18. *Mixed Problems of Hyperbolic Equations in a General Domain*

By Junko KATO
Department of Mathematics, Osaka University

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§0. **Introduction.** Concerning to hyperbolic mixed problems with uniform Lopatinski's conditions, a priori estimates have been obtained by R. Sakamoto [5]. Our aim is to obtain existence theorems in general domains by making use of a priori estimates in a half space. To prove it, we make use of the method of partition of unity. So we must be sure two typical properties in a half spaces, that is, Huygens' principle and finite propagation speed. We sketch of the former in §1 and the latter in §2. Finally we state some results in general domains in §3.

Let us consider the problem $[A, \{B_i\}, \{D_t^i\}, f, \{g_j\}, \{u_j\}]$ in Ω_T;

$$(P)\begin{cases} A(D_x, D_t)u \equiv D_t^m u + \sum\limits_{\substack{|\nu|+j\leq m\\ j\leq m-1}} a_{\nu j} D_x^\nu D_t^j u = f \text{ in } \Omega_T = \Omega\times(0,T), \ (m=2m'), \\ B_i(D_x, D_t)u \equiv \sum\limits_{|\nu|+j=0}^{m_i} b_{\nu j}^i D_x^\nu D_t^j u = g_i \text{ on } S_T = S\times(0,T), \ (i=1,\cdots,m'), \\ D_t^j u = u_j \text{ on } \Omega_0 \ (j=0,\cdots,m-1), \end{cases}$$

$\left(D_{x_j} = \dfrac{1}{i}\dfrac{\partial}{\partial x_j}, \ D_x^\nu = D_{x_1}^{\nu_1}, \cdots, D_{x_n}^{\nu_n}, |\nu| = \nu_1 + \cdots + \nu_n \text{ for multi index } \nu = (\nu_1, \cdots, \nu_n)\right)$ where Ω is a domain in R^n whose boundary S is a C^∞ hypersurface. For $\{A, B_i\}$, we set following assumptions;

(1) $\quad a_{\nu_j} \in \mathcal{B}(\Omega_T), \qquad b_{\nu_j}^i \in \mathcal{B}(S_T).$

(2) A is regularly hyperbolic with respect to t, that is, m roots in λ of the characteristic polynomial $A_0(\xi, \lambda) = 0$ are real for parameter $\xi \in R^n$ and

$$d = \inf_{\substack{(x,t)\in\Omega_T\\ \xi\in R^n, |\xi|=1\\ i\neq j}} |\lambda_i(x,t,\xi) - \lambda_j(x,t,\xi)| > 0. \quad (\lambda_1, \cdots, \lambda_m; \text{ roots}).$$

In addition, A has the ellipticity with respect to the x-direction on S, that is,

$$|A_0(\xi, x)| \geqslant c|\xi|^m \text{ where } (x,t)\in S_T.$$

Let $N(x)$ and $T(x)$ be the unit inner normal and the tangent space to S at $x\in S$ respectively. From (2), $A_0(z) = A_0(\zeta + zN, \lambda) = 0$ has m' roots $z_1^+, \cdots, z_{m'}^+$ whose imaginary parts are positive for $(x,t)\in S_T$, $(\zeta, \lambda) \in T\times C^1$, $\operatorname{Im}\lambda < 0$.

(3) $\{B_i\}$ is a normal set, that is, S is not a characteristic surface of any B_i and $m_i \leqslant m-1$, $m_i \neq m_j$ $(i \neq j)$.

(4) $\{A, B_i\}$ satisfies the uniform Lopatinski's condition, that is,

$$l = \inf_{\substack{(x,t)\in S_T \\ (\zeta,\lambda)\in T\times C^1-\{0\} \\ \operatorname{Im}\lambda\leq 0 \\ |\zeta|^2+|\lambda|^2=1}} \left| \det\left(\oint \frac{B_{i0}(z)z^{j-1}}{\prod_{k=1}^{m'}(z-z_k^+)} dz \right)_{i,j=1,\cdots,m'} \right| > 0$$

where integration are taken along a closed curve in the z-plane enclosing all of $\{z_i^+\}_{i=1,\cdots,m'}$.

