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MIXED PROBLEM WITH A WEIGHTED INTEGRAL CONDITION FOR A PARABOLIC EQUATION WITH THE BESSEL OPERATOR

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In this paper, we prove the existence, uniqueness and continuous dependence on the data of a solution of a mixed problem with a weighted integral condition for a parabolic equation with the Bessel operator. The proof uses a functional analysis method based on an a priori estimate and on the density of the range of the operator generated by the considered problem.

Key words: Singular Parabolic Equations, Weighted Integral Condition, A Priori Estimate.

AMS subject classifications: 35K.

1. Introduction

We consider the problem of finding a solution $v = v(r,t)$ of the problem:

$$\mathcal{L}v: = v_t - \tfrac{1}{r}(rv_r)_r = f(r,t), \quad r \in (0,R), \ \ t \in (0,T). \tag{1.1}$$

where R, T are given positive constants.

We adjoin to equation (1.1) the initial condition

$$\ell v: = v(r,0) = \Phi(r), \ \ r \in (0,R), \tag{1.2}$$

the Neumann condition

$$v_r(R,t) = \mu(t), \quad t \in (0,T), \tag{1.3}$$

and the weighted integral condition

$$\int\limits_0^R r\, v(r,t)\, dr = m(t), \tag{1.4}$$

where f, Φ, μ and m are given functions.

Problem (1.1)-(1.4) can be viewed as a nonlocal problem for parabolic equation with the Bessel operator. The analogous problem which combines a homogeneous Dirichlet condition and the linear constraint

$$\int_0^R v(r,t)\,dr = 0,$$

has been studied by Benouar-Yurchuk [1]. In the case when in Equation (1.1) instead of the Bessel operator, we have the operator $\frac{\partial}{\partial r}\left(a(r,t)\frac{\partial \cdot}{\partial r}\right)$, with Neumann and the linear constraint defined above, we refer the reader to Cannon [5], Cannon-Esteva-van der Hoek [6], Cannon-van der Hoek [7], Muravey-Filinovskii [8], Shi [9], and Bouziani [2]. For other problems with integral conditions, see Bouziani [3, 4] and references therein.

2. Preliminaries

We first transform problem (1.1)-(1.4) with inhomogeneous boundary conditions (1.3) and (1.4) to an equivalent problem with homogeneous conditions by introducing a new unknown function u defined as follows

$$u(r,t) = v(r,t) - U(r,t),$$

where

$$U(r,t) \colon = \mu(t)\cdot r + \tfrac{12(r-R)^2}{R^4}\Big(m(t) - \tfrac{R^3}{3}\mu(t)\Big).$$

Then problem (1.1)-(1.4) can be formulated as follows:

$$\mathcal{L}u = f(r,t) - \mathcal{L}U = f(r,t), \tag{2.1}$$

$$\ell u = u(r,0) = \Phi(r) - \ell U = \varphi(r), \tag{2.2}$$

$$u_r(R,t) = 0, \tag{2.3}$$

$$\int_0^R r\, u(r,t)\,dr = 0. \tag{2.4}$$

Instead of searching for the function v, we search for the function u. So the solution of problem (1.1)-(1.4) will be given by $v(r,t) = u(r,t) + U(r,t)$.

For the investigation of the problem (2.1)-(2.4), we use some function spaces. Let

$L^2_r(Q)$ and $L^2_{\sqrt{r}}(Q)$ be weighted spaces of square integrable functions with finite norms

$$\| u \|^2_{L^2_r(Q)} \colon = \| ru \|^2_{L^2(Q)},$$

$$\| u \|^2_{L^2_{\sqrt{r}}(Q)} \colon = \| \sqrt{r}u \|^2_{L^2(Q)},$$

respectively, and let $V_r^{1,0}(Q)$ be the subspace of $L_r^2(Q)$ with the finite norm

$$\| u \|^2_{V_r^{1,0}(Q)} := \| u \|^2_{L_r^2(Q)} + \| u_r \|^2_{L_r^2(Q)}.$$

We also use weighted spaces on the interval $(0, R)$ such as $L_r^2(0, R)$, $L^2_{\sqrt{r}}(0, R)$ and $V_r^1(0, R)$, whose definitions are analogous to those for functions defined on Q. For example, $V_r^1(0, R)$ is the subspace of $L_r^2(0, R)$ with the finite norm

