does not contain triangles and $\alpha = 3$ otherwise.

We will prove this theorem in Sect. 4 using a macro-element technique (cf. Proposition 4.11 ahead). The main difficulty is to establish local stability results on the reference meshes.

Remark 3.2 Although a geometric boundary layer mesh $\mathcal{T}$ may contain anisotropic mesh-patches, the inf-sup constant in Theorem 3.1 is independent of the element aspect-ratio in such a patch.

Remark 3.3 We could also allow for different geometric grading factors σ in the geometrically refined patches. As long as σ is bounded away from 1 and 0, Theorem 3.1 still holds true. This is for example satisfied if only finitely many macro-elements are refined geometrically. More general families of reference meshes are of course admissible for the local refinement of the macro-elements, provided they are patchwise divergence stable as will be explained in Sect. 4.

Remark 3.4 In particular, Theorem 3.1 states divergence stability on κ-uniform regular meshes consisting of affine triangles and quadrilaterals, which is already well known (cf. [21] for the hp-version).

Remark 3.5 The inf-sup constant in Theorem 3.1 depends on the geometric grading factor σ. The following numerical estimate indicates that one can not expect to remove this dependence. We calculated inf-sup constants for $[\mathcal{Q}_2]^2 \times \mathcal{Q}_0$ elements (that is piecewise quadratic velocities and piecewise constant pressure) on the basic geometric mesh $\Delta_{1,\sigma}$ which consists (with the numbering in (2.11)) of the four quadrilaterals

$$\Omega_{11} = (0,\sigma) \times (0,\sigma), \quad \Omega_{22} = (\sigma,1) \times (0,\sigma),$$
$$\Omega_{12} = (\sigma,1) \times (\sigma,1), \quad \Omega_{32} = (0,\sigma) \times (\sigma,1).$$

In Fig. 9 the inf-sup constants are plotted for $\sigma \in (0,1)$. The inf-sup constants $C(\sigma)$ deteriorate as σ approaches $\sigma = 0$ or $\sigma = 1$. The graph indicates clearly that one can not bound the inf-sup constant uniformly in $\sigma \in (0,1)$

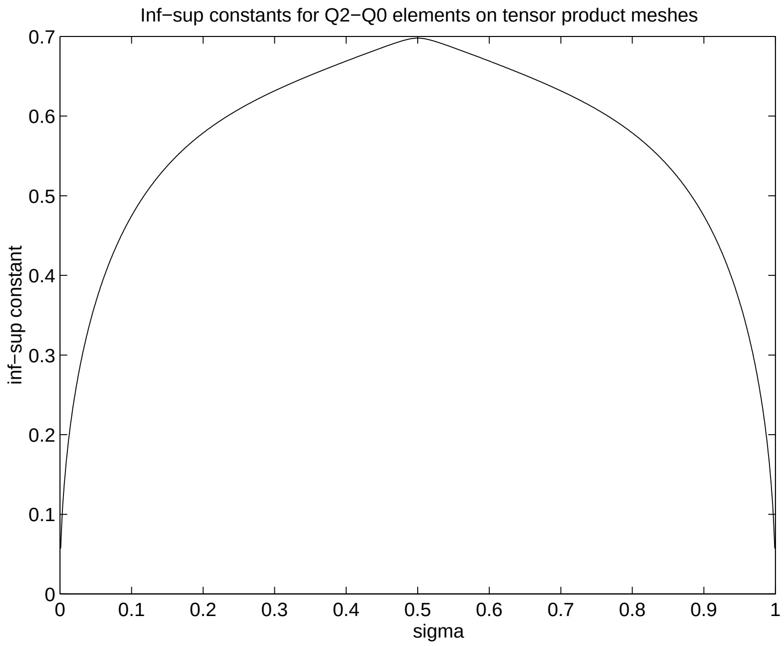

Fig. 9. Inf-sup constants for $S_0^{2,1} \times S_0^{0,0}$ elements on $\Delta_{1,\sigma}$ for varying σ

although the boundary layer meshes $\Delta_{\mathcal{T}_x}$ are stable independently of the aspect ratio [18]. In that sense we expect our results to be sharp. Figure 9 suggests in fact that $C(\sigma) \geq K\sqrt{\sigma(1-\sigma)}$ with $K \approx 1.4$ independent of $\sigma \in (0,1)$.

3.2 Consistency

Theorem 3.1 establishes divergence stability for general hp-FE spaces on geometric meshes such as $\Delta_{n,\sigma}$, $\Delta^2_{n,\sigma}$, $\widetilde{\Delta}_{n,\sigma}$ and combinations thereof. With the subsequent arguments we wish to illustrate in this section that two solution features which typically arise in viscous, incompressible flow mandate the meshes considered here and can be approximated at a robust exponential rate of convergence. In contrast to the inviscid case, the Navier-Stokes Equations for viscous flows with viscosity $\nu = 1/\mathrm{Re} > 0$ are known to exhibit, due to the elliptic spatial operator $-\nu\Delta\boldsymbol{u}$, corner singularities and boundary layers governing the laminar behaviour (for corner singularities we refer to [14,16,26] and for boundary layers to [11,12,27]).

Corner singularities basically take the form (1.1) where (r,φ) denote polar coordinates at the corner as in Fig. 1 (more precisely, the solutions belong to certain weighted Sobolev spaces). *Boundary layers* are solution components that show in boundary fitted coordinates as in Fig. 1 a rapid variation in the coordinate ρ normal to the boundary $\partial\Omega$, but a smooth behaviour independent of ν in the coordinate s tangential to $\partial\Omega$. Generically, they assume the form

$$u_{\mathrm{b}}(\rho, s) = C(s)U(\rho/d) \tag{3.1}$$

where U is the so-called boundary layer profile which is independent of ν. For singularly perturbed linear reaction-diffusion equations where the viscosity ν tends to zero, we have $d = \sqrt{\nu}$ and $U(\hat{\rho}) = \exp(-\hat{\rho})$. The same type of *exponential boundary layers* appears in the linear Oseen approximation of the NSE (see [27] where boundary layers in Oseen type equations are studied in a two dimensional channel). For the full NSE, the laminar boundary layer is related to an analytic solution of a certain nonlinear ordinary differential equation (as, e.g., the Blasius or the Falkner-Skan equations) and we have $d = \sqrt{\nu}\hat{d}(s)$, $\hat{d}(s) > 0$ smooth. Note that although in the Stokes equation (2.1) there appears the viscosity ν as well, the Stokes-solutions do not exhibit boundary layers - these phenomena are strictly related to the presence of additional advective terms in the equations as they arise for example in the Oseen linearization. However, our stability analysis is only concerned with the incompressibility constraint and does not deal with such advective effects where an additional stabilization of the scheme is necessary at small ν (see, e.g., [28] for an analysis of a low order h-version FEM).

Nevertheless, a rigorous asymptotic expansion of laminar solutions of the NSE near walls seems not to be available, and the interaction of boundary layers and corner singularities at high Reynolds number seems not to be completely understood yet. Therefore, we confine ourselves to a very simple model situation where the hp approximation of singular behaviour as in (1.1) or (1.2) is considered. We focus on the approximation of one velocity component, similar statements hold also for the pressure [17].