We use following notations; Let $\mathfrak{H}_{t_0}^{m-1+r}$ $(r=0,1,\cdots)$ be the Hilbert space defined by the completion of $H^{m+r}(\Omega_{t_0})$ with the norm

$$\|u\|^2_{\mathfrak{H}_{t_0}^{m-1+r}} = \|Au\|^2_{r,(t_0)} + \sum_{i=1}^{m'} \langle B_i u\rangle^2_{m-1-m_i+r,(t_0)} + \sum_{j=0}^{m-1} [D_t^j u(0)]^2_{m-1-j+r}$$

where

$$\|\varphi\|^2_{r,(t_0)} = \sum_{|\nu|+j\leq r} \int_{\Omega_{t_0}} |D_x^\nu D_t^j \varphi|^2 dx dt \quad (\varphi \in H^r(\Omega_{t_0})),$$

$$\langle\varphi\rangle^2_{r,(t_0)} = \sum_{|\nu|+j\leq r} \int_{S_{t_0}} |D_x^\nu D_t^j \varphi|^2 dx dt \quad (\varphi \in H^r(S_{t_0})),$$

$$[\varphi]^2_r = \sum_{|\nu|\leq r} \int_{\Omega} |D_x^\nu \varphi|^2 dx \qquad (\varphi \in H^r(\Omega)).$$

I would like to thank Professor Sakamoto for her kind advices and encouragement.

§1. Existence theorems in a half space.

Let $\Omega \equiv R_+^n = \{(x', x_n) \in R^{n-1} \times R_+^1\}$ in this section. We add an assumption:

(5) Coefficients of $\{A, B_i\}$ are constant where $x' \in R^{n-1} - K$ for some compact set K in R^{n-1}.

Theorem 1. *For*

$$(f, g_i, u_j) \in H^r(\Omega_T) \times \prod_{i=1}^{m'} H^{m-1-m_i+r}(S_T) \times \prod_{j=0}^{m-1} H^{m-1-j+r}(\Omega)$$

satisfying the compatibility condition of order $m+r$, *there exists unique solution* $u \in \mathfrak{H}_T^{m-1+r}$ *of* (P). *It is valid for this* u,

$$\|u\|^2_{m-1+r,(t_0)} + \sum_{j=0}^{m-1+r} \langle D_x^j u\rangle^2_{m-1-j+r,(t_0)} + \sum_{j=0}^{m-1+r} [D_t^j u(t_0)]^2_{m-1-j+r} \leqslant C \|u\|^2_{\mathfrak{H}_{t_0}^{m-1+r}}$$
$$(t_0 \in [0, T]).$$

Note. Put
$$u_{m+s}(x) = -\sum_{k=0}^{s} \sum_{\nu,j} (D_t^k a_{\nu_j})(x,0) D_x^\nu u_{j+s-k}(x) + D_t^s f(x,0)$$
$$= A^{(s)}(f, u_1, \cdots, u_{m+s-1}),$$
$$\sum_{k=0}^{s} \sum_{\nu,j} (D_t^k b_{\nu_j}^i)(x,0) D_t^\nu u_{j+s-k}(x) = B_i^{(s)}(u_1, \cdots, u_{m_i+s}).$$

We say (f, g_i, u_j) satisfies the compatibility condition of order $m+r$ if
$$B_i^{(s)}(u_1, \cdots, u_{m_i+s})|_{x_n=0} = D_t^s g_i|_{t=0}$$
$$(s=0, \cdots, m-2-m_i+r, i=1, \cdots, m').$$

Proof. Let us denote
$$\boldsymbol{H}^r = H^r(\Omega_T) \times \prod_{i=1}^{m'} H^{m-1-m_i+r}(S_T) \times \prod_{j=0}^{m-1} H^{m-1-j+r}(\Omega).$$

