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Mixed problems for hyperbolic equations of second order

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§ 1. Introduction.

This paper is concerned with the mixed problems for hyperbolic equations of second order. Let S be a sufficiently smooth compact hypersurface in R^n, and let Ω be the interior or exterior domain of S.

Consider the hyperbolic equation of second order

$$(1.1)\qquad L[u]=\frac{\partial^2}{\partial t^2}u+a_1(x,t:D)\frac{\partial}{\partial t}u+a_2(x,t:D)u=f$$

$$a_1(x,t:D)=\sum_{i=1}^{n}2h_i(x,t)\frac{\partial}{\partial x_i}+h(x,t)$$

$$a_2(x,t:D)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\Big)+\sum_{j=1}^{n}b_j(x,t)\frac{\partial}{\partial x_j}+c(x,t)$$

where the coefficients belong to $\mathscr{B}^2(\Omega\times(-\delta_0,\infty))$[1]. We assume that $a_2(x,t:D)$ is an elliptic operator satisfying

$$(1.2)\qquad \sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j>d\sum_{i=1}^{n}\xi_i^2\qquad(d>0)$$

$$a_{ij}(x,t)=a_{ji}(x,t)$$

for all $(x,t)\in\Omega\times(-\delta_0,\infty)$ and $\xi=(\xi_1,\xi_2,\cdots,\xi_n)\in R^n$, and that $h_i(x,t)$ $(i=1,2,\cdots,n)$ are real-valued. For this equation we consider the following boundary conditions

$$(1.3)\qquad B_1u(x,t)=u(x,t)=0\quad\text{on}\quad S,$$

$$(1.4)\qquad B_2u(x,t)=\frac{\partial}{\partial n_t}u(x,t)-\langle h,\nu\rangle\frac{\partial}{\partial t}u(x,t)+\sigma(s,t)u(x,t)=0\quad\text{on}\quad S$$

where

$$\frac{\partial}{\partial n_t}=\sum_{i,j=1}^{n}a_{ij}(s,t)\nu_i\frac{\partial}{\partial x_j},\qquad\langle h,\nu\rangle=\sum_{i=1}^{n}h_i(s,t)\nu_i,$$

$\nu=(\nu_1,\cdots,\nu_n)$ is the outer unit normal of S at $s\in S$, and $\sigma(s,t)$ is a real-valued

1) $\mathscr{B}^k(\omega)$, ω being an open set, is the set of all functions defined in ω such that their partial derivatives of order $\leq k$ all exist and are continuous and bounded.

sufficiently smooth function defined on $S\times(-\delta_0, \infty)$.

Our problem is to obtain $u(x, t)\in \mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$[2], for any given initial data $\{u_0, u_1\}$ and any second member $f(x, t)$, satisfying

(i) $L[u]=f(x, t)$ in $\Omega\times(0, T)$,

(ii) $u(x, 0)=u_0$, $\frac{\partial u}{\partial t}(x, 0)=u_1$,

(iii) the boundary condition (1.3) or (1.4) for all $t\in[0, T]$.

We treat this problem as evolution equation

$$\frac{d}{dt}U(t)=\mathcal{A}(t)U(t)+F(t) \tag{1.5}$$

where

$$\mathcal{A}(t)=\begin{bmatrix} 0 & 1 \\ -a_2(x, t:D) & -a_1(x, t:D)\end{bmatrix}$$

and

$$U(t)=\begin{bmatrix} u(x, t) \\ v(x, t)\end{bmatrix}, \qquad F(t)=\begin{bmatrix} 0 \\ f(x, t)\end{bmatrix}.$$

In our treatment we introduce the spaces $\mathcal{H}_i(t)$ $(\subseteqq H^1(\Omega)\times L^2(\Omega))$ $(i=1, 2)$ equipped with the norms equivalent to $\|u\|_{1,L^2(\Omega)}+\|v\|_{L^2(\Omega)}$, which are determined according to $a_2(x, t:D)$ and the boundary condition, and in the space $\mathcal{H}_i(t)$ the semi-group theory is applied to prove the existence of the solution to the equation (1.5). However it seems difficult to apply it to the operators with the definition domain depending on t. Therefore for the boundary condition (1.4), since the definition domain of $\mathcal{A}(t)$ varies with t when the coefficients of the principal part of L are not independent of t on the boundary S, another treatment is needed. In this case we extend the operator $\mathcal{A}(t)$ to the operator from $H^1(\Omega)\times L^2(\Omega)$ into $L^2(\Omega)\times H^1(\Omega)'$, under the additional condition about L that $a_1(x, t:D)=h(x, t)$ and $b_i(x, t)$ are real-valued on S.

In each case the regularity of the solution is considered. At first the regularity with respect to t is shown by the method of successive approximation, next the regularity with respect to x by using the above result and the ellipticity of $a_2(x, t:D)$.

Our problem has been already studied by M. Krzyzański and J. Schauder [7], G. F. D. Duff [3], L. Hörmander [5] and by O. A Ladyzhenskaya [8]. The materials of Chapter viii and ix of Lions [9] have a close connection with this paper. In [3], [7], the analytic case is first treated, and in non-analytic case, with the aid of estimates of L^2-type of the solutions, the analytic approximation is used. Their treatments are complicated. And [5] derives the uniqueness theorem and estimates of solutions. [9] does not consider the regularity of

2) $u(x, t)\in\mathcal{E}_t^m(E)$ means that $u(x, t)$ is m times continuously differentiable in t as E-valued function.

the solution, and [8] does it only in the case of the boundary condition (1.3), but its proof is also complicated.

Our treatment is an extension of those of [14] and [11] and different from those of [3] and [7], and our proof on the regularity of the solution seems to be simple and natural. The essential part of this paper is Section 3. However to make our exposition easy and complete, in Section 2 we show our treatment in detail in the case where the boundary condition is independent of t. The results written in Section 2 have been obtained under the correspondence with Professor S. Mizohata.

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§ 2. Cases where the boundary condition is invariant with t.

Throughout this section we consider only the case where the boundary condition is independent of t, namely the case (1.3) and also the case (1.4) with the additional condition that the coefficients of the principal part of L and σ are independent of t at the boundary[3].

Definitions and lemma.

In order to treat the case of the Dirichlet type boundary condition (1.3), we consider the space $\mathcal{H}_1(t)$, which is $\mathcal{D}^1_{L^2}(\Omega)\times L^2(\Omega)$ with the norm

$$\|U\|^2_{\mathcal{H}_1(t)}=(U,\,U)_{\mathcal{H}_1(t)}=\sum_{i,j=1}^{n}\left(a_{ij}(x,\,t)\frac{\partial u}{\partial x_i},\,\frac{\partial u}{\partial x_j}\right)+(u,\,u)+(v,\,v) \tag{2.1}$$

where

$$U=\{u,\,v\}\in\mathcal{D}^1_{L^2}(\Omega)\times L^2(\Omega)\,.$$

$\mathcal{A}(t)$ is the operator from the definition domain

$$D_1=H^2(\Omega)\cap\mathcal{D}^1_{L^2}(\Omega)\times\mathcal{D}^1_{L^2}(\Omega) \tag{2.2}$$

into $\mathcal{H}_1(t)$.

In the case of the Neumann type boundary condition (1.4), $\mathcal{H}_2(t)$ is $H^1(\Omega)\times L^2(\Omega)$ with the norm

$$\begin{aligned}\|U\|^2_{\mathcal{H}_2(t)}&=(U,\,U)_{\mathcal{H}_2(t)}\\&=\sum_{i,j=1}^{n}\left(a_{ij}(x,\,t)\frac{\partial u}{\partial x_i},\,\frac{\partial u}{\partial x_j}\right)+\int_S\sigma(s)u\bar{u}\,dS+\beta(u,\,u)+(v,\,v)\end{aligned} \tag{2.3}$$

where

$$U=\{u,\,v\}\in H^1(\Omega)\times L^2(\Omega)\,,$$

and $\mathcal{A}(t)$ is the operator from the definition domain

3) The condition can be relaxed partially, see Remark 2.6.

(2.4) $\qquad D_2=\left\{\{u, v\} \mid u \in H^2(\Omega),\ v \in H^1(\Omega) \text{ and } \frac{\partial}{\partial n}u-\langle h, \nu\rangle v+\sigma(s)u=0 \text{ on } S\right\}$

into $\mathcal{H}_2(t)$.

REMARK 2.1. From the additional condition posed on a_{ij} and h_i, D_2 is invariant with t.

REMARK 2.2. According to (1.2), for some $M_1>0$ we have

$$\frac{1}{M_1}(\|u\|_{1,L^2(\Omega)}^2+\|v\|_{L^2(\Omega)}^2) \leqq \|U\|_{\mathcal{H}_1(t)}^2 \leqq M_1(\|u\|_{1,L^2(\Omega)}^2+\|v\|_{L^2(\Omega)}^2)$$

for all $U=\{u, v\} \in \mathcal{H}_1(t)$. Next, from the inequality

$$\int_S |u|^2 dS \leqq \varepsilon\|u\|_{1,L^2(\Omega)}^2+c(\varepsilon)\|u\|_{L^2(\Omega)}^2$$

where ε is an arbitrary positive constant, by taking β sufficiently large it follows for some $M_2>0$

$$\frac{1}{M_2}(\|u\|_{1,L^2(\Omega)}^2+\|v\|_{L^2(\Omega)}^2) \leqq \|U\|_{\mathcal{H}_2(t)}^2 \leqq M_2(\|u\|_{1,L^2(\Omega)}^2+\|v\|_{L^2(\Omega)}^2)$$

for all $U=\{u, v\} \in \mathcal{H}_2(t)$. We fix such a β.

REMARK 2.3. D_i is dense in $\mathcal{H}_i(t)$. In the case $i=1$, this is evident. For $i=2$, since

$$N=\left\{u \mid u \in H^2(\Omega),\ \frac{\partial u}{\partial n}+\sigma(s)u=0 \text{ on } S\right\}$$

is dense in $H^1(\Omega)$ and $N \times \mathcal{D}(\Omega) \subseteqq D_2$, D_2 is dense in $H^1(\Omega) \times L^2(\Omega)$. By Remark 2.2 it follows that D_2 is dense in $\mathcal{H}_2(t)$.

LEMMA 2.1. *There exists a constant $c>0$ such that for any $U \in D_i$*

(2.5) $\qquad |(\mathcal{A}(t)U, U)_{\mathcal{H}_i(t)}+(U, \mathcal{A}(t)U)_{\mathcal{H}_i(t)}| \leqq c(U, U)_{\mathcal{H}_i(t)} \qquad (i=1, 2)$

holds.

PROOF. Let $U=\{u, v\} \in D_1$.

(2.6) $\qquad (\mathcal{A}(t)U, U)_{\mathcal{H}_1(t)}+(U, \mathcal{A}(t)U)_{\mathcal{H}_1(t)}$

$$=\sum_{i,j=1}^n\left(a_{ij}(x, t)\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_j}\right)+(v, u)+(-a_2(x, t: D)u-a_1(x, t: D)v, v)$$

$$+\sum_{i,j=1}^n\left(a_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j}\right)+(u, v)+(v, -a_2(x, t: D)u-a_1(x, t: D)v).$$

By integration by parts, we get

(2.7) $\qquad \sum_{i,j=1}^n\left(a_{ij}(x, t)\frac{\partial v}{\partial x_i}, \frac{\partial u}{\partial x_j}\right)$

$$=\int_S v\overline{\frac{\partial u}{\partial n}}dS+\left(v, -\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\frac{\partial u}{\partial x_j}\right)\right),$$

$$(2.8)\qquad (a_1(x,t:D)v,v)=\int_S 2\sum_{i=1}^n h_i(x,t)\nu_i v\bar{v}\,dS$$
$$-\Big(v,\sum_{i=1}^n 2\frac{\partial}{\partial x_i}h_i(x,t)v\Big)+(h(x,t)v,v)$$
$$=2\int_S\langle h,\nu\rangle v\bar{v}\,dS-\Big(v,2\sum_{i=1}^n h_i(x,t)\frac{\partial v}{\partial x_i}\Big)$$
$$+\Big(\Big(h(x,t)-2\sum_{i=1}^n\frac{\partial h_i}{\partial x_i}(x,t)\Big)v,v\Big).$$

All the surface integrals vanish since $v\in\mathscr{D}^1_{L^2(\Omega)}$, then the right side of (2.6) equals

$$2\,\mathrm{Re}(u,v)-2\,\mathrm{Re}\Big(\sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u,v\Big)$$
$$+2\,\mathrm{Re}\Big(\Big(h(x,t)-\sum_{i=1}^n\frac{\partial h_i}{\partial x_i}(x,t)\Big)v,v\Big)$$

and this is estimated by

$$c'(\|u\|_{1,L^2(\Omega)}\|v\|_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)})\leqq c''(\|u\|^2_{1,L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}).$$

By Remark 2.2, we see (2.5) holds for $i=1$.