Let $\varDelta^2_{n,\sigma}$ be the tensor product mesh on the unit square $\hat{Q}$ geometrically refined towards the origin (cf. Definition 2.3 and Fig. 3). We assume that the velocity component $u \in H^1(\hat{Q})$ consists of two exponential boundary layers and one corner singularity component, i.e. u is, up to smooth terms, of the form

$$\begin{aligned} u(x_1, x_2) &= u_{\mathrm{c}}(x_1, x_2) + u_{\mathrm{b}_1}(x_1, x_2) + u_{\mathrm{b}_2}(x_1, x_2) \\ &= u_{\mathrm{c}}(x_1, x_2) + C_1(x_2)\exp(-x_1/d) + C_2(x_1)\exp(-x_2/d). \end{aligned} \tag{3.2}$$

Here, C_1 and C_2 are analytic functions on $[0,1]$ and $d = \sqrt{\nu} = 1/\sqrt{\mathrm{Re}} \in (0,1]$ is the small parameter in (3.1) related to the viscosity ν and the Reynolds number Re. $u_{\mathrm{c}}(x,y)$ is a corner singularity function independent of d which belongs to the countably weighted space $\mathcal{B}^2_\beta(\hat{Q})$ (we refer to [1,10] for the exact definition of this space). In polar coordinates (r,φ) near the corner (in our example the origin) the function u_{c} is of the form

(1.1) with analytic Φ. The analyticity of Φ holds true if *the input data is piecewise analytic* as is indicated by closely related elasticity and potential problems (cf. [1, 10]). Decompositions as in (3.2) with $u_{\mathrm{b}_1} = u_{\mathrm{b}_2} = 0$, u_{c} of the form (1.1) and a (smooth) remainder u_{r} have been given for the Stokes and Navier-Stokes Equations in [16]. However, $u_{\mathrm{r}} = u_{\mathrm{r}}(\nu)$ and a further decomposition of u_{r} into boundary layers and a remainder (with bounds uniform in ν) does not seem to be available yet.

If in the geometric mesh $\Delta_{n,\sigma}$ with hanging nodes the number n of layers is related linearly to the polynomial degree k, i.e. $k = [Cn]$ for some $C > 0$, we have the following approximation property for the corner singularity $u_{\mathrm{c}} \in \mathcal{B}^2_\beta(\hat{Q})$ (see [9, 19]):

$$\inf_{v \in S^{k,1}(\hat{Q}, \Delta_{n,\sigma})} \|u_{\mathrm{c}} - v\|_{1,\hat{Q}} \leq K \exp(-bk) \tag{3.3}$$

where K and b are independent of k (and of course of d). Since $\Delta^2_{n,\sigma}$ is finer than $\Delta_{n,\sigma}$, (3.3) holds true also for the space $S^{k,1}(\hat{Q}, \Delta^2_{n,\sigma})$.

In [18] we investigated with the aid of [20] the approximation properties for an exponential boundary layer function u_{b} of the form (3.2) on a boundary layer mesh $\Delta_{\mathcal{T}_{n,\sigma}}$ where $\mathcal{T}_{n,\sigma}$ is the one dimensional geometric mesh as in Definition 2.3. If the grading factor σ and the number n of layers is such that $\sigma^n \leq Cd$ for some $C > 0$ then

$$\inf_{v \in S^{k,1}(\hat{Q}, \Delta_{\mathcal{T}_{n,\sigma}})} \left(\|u_{\mathrm{b}} - v\|_{0,\hat{Q}} + d\, |u_{\mathrm{b}} - v|_{1,\hat{Q}} \right) \leq K \exp(-bk) \tag{3.4}$$

for K and b independent of k and d. Since the mesh $\Delta_{\mathcal{T}_{n,\sigma}}$ is also contained in $\Delta^2_{n,\sigma}$, (3.4) remains equally valid for $S^{k,1}(\hat{Q}, \Delta^2_{n,\sigma})$. From (3.3) and (3.4) we conclude with the triangle inequality that the spaces $S^{k,1}(\hat{Q}, \Delta^2_{n,\sigma})$ can approximate functions u of the form (3.2) at a robust exponential rate:

Proposition 3.6 *Let u be of the specific form (3.2). Let the polynomial degree k be related linearly to the number n of layers and let n be such that $\sigma^n \leq Cd$ for some $C > 0$. Then*

$$\inf_{v \in S^{k,1}(\hat{Q}, \Delta^2_{n,\sigma})} \left(\|u - v\|_{0,\hat{Q}} + d\, |u - v|_{1,\hat{Q}} \right) \leq K \exp(-bN^{\frac{1}{3}}) \tag{3.5}$$

where $K, b > 0$ are independent of $N = \dim(S^{k,1}(\hat{Q}, \Delta^2_{n,\sigma}))$ and d.

Remark 3.7 The above analysis is strongly based on the exponential form of the boundary layer. However, similar arguments may be applied to any other separable form of the boundary layer given in (3.1).

Remark 3.8 We point out that in the mixed setting the a-priori estimates (2.8)-(2.9) are not uniform in $\nu > 0$. Nevertheless, the dependence of the constants on ν is algebraic. The exponential convergence estimate (3.5) indicates that the ν-dependence in (2.8) and (2.9) can be compensated at a modest number of degrees of freedom in the hp-FEM, at least for laminar flows.

4 Proof of the stability result

This section is devoted to the proof of Theorem 3.1. The proof will proceed in analogy to the definition of geometric boundary layer meshes. First we present local stability results, then we give in Sect. 4.2 a general stability result for some low order elements on the irregular reference mesh $\Delta_{n,\sigma}$ which is of independent interest. These results are combined with the aid of a macro-element technique presented in Sect. 4.3 in order to obtain the proof of Theorem 3.1.

4.1 Local stability results

For the stability proof, we recapitulate some results on the stability of spectral elements on the reference square and triangle.

Theorem 4.1 *Let $\hat{K} = \hat{Q}$ and $k \geq 2$. Then there exists a constant $C > 0$ independent of k such that*

$$\inf_{0\neq p\in M_N} \sup_{0\neq \boldsymbol{v}\in \boldsymbol{V}_N} \frac{(\nabla\cdot\boldsymbol{v}, p)_{\hat{Q}}}{|\boldsymbol{v}|_{1,\hat{Q}}\,\|p\|_{0,\hat{Q}}} \geq Ck^{-\frac{1}{2}} \tag{4.1}$$

where $\boldsymbol{V}_N = \mathcal{Q}_k(\hat{Q})^2 \cap H_0^1(\hat{Q})^2$, $M_N = \mathcal{Q}_{k-2}(\hat{Q}) \cap L_0^2(\hat{Q})$.
If $\hat{K} = \hat{T}$ and $k \geq 2$ then there holds

$$\inf_{0\neq p\in M_N} \sup_{0\neq \boldsymbol{v}\in \boldsymbol{V}_N} \frac{(\nabla\cdot\boldsymbol{v}, p)_{\hat{T}}}{|\boldsymbol{v}|_{1,\hat{T}}\,\|p\|_{0,\hat{T}}} \geq Ck^{-3} \tag{4.2}$$

with C independent of k, $\boldsymbol{V}_N = \mathcal{P}_k(\hat{T})^2 \cap H_0^1(\hat{T})^2$ and $M_N = \mathcal{P}_{k-2}(\hat{T}) \cap L_0^2(\hat{T})$.

Proof (4.1) is proved in [4] or in [24] and (4.3) in [21]. □

Remark 4.2 While (4.1) is known to be optimal, (4.3) is likely suboptimal.

Remark 4.3 As in [21], Theorem 4.1 and the macro-element technique ahead (cf. Proposition 4.11) imply immediately Theorem 3.1 on κ-uniform regular meshes of affine elements.