$$\| u \|^2_{V_r^1(0,R)} := \| u \|^2_{L_r^2(0,R)} + \| u_r \|^2_{L_r^2(0,R)}.$$

Problem (2.1)-(2.4) is equivalent to the operator equation

$$Lu = \mathcal{F},$$

where $L = (\mathcal{L}, \ell)$ and $\mathcal{F} = (f, \varphi)$. The domain $D(L)$ of operator L is the set of all functions $u \in L_r^2(Q)$ satisfying conditions (2.3) and (2.4) for which $u_t, u_r, u_{tr}, u_{rr} \in L_r^2(Q)$. The operator L is viewed as being from E to F, where E is the Banach space consisting of functions $u \in L_r^2(Q)$, satisfying (2.3) and (2.4), with finite norm

$$\| u \|^2_E : - \| u_t \|^2_{L_r^2(Q)} + \| (ru_r)_r \|^2_{L^2(Q)} + \sup_{0 \le \tau \le T} \| u(r, \tau) \|^2_{V_r^1(0,R)},$$

and F is the Hilbert space of vector valued functions $\mathcal{F} = (f, \varphi) \in L_r^2(Q) \times V_r^1(0, R)$ with finite norm

$$\| \mathcal{F} \|^2_F := \| f \|^2_{L_r^2(Q)} + \| \varphi \|^2_{V_r^1(0,R)}.$$

We assume that φ satisfies the compatibility conditions of the form (2.3) and (2.4):

$$\varphi_r(R) = 0 \text{ and } \int_0^R r\varphi(r)dr = 0.$$

3. Two-Sided A Priori Estimates

Theorem 1: *For any function $u \in D(L)$, we have a priori estimate*

$$\| Lu \|_F \le 2 \| u \|_E. \tag{3.1}$$

Proof: From equation (2.1), it follows that

$$\| f \|^2_{L_r^2(Q)} \le 2 \| u_t \|^2_{L_r^2(Q)} + \| (ru_r)_r \|^2_{L^2(Q)}. \tag{3.2}$$

The initial condition (2.2) implies that

$$\| \varphi \|^2_{V_r^1(0,R)} \le \sup_{0 \le \tau \le T} \| u(r, \tau) \|^2_{V_r^1(0,R)}. \tag{3.3}$$

Addition of (3.2) and (3.3) yields the desired inequality (3.1), and this proves Theorem 1.

Theorem 2: *For any function $u \in D(L)$, we have*

$$\| u \|_E \leq c \| Lu \|_F, \tag{3.4}$$

where c is a positive constant independent of the solution u.

Proof: Taking the scalar product in $L^2(Q^\tau)$ of equation (2.1) and the operator

$$Mu = r\mathcal{F}_r(\rho u_t) + r^2 u_t - r u_r,$$

where

$$\mathcal{F}_r(\rho u) = \int_0^r \rho\, u(\rho, t)\, d\rho,$$

and

$$Q^\tau = (0, R) \times (0, \tau),$$

with $0 \leq \tau \leq T$, we get

$$\int_{Q^\tau} r u_t \mathcal{F}_r(\rho u_t) drdt + \int_{Q^\tau} r^2 u_t^2 drdt$$

$$- \int_{Q^\tau} r u_r u_t drdt - \int_{Q^\tau} (r u_r)_r \mathcal{F}_r(\rho u_t) drdt \tag{3.5}$$

$$- \int_{Q^\tau} r u_t (r u_r)_r drdt + \int_{Q^\tau} u_r (r u_r)_r drdt = (f, Mu)_{L^2_r(Q^\tau)}.$$

Using condition (2.2)-(2.4) and integrating by parts, we obtain the following relations:

$$\int_{Q^\tau} r u_t \mathcal{F}_r(\rho u_t) drdt = \frac{1}{2} \int_{Q^\tau} (\mathcal{F}_r(\rho u_t))^2 \Big|_{r=0}^{r=R} dt = 0, \tag{3.6}$$

$$- \int_{Q^\tau} (r u_r)_r \mathcal{F}_r(\rho u_t) drdt = -\int_0^\tau r u_r \mathcal{F}_r(\rho u_t) \Big|_{r=0}^{r=R} dt + \int_{Q^\tau} r^2 u_r u_t drdt$$

$$= \int_{Q^\tau} r^2 u_r u_t drdt, \tag{3.7}$$

$$-\int_{Q^\tau} ru_t(ru_r)_r drdt = -\int_0^\tau r^2u_ru_t\Big|_{r=0}^{r=R} dt + \int_{Q^\tau} r^2u_ru_{tr}drdt + \int_{Q^\tau} ru_ru_tdrdt$$

$$= \frac{1}{2}\int_0^R r^2(u_r(r,\tau))^2dr - \frac{1}{2}\int_0^R r^2\varphi_r^2dr + \int_{Q^\tau} ru_ru_tdrdt, \tag{3.8}$$