Now let $U=\{u,v\}\in D_2$.

$$(2.9)\qquad (\mathcal{A}(t)U,U)_{\mathcal{H}_2(t)}+(U,\mathcal{A}(t)U)_{\mathcal{H}_2(t)}$$
$$=\sum_{i,j=1}^n\Big(a_{ij}(x,t)\frac{\partial v}{\partial x_i},\frac{\partial u}{\partial x_j}\Big)+\int_S\sigma(s)v\bar{u}\,dS+\beta(v,u)$$
$$+(-a_2(x,t:D)u-a_1(x,t:D)v,v)$$
$$+\sum_{i,j=1}^n\Big(a_{ij}(x,t)\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_j}\Big)+\int_S\sigma(s)u\bar{v}\,dS+\beta(u,v)$$
$$+(v,-a_2(x,t:D)u-a_1(x,t:D)v)$$

by (2.7) and (2.8)

$$=\int_S\Big(\frac{\partial u}{\partial n}+\sigma(s)u\Big)\bar{v}\,dS+\int_S v\overline{\Big(\frac{\partial u}{\partial n}+\sigma(s)u\Big)}dS-2\int_S\langle h,\nu\rangle v\bar{v}\,dS$$
$$+2\,\mathrm{Re}\Big[\beta(u,v)-\Big(\sum_{j=1}^n b_j(x,t)\frac{\partial u}{\partial x_j}+c(x,t)u,v\Big)+\Big(\Big(h(x,t)-\sum_{i=1}^n\frac{\partial h_i}{\partial x_i}(x,t)\Big)v,v\Big)\Big].$$

Since $U\in D_2$, the surface integral vanishes, and by the similar way to the case $i=1$, we get (2.5).

COROLLARY 2.1. *For all real λ such that $|\lambda|>c$, the estimate*

$$(2.10)\qquad \|(\lambda I-\mathcal{A}(t))U\|_{\mathcal{H}_i(t)}\geqq(|\lambda|-c)\|U\|_{\mathcal{H}_i(t)}\qquad(i=1,2)$$

holds for any $U\in D_i$.

PROOF.

$$((\lambda I-\mathcal{A}(t))U,\ (\lambda I-\mathcal{A}(t))U)_{\mathcal{H}_i(t)}$$
$$\geqq \lambda^2(U,U)_{\mathcal{H}_i(t)}-\lambda\{(u,\ \mathcal{A}(t)U)_{\mathcal{H}_i(t)}+(\mathcal{A}(t)U,\ U)_{\mathcal{H}_i(t)}\}$$

according to (2.5)

$$\geqq \lambda^2(U,\ U)_{\mathcal{H}_i(t)}-|\lambda|c(U,\ U)_{\mathcal{H}_i(t)}=\{(|\lambda|-c)^2+c(|\lambda|-c)\}\|U\|^2_{\mathcal{H}_i(t)}$$

then we obtain (2.10) if $|\lambda|>c$.

Resolvent.

LEMMA 2.2. *There exists a constant $\delta>0$ such that for all λ real and $|\lambda|>\delta$, $\lambda I-\mathcal{A}(t)$ is a bijective mapping from D_i onto $\mathcal{H}_i(t)$. Moreover we have*

(2.11)
$$\|(\lambda I-\mathcal{A}(t))^{-1}\|_{\mathcal{H}_i(t)}\leqq\frac{1}{|\lambda|-\delta}.$$

PROOF. Consider the equation in U

(2.12)
$$(\lambda I-\mathcal{A}(t))U=F$$

namely

(2.13)
$$\begin{cases}\lambda u-v=f_1\\ a_2u+(a_1+\lambda)v=f_2\end{cases}$$

where $\{f_1,f_2\}\in\mathcal{D}^1_{L^2}(\Omega)\times L^2(\Omega)$ in the case of the Dirichlet type boundary condition and $\in H^1(\Omega)\times L^2(\Omega)$ in the case of the Neumann type. For convenience, we call the former case the first case and the latter one the second case.

The substitution of the first relation

(2.14)
$$v=\lambda u-f_1$$

in the second of (2.13) gives

(2.15)
$$a_\lambda u\equiv(a_2+\lambda a_1+\lambda^2)u=(a_1+\lambda)f_1+f_2\in L^2(\Omega).$$

Thus we are led to consider an elliptic equation containing the parameter λ

(2.16)
$$a_\lambda u=f\in L^2(\Omega).$$

Consider the first case. The solvability of (2.16) means the existence of $u\in H^2(\Omega)\cap\mathcal{D}^1_{L^2}(\Omega)$ for any $f\in L^2(\Omega)$. Then if (2.16) is solvable, defining v by (2.14), we have a solution $\{u,v\}\in H^2(\Omega)\cap\mathcal{D}^1_{L^2}(\Omega)\times\mathcal{D}^1_{L^2}(\Omega)$ of (2.12). Now the solvability of (2.16) is seen by the well-known variation method.

Consider the second case. If $\{u,v\}\in D_2$ is a solution of (2.13), then for $x\in S$

$$\langle h,\nu\rangle v=\langle h,\nu\rangle(\lambda u-f_1)=\frac{\partial u}{\partial n}+\sigma u.$$

Namely, u satisfies the boundary condition

(2.17)
$$\frac{\partial}{\partial n}u-\lambda\langle h,\nu\rangle u+\sigma u=-\langle h,\nu\rangle f_1.$$

Conversely, if $u \in H^2(\Omega)$ satisfies this boundary condition, then by defining $v=\lambda u-f_1$, we see $\{u, v\} \in D_2$. Here the solvability of (2.16) means the existence of the solution $u \in H^2(\Omega)$ of (2.16) satisfying (2.17) for any given $f \in L^2(\Omega)$, $f_1 \in H^1(\Omega)$.

Let us assume the existence of such a solution u. Then

$$\langle (a_2+\lambda a_1+\lambda^2)u, \bar{\phi}\rangle = \langle f, \bar{\phi}\rangle, \qquad \phi \in H^1(\Omega)$$

gives, by integration by parts,

$$\sum_{i,j=1}^{n}\Big(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial \phi}{\partial x_j}\Big)+(e_1 u, \phi)+\frac{\lambda}{2}(a_1 u, \phi)+\frac{\lambda}{2}(u, a_1^*\phi)+\lambda^2(u, \phi)$$
$$-\int_S\Big(\frac{\partial u}{\partial n}-\lambda\langle h, \nu\rangle u\Big)\bar{\phi}\,dS=(f, \phi),$$

where e_1 is the first order and a_1^* is the formal adjoint of a_1. Taking account of (2.17), the surface integral is equal to

$$\int_S \sigma u\bar{\phi}\,dS+\int_S\langle h, \nu\rangle f_1\bar{\phi}\,dS.$$

Thus, u is a solution of the variational equation:

$$\text{(2.18)} \qquad \sum_{i,j=1}^{n}\Big(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial \phi}{\partial x_j}\Big)+(e_1 u, \phi)+\frac{\lambda}{2}\{(a_1 u, \phi)+(u, a_1^*\phi)\}+\lambda^2(u, \phi)$$
$$+\int_S \sigma u\bar{\phi}\,dS=(f, \phi)-\int_S\langle h, \nu\rangle f_1\bar{\phi}\,dS.$$

Since h_i $(i=1, 2, \cdots, n)$ are real-valued, we see that

$$|\mathrm{Re}\,\{(a_1 u, u)+(u, a_1^* u)\}| \leq c\|u\|^2.$$

Now we see that there exists some positive λ_0 such that for any $|\lambda|>\lambda_0$ (λ real) the variational equation (2.18) has a unique solution $u \in H^1(\Omega)$. Moreover, one can prove that $u \in H^2(\Omega)$. This implies that u is a solution of (2.16) satisfying the boundary condition (2.17).

From the solvability of (2.12) and the estimate of (2.10) it follows that the existence of $(\lambda I-\mathcal{A}(t))^{-1}$ and the estimate (2.11).

For $U=\{u, v\} \in H^p(\Omega)\times H^{p-1}(\Omega)$, we define the following norm

$$\text{(2.19)} \qquad |||U|||_p^2=\|u\|_{p,L^2(\Omega)}^2+\|v\|_{p-1,L^2(\Omega)}^2.$$

Suppose that the coefficients of L belong to $\mathcal{B}^p(\Omega\times(-\delta_0, \infty))$, then we have

COROLLARY. *For $\lambda_0>\delta$ (λ_0 fixed), there exists $d_p>0$ such that for any $U \in D_i \cap H^p(\Omega)\times H^{p-1}(\Omega)$*

$$\text{(2.20)} \qquad |||U|||_p < d_p|||(\lambda_0 I-\mathcal{A}(t))U|||_{p-1}.$$

PROOF. From the ellipticity of $a_2(x, t: D)$ and Lemma 2.2, $\lambda_0 I-\mathcal{A}(t)$ is a bijective continuous mapping from $D_i \cap H^p(\Omega)\times H^{p-1}(\Omega)$ onto $\mathcal{H}_i(t)\cap H^{p-1}(\Omega)$

$\times H^{p-2}(\Omega)$, then by Banach's closed graph theorem we get (2.20).

Energy inequality.

Now we show the energy inequalities for our problems. These inequalities play an important rôle not only in the proof of the existence of the solution but also in that of the regularity of solutions.

First, we state an elementary lemma without proof.

LEMMA 2.3. *Let $\gamma(t)$ and $\rho(t)$ be positive, and defined on $[0, a]$ $(a>0)$. If $\gamma(t)$ is summable on $[0, a]$ and $\rho(t)$ is increasing, and*

$$\gamma(t) \leqq c\int_0^t \gamma(s)ds+\rho(t)$$

holds, then we have

$$\gamma(t) \leqq e^{ct}\rho(t) .$$

We prove the required energy inequality.

PROPOSITION 2.6. *Let $u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$ for $t \in [-\delta_0, T+\delta_0]$ $(\delta_0>0)$, and satisfying the boundary condition. If $L[u]=f(x, t) \in \mathcal{E}_t^1(L^2(\Omega))$, then*

$$\|u(t)\|_{2, L^2(\Omega)}+\|u'(t)\|_{1, L^2(\Omega)}+\|u''(t)\|_{L^2(\Omega)} \tag{2.21}$$

$$\leqq C(T)\Big[\|u(0)\|_{2, L^2(\Omega)}+\|u'(0)\|_{1, L^2(\Omega)}+\|f(0)\|_{L^2(\Omega)}$$

$$+\int_0^t \|f'(s)\|_{L^2(\Omega)}ds\Big] \qquad \text{for all} \quad t \in [0, T]$$

holds. ($C(T)$ depends on T, but is independent of $u(x, t)$.)