Divergence stability on boundary layer patches (as shown in Fig. 3) is established in [18]:

Theorem 4.4 *Let $\mathcal{T} = \Delta_{\mathcal{T}_x}$ be a boundary layer mesh as in Definition 2.2. Then there exists a constant $C > 0$ independent of $\mathcal{T}_x$ and $k \geq 2$ such that the spaces*

$$\boldsymbol{V}_N = S_0^{k,1}(\Omega, \Delta_{\mathcal{T}_x})^2, \quad M_N = S_0^{k-2,0}(\Omega, \Delta_{\mathcal{T}_x})$$

satisfy the inf-sup condition (2.7) with $\gamma(N) \geq Ck^{-\frac{1}{2}}$.

Proof This is proved in [18]. □

4.2 Stability of some low order elements on geometric meshes with hanging nodes

In this subsection we establish divergence stability of low order elements on the irregular geometric meshes $\Delta_{n,\sigma}$.

4.2.1 A Clément type interpolant on $\Delta_{n,\sigma}$ We first present a result which is of independent interest, namely a Clément type interpolant $I : H_0^1(\hat{Q}) \to S_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$ on geometric meshes with hanging nodes. We remark that such irregular meshes are frequently generated by adaptive FE codes and our interpolant I allows one to derive residual a-posteriori error estimates along the lines of [30]. This will be elaborated elsewhere. The degrees of freedom of the FE-space $S_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$ are given by the nodes $\{N_i\}_{i=1}^M$ shown in Fig. 10. Let $\{\varphi_i\}_{i=1}^M$ be the usual Lagrange basis functions for these nodes, i.e. $\varphi_i \in S_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$, $|\varphi_i| \leq 1$ and $\varphi_i(N_j) = \delta_{ij}$. The support of φ_i consists of the layers i and $i+1$ (cf. Fig. 10). We define an interpolant Iu by

$$I : H_0^1(\hat{Q}) \to S_0^{1,1}(\hat{Q}, \Delta_{n,\sigma}), \quad Iu = \sum_{i=1}^{M} \alpha_i \varphi_i$$

where

$$\alpha_i = \frac{\int_{\mathrm{supp}(\varphi_i)} u dx}{\mathrm{area}(\mathrm{supp}(\varphi_i))}.$$

The next proposition states that I is essentially an interpolant of Clément type. Let

$$\mathcal{E}(\Delta_{n,\sigma}) = \{e : e \text{ edge of } K,\, K \in \Delta_{n,\sigma}\}$$

be the set of all edges of elements in $\Delta_{n,\sigma}$. The length of the edge e is denoted by h_e.

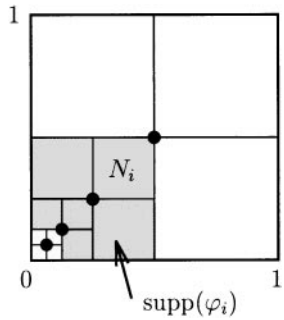

Fig. 10. Nodes in $S_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$ and $\mathrm{supp}(\varphi_i)$

Proposition 4.5 *There exists a constant $C > 0$ just depending on the grading factor σ such that*

$$\sum_{K\in\Delta_{n,\sigma}} \frac{1}{h_K^2} \|u - Iu\|_{0,K}^2 + \sum_{K\in\Delta_{n,\sigma}} |u - Iu|_{1,K}^2 + \sum_{e\in\mathcal{E}(\Delta_{n,\sigma})} h_e^{-1} \|u - Iu\|_{0,e}^2 \le C\, |u|_{1,\hat{Q}}^2 .$$

In particular, $\|Iu\|_{1,\hat{Q}}^2 \le C\, |u|_{1,\hat{Q}}^2$.

Proof Define

$$\Omega_i = \mathrm{supp}(\varphi_i), \quad d_i = \mathrm{diam}(\mathrm{supp}(\varphi_i)).$$

Ω_i is affine equivalent to a reference support $\hat{\Omega}$ which is either an L-shaped patch as in Fig. 10 or a square. As usual, the following scaling property holds [6]

$$\left|\widehat{f}\right|_{k,\hat{\Omega}} \sim d_i^{k-1}\, |f|_{k,\Omega_i}\,, \quad k = 0, 1. \tag{4.3}$$

Here, we use $f \mapsto \widehat{f}$ for the pullback operators which are defined on functions via composition with the affine mappings $\hat{\Omega} \to \Omega_i$. Now, write $u_i = u|_{\Omega_i}$ and fix an element $K \in \Delta_{n,\sigma}$. Let

$$J_K = \{i : K \subseteq \Omega_i\}\,.$$

Clearly, the cardinality of J_K is bounded by a constant C independently of K. Further, there exist constants C_1 and C_2 just depending on σ such that

$$C_2 \le \frac{d_i}{h_K} \le C_1 \qquad \forall i \in J_K. \tag{4.4}$$

Now, since $|\varphi_i| \leq 1$ and $|J_K| \leq C$

$$\frac{1}{h_K^2} \|u - Iu\|_{0,K}^2 = \frac{1}{h_K^2} \left\| u - \sum_{i \in J_K} u_i \varphi_i + \sum_{i \in J_K} u_i \varphi_i - \sum_{i \in J_K} \alpha_i \varphi_i \right\|_{0,K}^2$$
$$\leq \frac{C}{h_K^2} \sum_{i \in J_K} \left(\|u_i\|_{0,\Omega_i}^2 + \|u_i - \alpha_i\|_{0,\Omega_i}^2 \right).$$

Scaling and applying (4.4) yields

$$\frac{1}{h_K^2} \|u - Iu\|_{0,K}^2 \leq C \sum_{i \in J_K} \left(\frac{d_i^2}{h_K^2} \|\widehat{u}_i\|_{0,\hat{\Omega}}^2 + \frac{d_i^2}{h_K^2} \|\widehat{u}_i - \widehat{\alpha}_i\|_{0,\hat{\Omega}}^2 \right)$$
$$\leq C \sum_{i \in J_K} \|\widehat{u}_i\|_{1,\hat{\Omega}}^2 + \|\widehat{u}_i - \widehat{\alpha}_i\|_{0,\hat{\Omega}}^2$$

where

$$\widehat{\alpha}_i = \frac{\int_{\hat{\Omega}} \widehat{u}_i dx}{\int_{\hat{\Omega}} dx} \; (= \alpha_i).$$

With the aid of the first and the second Poincaré inequality we get

$$\frac{1}{h_K^2} \|u - Iu\|_{0,K}^2 \leq C \sum_{i \in J_K} |\widehat{u}_i|_{1,\hat{\Omega}}^2 .$$

The right hand side is scaled back to Ω_i which gives the desired result:

$$\frac{1}{h_K^2} \|u - Iu\|_{0,K}^2 \leq C \sum_{i \in J_K} |u_i|_{1,\Omega_i}^2 . \tag{4.5}$$

Further,

$$|u - Iu|_{1,K}^2 = \left| u - \sum_{i \in J_K} u_i \varphi_i + \sum_{i \in J_K} u_i \varphi_i - \sum_{i \in J_K} \alpha_i \varphi_i \right|_{1,K}^2$$
$$\leq C \left\{ |u|_{1,K}^2 + \sum_{i \in J_K} |u_i \varphi_i|_{1,\Omega_i}^2 + \sum_{i \in J_K} |u_i \varphi_i - \alpha_i \varphi_i|_{1,\Omega_i}^2 \right\}.$$