$$\int_{Q^\tau} u_r(ru_r)_r drdt = \frac{1}{2}\int_0^\tau ru_r^2\Big|_{r=0}^{r=R} dt + \frac{1}{2}\int_{Q^\tau} u_r^2drdt$$

$$= \frac{1}{2}\int_{Q^\tau} u_r^2drdt. \tag{3.9}$$

Substitution of relations (3.6)-(3.9) into (3.5) gives

$$\int_{Q^\tau} r^2u_t^2drdt + \frac{1}{2}\int_0^R r^2(u_r(r,\tau))^2dr + \frac{1}{2}\int_{Q^\tau} u_r^2drdt$$

$$= \int_{Q^\tau} rf\mathcal{F}_r(\rho u_t)drdt + \int_{Q^\tau} r^2fu_tdrdt - \int_{Q^\tau} rfu_rdrdt \tag{3.10}$$

$$-\int_{Q^\tau} r^2u_ru_tdrdt + \frac{1}{2}\int_0^R r^2\varphi_r^2dr.$$

We estimate the first four terms on the right-hand side of (3.10). By applying Cauchy's inequality we have

$$\int_{Q^\tau} rf\mathcal{F}_r(\rho u_t)drdt \leq \frac{1}{8}\parallel u_t\parallel^2_{L_r^2(Q^\tau)} + R^2\parallel f\parallel^2_{L_r^2(Q^\tau)}, \tag{3.11}$$

$$\int_{Q^\tau} r^2u_t\cdot fdrdt \leq \frac{1}{8}\parallel u_t\parallel^2_{L_r^2(Q^\tau)} + 2\parallel f\parallel^2_{L_r^2(Q^\tau)}, \tag{3.12}$$

$$-\int_{Q^\tau} ru_r \cdot f drdt \leq \frac{1}{2} \parallel u_r \parallel^2_{L^2(Q^\tau)} + \frac{1}{2} \parallel f \parallel^2_{L^2_r(Q^\tau)}, \tag{3.13}$$

$$=\int_{Q^\tau} r^2 u_r \cdot u_t drdt \leq \frac{1}{4} \parallel u_t \parallel^2_{L^2_r(Q^\tau)} + \parallel u_r \parallel^2_{L^2_r(Q^\tau)}. \tag{3.14}$$

Therefore, from equality (3.10), by virtue of inequalities (3.11)-(3.14), we obtain

$$\begin{aligned} &\frac{1}{2} \parallel u_t \parallel^2_{L^2_r(Q^\tau)} + \frac{1}{2} \parallel u_r(r,\tau) \parallel^2_{L^2_r(0,R)} \\ &\leq \left(R^2 + \frac{5}{2}\right) \parallel f \parallel^2_{L^2_r(Q^\tau)} + \frac{1}{2} \parallel \varphi_r \parallel^2_{L^2_r(0,R)} + \parallel u \parallel_r \parallel^2_{L^2_r(Q^\tau)}. \end{aligned} \tag{3.15}$$

From equation (2.1) we have

$$\frac{1}{16} \parallel (ru_r)_r \parallel^2_{L^2(Q^\tau)} \leq \frac{1}{8} \parallel f \parallel^2_{L^2_r(Q^\tau)} + \frac{1}{8} \parallel u_t \parallel^2_{L^2_r(Q^\tau)}. \tag{3.16}$$

We have the elementary inequality

$$\frac{3}{16} \parallel u(r,\tau) \parallel^2_{L^2_r(0,R)} \leq \frac{3}{16} \parallel \varphi \parallel^2_{L^2_r(0,R)} + \frac{3}{16} \parallel u \parallel^2_{L^2_r(Q^\tau)} + \frac{3}{16} \parallel u_t \parallel^2_{L^2_r(Q^\tau)}. \tag{3.17}$$

Adding (3.15)-(3.17) yields the inequality

$$\begin{aligned} &\parallel u_t \parallel^2_{L^2_r(Q^\tau)} + \parallel (ru_r)_r \parallel^2_{L^2(Q^\tau)} + \parallel u(r,\tau) \parallel^2_{V^2_r(0,R)} \\ &\leq (42 + 16R^2) \cdot (\parallel f \parallel^2_{L^2_\sigma(Q^\tau)} + \parallel \varphi \parallel^2_{V^1_r(0,R)}) \\ &\quad + (42 + 16R^2) \parallel u \parallel^2_{V^1_r(Q^\tau)}. \end{aligned} \tag{3.18}$$

Applying Lemma 3.1 from [2] to (3.18), we get

$$\begin{aligned} &\parallel u_t \parallel^2_{L^2_r(Q^\tau)} + \parallel (ru_r)_r \parallel^2_{L^2(Q^\tau)} + \parallel u(r,\tau) \parallel^2_{V^1_r(0,R)} \\ &\leq k_1 e^{k_1 T} (\parallel f \parallel^2_{L^2_r(Q)} + \parallel \varphi \parallel^2_{V^1_r(0,R)}), \end{aligned} \tag{3.19}$$

where $k_1 = 42 + 16R^2$.