PROOF. First, let us consider the case of the boundary condition (1.3). Put $U(t)=\{u(t), u'(t)\}$, then $U(t) \in D_1$ and satisfies the equation

$$\frac{d}{dt}U(t)=\mathcal{A}(t)U(t)+F(t) \tag{1.5}$$

where $F(t)=\{0, f(t)\}$.

$$\begin{aligned} \frac{d}{dt}(U(t), U(t))_{\mathcal{H}_1(t)} &= (U'(t), U(t))_{\mathcal{H}_1(t)} \\ &\quad +(U(t), U'(t))_{\mathcal{H}_1(t)}+(U(t), U(t))_{\dot{\mathcal{H}}_1(t)} \\ &= (\mathcal{A}(t)U(t)+F(t), U(t))_{\mathcal{H}_1(t)} \\ &\quad +(U(t), \mathcal{A}(t)U(t)+F(t))_{\mathcal{H}_1(t)} \\ &\quad +(U(t), U(t))_{\dot{\mathcal{H}}_1(t)} \end{aligned} \tag{2.22}$$

where

$$(U, U)_{\dot{\mathcal{H}}_1(t)}=\sum_{i,j=1}^n\Big(a'_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big) \qquad \text{for} \quad U=\{u, v\} \in \mathcal{H}_1(t) .$$

By (2.5) and

$$|(U(t), U(t))_{\dot{\mathcal{H}}_1(t)}| \leqq \text{const}\,\|U(t)\|^2_{\mathcal{H}_1(t)} , \tag{2.23}$$

$$|(F(t), U(t))_{\mathcal{H}_1(t)}| \leqq \|F(t)\|_{\mathcal{H}_1(t)} \|U(t)\|_{\mathcal{H}_1(t)},$$

the rigth side of (2.22) is estimated by

$$2c_1\|U(t)\|^2_{\mathcal{H}_1(t)} + 2\|F(t)\|_{\mathcal{H}_1(t)} \|U(t)\|_{\mathcal{H}_1(t)}.$$

Thus

$$\frac{d}{dt}\|U(t)\|^2_{\mathcal{H}_1(t)} \leqq 2\|U(t)\|_{\mathcal{H}_1(t)}(c_1\|U(t)\|_{\mathcal{H}_1(t)} + \|F(t)\|_{\mathcal{H}_1(t)})$$

and

$$\frac{d}{dt}\|U(t)\|_{\mathcal{H}_1(t)} \leqq c_1\|U(t)\|_{\mathcal{H}_1(t)} + \|F(t)\|_{\mathcal{H}_1(t)}.$$

From this it follows

$$\|U(t)\|_{\mathcal{H}_1(t)} \leqq e^{c_1 t}\Big(\|U(0)\|_{\mathcal{H}_1(0)} + \int_0^t \|F(s)\|_{\mathcal{H}_1(s)} ds\Big). \tag{2.24}$$

In addition, assume that $u'(x, t) \in \mathcal{E}^0_t(H^2(\Omega)) \cap \mathcal{E}^1_t(H^1(\Omega))$. Then we see

$$U'(t) = \{u'(t), u''(t)\} \in D_1.$$

Differentiation of (1.5) with respect to t gives

$$\frac{d}{dt} U'(t) = \mathcal{A}(t)U'(t) + \mathcal{A}'(t)U(t) + F'(t).$$

Applying (2.24) for $U'(t)$, we get

$$\|U'(t)\|_{\mathcal{H}_1(t)} \leqq e^{c_1 t}\Big(\|U'(0)\|_{\mathcal{H}_1(0)} + \int_0^t \|\mathcal{A}'(s)U(s) + F'(s)\|_{\mathcal{H}_1(s)} ds\Big). \tag{2.25}$$

According to (2.20)

$$\begin{aligned}|||U(t)|||_2 + |||U'(t)|||_1 &\leqq d_2\|(\lambda_0 I - \mathcal{A}(t))U(t)\|_{\mathcal{H}_1(t)} + c_2\|U'(t)\|_{\mathcal{H}_1(t)} \\ &\leqq d_2\{\lambda_0\|U(t)\|_{\mathcal{H}_1(t)} + \|U'(t)\|_{\mathcal{H}_1(t)} + \|F(t)\|_{\mathcal{H}_1(t)}\} \\ &\quad + c_2\|U'(t)\|_{\mathcal{H}_1(t)}\end{aligned}$$

by (2.24) and (2.25)

$$\begin{aligned}&\leqq c''\Big\{\Big(\|U(0)\|_{\mathcal{H}_1(0)} + \int_0^t \|F(s)\|_{\mathcal{H}_1(s)} ds\Big) \\ &\quad + \|F(t)\|_{\mathcal{H}_1(t)} + \|U'(0)\|_{\mathcal{H}_1(0)} \\ &\quad + \int_0^t \|\mathcal{A}'(s)U(s)\|_{\mathcal{H}_1(s)} ds + \int_0^t \|F'(s)\|_{\mathcal{H}_1(s)} ds\Big\}.\end{aligned}$$

And for all $t \in [0, T]$

$$\int_0^t \|F(s)\|_{\mathcal{H}_1(s)} ds \leqq T\Big(\|f(0)\|_{L^2(\Omega)} + \int_0^t \|f'(s)\|_{L^2(\Omega)} ds\Big),$$

moreover we have

$$\begin{aligned}\|U'(0)\|_{\mathcal{H}_1(0)} &\leqq \|\mathcal{A}(0)U(0)\|_{\mathcal{H}_1(0)} + \|F(0)\|_{\mathcal{H}_1(0)} \\ &\leqq \text{const}\,|||U(0)|||_2 + \|f(0)\|_{L^2(\Omega)}.\end{aligned}$$

Inserting these estimates to the above inequality, we get

$$|||U(t)|||_2+|||U'(t)|||_1 \leqq C'(T)\Big[|||U(0)|||_2+\|f(0)\|_{L^2(\Omega)} + \int_0^t \|f'(s)\|_{L^2(\Omega)}ds+\int_0^t |||U(s)|||_2 ds\Big].$$

Applying Lemma 2.5 by taking $\gamma(t)=|||U(t)|||_2+|||U'(t)|||_1$

$$\rho(t)=C'(T)\Big(|||U(0)|||_2+\|f(0)\|_{L^2(\Omega)}+\int_0^t \|f'(s)\|_{L^2(\Omega)}ds\Big),$$

it follows

$$|||U(t)|||_2+|||U'(t)|||_1 \leqq C'(T)e^{C'(T)t}\Big(|||U(0)|||_2+\|f(0)\|_{L^2(\Omega)}+\int_0^t \|f'(s)\|_{L^2(\Omega)}ds\Big).$$

Thus we obtain (2.21).

To remove the additional assumption that $u'(x, t)\in \mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))$, we make use of the mollifier with respect to t. Denote by $u_\delta(x, t)$ the function $(\phi_{\delta \underset{(t)}{*}} u)(x, t)$, where $\phi_{\delta \underset{(t)}{*}}$ is Friedrichs' mollifier.

Applying $\phi_{\delta \underset{(t)}{*}}$ to (1.1), we get

$$L[u_\delta]=f_\delta-C_\delta u$$

where

$$(C_\delta u)(x, t)=[\phi_{\delta \underset{(t)}{*}}, a_1(x, t: D)]\frac{\partial}{\partial t}u+[\phi_{\delta \underset{(t)}{*}}, a_2(x, t: D)]u$$

for all $t\in[0, T]$ if $0<\delta<\delta_0$.

Since $u_\delta\in\mathcal{E}_t^\infty(H^2(\Omega)\cap\mathcal{D}_{L^2}^1(\Omega))$, from the obtained results it follows

$$\text{(2.26)}\qquad \|u_\delta(t)\|_2+\|u_\delta'(t)\|_1+\|u_\delta''(t)\| \leqq C(T)\Big[\|u_\delta(0)\|_2+\|u_\delta'(0)\|_1+\|f_\delta(0)\| + \int_0^t \|f_\delta'(s)\|ds+\|(C_\delta u)(0)\|+\int_0^t \Big\|\Big(\frac{\partial}{\partial s}C_\delta u\Big)(s)\Big\|ds\Big].$$

Now we know that

$$\|u_\delta(t)\|_2\to\|u(t)\|_2,$$
$$\|u_\delta'(t)\|_1\to\|u'(t)\|_1,$$
$$\|u_\delta''(t)\|\to\|u''(t)\|,$$
$$\|f_\delta(0)\|\to\|f(0)\|,$$
$$\int_0^t\|f_\delta'(s)\|ds\to\int_0^t\|f'(s)\|ds$$

when $\delta\to0$. Moreover we have

$$\text{(2.27)}\qquad \|(C_\delta u)(0)\|\to 0.$$

$$\text{(2.28)}\qquad \int_0^t \Big\|\Big(\frac{\partial}{\partial s}C_\delta u\Big)(s)\Big\|ds\to 0$$

when $\delta \to 0$. In fact (2.27) is evident, so we prove (2.28). In view of the explicit form of $C_\delta u$, it suffices to show the following fact: Let

$$a(x, t) \in \mathscr{B}^2(\Omega \times (-\delta_0, T+\delta_0))$$

and

$$v(x, t) \in L^2(\Omega \times (-\delta_0, T+\delta_0)).$$

Then putting

$$\psi_\delta(x, t) = \frac{\partial}{\partial t}[\phi_{\delta(t)}^{*}, a(x, t)]v(x, t),$$

we have

$$\int_0^T \|\psi_\delta(x, t)\| dt \to 0 \qquad \text{when} \quad \delta \to 0.$$

Now

$$\text{(2.29)} \qquad \frac{\partial}{\partial t}\{\phi_\delta(t-\tau)[a(x, \tau)-a(x, t)]\} = -\frac{\partial}{\partial \tau}\{\phi_\delta(t-\tau)[a(x, \tau)-a(x, t)]\} + \phi_\delta(t-\tau)[a'(x, \tau)-a'(x, t)].$$

Thus

$$\psi_\delta = \int \frac{\partial}{\partial \tau}\{\phi_\delta(t-\tau)[a(x, \tau)-a(x, t)]\}(v(x, \tau)-v(x, t))d\tau + \int \phi_\delta(t-\tau)[a'(x, \tau)-a'(x, t)]v(x, \tau)d\tau.$$

Then, by ordinary calculus, we see easily the desired property of ψ_δ.

Thus, the passage to the limit of (2.26) when $\delta \to 0$ proves Proposition in the case $i=1$.

Now we consider the case $i=2$. At first we prove the inequality under the additional assumption:

$$u'(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)).$$

Since B_2 is assumed to be independent of t, u' satisfies also the same boundary condition as u. Thus by the same reasoning as before we get (2.21). Next we remove the additional assumption by using the mollifier $\phi_{\delta(t)}^{*}$.

Let us remark

$$B_2[\phi_{\delta(t)}^{*} u(x, t)] = \phi_{\delta(t)}^{*}[B_2 u(x, t)] \quad \text{in} \quad L^2(S \times [0, T]).$$

In fact, this relation is true when u is assumed moreover twice continuously differentiable in $\bar{\Omega} \times \left[-\frac{\delta_0}{2}, T+\frac{\delta_0}{2}\right]$. Then, by taking a sequence $\{u_j(x, t)\}$ satisfying these conditions and tending to $u(x, t)$ in $H^2\left(\Omega \times \left(-\frac{\delta_0}{2}, T+\frac{\delta_0}{2}\right)\right)$, we affirm this relation. In view of the fact that $u_\delta(x, t)$ is continuously differentiable in t with values in $H^2(\Omega)$, and $B_2[u(x, t)]=0$, we see that for every t $B_2[u_\delta(x, t)]=0$.

Existence of the solution.

The straightforward application of the semi-group theory to the equation (1.5) gives the following Proposition, whose proof will be given later.

PROPOSITION 2.2. *Given* $U_0 \in D_i$ *and* $F(t) \in D_i$ *such that* $F(t)$ *and* $\mathcal{A}(t)F(t)$ *are continuous in* $H^1(\Omega)\times L^2(\Omega)$, *then there exists a solution* $U(t) \in \mathcal{E}_t^1(H^1(\Omega)\times L^2(\Omega))$ *of* (1.5) *such that* $U(0)=U_0$ *and* $U(t) \in D_i$ *for all* $t \in (-\delta_0, T+\delta_0)$.