We have

$$|u_i \varphi_i|_{1,\Omega_i}^2 \leq C \, |\widehat{u}_i \widehat{\varphi}_i|_{1,\hat{\Omega}}^2$$
$$\leq C \, \|(\nabla \widehat{u}_i) \widehat{\varphi}_i\|_{0,\hat{\Omega}}^2 + C \, \|\widehat{u}_i (\nabla \widehat{\varphi}_i)\|_{0,\hat{\Omega}}^2$$
$$\leq C \, \|\widehat{u}_i\|_{1,\hat{\Omega}}^2 \leq C \, |\widehat{u}_i|_{1,\hat{\Omega}}^2 \leq C \, |u_i|_{1,\Omega_i}^2$$

and

$$\begin{aligned}|u_i\varphi_i - \alpha_i\varphi_i|^2_{1,\Omega_i} &\le C\,\|(\nabla\widehat{\varphi}_i)(\widehat{u}_i - \widehat{\alpha}_i)\|^2_{0,\hat{\Omega}} + C\,\|(\nabla\widehat{u}_i - \nabla\widehat{\alpha}_i)\widehat{\varphi}_i\|^2_{0,\hat{\Omega}} \\ &\le C\,\|\widehat{u}_i - \widehat{\alpha}_i\|^2_{0,\hat{\Omega}} + C\,\|\nabla\widehat{u}_i\|^2_{0,\hat{\Omega}} \\ &\le C\,|\widehat{u}_i|^2_{1,\hat{\Omega}} \le C\,|u_i|^2_{1,\Omega_i}\end{aligned}$$

where we used again scaling and the inequalities of Poincaré. Together we get

$$|u - Iu|^2_{1,K} \le C \sum_{i\in J_K} |u_i|^2_{1,\Omega_i}\,. \tag{4.6}$$

Let now e be an edge of the element K and $\hat{e}$ the corresponding egde in the reference element $\hat{K}$. We use now the notation $f \mapsto \widehat{f}$ for the pullback operator induced by the affine equivalence of K and $\hat{K}$. We get with the trace theorem

$$\begin{aligned}\frac{1}{h_e}\,\|u - Iu\|^2_{0,e} &\le C\,\left\|\widehat{u} - \widehat{Iu}\right\|^2_{0,\hat{e}} \le C\,\left\|\widehat{u} - \widehat{Iu}\right\|^2_{1,\hat{K}} \\ &\le \frac{C}{h_K^2}\,\|u - Iu\|^2_{0,K} + C\,|u - Iu|^2_{1,K}\,.\end{aligned}$$

Referring to (4.5) and (4.6) gives

$$\frac{1}{h_e}\,\|u - Iu\|^2_{0,e} \le C \sum_{i\in J_K} |u_i|^2_{1,\Omega_i}\,. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) is the assertion (since $|J_K| \le C$). □

Remark 4.6 An analogous interpolant can be constructed for the geometric mesh $\Delta_{n,\sigma}$ on the triangle $\hat{T}$.

4.2.2 The space $\mathcal{L}^1(K)$ In this subsection we introduce a low order velocity space which is also used e.g. in [8]. To define this space, consider a parallelogram K with vertices a_1, a_2, a_3, $a_4 = a_0$. We denote by f_i the edge $[a_{i-1}, a_i]$ and by $\boldsymbol{n}_i$ its unit outward normal as shown in Fig. 11. K is affine equivalent to the reference unit square $\hat{Q}$ in the $(\hat{x}_1, \hat{x}_2)$ reference space. The vertices, edges and normals of $\hat{Q}$ are denoted by $\hat{f}_i$, $\hat{a}_i$ and $\widehat{\boldsymbol{n}_i}$, respectively. We introduce the reference variables

$$\hat{x}_1, \quad \hat{x}_2, \quad \hat{x}_3 := 1 - \hat{x}_1, \quad \hat{x}_4 := 1 - \hat{x}_2$$

and set

$$\hat{q}_1 := \hat{x}_2\hat{x}_3\hat{x}_4, \quad \hat{q}_2 := \hat{x}_1\hat{x}_3\hat{x}_4, \quad \hat{q}_3 := \hat{x}_1\hat{x}_2\hat{x}_4, \quad \hat{q}_4 := \hat{x}_1\hat{x}_2\hat{x}_3.$$

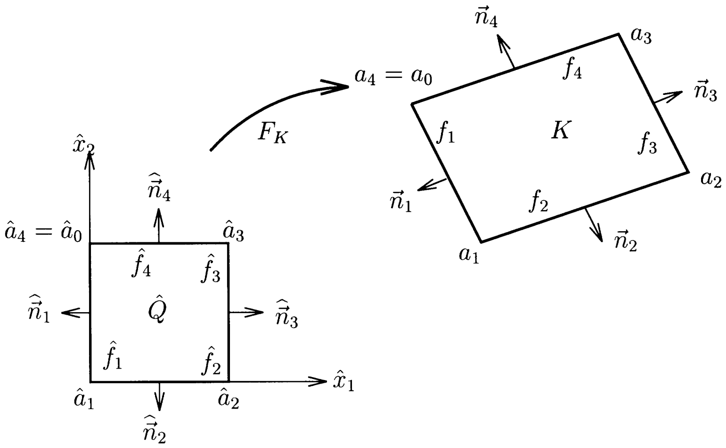

Fig. 11. Notation for K and $\hat{Q}$

For example, the polynomial $\hat{q}_1$ vanishes on the sides $\hat{f}_2$, $\hat{f}_3$ and $\hat{f}_4$. Finally, we let

$$\boldsymbol{p}_i := \boldsymbol{n}_i \left(\hat{q}_i \circ F_K^{-1}\right) \quad i = 1, \ldots, 4.$$

The velocity space $\mathcal{L}^1(K)$ is then defined as

$$\mathcal{L}^1(K) := \mathcal{Q}_1(K)^2 \oplus \operatorname{span}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4).$$

$\mathcal{L}^1(K)$ is of dimension 12 and $\mathcal{Q}_1(K)^2 \subset \mathcal{L}^1(K) \subset \mathcal{Q}_2(K)^2$ with strict inclusion.

Lemma 4.7 *A polynomial $\boldsymbol{p} \in \mathcal{L}^1(K)$ is uniquely determined by the 12 quantities:*

$$\boldsymbol{p}(a_i) \quad i = 1, \ldots, 4,$$
$$\int_{f_i} \boldsymbol{p} \cdot \boldsymbol{n}_i ds \quad i = 1, \ldots, 4.$$

Furthermore, the restriction of $\boldsymbol{p}$ to any side f_i of K depends only upon the degrees of freedom defined on that side.

Proof This is proved in [8, Sect. II.3.1]. □

Remark 4.8 If K is a triangle, we may define a space $\mathcal{K}^1(K)$ with $\mathcal{P}_1(K)^2 \subset \mathcal{K}^1(K) \subset \mathcal{P}_2(K)^2$ in complete analogy to the definition of $\mathcal{L}^1(K)$. For details, see [8, Sect. II.2.1].

For an affine mesh $\mathcal{T}$ on Ω consisting of quadrilaterals the space $\mathcal{L}^{1,1}(\Omega, \mathcal{T})$ is

$$\mathcal{L}^{1,1}(\Omega, \mathcal{T}) := \left\{\boldsymbol{u} \in H^1(\Omega)^2 : \boldsymbol{u}|_K \in \mathcal{L}^1(K)\ \forall K \in \mathcal{T}\right\} \tag{4.8}$$

and

$$\mathcal{L}_0^{1,1}(\Omega, \mathcal{T}) := \mathcal{L}^{1,1}(\Omega, \mathcal{T}) \cap H_0^1(\Omega)^2.$$

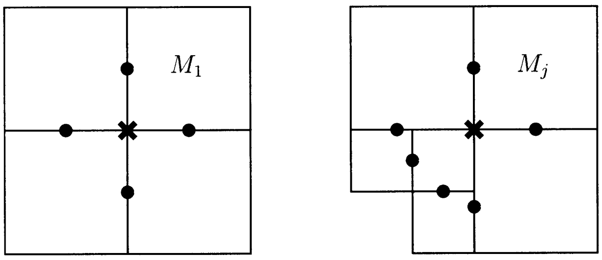

Fig. 12. The degrees of freedom of $\mathcal{L}_0^{1,1}(M_j)$

4.2.3 Divergence stability of $\mathcal{L}_0^{1,1} \times S_0^{0,0}$ on $\Delta_{n,\sigma}$ We are now able to show the inf-sup condition for $\mathcal{L}_0^{1,1} \times S_0^{0,0}$ elements on the irregular geometric mesh $\Delta_{n,\sigma}$. To do so, we apply the technique of overlapping macro-patches of [23].