Since the right-hand side of (3.19) is independent of τ, in the left-hand side, we take the least upper bound with respect to τ from 0 to T. We thus obtain inequality (3.4), with $c = k_1^{1/2} e^{k_1 T/2}$.

4. Solvability of the Problem

It follows from inequality (3.1) that the operator $L\colon E \to F$ is continuous. From inequality (3.4), it follows that the range $R(L)$ of L is closed in F. Therefore, there exists a continuous inverse operator L^{-1} yielding the solution. That is, L is a linear homeomorphism from the space E on the closed set $R(L) \subset F$. To prove that there exists a unique solution of problem (2.1)-(2.4), we have to show that $R(L) = F$.

Theorem 3: *For any $\Phi = (f, \varphi) \in F$, there is a unique solution $u = L^{-1}\Phi$ of the problem* (2.1)-(2.4)*, satisfying the estimate*

$$\parallel u \parallel_E \leq c \parallel \Phi \parallel_F,$$

where c is a positive constant independent of the solution u.

Proof: To establish the proof of this theorem, we need the following:

Proposition: *If, for all u in the set $D_0(L) \colon = \{u \in D(L) \colon \ell u = 0\}$ and for some function $\omega \in L_r^2(Q)$, we have*

$$(\mathcal{L}u, \omega)_{L_r^2(Q)} = 0, \tag{4.1}$$

then $\omega \equiv 0$ almost everywhere on Q.

Proof of Proposition: Using the fact that (4.1) holds for any function $u \in D_0(L)$, we can express it in a particular form. Let us define the function ϑ by the relation

$$\vartheta(r,t) = \int_t^T \omega(r,s)ds,$$

and let u_t be a solution of the equation

$$\tfrac{1}{r}(u_t - \mathcal{F}_r^2(\rho u_t)) = \vartheta(r,t), \tag{4.2}$$

where

$$\mathcal{F}_r^2(\rho u_t) = \int_0^r \int_0^\rho \eta u_t(\eta, t) d\eta d\rho.$$

Let the function u be given by

$$u = \begin{cases} \int_v^t u_\xi d\xi, & 0 \leq t \leq v, \\ 0, & v \leq t \leq T. \end{cases} \tag{4.3}$$

We now have

$$\omega(r,t) = \tfrac{1}{4}(\mathcal{F}_r^2(\rho u_t) - u_t)_t. \tag{4.4}$$

We now have

$$\omega(r,t) = \tfrac{1}{r}(\mathcal{F}_r^2(\rho u_t) - u_t)_t. \tag{4.4}$$

Relations (4.2) and (4.3) imply that u is in $D_0(L)$.

Lemma: *The function u defined by* (4.2) *and* (4.3) *had derivatives with respect to t up to the second order belonging to the space $L^2(Q_v)$, where $Q_v = (0,R) \times (v,T)$.*

Proof: The proof of this lemma has a purely technical character. We shall not reproduce it here, but instead refer the reader to [2].

Now replacing ω in (4.1) by its representation (4.4), we have

$$\int_Q ru_t\mathcal{F}_r^2(\rho u_{tt})drdt - \int_Q u_r\mathcal{F}_r^2(\rho u_{tt})drdt + \int_Q u_r u_{tt}drdt$$
$$- \int_Q ru_{rr}\mathcal{F}_r^2(\rho u_{tt})drdt - \int_Q ru_t u_{tt}drdt + \int_Q r_{rr}u_{tt}drdt = 0. \tag{4.5}$$