From this it follows:

THEOREM 1. *Given* $\{u_0, u_1\} \in D_i$ *and* $f(x, t) \in \mathcal{E}_t^1(L^2(\Omega))$, *then there exists one and only one solution* $u(x, t)$ *of* (1.1) *satisfying the boundary condition* $B_i u=0$ *and the initial condition*

$$u(x, 0)=u_0, \qquad \frac{\partial u}{\partial t}(x, 0)=u_1$$

such that

$$u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega)).$$

PROOF. At first, let us assume $f(x, t) \in \mathcal{E}_t^0(\mathcal{D}_{L^2}^1(\Omega))$, then we see that $F(t) = \{0, f(x, t)\} \in D_i$ and $F(t)$ and $\mathcal{A}(t)F(t)$ are continuous in $H^1(\Omega)\times L^2(\Omega)$. Thus, by setting $U_0=\{u_0, u_1\}$, Proposition 2.2 assures the existence of the solution $U(t) \in \mathcal{E}_t^1(H^1(\Omega)\times L^2(\Omega))$ of (1.5) such that $U(0)=U_0$ and $U(t) \in D_i$.

$$(\lambda_0 I-\mathcal{A}(t))U(t)=\lambda_0 U(t)-U'(t)+F(t) \in \mathcal{E}_t^0(H^1(\Omega)\times L^2(\Omega)),$$

from which it follows, with the aid of (2.20) taking $p=2$,

$$U(t) \in \mathcal{E}_t^0(H^2(\Omega)\times H^1(\Omega)).$$

Denote by $u(x, t)$ the first component of $U(t)$, then we can easily see $u(x, t)$ is the required solution of (1.1). When $f(x, t) \in \mathcal{E}_t^1(L^2(\Omega))$, let us choose a sequence $f_j(x, t) \in \mathcal{E}_t^1(\mathcal{D}_{L^2}^1(\Omega))$ $(j=1, 2, \cdots)$ tending to $f(x, t)$ in $\mathcal{E}_t^1(L^2(\Omega))$ and denote by $u_j(x, t)$ the solution of (1.1) for the initial data $\{u_0, u_1\}$ and the second member $f_j(x, t)$, then from (2.21) we see

$$\begin{aligned}&\|u_k(t)-u_l(t)\|_2+\|u_k'(t)-u_l'(t)\|_1+\|u_k''(t)-u_l''(t)\| \\ &\qquad \leqq C(T)\Big(\|f_k(0)-f_l(0)\|+\int_0^t \|f_k'(s)-f_l'(s)\|ds\Big).\end{aligned}$$

This shows that $\{u_j(x, t)\}$ converges to some $u(x, t)$ in $\mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$.

Then the passage to the limit when $j \to 0$ of

$$L[u_j]=f_j, \qquad u_j(x, 0)=u_0, \qquad \frac{\partial u_j}{\partial t}(x, 0)=u_1,$$
$$B_i u_j=0,$$

proves that $u(x, t)$ is the required solution. The proof is thus complete.

In order to prove Proposition 2.2, we mention the following theorem essentially due to T. Kato.

L: a Banach space, and we denote its norm by $\|\cdot\|_L$. $L(t)$, $a \leqq t \leqq b$: we give a family of norms equivalent to that of L, and denote it by $\|\cdot\|_{L(t)}$. $L(t)$ is the space L equipped with the norm $\|\cdot\|_{L(t)}$. Then

THEOREM. *Hypothesis:*

c_1) *For all* $t \in [a, b]$, *the operator* $A(t)$ *is a closed operator with the dense definition domain* D *independent of* t, *and we have for all* $|\lambda| > \delta$ (λ: *real*)

$$\|(\lambda I - A(t))^{-1}\|_{L(t)} < \frac{1}{|\lambda| - \delta}.$$

c_2) $B(t, s) = (\lambda_0 I - A(t))(\lambda_0 I - A(s))^{-1}$ *is differentiable in* t *for some* s *in* $\mathcal{L}(L, L)$ *equipped by the simple topology, moreover* $B'(t, s)$ *is continuous in* t *for the topology of* $\mathcal{L}(L, L)$.

c_3) *There exists a constant* $\delta > 0$ *and non-increasing function* $\phi(t)$ *such that*

$$\|x\|_{L(t)} \geqq \delta \|x\|_L \qquad \text{for all} \quad t \in [a, b],$$

$$|\, \|x\|_{L(t)} - \|x\|_{L(s)} | < (\phi(t) - \phi(s)) \|x\|_L \qquad \text{for all} \quad t > s \text{ and } x \in L.$$

Conclusion:

For any $x \in D$ *and* $f(t) \in D$ *such that* $f(t)$ *and* $A(t)f(t)$ *are continuous in* t *for* $t \in [a, b]$, *then there exists one and only one solution of the equation*

$$\frac{d}{dt} x(t) = A(t)x(t) + f(t)$$

such that $x(t) \in D$ *and* $\in \mathcal{E}_t^1(L)$ *and* $x(t_0) = x$ (t_0 *is any fixed point of* $[a, b]$).

REMARK 2.4. The above Theorem is stated in a slightly different form in S. Mizohata, Le problèmes de Cauchy pour les équations paraboliques, J. Math. Soc. Japan, 8 (1956)[4]. In fact, he assumes instead of c_3) the following condition $c_3)'$: There exists a family of operators $T(t) \in \mathcal{L}(L, L)$ such that

1) $T(t)$ is isomorphism of L onto L: $T(t)^{-1} \in \mathcal{L}(L, L)$.

2) For any $x \in L$

$$\|x\|_{L(t)} = \|T(t)x\|_L.$$

3) $\|T(t)^{-1}\|$ is bounded for $t \in [a, b]$.

4) $T(t)$ is bounded variation, i.e. there exists $N > 0$ such that for all partitions

$$\sum_{i=1}^{n} \|T(t_i) - T(t_{i-1})\|_L \leqq N.$$

In fact, let us observe that, in view of his reasoning, $T(t)$ need not be linear. It suffices to assure $T(t)\lambda x = \lambda T(t)x$. Then c_3) is equivalent to $c_3)'$. To show $c_3)'$ follows from c_3), we define the operator $T(t)$ by

4) See also T. Kato, On linear differential equations in Banach spaces, Comm. Pure Appl. Math., 9 (1956). The above formulation was pointed out by S. Miyatake.

$$T(t)x=\frac{\|x\|_{L(t)}}{\|x\|_L}\cdot x\,,$$

then $T(t)$ satisfies the condition $c_3)'$. And it is evident that $c_3)$ follows from $c_3)'$.

REMARK 2.5. Suppose L is a Hilbert space. Denote the inner product of L and $L(t)$ by $(\ ,\ )_L$ and $(\ ,\ )_{L(t)}$ respectively. If for all $x\in L$, $(x, x)_{L(t)}$ is continuously differentiable and for some constant $c>0$ the inequality

$$\left|\frac{d}{dt}(x, x)_{L(t)}\right|\leqq c(x, x)_L$$

holds, then $c_3)$ is satisfied.

PROOF. Since $(x, x)_{L(t)}$ is uniformly continuous in t on $\|x\|=1$ and $L(t)$ is equivalent to L, there exists a constant $\delta>0$ such that

$$\|x\|_{L(t)}>\delta\|x\|_L\,,$$

$$\frac{d}{dt}(x, x)_{L(t)}=\frac{d}{dt}\|x\|_{L(t)}^2=2\|x\|_{L(t)}\frac{d}{dt}\|x\|_{L(t)}$$

then

$$\frac{d}{dt}\|x\|_{L(t)}\leqq c_2\frac{\|x\|_L}{\|x\|_{L(t)}}\|x\|_L\leqq\frac{c}{2\delta}\|x\|_L\,.$$

Hence

$$|\,\|x\|_{L(t)}-\|x\|_{L(s)}\,|\leqq\int_s^t\left|\frac{d}{dl}\|x\|_{L(l)}dl\right|\leqq\frac{c}{2\delta}(t-s)\|x\|_L\,.$$

Thus we can take as $\phi(t)=\frac{c}{2\delta}t$.

PROOF OF PROPOSITION 2.2. We take $L(t)=\mathcal{H}_i(t)$, then we can easily see that $\mathcal{A}(t)$ satisfies the condition $c_1)\sim c_3)$. In fact Lemma 2.2 assures $c_1)$ and from its corollary (taking $p=2$) and the differentiability of $\mathcal{A}(t)$ in $\mathcal{L}(D_i, H^1(\Omega)\times L^2(\Omega))$ we see that $c_2)$ is satisfied. Finally $c_3)$ is satisfied since $(U, U)_{\mathcal{H}_i(t)}$ satisfies the condition stated in the above Remark.

Regularity.

When the coefficients of L are sufficiently smooth, if we suppose the regularity of the initial data and the second member, then the solution of (1.1) becomes regular. Of course, since the equation is hyperbolic, we should assume the compatibility condition. This condition is formulated as follows: Define successively $u_p(x)$ by

$$u_p=-\sum_{k=0}^{p-2}\binom{p-2}{k}\{a_1^{(k)}(x, 0:D)u_{p-k-1}+a_2^{(k)}(x, 0:D)u_{p-k-2}\}+f^{(p-2)}(x, 0)\qquad(2.30)$$

$$(p=2, 3, \cdots, m+1)$$

then

$$\{u_p, u_{p+1}\}\in D_i\qquad(p=1, 2, \cdots, m)\,.$$

THEOREM 2. *Suppose that the coefficients of L belong to $\mathcal{B}^{\max[m,2]}$ $(\Omega\times(-\delta_0,$*

$T+\delta_0)$) *and*

$$\{u_0, u_1\} \in D_i \cap (H^{m+2}(\Omega) \times H^{m+1}(\Omega)) \qquad (m \geqq 1)$$

(2.31) $$f(x, t) \in \mathcal{E}_t^0(H^m(\Omega)) \cap \mathcal{E}_t^1(H^{m-1}(\Omega)) \cap \cdots \cap \mathcal{E}_t^{m-1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+1}(L^2(\Omega))$$

then, if the above compatibility condition is satisfied, the solution of the equation (1.1) *satisfies*

(2.32) $$u(x, t) \in \mathcal{E}_t^0(H^{m+2}(\Omega)) \cap \mathcal{E}_t^1(H^{m+1}(\Omega)) \cap \cdots \cap \mathcal{E}_t^{m+2}(L^2(\Omega)).$$

PROOF. At first we prove

(2.33) $$u(x, t) \in \mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega)).$$

For this purpose we consider the solution $\tilde{u}$ of the equation

(2.34) $$L[\tilde{u}^{(m)}] = -\sum_{k=0}^{m-1} \binom{m}{k} L^{(m-k)}[\tilde{u}^{(k)}] + f^{(m)}(x, t).$$

Its existence can be shown by the method of successive approximation. In fact, we can obtain $\tilde{u}_j(x, t)$ $(j = 1, 2, \cdots)$ successively by

(2.35) $$L[v_j] = -\sum_{k=0}^{m-1} \binom{m}{k} L^{(m-k)}[\tilde{u}_{j-1}^{(k)}] + f^{(m)}(x, t).$$

$$v_j(x, 0) = u_m, \qquad v_j'(x, 0) = u_{m+1},$$

(2.36) $$\tilde{u}_j(x, t) = u_0 + tu_1 + \cdots + \frac{t^{m-1}}{(m-1)!} u_{m-1} + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} v_j(x, s) ds$$

here $\tilde{u}_0 \equiv 0$, because if $v_j \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega))$, from (2.36) $\tilde{u}_j \in \mathcal{E}_t^m(H^2(\Omega)) \cap \mathcal{E}_t^{m+1}(H^1(\Omega))$, thus the right side of (2.35) $\in \mathcal{E}_t^1(L^2(\Omega))$ and $\{u_m, u_{m+1}\} \in D_i$, then Theorem 1 can be applied to (2.35). Evidently $B_i \tilde{u}_j = 0$. Now we show that $\{v_j\}$ is a Cauchy sequence in $\mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$.