Theorem 4.9 *The spaces $\mathcal{L}_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$ and $S_0^{0,0}(\hat{Q}, \Delta_{n,\sigma})$ are divergence stable, that is the inf-sup condition (2.7) holds with a constant just depending on the grading factor σ.*

Proof We introduce the patches $\{M_j\}_{j=1}^{M}$ using the numbering in (2.11):

$$M_1 = \Omega_{11} \cup \Omega_{22} \cup \Omega_{12} \cup \Omega_{32},$$
$$M_j = \cup \{\Omega_{ik} : 1 \leq i \leq 3,\ j \leq k \leq j+1\} \quad 2 \leq j \leq n.$$

M_1 is built of the four elements near the origin, whereas M_j for $j \geq 2$ consists of the elements in the layers j and $j+1$. As in (2.14) and (4.8) we let

$$S^{0,0}(M_j) = \left\{p \in L^2(M_j) : p|_K \in \mathcal{Q}_0(K),\ K \subset M_j\right\},$$
$$\mathcal{L}_0^{1,1}(M_j) = \left\{\boldsymbol{v} \in H_0^1(M_j)^2 : \boldsymbol{v}|_K \in \mathcal{L}^1(K),\ K \subset M_j\right\}$$

and

$$N_{M_j} = \left\{p \in S^{0,0}(M_j) :\ (\nabla \cdot \boldsymbol{v}, p)_{M_j} = 0\ \forall \boldsymbol{v} \in \mathcal{L}_0^{1,1}(M_j)\right\}.$$

The degrees of freedom of $\mathcal{L}_0^{1,1}(M_j)$ are shown on Fig. 12. The circles indicate the values of $\boldsymbol{v} \cdot \boldsymbol{n}$ and the crosses the nodal values (cf. Lemma 4.7). Now, it holds

(4.9) $$N_{M_j} = \{p = \text{const on } M_j\},$$

since by our choice of the velocity spaces a pressure in N_{M_j} is not allowed to have jumps over the interelement edges. We can split $S^{0,0}(M_j)$ orthogonally in $L^2(M_j)$ into

(4.10) $$S^{0,0}(M_j) = N_{M_j} \oplus W_{M_j}.$$

Let

$$\mathcal{E}(M_j) = \{e : e \text{ edge of an element } K \subset M_j,\, e \not\subset \partial M_j\}$$

denote the set of all interelement edges in the patch M_j. Extra care must be taken due to the presence of hanging nodes. Therefore, we define

$$\mathcal{E}_0(M_j) = \{e \in \mathcal{E}(M_j) : e \text{ has no hanging node in the mid-point}\}\,.$$

Globally, $\mathcal{E}(\Delta_{n,\sigma})$ and $\mathcal{E}_0(\Delta_{n,\sigma})$ are defined completely analogous. Recall that the length of an edge e is h_e. We denote by $[f]_e$ the jump of a piecewise continuous function f across the edge e of an element K:

$$[f]_e(x) = \lim_{t\to 0^+} f(x + t\boldsymbol{n}_e) - \lim_{t\to 0^+} f(x - t\boldsymbol{n}_e) \quad x \in e$$

where $\boldsymbol{n}_e$ is the unit outward normal to the element K. On each patch M_j we introduce a mesh-dependent seminorm

$$|p|^2_{M_j} = \sum_{K\subset M_j} h_K^2 \,\|\nabla p\|^2_{0,K} + \sum_{e\in\mathcal{E}_0(M_j)} h_e \int_e |[p]_e|^2\, ds.$$

For $p \in S^{0,0}(M_j)$ only the jump terms contribute to this seminorm. Globally, we define analogously

$$|p|^2_{h,\hat{Q}} = \sum_{K\in\Delta_{n,\sigma}} h_K^2 \,\|\nabla p\|^2_{0,K} + \sum_{e\in\mathcal{E}_0(\Delta_{n,\sigma})} h_e \int_e |[p]_e|^2\, ds.$$

Hence, a scaling argument gives the local stability condition

$$\sup_{0\neq \boldsymbol{v}\in\mathcal{L}_0^{1,1}(M_j)} \frac{(\nabla\cdot\boldsymbol{v}, p)_{M_j}}{|\boldsymbol{v}|_{1,M_j}\,|p|_{M_j}} \geq \gamma > 0 \qquad \forall p \in W_{M_j}\backslash\{0\} \tag{4.11}$$

where γ is independent of j (and thus of the meshwidth h) but depends on the grading factor σ.
Now, let $0 \neq p \in S_0^{0,0}(\hat{Q}, \Delta_{n,\sigma})$. We write $p_j := p|_{M_j}$. According to (4.9) and (4.10) we decompose p_j into

$$p_j = c_j + q_j$$

where $c_j \in N_{M_j}$ is constant on M_j and $q_j \in W_{M_j}$. (4.11) implies that for each q_j there exists a velocity $\boldsymbol{v}_j \in \mathcal{L}_0^{1,1}(M_j)$ (choose $\boldsymbol{v}_j = 0$ if $q_j = 0$) such that

$$(\nabla\cdot\boldsymbol{v}_j, q_j)_{M_j} \geq \gamma\, |q_j|^2_{M_j}\,, \quad |\boldsymbol{v}_j|_{1,M_j} \leq |q_j|_{M_j}\,,$$

and therefore also

$$(\nabla\cdot\boldsymbol{v}_j, p_j)_{M_j} \geq \gamma\, |p_j|^2_{M_j}\,, \quad |\boldsymbol{v}_j|_{1,M_j} \leq |p_j|_{M_j}\,.$$

We set now $\boldsymbol{v} := \sum_{j=1}^{M} \boldsymbol{v}_j$ and have $\boldsymbol{v} \in \mathcal{L}_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$. Then

$$(\nabla \cdot \boldsymbol{v}, p)_{\hat{Q}} \geq \gamma \sum_{j=1}^{M} |p_j|^2_{M_j} \geq C\, |p|^2_{h,\hat{Q}} \tag{4.12}$$

and

$$|\boldsymbol{v}|^2_{1,\hat{Q}} \leq \sum_{j=1}^{M} |\boldsymbol{v}_j|^2_{1,M_j} \leq C\, |p|^2_{h,\hat{Q}} \,. \tag{4.13}$$

(4.12) and (4.13) imply

$$\sup_{0 \neq \boldsymbol{v} \in \mathcal{L}_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})} \frac{(\nabla \cdot \boldsymbol{v}, p)_{\hat{Q}}}{|\boldsymbol{v}|_{1,\hat{Q}}} \geq C_1\, |p|_{h,\hat{Q}} = C_1\, \|p\|_{0,\hat{Q}} \frac{|p|_{h,\hat{Q}}}{\|p\|_{0,\hat{Q}}}. \tag{4.14}$$