Using conditions (2.3) and (2.4), and the special form of u given by (4.2) and (4.3), integration by parts of the first and last three terms on the right-hand side of (4.5) give

$$\int_Q ru_t\mathcal{F}_r^2(\rho u_{tt})drdt = \frac{1}{2} \parallel \mathcal{F}_r(\rho u_t(r,v) \parallel^2_{L^2(0,R)}, \tag{4.6}$$

$$- \int_Q ru_{rr}\mathcal{F}_r^2(\rho u_{tt})drdt = - \int_{Q_v} ru_{tr}\mathcal{F}_r(\rho u_t)drdt + \int_{Q_v} u_r\mathcal{F}_r^2(\rho u_{tt})drdt, \tag{4.7}$$

$$- \int_Q ru_t u_{tt}drdt = \frac{1}{2} \parallel u(r,v) \parallel^2_{L^2_{\sqrt{r}}(0,R)}, \tag{4.8}$$

$$\int_Q ru_{rr}u_{tt}drdt = \parallel u_{tr} \parallel^2_{L^2_{\sqrt{r}}(Q_v)} = \int_{Q_v} u_r u_{tt}drdt. \tag{4.9}$$

Substituting equalities (4.6)-(4.9) into (4.5), we obtain

$$\parallel u_{tr} \parallel^2_{L^2_{\sqrt{r}}(Q_v)} + \frac{1}{2} \parallel \mathcal{F}_r(\rho u_t(\rho,v) \parallel^2_{L^2(0,R)} + \frac{1}{2} \parallel u_t(r,v) \parallel^2_{L^2_{\sqrt{r}}(0,R)}$$
$$= \int_{Q_v} ru_{tr}\mathcal{F}_r(\rho u_t)drdt. \tag{4.10}$$

Further, by virtue of an elementary inequality, we estimate the right-hand side of (4.10) to get

$$\parallel u_{tr} \parallel^2_{L^2_{\sqrt{r}}(Q_v)} + \parallel \mathcal{F}_r(\rho u_t(\,\cdot\,,v) \parallel^2_{L^2(0,R)} + \parallel u_t(\,\cdot\,,v) \parallel^2_{L^2_{\sqrt{r}}(0,R)}$$
$$\leq k_2(\parallel \mathcal{F}_r(\rho u_t) \parallel^2_{L^2(Q_v)} + \parallel u_t \parallel^2_{L^2_{\sqrt{r}}(Q_v)}, \tag{4.11}$$

where $k_2 = \max(R,1)$. If we put

$$y(v) = \parallel \mathcal{F}_r(\rho u_t) \parallel^2_{L^2(Q_v)} + \parallel u_t \parallel^2_{L^2_{\sqrt{r}}(Q_v)},$$

then from (4.11) we have

$$-\frac{dy}{dv} \leq k_2 y(v),$$

and consequently,

$$-\frac{d}{dv}(y(v)e^{k_2 v}) \leq 0. \tag{4.12}$$

Integrating (4.12) over the interval (v, T) and taking into account that $y(T) = 0$, we obtain

$$y(v)e^{k_2 v} \leq 0. \tag{4.13}$$

It follows from (4.13), that $\omega \equiv 0$ almost everywhere in Q_v. The length v does not depend on the choice of the origin. Proceeding in this way step by step, we prove that $\omega \equiv 0$ almost everywhere in Q. Thus, the proposition is proved.

Now, we return to the proof of Theorem 3. It is sufficient to prove that the set $R(L)$ is dense in F. Let $\Psi = (\omega, \omega_0) \in R(L)^{\perp}$ such that

$$(\mathcal{L}u, \omega)_{L_r^2(Q)} + (\ell u, \omega_0)_{V_r^1(0,R)} = 0. \tag{4.14}$$

Putting $u \in D_0(L)$ in the equation (4.14), we get

$$(\mathcal{L}u, \omega)_{L_r^2(Q)} = 0, \quad \forall u \in D_0(L).$$

By virtue of the proposition, we deduce that $\omega \equiv 0$. Thus (4.14) becomes

$$(\ell u, \omega_0)_{V_r^1(0,R)} = 0. \tag{4.15}$$

Since the range $R(\ell)$ of the operator ℓ is everywhere dense in $V_r^1(0, R)$, (4.15) implies that $\omega_0 \equiv 0$. Hence $\Psi \equiv 0$. This completes the proof of Theorem 3.

5. Conclusion

In summary, we have established the existence, uniqueness and continuous dependence on given data for solutions to a mixed problem for a parabolic equation with the Bessel operator which combines inhomogeneous Neumann condition and integral conditions. More precisely, we have constructed a sufficiently smooth function satisfying these conditions. Then we are lead to study an equivalent problem with homogeneous boundary conditions. Thus, we have established two sided a priori estimates for the operator $L: E \to F$ generated by the considered problem. Then we concluded that the operator L realizes a linear homeomorphism of the space E on the closed set $R(L) \subset F$. To prove that the studied problem possesses a unique solution, we proved that $R(L)$ is dense in F.

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