$$L[v_{j+1} - v_j] = -\sum_{k=0}^{m-1} \binom{m}{k} a_1^{(m-k)}(x, t : D) \int_0^t \frac{(t-s)^{m-k-1}}{(m-k-1)!} (v_j'(x, s) - v_{j-1}'(x, s)) ds$$
$$-\sum_{k=0}^{m-1} \binom{m}{k} a_2^{(m-k)}(x, t : D) \int_0^t \frac{(t-s)^{m-k-1}}{(m-k-1)!} (v_j(x, s) - v_{j-1}(x, s)) ds,$$

then, by (2.21) for some constant $K(T)$

$$\|v_{j+1}(t) - v_j(t)\|_2 + \|v_{j+1}'(t) - v_j'(t)\|_1 + \|v_{j+1}''(t) - v_j''(t)\|$$
$$\leqq K(T) \Big[\int_0^t \|v_j(s) - v_{j-1}(s)\|_2 ds + \int_0^t \|v_j'(s) - v_{j-1}'(s)\|_1 ds \Big]$$
$$(j = 2, 3, \cdots)$$

holds, and

$$\|v_1(t)\|_2 + \|v_1'(t)\|_1 \leqq K,$$

thus

$$\|v_{j+1}(t) - v_j(t)\|_2 + \|v_{j+1}'(t) - v_j'(t)\|_1 + \|v_{j+1}''(t) - v_j''(t)\| \leqq K \frac{(K(T)t)^j}{j!}.$$

This implies that v_j converges to some v in $\mathcal{E}_t^0(H^2(\Omega))\cap\mathcal{E}_t^1(H^1(\Omega))\cap\mathcal{E}_t^2(L^2(\Omega))$, and set

$$\tilde{u}(x, t)=u_0+tu_1+\cdots+\frac{t^{m-1}}{(m-1)!}u_{m-1}+\int_0^t\frac{(t-s)^{m-1}}{(m-1)!}v(x, s)ds\,,$$

then $\tilde{u}_j$ tends to $\tilde{u}$ in $\mathcal{E}_t^m(H^2(\Omega))\cap\mathcal{E}_t^{m+1}(H^1(\Omega))\cap\mathcal{E}_t^{m+2}(L^2(\Omega))$. The passage to the limit of (2.35) gives (2.34) which shows

$$\frac{d^m}{dt^m}(L[\tilde{u}])=f^{(m)}(x, t)\,.$$

Taking account of the definition of u_p, we see

$$\frac{d^p}{dt^p}(L[\tilde{u}])\Big|_{t=0}=f^{(p)}(x, 0)\qquad p=0, 1, 2, \cdots, m-1\,.$$

Therefore we get

$$L[\tilde{u}]=f(x, t)\,,$$

$$\tilde{u}(x, t)\in\mathcal{E}_t^m(H^2(\Omega))\cap\mathcal{E}_t^{m+1}(H^1(\Omega))\cap\mathcal{E}_t^{m+2}(L^2(\Omega))\,,$$

$$B_i\tilde{u}=0\,,\qquad \tilde{u}(x, 0)=u_0\,,\qquad \frac{\partial\tilde{u}}{\partial t}(x, 0)=u_1\,.$$

From the uniqueness of the solution, it follows (2.33).

Set $U(t)=\{u(t), u'(t)\}$, then $U(t)$ is the solution of (1.5) and $\in\mathcal{E}_t^m(H^2(\Omega)\times H^1(\Omega))$. Now

$$(2.37)\qquad (\lambda_0 I-\mathcal{A}(t))U(t)=\lambda_0 U(t)-U'(t)+F(t)\in\mathcal{E}_t^0(H^2(\Omega)\times H^1(\Omega))\,,$$

then by (2.20) (taking $p=3$) we see

$$U(t)\in\mathcal{E}_t^0(H^3(\Omega)\times H^2(\Omega))\,.$$

Differentiation of (2.37) with respect to t gives

$$(2.38)\qquad (\lambda_0 I-\mathcal{A}(t))U'(t)=\lambda_0 U'(t)+F'(t)-U''(t)+\mathcal{A}'(t)U(t)$$

and by the above result $\mathcal{A}'(t)U(t)\in\mathcal{E}_t^0(H^2(\Omega)\times H^1(\Omega))$, the right side of (2.38) $\in\mathcal{E}_t^0(H^2(\Omega)\times H^1(\Omega))$, from which it follows

$$U'(t)\in\mathcal{E}_t^0(H^3(\Omega)\times H^2(\Omega))\,.$$

Repeating this process, we get

$$(2.39)\qquad U(t)\in\mathcal{E}_t^{m-1}(H^3(\Omega)\times H^2(\Omega))\,.$$

Using this, we see the right side of (2.37) $\in\mathcal{E}_t^0(H^3(\Omega)\times H^2(\Omega))$, then by (2.20) (taking $p=4$)

$$U(t)\in\mathcal{E}_t^0(H^4(\Omega)\times H^3(\Omega))\,.$$

This assures the right side of (2.38) $\in\mathcal{E}_t^0(H^3(\Omega)\times H^2(\Omega))$, then

$$U'(t) \in \mathcal{E}_t^0(H^4(\Omega)\times H^3(\Omega)).$$

Step by step, finally we get

$$U(t) \in \mathcal{E}_t^0(H^{m+2}(\Omega)\times H^{m+1}(\Omega)) \cap \mathcal{E}_t^1(H^{m+1}(\Omega)\times H^m(\Omega)) \cap \cdots \cap \mathcal{E}_t^m(H^2(\Omega)\times H^1(\Omega)).$$

This shows (2.32) holds.

REMARK 2.6. Theorem 1 can be extended to the case when $\sigma(s, t)$ varies with t. Let $a(x, t)$ be the sufficiently smooth function defined on $\bar{\Omega}\times(-\delta_0, \infty)$ such that

$$\frac{\partial a}{\partial n} - \langle h, \nu\rangle \frac{\partial a}{\partial t} + \sigma(s, t) = 0 \quad \text{and} \quad a = 1 \text{ on } S$$

for all t, and a, a^{-1} are uniformly bounded. If we define L_a by

$$L[av] = aL_a[v].$$

L_a has the same principal part as L. Therefore we can obtain the solution $v(x, t)$ of the equation

$$L_a[v] = a^{-1}f$$

for the initial data

$$v(x, 0) = a^{-1}(x, 0)u_0(x),$$

$$\frac{\partial v}{\partial t}(x, 0) = a^{-1}(x, 0)\Big[u_1(x) - a^{-1}(x, 0)\frac{\partial a}{\partial t}(x, 0)\, u_0(x)\Big],$$

satisfying the boundary condition

$$\frac{\partial v}{\partial n} - \langle h, \nu\rangle \frac{\partial v}{\partial t} = 0 \quad \text{on} \quad S,$$

since the initial data for v satisfies the boundary condition. Then $u = av$ is the solution of (1.1) for initial data $\{u_0, u_1\}$ and the boundary condition (1.4).

§ 3. Case where B_2 varies with t.

In this section we consider the case where the boundary condition B_2 varies with t under the additional condition that $a_1(x, t: D) = h(x, t)$ and $b_i(x, t)$ $(i = 1, 2, \cdots, n)$ are real valued on the boundary S. We shall make use of the operator $A(t)$ from $H^1(\Omega)$ into $H^1(\Omega)'$, which is an extension of the operator $a_2(x, t: D)$ defined on the domain

(3.1) $$D(t) = \Big\{u \,|\, u \in H^2(\Omega),\ \frac{\partial u}{\partial n_t} + \sigma(s, t)u = 0 \text{ on } S\Big\}.$$

We introduce the quadratic form for $u, \phi \in H^1(\Omega)$

(3.2) $$a_0(t; u, \phi) = \sum_{i,j=1}^{n}\Big(a_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial \phi}{\partial x_j}\Big)$$

$$+\int_S \sigma(s, t)u\bar{\phi}\, dS + \int_S \langle b, \nu\rangle u\bar{\phi}\, dS + \beta(u, \phi),$$

where $\beta > 0$ is sufficiently large such that for some $M > 0$

$$\frac{1}{M}\|u\|^2_{1, L^2(\Omega)} < a_0(t; u, u) < M\|u\|^2_{1, L^2(\Omega)}. \tag{3.3}$$

Thus $\sqrt{a_0(t; u, u)}$ defines on $H^1(\Omega)$ an equivalent norm. Let us denote the Hilbert space $H^1(\Omega)$ equipped with the scalar product (3.2) by $H(t)$ and its dual Hilbert space by $H(t)'$.

Using the quadratic form (3.2) we define $A_0(t) \in \mathcal{L}(H^1(\Omega), H^1(\Omega)')$ by

$$\langle A_0(t)u, \bar{\phi}\rangle = a_0(t; u, \phi) \qquad \forall\phi \in H^1(\Omega). \tag{3.4}$$

Then, by Riesz's theorem $\|A_0(t)u\|_{H(t)'} = \|u\|_{H(t)}$. In other words, $A_0(t)$ is an isomorphic operator from $H(t)$ to $H(t)'$. Thus, let us define the scalar product in $H(t)'$:

$$(u, v)_{H(t)'} = (A_0(t)^{-1}u, A_0(t)^{-1}v)_{H(t)} = a_0(t; A_0(t)^{-1}u, A_0(t)^{-1}v) \tag{3.5}$$

for $u, v \in H(t)'$.

From the equivalence of $H(t)$ and $H^1(\Omega)$, it follows that the dual norms $(v, v)_{H(t)'}$ and $(v, v)_{H^1(\Omega)'}$ are equivalent.

For $u \in L^2(\Omega)$, $\phi \in H^1(\Omega)$, set

$$b_0(t; u, \phi) = \Big(u, \Big(-\sum_{j=1}^n \frac{\partial}{\partial x_j}\overline{b_j(x, t)} + \overline{c(x, t)} - \beta\Big)\phi\Big) \tag{3.6}$$

and we define $B_0(t) \in \mathcal{L}(L^2(\Omega), H^1(\Omega)')$ by

$$\langle B_0(t)u, \bar{\phi}\rangle = b_0(t; u, \phi), \qquad \forall\phi \in H^1(\Omega). \tag{3.7}$$

REMARK 3.1. From the definition of $A_0(t)$, we have

$$\langle u, \bar{\phi}\rangle = a_0(t; A_0(t)^{-1}u, \phi) \tag{3.8}$$

for all $u \in H^1(\Omega)'$, $\phi \in H^1(\Omega)$.

LEMMA 3.1. *$A_0(t)$ is differentiable in $\mathcal{L}(H^1(\Omega), H^1(\Omega)')$ and $B_0(t)$ is also in $\mathcal{L}(L^2(\Omega), H^1(\Omega)')$.*

PROOF.

$$\Big\langle \frac{A_0(t+h) - A_0(t)}{h}u, \bar{\phi}\Big\rangle = \sum_{i,j=1}^n \Big(\frac{a_{ij}(x, t+h) - a_{ij}(x, t)}{h}\frac{\partial u}{\partial x_i}, \frac{\partial \phi}{\partial x_j}\Big)$$

$$+\int_S \sum_{j=1}^n \frac{b_j(x, t+h) - b_j(x, t)}{h}\nu_j u\bar{\phi}\, dS + \int_S \frac{\sigma(s, t+h) - \sigma(s, t)}{h}u\bar{\phi}\, dS$$

$$\longrightarrow \sum_{i,j=1}^n \Big(a'_{ij}(x, t)\frac{\partial u}{\partial x_i}, \frac{\partial \phi}{\partial x_j}\Big) + \int_S (\langle b', \nu\rangle + \sigma'(s, t))u\bar{\phi}\, dS$$

$$= a_0'(t; u, \phi) \qquad \text{when} \quad h \to 0.$$

Since the convergence is uniform in $u, \phi, A_0(t)$ is differentiable in $\mathcal{L}(H^1(\Omega), H^1(\Omega)')$ and

$$\langle A_0(t)'u, \bar{\phi}\rangle = a_0'(t; u, \phi).$$

The higher differentiability of $A_0(t)$ and that of $B_0(t)$ can be shown by the similar way.