Following still [23], we show that in (4.14) the semi-norm can be replaced by the full L^2-norm. By the continuous inf-sup condition (2.6) there is a velocity $\boldsymbol{v} \in H_0^1(\hat{Q})^2$ such that

$$(\nabla \cdot \boldsymbol{v}, p)_{\hat{Q}} \geq C\, \|p\|^2_{0,\hat{Q}} \,, \quad |\boldsymbol{v}|_{1,\hat{Q}} \leq \|p\|_{0,\hat{Q}} \,.$$

Let $\boldsymbol{v}_h = \boldsymbol{I}\boldsymbol{v} := (Iv_1, Iv_2) \in S_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})^2$ where I is the Clément type interpolant of Proposition 4.5. We integrate by parts, apply Cauchy-Schwarz and Proposition 4.5 to get

$$\begin{aligned}
(\nabla \cdot \boldsymbol{v}_h, p)_{\hat{Q}} &= (\nabla \cdot (\boldsymbol{v}_h - \boldsymbol{v}), p)_{\hat{Q}} + (\nabla \cdot \boldsymbol{v}, p)_{\hat{Q}} \\
&= \sum_{K \in \Delta_{n,\sigma}} \int_K (\boldsymbol{v} - \boldsymbol{v}_h) \cdot \nabla p \; dx \\
&\quad + \sum_{e \in \mathcal{E}_0(\Delta_{n,\sigma})} \int_e ((\boldsymbol{v}_h - \boldsymbol{v}) \cdot \boldsymbol{n})\, [p]_e ds + C\, \|p\|^2_{0,\hat{Q}} \\
&\geq - \left\{ \sum_{K \in \Delta_{n,\sigma}} h_K^{-2}\, \|\boldsymbol{v}_h - \boldsymbol{v}\|^2_{0,K} \right. \\
&\quad \left. + \sum_{e \in \mathcal{E}(\Delta_{n,\sigma})} h_e^{-1}\, \|\boldsymbol{v}_h - \boldsymbol{v}\|^2_{0,e} \right\}^{\frac{1}{2}} |p|_{h,\hat{Q}} + C\, \|p\|^2_{0,\hat{Q}} \\
&\geq -C_2\, |\boldsymbol{v}|_{1,\hat{Q}}\, |p|_{h,\hat{Q}} + C_3\, \|p\|^2_{0,\hat{Q}} \\
&\geq \|p\|^2_{0,\hat{Q}} \left(C_3 - C_2 \frac{|p|_{h,\hat{Q}}}{\|p\|_{0,\hat{Q}}} \right).
\end{aligned}$$

Further, $|\boldsymbol{v}_h|_{1,\hat{Q}} \leq C\,\|p\|_{0,\hat{Q}}$, such that we established

$$\sup_{0\neq\boldsymbol{v}\in\mathcal{L}_0^{1,1}(\hat{Q},\Delta_{n,\sigma})} \frac{(\nabla\cdot\boldsymbol{v},p)_{\hat{Q}}}{|\boldsymbol{v}|_{1,\hat{Q}}} \geq \|p\|_{0,\hat{Q}}\left(C_4 - C_5\frac{|p|_{h,\hat{Q}}}{\|p\|_{0,\hat{Q}}}\right) \tag{4.15}$$

We write t for the ratio $|p|_{h,\hat{Q}} \,/\, \|p\|_{0,\hat{Q}}$ and combine (4.14) and (4.15) into

$$\sup_{0\neq\boldsymbol{v}\in\mathcal{L}_0^{1,1}(\hat{Q},\Delta_{n,\sigma})} \frac{(\nabla\cdot\boldsymbol{v},p)_{\hat{Q}}}{|\boldsymbol{v}|_{1,\hat{Q}}} \geq \|p\|_{0,\hat{Q}}\min_{t\geq 0} f(t)$$

with $f(t) = \max(C_4 - C_5 t, C_1 t)$. Since $\min_{t\geq 0} f(t) = \frac{C_1 C_4}{C_1+C_5}$, the assertion follows. $\square$

Remark 4.10 A similar construction yields stability of low order elements for the geometric mesh $\Delta_{n,\sigma}$ on the triangle $\hat{T}$. This holds for example for $S_0^{2,1}\times S_0^{0,0}$ elements or one could use the velocity space $\mathcal{K}^1(K)$ mentioned in Remark 4.8.

4.3 A macro-element technique

A useful tool to prove divergence stability is the macro-element technique introduced for example in [24]. It is stated in a very general form in the next proposition whose proof is given for the sake of completeness.

Proposition 4.11 *Let $\mathcal{F}$ be a family of irregular or regular affine meshes on the reference element $\hat{K}$. On a bounded polygon $\Omega\subset\mathbb{R}^2$ let $\mathcal{T}$ be an affine mesh which is obtained from a (coarser) affine κ-uniform macro-element mesh $\mathcal{T}_{\rm m}$ in the following way: Some elements of $\mathcal{T}_{\rm m}$ are further partitioned into $F_K(\hat{\mathcal{T}})$ where $\hat{\mathcal{T}}\in\mathcal{F}$ and F_K is the affine mapping between $\hat{K}$ and K. Let $\underline{k}$ be a polynomial degree distribution on $\mathcal{T}$ and $|\underline{k}| := \max\{k_K : K\in\mathcal{T}\}$. Assume that there exists a space $\boldsymbol{X}_N\subseteq S_0^{\underline{k},1}(\Omega,\mathcal{T})^2\subset H_0^1(\Omega)^2$ such that*

$$\inf_{0\neq p\in S_0^{0,0}(\Omega,\mathcal{T}_{\rm m})}\ \sup_{0\neq\boldsymbol{v}\in\boldsymbol{X}_N} \frac{(\nabla\cdot\boldsymbol{v},p)_\Omega}{|\boldsymbol{v}|_{1,\Omega}\,\|p\|_{0,\Omega}} \geq C_1 \tag{4.16}$$

with a constant $C_1 > 0$ independent of $\underline{k}$. Assume that on the reference element $\hat{K}$ the local stability condition

$$\inf_{0\neq p\in S_0^{k-2,0}(\hat{K})}\ \sup_{0\neq\boldsymbol{v}\in S_0^{k,1}(\hat{K})^2} \frac{(\nabla\cdot\boldsymbol{v},p)_\Omega}{|\boldsymbol{v}|_{1,\Omega}\,\|p\|_{0,\Omega}} \geq C_2 k^{-\alpha} \qquad \forall k\geq 2 \tag{4.17}$$

is valid with $C_2 > 0$ and $\alpha > 0$ independent of k. Assume further that the family $\mathcal{F}$ is uniformly stable in the sense that there holds

$$\inf_{0 \neq p \in S_0^{\underline{k}-2,0}(\hat{K},\hat{\mathcal{T}})} \sup_{0 \neq \boldsymbol{v} \in S_0^{\underline{k},1}(\hat{K},\hat{\mathcal{T}})^2} \frac{(\nabla \cdot \boldsymbol{v}, p)_{\hat{K}}}{|\boldsymbol{v}|_{1,\hat{K}} \, \|p\|_{0,\hat{K}}} \geq C_2 \, |\underline{k}|^{-\alpha} \tag{4.18}$$

for all $\hat{\mathcal{T}} \in \mathcal{F}$ and all polynomial degree vectors $\underline{k}$ on $\hat{\mathcal{T}}$ that appear in the correspondingly refined macro-elements.

Then there exists a constant $C > 0$ only depending on C_1, C_2 and κ such that the spaces

$$\boldsymbol{V}_N = S_0^{\underline{k},1}(\Omega,\mathcal{T})^2, \quad M_N = S_0^{\underline{k}-2,0}(\Omega,\mathcal{T}) \tag{4.19}$$

satisfy the inf-sup condition (2.7) with $\gamma(N) \geq C \, |\underline{k}|^{-\alpha}$.