LEMMA 3.2. *$A_0(t)^{-1}$ is continuously differentiable in $\mathcal{L}(H^1(\Omega)', H^1(\Omega))$.*

PROOF. This is consequent of the previous Lemma. Indeed, the inverse of the operator which is differentiable in t is also differentiable in t and we have

$$(A_0(t)^{-1})_t' = -A_0(t)^{-1}A_0(t)'A_0(t)^{-1}.$$

Then, from the continuity of $A_0(t)'$, that of $(A_0(t)^{-1})'$ follows.

We define the operator $A(t)$ by

$$A(t) = A_0(t) + B_0(t). \tag{3.9}$$

REMARK 3.2.[5] According to the theory of elliptic equation, if $u \in H^1(\Omega)$ and $A(t)u \in L^2(\Omega)$, then $u \in D(t)$ and $A(t)u = a_2(x, t: D)u$. Conversely if $u \in D(t)$, $A(t)u = a_2(x, t: D)u \in L^2(\Omega)$. This shows that $A(t)$ is an extension of $a_2(x, t: D)$.

We now solve the equation

$$\tilde{L}[u] = u''(x, t) + h(x, t)u'(x, t) + A(t)u(x, t) = f(x, t) \tag{3.10}$$

in the space $u(x, t) \in \mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)')$. Let us remark that this equation is considered in $H^1(\Omega)'$, by taking $h(x, t)u'(x, t) \in \mathcal{E}_t^0(L^2(\Omega))$ as its natural injection into $\mathcal{E}_t^0(H^1(\Omega)')$[6].

For this, we consider the equation

$$\frac{d}{dt}U(t) = \mathcal{A}(t)U(t) + F(t) \tag{3.11}$$

where

$$\mathcal{A}(t) = \begin{bmatrix} 0 & 1 \\ -A(t) & -h(x, t) \end{bmatrix},$$

in the space $L^2(\Omega) \times H^1(\Omega)'$.

$\mathcal{H}_0(t)$ is $L^2(\Omega) \times H^1(\Omega)'$ with the norm

$$\begin{aligned} \|U\|_{\mathcal{H}_0(t)}^2 = (U, U)_{\mathcal{H}_0(t)} &= (u, u) + (v, v)_{H(t)'} \\ &= (u, u) + a_0(t; A_0(t)^{-1}v, A_0(t)^{-1}v) \end{aligned} \tag{3.12}$$

for $\{u, v\} \in L^2(\Omega) \times H^1(\Omega)'$.

$\mathcal{A}(t)$ is the operator from

5) See, for example, Mizohata [10] chap. 3.

6) To $u \in L^2(\Omega)$, we make correspond $w \in H^1(\Omega)'$ by the formula $\langle w, \bar{\phi}\rangle = (u, \phi)_{L^2(\Omega)}$, $\forall \phi \in H^1(\Omega)$. Conversely $\tilde{u} \in H^1(\Omega)'$ belongs to $L^2(\Omega)$ means that for some $u \in L^2(\Omega)$ we have $\langle \tilde{u}, \bar{\phi}\rangle = (u, \phi)_{L^2(\Omega)}$, $\forall \phi \in H^1(\Omega)$.

(3.13) $$D=H^1(\Omega)\times L^2(\Omega)$$

into $\mathcal{H}_0(t)$.

REMARK 3.3. $\|U\|_{\mathcal{H}_0(t)}$ is equivalent to $\|u\|+\|v\|_{-1}$, where $\|v\|_{-1}$ denotes the norm of $H^1(\Omega)'$.

REMARK 3.4. D is dense in $\mathcal{H}_0(t)$. Because $H^1(\Omega)$ is dense in $L^2(\Omega)$, and since $H^1(\Omega)$ and $L^2(\Omega)$ are reflexive, $L^2(\Omega)$ is dense in $H^1(\Omega)'$.

We prove some lemmas about $\mathcal{A}(t)$ which correspond to Lemmas 2.1, 2.2.

LEMMA 3.3. *There exists a constant $c>0$ such that for every $U\in D$*

(3.14) $$|(\mathcal{A}(t)U, U)_{\mathcal{H}_0(t)}+(U, \mathcal{A}(t)U)_{\mathcal{H}_0(t)}|\leqq c\|U\|^2_{\mathcal{H}_0(t)}.$$

PROOF. Let $U=\{u, v\}\in D$, then

(3.15) $$\begin{aligned}&(\mathcal{A}(t)U, U)_{\mathcal{H}_0(t)}+(U, \mathcal{A}(t)U)_{\mathcal{H}_0(t)}\\ &=(v, u)+a_0(t\,;A_0(t)^{-1}(-A_0(t)u-B_0(t)u-h(x, t)v), A_0(t)^{-1}v)\\ &\quad+(u, v)+a_0(t\,;A_0(t)^{-1}v, A_0(t)^{-1}(-A_0(t)u-B_0(t)u-h(x, t)v))\\ &=(u, v)+(v, u)-a_0(t\,;u, A_0(t)^{-1}v)-a_0(t\,;A_0(t)^{-1}v, u)\\ &\quad-2\,\mathrm{Re}\,a_0(t\,;A_0(t)^{-1}v, A_0(t)^{-1}(B_0(t)u+h(x, t)v))\end{aligned}$$

by (3.8)[7]

$$=-2\,\mathrm{Re}\,a_0(t\,;A_0(t)^{-1}v, A_0(t)^{-1}(B_0(t)u+h(x, t)v)).$$

This is estimated by

$$\begin{aligned}&\mathrm{const}\,\|v\|_{-1}\|B_0(t)u+h(x, t)v\|_{-1}\\ &\leqq\mathrm{const}\,\|v\|_{-1}(\|u\|_{L^2(\Omega)}+\|v\|_{-1}).\end{aligned}$$

From Remark 3.3 it follows (3.14).

COROLLARY. *For all real λ such that $|\lambda|>c$ we have*

(3.16) $$\|(\lambda I-\mathcal{A}(t))U\|_{\mathcal{H}_0(t)}>(|\lambda|-c)\|U\|_{\mathcal{H}_0(t)}\qquad(U\in D).$$

LEMMA 3.4. *There exists a constant $\delta>0$ such that for all $|\lambda|>\delta$ (λ real), $\lambda I-\mathcal{A}(t)$ is a bijective mapping from D onto $\mathcal{H}_0(t)$. Moreover*

(3.17) $$\|(\lambda I-\mathcal{A}(t))^{-1}\|_{\mathcal{H}_0(t)}<\frac{1}{|\lambda|-\delta}$$

holds.

PROOF. As the proof of Lemma 2.2, we consider the equation

(3.18) $$A_\lambda u\equiv(A(t)+\lambda h(x, t)+\lambda^2)u=f\in H^1(\Omega)'.$$

By the definition of $A(t)$, for any $u\in H^1(\Omega)$

$$\mathrm{Re}\langle A_\lambda u, \bar{u}\rangle=\mathrm{Re}\Big[\sum_{i,j=1}^n\Big(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big)+\Big(\sum_{j=1}^n b_j\frac{\partial u}{\partial x_j}, u\Big)+(cu, u)$$

7) Let us remark

$$a_0(t\,;u, A_0(t)^{-1}v)=\overline{a_0(t\,;A_0(t)^{-1}v, u)}=\overline{(v, u)}=(u, v).$$

$$+\int_S \sigma(s, t) u\bar{u} dS + \lambda(hu, u) + \lambda^2(u, u)\Big]$$

$$\geqq \text{const} \|u\|_{1, L^2(\Omega)}^2$$

holds if λ^2 is sufficiently large. And

$$|\text{Im} \langle A_\lambda u, u \rangle| \leqq \text{const} \|u\|_{1, L^2(\Omega)}^2 .$$

Thus (3.18) is solvable for $|\lambda| > \delta$ (λ real), namely for any $f \in H^1(\Omega)'$, there exists a unique solution $u \in H^1(\Omega)$. Then for any $F = \{f, g\} \in \mathcal{H}_0(t)$, set $u = A_\lambda^{-1}((\lambda + h)f + g)$ and $v = \lambda u - f$, then $U = \{u, v\} \in D$ and

$$(\lambda I - \mathcal{A}(t))U = F .$$

This proves $(\lambda I - \mathcal{A}(t))D = \mathcal{H}_0(t)$. (3.17) follows from (3.16).

COROLLARY 3.1. *$\lambda_0 > \delta$ (fixed), then for all $U \in D$ we have*

$$|||U|||_1 < d_0 \|(\lambda_0 I - \mathcal{A}(t))U\|_{\mathcal{H}_0(t)} \qquad d_0 > 0 . \tag{3.19}$$

PROOF. $\lambda_0 I - \mathcal{A}(t)$ is a bijective continuous mapping from $H^1(\Omega) \times L^2(\Omega)$ onto $\mathcal{H}_0(t)$. By Banach's closed graph theorem we get (3.19).

With the aid of these Lemmas, the existence of the solution of (3.10) is proved by the same argument as that of Section 2.

PROPOSITION 3.1.[8)] *Let $u(x, t) \in \mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)')$ $t \in [-\delta_0, T + \delta_0]$ ($\delta_0 > 0$) and*

$$\tilde{L}[u] = f(x, t) \in \mathcal{E}_t^1(H^1(\Omega)') ,$$

then the following energy inequality holds

$$\begin{aligned} &\|u(t)\|_1 + \|u'(t)\| + \|u''(t)\|_{-1} \\ &\leqq C(T)\Big[\|u(0)\|_1 + \|u'(0)\| + \|f(0)\|_{-1} + \int_0^t \|f'(s)\|_{-1} ds\Big] . \end{aligned} \tag{3.20}$$

PROOF. For $U = \{u, v\} \in \mathcal{H}_0(t)$,

$$(U, U)_{\dot{\mathcal{H}}_0(t)} = a_0'(t; A_0(t)^{-1}v, A_0(t)^{-1}v) + 2 \,\text{Re}\, a_0(t; (A_0(t)^{-1})'v, A_0(t)^{-1}v)$$

then by Lemma 3.2

$$|(U, U)_{\dot{\mathcal{H}}_0(t)}| \leqq \text{const} (U, U)_{\mathcal{H}_0(t)} , \tag{3.21}$$

and from Lemma 3.1

$$\|\mathcal{A}'(t)U\|_{\mathcal{H}_0(t)} \leqq \text{const} |||U|||_1 .$$

Therefore by using (3.14) and (3.19) instead of (2.5) and (2.19) respectively,

8) For $u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$ such that $u(x, t) \in D(t)$, we see that the following energy inequality holds

$$\begin{aligned} &\|u(t)\|_2 + \|u'(t)\|_1 + \|u''(t)\| \\ &\leqq C(T)(\|u(0)\|_2 + \|u'(0)\|_1 + \|f(0)\| + \|f'(0)\|_{-1} + \int_t^0 \|f'(s)\| ds + \int_0^t \|f''(s)\|_{-1} ds) . \end{aligned}$$

the entirely same procedure as that of Proposition 2.1 can be carried out, then for

$$u(x, t) \in \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$$

(3.20) is proved. We also use the mollifier to remove the additional assumption that $u(x, t) \in \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$. Let $u(x, t) \in \mathcal{E}_t^0(H^1(\Omega)')$. We define the regularized function $\phi_{\delta \overset{*}{(t)}} u \in \mathcal{E}_t^\infty(H^1(\Omega)')$ by

$$\langle \phi_{\delta \overset{*}{(t)}} u(x, t), \psi \rangle = \phi_{\delta \overset{*}{(t)}} \langle u(t), \psi \rangle \qquad \text{for all} \quad \psi \in H^1(\Omega). \tag{3.22}$$

Then if $0 < \delta < \delta_0$ and $t \in [0, T]$,

$$(\phi_{\delta \overset{*}{(t)}} u)_t'' = \phi_{\delta \overset{*}{(t)}} u''$$

and when $u(x, t) \in \mathcal{E}_t^0(L^2(\Omega))$, the above definition coincides with the ordinary one.

Now it suffices to show

$$\int_0^T \left\| \left(\frac{\partial}{\partial s} C_\delta u \right)(s) \right\|_{-1} ds \to 0. \tag{3.23}$$

$$\|(C_\delta u)(0)\|_{-1} \to 0 \qquad \text{when} \quad \delta \to 0, \tag{3.24}$$

where

$$\begin{aligned} C_\delta u = & [\phi_{\delta \overset{*}{(t)}} A(t) u(t) - A(t)(\phi_{\delta \overset{*}{(t)}} u)(t)] \\ & + [\phi_{\delta \overset{*}{(t)}} h(x, t) u'(t) - h(x, t) \phi_{\delta \overset{*}{(t)}} u'(t)]. \end{aligned}$$

$\left\langle \frac{\partial}{\partial t}(C_\delta u)(t), \psi \right\rangle$ is the sum of the terms in the form

$$\int_\Omega \left\{ \frac{d}{dt} \int \phi_\delta(t-\tau)[a(x, \tau) - a(x, t)] w(x, \tau) d\tau \right\} \overline{\left(\frac{\partial}{\partial x} \right)^\alpha \psi} dx, \tag{3.25}$$

$$\int_S \left\{ \frac{d}{dt} \int \phi_\delta(t-\tau)[a(x, \tau) - a(x, t)] v(x, \tau) d\tau \right\} \bar{\psi} dS, \tag{3.26}$$

where $w(x, t) \in \mathcal{E}_t^0(L^2(\Omega))$, $v(x, t) \in \mathcal{E}_t^0(H^1(\Omega))$, $|\alpha| \leq 1$ and $a(x, t) \in \mathcal{B}^2(\Omega \times (-\delta_0, T+\delta_0))$.

Using (2.29) and Schwarz inequality we see (3.25) is estimated by

$$\text{const}\ \Phi_\delta(t) \left\| \left(\frac{\partial}{\partial x} \right)^\alpha \psi \right\|_{L^2(\Omega)},$$

where

$$\begin{aligned} \Phi_\delta(t)^2 = \int_{|t-\tau|<\delta} d\tau \int_\Omega \Big[& \{\phi_\delta(t-\tau) + |t-\tau| \, |\phi_\delta'(t-\tau)|\} \, |w(x, t) - w(x, \tau)|^2 \\ & + \{\phi_\delta(t-\tau) |\tau - t| \, |w(x, \tau)|^2\} \Big] dx \end{aligned}$$

and

$$\int_0^T \Phi_\delta(t) dt \to 0 \qquad \text{when} \quad \delta \to 0.$$

Similarly (3.26) is estimated and using $\|\cdot\|_{L^2(S)} \leq \text{const}\ \|\cdot\|_{1, L^2(\Omega)}$ we get

$$\int_0^T \left\| \frac{\partial}{\partial t} C_\delta(t) \right\|_{-1} dt \to 0 \qquad \text{when} \quad \delta \to 0 .$$

(3.24) is also shown by the same way. Thus Proposition is proved.

Existence of the solution.

PROPOSITION 3.2. *Given* $U_0 \in D$ *and* $F(t) \in D$ *such that* $F(t)$ *and* $\mathcal{A}(t)F(t)$ *are continuous in* $L^2(\Omega) \times H^1(\Omega)'$, *then there exists a solution* $U(t) \in \mathcal{E}_t^1(L^2(\Omega) \times H^1(\Omega)')$ *of* (3.11) *such that* $U(0) = U_0$ *and* $U(t) \in D$ *for all* $t \in [-\delta_0, T+\delta_0]$.

PROOF. As the proof of Proposition 2.2, we apply Theorem stated before the proof of Proposition 2.2. Lemma 3.4 and Remark 3.4 assure the condition c_1), and from Lemma 3.1 and Corollary 3.1 it follows that $\mathcal{A}(t)$ satisfies c_2). The condition c_3) is already shown in the proof of Proposition 3.1, say (3.21). Thus we get Proposition 3.2.

Using Proposition 3.1 and 3.2 instead of Proposition 2.1 and 2.2 respectively, by the same argument as Theorem 1 it follows:

PROPOSITION 3.3. *Given* $\{u_0, u_1\} \in D$ *and* $f(x, t) \in \mathcal{E}_t^1(H^1(\Omega)')$, *then there exists one and only one solution* $u(x, t)$ *of* (3.10) *satisfying the initial condition*

$$u(x, 0) = u_0 , \qquad \frac{\partial u}{\partial t}(x, 0) = u_1$$

such that

$$u(x, t) \in \mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)') .$$

With the aid of these Propositions, we now prove the existence theorem.

THEOREM 3. *Given* $u_0 \in D(0)$, $u_1 \in H^1(\Omega)$ *and* $f(x, t) \in \mathcal{E}_t^0(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)')$, *there exists one and only one solution* $u(x, t)$ *of* (1.1) (*under the additional condition* $a_1(x, t: D) = h(x, t)$ *and* $b_i(x, t)$ *are real on* S) *satisfying the boundary condition*

$$B(t)u(x, t) = \frac{\partial}{\partial n_t} u + \sigma(s, t)u = 0 \qquad \text{on} \quad S$$

such that

$$u(x, t) \in \mathcal{E}_t^0(H^2(\Omega)) \cap \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega)) .$$

PROOF. $u_0 \in D(0)$ assures $A(0)u_0 \in L^2(\Omega)$ and since $u_1, f(0) \in L^2(\Omega)$ then

$$u_2 = -A(0)u_0 - h(x, t)u_1 + f(0) \in L^2(\Omega) .$$

We solve the equation

$$\tilde{L}[u'(x, t)] = -A'(t)u(t) - h'(x, t)u'(t) + f'(x, t) , \tag{3.27}$$

$$u(0) = u_0 , \qquad u'(0) = u_1 , \qquad u''(0) = u_2$$

by the method of successive approximation. Namely,

$$\tilde{L}[v_{j+1}] = -A'(t)\tilde{u}_j(t) - h'(x, t)\tilde{u}_j'(t) + f'(x, t) , \tag{3.28}$$

$$\tilde{u}_j(x, t) = u_0 + \int_0^t v_j(x, s)ds\,,$$
$$v_{j+1}(0) = u_1\,, \qquad v'_{j+1}(0) = u_2\,,$$
$$\tilde{u}_0 \equiv 0\,.$$

If $\tilde{u}_j(x, t) \in \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega))$, then the right side of (3.28) $\in \mathcal{E}_t^1(H^1(\Omega)')$, and since $\{u_1, u_2\} \in D$, Proposition 3.3 assures the existence of the solution v_{j+1} of (3.28) such that

$$v_{j+1}(x, t) \in \mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)')\,.$$

Therefore we can get $\tilde{u}_j(x, t)$ successively and using the energy inequality (3.20), the same reasoning as we used in the proof of Theorem 2 shows that $\tilde{u}_j(x, t)$ converges to

$$u(x, t) \in \mathcal{E}_t^1(H^1(\Omega)) \cap \mathcal{E}_t^2(L^2(\Omega)) \cap \mathcal{E}_t^3(H^1(\Omega)')\,, \tag{3.29}$$

which satisfies the equation (3.27), then also the equation (3.10). Thus we see

$$A(t)u(x, t) = -u''(t) - h(x, t)u'(t) + f(x, t) \in \mathcal{E}_t^0(L^2(\Omega))\,. \tag{3.30}$$

By Remark 3.2, $u(x, t) \in D(t)$ and

$$A(t)u(x, t) = a_2(x, t : D)u(x, t)\,.$$

Thus $u(x, t)$ is the solution of (1.1) satisfying the boundary and initial condition. Then it suffices to show only

$$u(x, t) \in \mathcal{E}_t^0(H^2(\Omega))\,. \tag{3.31}$$

Applying the apriori estimate concerning the elliptic operator

$$\|u\|_{2,L^2(\Omega)} \leqq K(\|a_2(x, t : D)u\|_{L^2(\Omega)} + \langle B(t)u\rangle_{\frac{1}{2}} + \|u\|_{L^2(\Omega)})\,, \tag{3.32}$$

we see first

$$\|u(x, t)\|_{2,L^2(\Omega)} \leqq \text{const} \qquad \forall t \in [0, T] \tag{3.33}$$

by taking $u = u(x, t)$. Next,

$$\begin{aligned}\|u(t) - u(t')\|_{2,L^2(\Omega)} \leqq K(&\|a_2(x, t : D)(u(t) - u(t'))\|_{L^2(\Omega)} \\ &+ \langle B(t)(u(t) - u(t'))\rangle_{\frac{1}{2}} + \|u(t) - u(t)\|_{L^2(\Omega)})\end{aligned} \tag{3.34}$$

by taking $u = u(t) - u(t')$.

Here

$$a_2(x, t : D)(u(t) - u(t'))\,,$$
$$= g(t) - g(t') + (a_2(x, t' : D) - a_2(x, t : D))u(t')\,,$$
$$B(t)(u(t) - u(t')) = (B(t') - B(t))u(t')\,,$$

where $g(t)$ is the right side of (3.30). From this (3.31) follows, because the right side of (3.34) tends to zero when t' tends to t. The uniqueness has been shown in Proposition 3.3.

Regularity.

Let us show that even in the actual case, we also obtain the regularity of the solution.

LEMMA 3.5. *Suppose that the coefficients of* $a_2(x, t: D)$ *belong to* $\mathcal{B}^{p+k}(\Omega \times(-\delta_0, T+\delta_0))$ *and*

$$u(x, t) \in H^{p+2}(\Omega) \quad \forall t, \quad \text{and} \quad \in \mathcal{E}_t^k(L^2(\Omega)), \tag{3.35}$$

$$a_2(x, t: D)u(x, t)=f(x, t) \in \mathcal{E}_t^k(H^p(\Omega)), \tag{3.36}$$

$$B(t)u(x, t)=g(x, t) \in \mathcal{E}_t^k(H^{p+\frac{1}{2}}(S)), \tag{3.37}$$

where $p \geqq 0$, $k \geqq 0$, *then*

$$u(x, t) \in \mathcal{E}_t^k(H^{p+2}(\Omega)). \tag{3.38}$$

PROOF. Let us remark that for all $w \in H^{p+2}(\Omega)$

$$\|w\|_{p+2} \leqq K_p(\|a_2(x, t: D)w\|_p+\langle B(t)w\rangle_{p+\frac{1}{2}}+\|w\|_0) \tag{3.39}$$

holds. If we take $w=u(x, t)$, then it follows

$$\|u(x, t)\|_{p+2} \leqq K_p(\|f(x, t)\|_p+\langle g(x, t)\rangle_{p+\frac{1}{2}}+\|u(x, t)\|_0),$$

therefore $\|u(x, t')\|_{p+2}<M$ for all $t' \in(t-\delta, t+\delta)$.

$$\begin{aligned} a_2(x, t: D)&(u(x, t')-u(x, t)) \\ &=-(a_2(x, t': D)-a_2(x, t: D))u(x, t')+f(x, t')-f(x, t), \end{aligned} \tag{3.40}$$

$$B(t)(u(x, t')-u(x, t))=-(B(t')-B(t))u(x, t')+g(x, t')-g(x, t), \tag{3.41}$$

and the right sides of (3.40) and (3.41) tend to 0 in the space $H^p(\Omega)$ and $H^{p+\frac{1}{2}}(S)$ respectively, then applying (3.39) by taking $w=u(x, t')-u(x, t)$, we have

$$\|u(x, t')-u(x, t)\|_{p+2} \rightarrow 0 \quad \text{when} \quad t' \rightarrow t.$$

Thus we have $u(x, t) \in \mathcal{E}_t^0(H^{p+2}(\Omega))$.