Proof Let $p \in S_0^{\underline{k}-2,0}(\Omega,\mathcal{T})$. We decompose p into $p = p^* + p_{\rm m}$ where $p_{\rm m}$ is the $L^2(\Omega)$-projection of p onto $S_0^{0,0}(\Omega,\mathcal{T}_{\rm m})$, the space of piecewise constant pressures with vanishing mean value on the macro-element mesh $\mathcal{T}_{\rm m}$. Because of (4.16) there exists $\boldsymbol{v}_{\rm m} \in \boldsymbol{X}_N \subseteq S_0^{\underline{k},1}(\Omega,\mathcal{T})^2$ such that

$$(\nabla \cdot \boldsymbol{v}_{\rm m}, p_{\rm m})_\Omega \geq C_1 \, \|p_{\rm m}\|_{0,\Omega}^2 \, , \quad |\boldsymbol{v}_{\rm m}|_{1,\Omega} \leq \|p_{\rm m}\|_{0,\Omega} \, . \tag{4.20}$$

Next, consider $p^* \in S_0^{\underline{k}-2,0}(\Omega,\mathcal{T})$. Therefore, fix a macro-element $K \in \mathcal{T}_{\rm m}$ and set $p_K^* := p^*|_K$. By construction, $p_K^* \in S_0^{\underline{k}-2,0}(K,\mathcal{T}_K)$ where $\mathcal{T}_K$ is the restriction of $\mathcal{T}$ to the macro-element K. We transform p_K^* back to the reference element $\hat{K}$ via the affine transformation F_K, that is we put

$$p_{\hat{K}}^* = p_K^* \circ F_K.$$

We have $\mathcal{T}_K = F_K(\hat{\mathcal{T}})$ for some $\hat{\mathcal{T}} \in \mathcal{F}$ if K is further refined or $\mathcal{T}_K = F_K(\hat{\mathcal{T}})$ with $\hat{\mathcal{T}} = \hat{K}$ if K is not locally refined. By (4.17) or (4.18) there exists $\boldsymbol{v}_{\hat{K}}^* \in S_0^{\underline{k},1}(\hat{K},\hat{\mathcal{T}})^2$ such that

$$\begin{aligned} &\left(\nabla \cdot \boldsymbol{v}_{\hat{K}}^*, p_{\hat{K}}^*\right)_{\hat{K}} \geq C_2 \, |\underline{k}|^{-\alpha} \left\|p_{\hat{K}}^*\right\|_{0,\hat{K}}^2 \, , \\ &\left|\boldsymbol{v}_{\hat{K}}^*\right|_{1,\hat{K}} \leq \left\|p_{\hat{K}}^*\right\|_{0,\hat{K}} \, . \end{aligned} \tag{4.21}$$

We can not use the usual pushforward operator to define $\boldsymbol{v}_K^*$ on K but rather the *Piola-transform*

$$\boldsymbol{v}_K^* = P_K(\boldsymbol{v}_{\hat{K}}^*) = |J_K|^{-1} \, J_K \boldsymbol{v}_{\hat{K}}^* \circ F_K^{-1}.$$

Here, J_K is the Jacobian of F_K and $|J_K| = \det(J_K)$. J_K is constant and thus $\boldsymbol{v}_K^* \in S_0^{\underline{k},1}(K,\mathcal{T}_K)^2$. Moreover, there holds (cf. [5])

$$\left(\nabla \cdot \boldsymbol{v}_{\hat{K}}^*, p_{\hat{K}}^*\right)_{\hat{K}} = (\nabla \cdot \boldsymbol{v}_K^*, p_K^*)_K \,. \tag{4.22}$$

(4.21), (4.22) and scaling give

$$\begin{aligned}(\nabla \cdot \boldsymbol{v}_K^*, p_K^*)_K &\geq C_2 \, |\underline{k}|^{-\alpha} \left\| p_{\hat{K}}^* \right\|_{0,\hat{K}}^2 \\ &\geq \frac{C}{h_K^2} C_2 \, |\underline{k}|^{-\alpha} \, \|p_K^*\|_{0,K}^2 \,.\end{aligned} \tag{4.23}$$

By similar scaling properties for the Piola-transform (cf. [5]) we get

$$\begin{aligned}|\boldsymbol{v}_K^*|_{1,K} &\leq C \frac{h_K}{\rho_K^2} \left| \boldsymbol{v}_{\hat{K}}^* \right|_{1,\hat{K}} \leq C \frac{h_K}{\rho_K^2} \left\| p_{\hat{K}}^* \right\|_{0,\hat{K}} \\ &\leq C \frac{h_K}{\rho_K^3} \, \|p_K^*\|_{0,K}\end{aligned} \tag{4.24}$$

where we applied once again (4.21). (4.23) and (4.24) imply the existence of a $S_0^{\underline{k},1}(K,\mathcal{T}_K)^2$-velocity field on K also denoted by $\boldsymbol{v}_K^*$ such that

$$\begin{aligned}(\nabla \cdot \boldsymbol{v}_K^*, p_K^*)_K &\geq \frac{C}{\kappa^3} C_2 \, |\underline{k}|^{-\alpha} \, \|p_K^*\|_{0,K}^2 \,, \\ |\boldsymbol{v}_K^*|_{1,K} &\leq \|p_K^*\|_{0,K} \,.\end{aligned} \tag{4.25}$$

We define now $\boldsymbol{v}^* = \sum_{K \in \mathcal{T}_{\mathrm{m}}} \boldsymbol{v}_K^*$ which belongs to $S_0^{\underline{k},1}(\Omega,\mathcal{T})^2 \subset H_0^1(\Omega)^2$. (4.25) holds independently of K and hence the same estimate is valid for $\boldsymbol{v}^*$,

$$(\nabla \cdot \boldsymbol{v}^*, p^*)_\Omega \geq \underbrace{\frac{C}{\kappa^3} C_2}_{=:C_3} |\underline{k}|^{-\alpha} \, \|p^*\|_{0,\Omega}^2 \,, \quad |\boldsymbol{v}^*|_{1,K} \leq \|p^*\|_{0,\Omega} \,. \tag{4.26}$$

Select now $\boldsymbol{v} = \boldsymbol{v}^* + \delta \boldsymbol{v}_{\mathrm{m}}$ where $\delta > 0$ is still at our disposal. Then

$$\begin{aligned}(\nabla \cdot \boldsymbol{v}, p)_\Omega = (\nabla \cdot \boldsymbol{v}^*, p^*)_\Omega &+ \delta \, (\nabla \cdot \boldsymbol{v}_{\mathrm{m}}, p_{\mathrm{m}})_\Omega \\ &+ (\nabla \cdot \boldsymbol{v}^*, p_{\mathrm{m}})_\Omega + \delta \, (\nabla \cdot \boldsymbol{v}_{\mathrm{m}}, p^*)_\Omega \,.\end{aligned}$$