In the case $k=1$, by dividing (3.40) and (3.41) by $t'-t$ and since from the assumption

$$a_2(x, t: D)\frac{u(x, t')-u(x, t)}{t'-t},$$

$$B(t)\frac{u(x, t')-u(x, t)}{t'-t},$$

converges in $H^p(\Omega)$ and $H^{p+\frac{1}{2}}(S)$ respectively when $t' \rightarrow t$, using (3.39) we see the convergence of $\frac{u(x, t')-u(x, t)}{t'-t}$ in $H^{p+2}(\Omega)$.

Then we have relations

(3.42) $$a_2(x, t: D)u'(x, t) = -a_2'(x, t: D)u(x, t) + f'(x, t),$$

(3.43) $$B(t)u'(x, t) = -B'(t)u(x, t) + g'(x, t).$$

Now, since the right sides are continuous, by the above result for $k=0$, $u'(x, t) \in \mathcal{E}_t^0(H^{p+2}(\Omega))$ is proved. Next let us show that $u' \in \mathcal{E}_t^1(H^{p+2}(\Omega))$ in the case $k \geqq 2$. We see that $a_2(x, t: D)u'(x, t) \in \mathcal{E}_t^1(H^p(\Omega))$ by (3.42), and $B(t)u'(x, t) \in \mathcal{E}_t^1(H^{p+\frac{1}{2}}(S))$ by (3.43). Applying the just obtained result, we see that $u''(x, t) \in \mathcal{E}_t^0(H^{p+2}(\Omega))$. We can continue this reasoning and get the desired result.

Now consider a solution of (1.1) such that $u(x, t) \in \mathcal{E}_t^0(H^{m+2}(\Omega)) \cap \mathcal{E}_t^1(H^{m+1}(\Omega)) \cap \cdots \cap \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega))$. Then $u^{(p)}(x, 0)$ $(p=1, 2, \cdots, m+2)$ is represented by the initial data $\{u_0, u_1\}$ and by the second member $f(x, t)$. If $u(x, t)$ satisfies the boundary condition

$$B(t)u(x, t) = 0,$$

then

$$B(t)u'(x, t) + B'(t)u(x, t) = 0,$$

$$B(t)u''(x, t) + 2B'(t)u'(x, t) + B''(t)u(x, t) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

Thus putting $t=0$, we see that $u^{(p)}(x, 0)$ should satisfy

$$B(0)u_1 + B'(0)u_0 = 0,$$

$$B(0)u^{(2)}(x, 0) + 2B'(0)u_1(x) + B''(0)u_0(x) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

Therefore we introduce the compatibility condition as follows:

Let u_p $(p=2, 3, \cdots, m+2)$ be those of (2.30), then

(3.44) $$\sum_{k=0}^{l} \binom{l}{k} B^{(k)}(0)u_{l-k}(x) = 0 \quad \text{on } S \qquad (l=1, 2, \cdots, m).$$

THEOREM 4. *Suppose that coefficients of L belong to $\mathcal{B}^{m+1}(\Omega \times (-\delta_0, T+\delta_0))$ and*

$$\{u_0, u_1\} \in (D(0) \cap H^{m+2}(\Omega)) \times H^{m+1}(\Omega) \qquad (m \geqq 1),$$

$$f(x, t) \in \mathcal{E}_t^0(H^m(\Omega)) \cap \mathcal{E}_t^1(H^{m-1}(\Omega)) \cap \cdots \cap \mathcal{E}_t^m(L^2(\Omega)) \cap \mathcal{E}_t^{m+2}(H^1(\Omega)')$$

then, if the compatibility condition (3.44) *is satisfied, the solution $u(x, t)$ of* (1.1) (*under the additional condition that $a_1(x, t: D) = h(x, t)$ and $b_j(x, t)$ are real-valued on S*) *satisfies*

(3.45) $$u(x, t) \in \mathcal{E}_t^0(H^{m+2}(\Omega)) \cap \mathcal{E}_t^1(H^{m+1}(\Omega)) \cap \cdots \cap \mathcal{E}_t^{m+2}(L^2(\Omega)).$$

PROOF. At first we prove

(3.46) $$u(x, t) \in \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega)).$$

Since $u(x, t)$ is also the solution of (3.10), (3.46) is shown by using the analogous reasoning to that of Theorem 2. For this purpose, let us notice that

$$u_p=-\sum_{k=0}^{p-2}\binom{p-2}{k}[A^{(k)}(0)u_{p-k-2}+h^{(k)}(x, 0)u_{p-k-1}]+f^{(p-2)}(x, 0), \tag{3.47}$$

where u_p $(p=1, 2, \cdots, m+2)$ are defined by (2.30). In fact, since $u_p\in H^{m+2-p}(\Omega)$, for $\phi\in H^1(\Omega)$ we have

$$\begin{aligned}\langle A^{(k)}(0)u_{p-k-2}, \bar\phi\rangle=&\sum_{i,j=1}^{n}\Big(a_{ij}^{(k)}(x, 0)\frac{\partial u_{p-k-2}}{\partial x_i}, \frac{\partial\phi}{\partial x_j}\Big)\\&+\int_S\sigma^{(k)}(x, 0)u_{p-k-2}\bar\phi dS+\int_S\langle b^{(k)}(x, 0), \nu\rangle u_{p-k-2}\bar\phi dS\\&+\Big(u_{p-k-2}, \Big(-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\overline{b_j^{(k)}(x, 0)}+\overline{c^{(k)}(x, 0)}\Big)\phi\Big).\end{aligned}$$

The integration by parts gives

$$\begin{aligned}=&-\sum_{i,j=1}^{n}\Big(\frac{\partial}{\partial x_i}a_{ij}^{(k)}(x, 0)\frac{\partial}{\partial x_j}u_{p-k-2}, \phi\Big)+\int_S\sigma^{(k)}(x, 0)u_{p-k-2}\bar\phi dS\\&+\Big(\sum_{j=1}^{n}b_j^{(k)}(x, 0)\frac{\partial}{\partial x_j}u_{p-k-2}+c^{(k)}(x, 0)u_{p-k-2}, \phi\Big)\\&+\int_S\sum_{i,j=1}^{n}a_{ij}^{(k)}(x, 0)\nu_i\frac{\partial u_{p-k-2}}{\partial x_j}\bar\phi\, dS.\end{aligned}$$

Then

$$\begin{aligned}\Big\langle-\sum_{k=0}^{p-2}&\binom{p-2}{k}[A^{(k)}(0)u_{p-k-2}+h^{(k)}(x, 0)u_{p-k-1}]+f^{(p-2)}(x, 0), \bar\phi\Big\rangle\\&=-\sum_{k=0}^{p-2}\binom{p-2}{k}[\langle A^{(k)}(0)u_{p-k-2}, \bar\phi\rangle+(h^{(k)}(x, 0)u_{p-k-1}, \phi)]+(f^{(p-2)}(x, 0), \phi)\\&=\Big(-\sum_{k=0}^{p-2}\binom{p-2}{k}(a_2^{(k)}(x, 0: D)u_{p-k-2}+h^{(k)}(x, 0)u_{p-k-1})+f^{(p-2)}(x, 0), \phi\Big)\\&\quad-\sum_{k=0}^{p-2}\binom{p-2}{k}\int_S B^{(k)}(0)u_{p-k-2}\bar\phi\, dS.\end{aligned}$$

Now by virtue of the compatibility condition the surface integral vanishes. Then the last member is nothing but $\langle u_p, \bar\phi\rangle$, so we get (3.47)[9].

Now let us show the existence of the solution to the equation

$$\tilde L[\tilde u^{(m+1)}]=-\sum_{k=0}^{m}\binom{m+1}{k}\tilde L^{(m+1-k)}[\tilde u^{(k)}]+f^{(m+1)}(x, t), \tag{3.48}$$

with the initial condition $\tilde u^{(j)}(x, 0)=u_j$ $(j=0, 1, 2, \cdots, m+2)$, by the method of

9) Conversely (3.47) assures the compatibility condition.

successive approximation. In fact, we get $\tilde{u}_j(x, t) \in \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega)) \cap \mathcal{E}_t^{m+3}(H^1(\Omega)')$ successively by

$$(3.49) \qquad \tilde{L}[v_{j+1}(x, t)] = -\sum_{k=0}^{m} \binom{m+1}{k} \tilde{L}^{(m+1-k)}[\tilde{u}_j^{(k)}] + f^{(m+1)}(x, t),$$

$$v_{j+1}(x, 0) = u_{m+1} \in H^1(\Omega), \qquad v'_{j+1}(x, 0) = u_{m+2} \in L^2(\Omega).$$

$$(3.50) \qquad \tilde{u}_{j+1}(x, t) = u_0 + tu_1 + \cdots + \frac{t^m}{m!} u_m + \int_0^t \frac{(t-s)^m}{m!} v_{j+1}(x, s) ds,$$

$$\tilde{u}_0(x, t) \equiv 0,$$

because if

$$v_j \in \mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)'),$$

then

$$\tilde{u}_j \in \mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega)) \cap \mathcal{E}_t^{m+3}(H^1(\Omega)'),$$

therefore the right side of (3.49) $\in \mathcal{E}_t^1(H^1(\Omega)')$ and $\{u_{m+1}, u_{m+2}\} \in D$, so Proposition 3.3 can be applied to (3.49), then we get

$$v_{j+1} \in \mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)').$$

By using the energy inequality (3.20), the convergence of $v_j(x, t)$ in $\mathcal{E}_t^0(H^1(\Omega)) \cap \mathcal{E}_t^1(L^2(\Omega)) \cap \mathcal{E}_t^2(H^1(\Omega)')$, and also that of $\tilde{u}_j(x, t)$ in $\mathcal{E}_t^{m+1}(H^1(\Omega)) \cap \mathcal{E}_t^{m+2}(L^2(\Omega)) \cap \mathcal{E}_t^{m+3}(H^1(\Omega)')$ are proved, and (3.47) assures that the limit $\tilde{u}(x, t)$ of $\tilde{u}_j(x, t)$ is the solution of (3.10)[10]. From the uniqueness of the solution of (3.10) it follows (3.46).

Now we can prove Theorem. Since $u(x, t) \in \mathcal{E}_t^0(H^2(\Omega))$, $B(t)u(t) = 0$, and that

$$(3.51) \qquad a_2(x, t: D)u(x, t) = -u''(x, t) - h(x, t)u'(x, t) + f(x, t) \in \mathcal{E}_t^m(L^2(\Omega)),$$

we have $u(x, t) \in \mathcal{E}_t^m(H^2(\Omega))$ by applying Lemma 3.5 taking $p=0$, $k=m$.

Next, since $u \in \mathcal{E}_t^{m+1}(H^1(\Omega))$, $m \geqq 1$, (3.51) shows $u \in H^3(\Omega)$ and $a_2(x, t: D)u(x, t) \in \mathcal{E}_t^{m-1}(H^1(\Omega))$. Applying Lemma 3.5, we see that $u(x, t) \in \mathcal{E}_t^{m-1}(H^3(\Omega))$. Suppose $m \geqq 2$. Then $u(x, t) \in \mathcal{E}_t^m(H^2(\Omega))$ implies that $a_2(x, t: D)u(x, t) \in \mathcal{E}_t^{m-2}(H^2(\Omega))$. We have then $u(x, t) \in H^4(\Omega)$ and applying again Lemma 3.5, it follows that $u(x, t) \in \mathcal{E}_t^{m-2}(H^4(\Omega))$. Step by step, we get the desired property of u.

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