Since p_{m} is piecewise constant on $\mathcal{T}_{\mathrm{m}}$ and $\boldsymbol{v}^*$ vanishes on ∂K for all $K \in \mathcal{T}_{\mathrm{m}}$ the third term $(\nabla \cdot \boldsymbol{v}^*, p_{\mathrm{m}})_\Omega$ is zero. With (4.20) and (4.26) one has for $\varepsilon > 0$

$$\begin{aligned}(\nabla \cdot \boldsymbol{v}, p)_\Omega &\geq C_3 \, |\underline{k}|^{-\alpha} \, \|p^*\|_{0,\Omega}^2 + \delta C_1 \, \|p_{\mathrm{m}}\|_{0,\Omega}^2 - \delta C_4 \, |\boldsymbol{v}_{\mathrm{m}}|_{1,\Omega} \, \|p^*\|_{0,\Omega} \\ &\geq C_3 \, |\underline{k}|^{-\alpha} \, \|p^*\|_{0,\Omega}^2 + \delta C_1 \, \|p_{\mathrm{m}}\|_{0,\Omega}^2 - \delta C_4 \, \|p_{\mathrm{m}}\|_{0,\Omega} \, \|p^*\|_{0,\Omega} \\ &\geq C_3 \, |\underline{k}|^{-\alpha} \, \|p^*\|_{0,\Omega}^2 + \delta C_1 \, \|p_{\mathrm{m}}\|_{0,\Omega}^2\end{aligned}$$

$$-\frac{\delta C_4}{4\varepsilon}\|p^*\|^2_{0,\Omega} - \delta\varepsilon C_4 \|p_{\mathrm{m}}\|^2_{0,\Omega}$$
$$= \left(C_3\,|\underline{k}|^{-\alpha} - \frac{\delta C_4}{4\varepsilon}\right)\|p^*\|^2_{0,\Omega} + \delta\,(C_1 - C_4\varepsilon)\,\|p_{\mathrm{m}}\|^2_{0,\Omega}\,.$$

Choosing $\varepsilon = \frac{C_1}{2C_4}$ and $\delta = \frac{2\varepsilon C_3|\underline{k}|^{-\alpha}}{C_4}$ yields

$$(\nabla\cdot\boldsymbol{v}, p)_\Omega \geq \frac{C_3}{2}\,|\underline{k}|^{-\alpha}\,\|p^*\|^2_{0,\Omega} + C_5\,|\underline{k}|^{-\alpha}\,\|p_{\mathrm{m}}\|^2_{0,\Omega} \geq C_6\,|\underline{k}|^{-\alpha}\,\|p\|^2_{0,\Omega}\,. \tag{4.27}$$

From (4.26) and (4.20) follows also

$$|\boldsymbol{v}|_{1,\Omega} \leq |\boldsymbol{v}^*|_{1,\Omega} + \delta\,|\boldsymbol{v}_{\mathrm{m}}|_{1,\Omega} \leq \|p^*\|_{0,\Omega} + C\,|\underline{k}|^{-\alpha}\,\|p_{\mathrm{m}}\|_{0,\Omega}$$
$$\leq C_7\,\|p\|_{0,\Omega} \tag{4.28}$$

with C_7 independent of $\underline{k}$. (4.27) and (4.28) imply (4.19) which finishes the proof of Proposition 4.11. □

4.4 Proof of the main result

Applying the macro-element technique in Proposition 4.11 gives immediately the following corollaries used in the proof of Theorem 3.1.

Corollary 4.12 *Let $\Delta_{n,\sigma}$ be the geometric mesh on $\hat{Q}$ (cf. Definition 2.1). Let $\underline{k}$ be a polynomial degree vector as in (2.12) and let $|\underline{k}| = \max\{k_K : K \in \Delta_{n,\sigma}\}$. Then there exists a constant $C > 0$ independent of n and $\underline{k}$ but depending on σ such that the pairs*

$$\boldsymbol{V}_N = S_0^{\underline{k},1}(\hat{Q}, \Delta_{n,\sigma})^2, \quad M_N = S_0^{\underline{k}-2,0}(\hat{Q}, \Delta_{n,\sigma})$$

fulfill the inf-sup condition (2.7) with $\gamma(N) \geq C\,|\underline{k}|^{-\frac{1}{2}}$.

Proof We apply Proposition 4.11 with

$$\mathcal{F} = \emptyset, \quad \mathcal{T}_{\mathrm{m}} = \Delta_{n,\sigma}$$

and $\kappa = \kappa(\sigma)$ is the uniformity constant of the mesh $\Delta_{n,\sigma}$ (which depends only on σ). Setting $\boldsymbol{X}_N = \mathcal{L}_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$, condition (4.16) is satisfied due to Theorem 4.9 with $C_1 = C_1(\sigma)$ independent of $\underline{k}$. (4.17) holds because of Theorem 4.1 with $\alpha = 1/2$. The assertion follows now from Proposition 4.11. □

Corollary 4.13 *Let $\Delta^2_{n,\sigma}$ be the geometric tensor product mesh on $\hat{Q}$ (cf. Definition 2.3) with underlying geometric mesh $\Delta_{n,\sigma}$. Let $\underline{k}$ be a polynomial distribution on $\Delta^2_{n,\sigma}$ which is constant on each element $K' \in \Delta_{n,\sigma}$. Let $|\underline{k}| =$*

$\max\left\{k_K : K \in \Delta^2_{n,\sigma}\right\}$. *Then there exists a constant* $C > 0$ *independent of* n *and* $\underline{k}$ *but depending on* σ *such that the spaces*

$$\boldsymbol{V}_N = S_0^{\underline{k},1}(\hat{Q}, \Delta^2_{n,\sigma})^2, \quad M_N = S_0^{\underline{k}-2,0}(\hat{Q}, \Delta^2_{n,\sigma})$$

satisfy the inf-sup condition (2.7) with $\gamma(N) \geq C\,|\underline{k}|^{-\frac{1}{2}}$.

Proof As in Corollary 4.12 above, we apply Proposition 4.11 with

$$\mathcal{F} = \{\Delta_{\mathcal{T}_x} : \mathcal{T}_x \text{ arbitrary}\}\,, \quad \mathcal{T}_{\mathrm{m}} = \Delta_{n,\sigma}$$

and $\kappa = \kappa(\sigma)$ is the uniformity constant of the mesh $\Delta_{n,\sigma}$ (which depends only on σ). Setting $\boldsymbol{X}_N = \mathcal{L}_0^{1,1}(\hat{Q}, \Delta_{n,\sigma})$, condition (4.16) is satisfied due to Theorem 4.9 with $C_1 = C_1(\sigma)$ independent of $\underline{k}$. (4.17) follows from Theorem 4.1 and (4.18) from Theorem 4.4 with $\alpha = 1/2$ (since the constant in Theorem 4.4 does not depend on the one dimensional mesh $\mathcal{T}_x$). Thus Proposition 4.11 can be applied and Corollary 4.13 follows. □

Remark 4.14 Corollaries 4.12 and 4.13 hold also for the meshes $\Delta_{n,\sigma}$ and $\Delta^2_{n,\sigma}$ on the reference triangle $\hat{T}$ with inf-sup constant $\gamma(N) \geq C\,|\underline{k}|^{-3}$. Divergence stability for the mixed meshes mentioned in Remark 2.4 is obtained in the same way using Proposition 4.11, Theorem 4.1 and Theorem 4.4. The inf-sup condition holds with $C\,|\underline{k}|^{-\alpha}$ where $\alpha = 1/2$ if the mesh contains no triangles and $\alpha = 3$ otherwise.

Proof of Theorem 3.1. The proof of Theorem 3.1 is now straightforward. We put $\boldsymbol{X}_N = S_0^{2,1}(\Omega, \mathcal{T}_{\mathrm{m}})^2$. By standard theory (see, e.g., [8,5]), (4.16) in Proposition 4.11 is satisfied. Due to Theorem 4.1, Theorem 4.4, Corollary 4.12, Corollary 4.13 and Remark 4.14, we see that (4.17) and (4.18) in Proposition 4.11 are valid with $\alpha = 1/2$ if the mesh does not contain triangles and with $\alpha = 3$ otherwise. Proposition 4.11 therefore gives the assertion of the theorem. □

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