If (f, g_i, u_j) belongs to $\boldsymbol{H}^\infty$ and satisfies the compatibility condition of order $m+r$, we can easily find unique solution $u \in H^{m+r-1}(\Omega_T)$ by reducing to [5] II, Theorem 1. If (f, g_i, u_j) belongs to $\boldsymbol{H}^r$ and satisfies the compatibility condition of order $m+r$, we shall make use of the approximation method. Let $\{(f^\mu, g_i^\mu, u_j^\mu)\} \subset \boldsymbol{H}^\infty$ be a sequence tending to (f, g_i, u_j) in $\boldsymbol{H}^r$. $\{(f^\mu, g_i^\mu, u_j^\mu)\}$ do not satisfy compatibility conditions in general, but we can construct another sequence $\{(f^\mu, \tilde{g}_i^\mu, u_j^\mu)\} \subset \boldsymbol{H}^\infty$ which satisfy compatibility conditions of order $m+r$ and tends to (f, g_i, u_j) in $\boldsymbol{H}^r$. In fact, let

$$u_{m+s}^\mu = A^{(s)}\,(f^\mu, u_1^\mu, \cdots, u_{m+s-1}^\mu),$$

$$\varphi_{i,s}^\mu = B^{(s)}\,(u_1^\mu, \cdots, u_{m_i+s}^\mu)|_{x_n=0} - D_t^s g_i^\mu|_{t=0}$$

and $\tilde{\varphi}_i^\mu \in H^\infty(S_T)$ be an extention of $\{\varphi_{i,s}^\mu\}_{s\leq m-2-m_i+r}$ such that

$$\langle \tilde{\varphi}_i^\mu \rangle_{m-1-m_i+r,(T)} \leqslant \text{Const.} \sum_{s=0}^{m-2-m_i+r} \langle\!\langle \varphi_{i,s}^\mu \rangle\!\rangle_{m-1-m_i+r-s-\frac{1}{2}}.$$

($\langle\!\langle\ \rangle\!\rangle$ denotes the norm on R^{n-1}).

Then $\{(f^\mu, \tilde{g}_i^\mu = g_i^\mu - \tilde{\varphi}_i^\mu, u_j^\mu)\}$ satisfy above conditions. Let u^μ be the solution for $(f^\mu, \tilde{g}_i^u, u_j^\mu)$, then $\{u^\mu\}$ converges to u in $\mathfrak{H}_T^{m-1+r}$ and u becomes a solution of (P). From a priori estimates for $\{u^\mu\}$ ([5] II, Theorem 2), we get the estimate for u.

§2. Finite propagation speed.

Let $D_{\delta(x_0,t_0)} = \{(x,t) \in \Omega_T\,;\, |x-x_0|^2 + |t-t_0|^2 < \delta^2,\ t_0 < t < t_0 + \delta^2\}$

$$(\delta > 0, (x_0, t_0) \in \bar{\Omega}_T,\ t_0 \in [0, T-\delta^2]),$$

$$D'_{\delta(x_0,t_0)} = \{(x,t) \in \Omega_T\,;\, |x-x_0| < \frac{1}{\delta}|t-t_0|,\ 0 < t < t_0\}$$

$$(\delta > 0, (x_0, t_0) \in \bar{\Omega}_T).$$

Lemma 1. *There exists a positive number δ_0 such that, if $\delta \leqslant \delta_0$ and $u \in \mathfrak{H}_T^{m-1+r}$ satisfies $[A, \{B_i\}, \{D_t^j\} : 0, \{0\}, \{0\}]$ in $D_{\delta(x_0,t_0)}$, then $u=0$ in $D_{\delta(x_0,t_0)}$, where δ_0 depends only on $m,n,d,l,M = \max\limits_{\substack{|\nu||\mu|+j=m\\ |\mu|+k=m_i\\ i=1,\cdots,m'}} (|a_{\nu_j}|_{\mathring{B}(\Omega_T)}, |b_{\mu k}^i|_{\mathring{B}(S_T)})$ and $L = \inf\limits_{(x,t)\in\Omega_T} (|a_{\nu_0,0}|)$ $(\nu_0 = (0, \cdots, 0, m))$.*

Proof. Let $\{\tilde{A}, \tilde{B}_i\}$ be the image of $\{A, B_i\}$ by Holmglen's transformation with (x_0, t_0) at the center $((x_0, t_0) \in \bar{\Omega}_T)$. From the continuity of roots of polynomials with respect to coefficients, there exists $\delta_0(m, n, d, l, M, L)$ such that $\{\tilde{A}, \tilde{B}_i\}$ satisfies conditions (1)$\sim$(4) restricted to $\tilde{D}_{\delta_0(x_0,t_0)}$. Let us modify $\{\tilde{A}, \tilde{B}_i\}$ outside of $\tilde{D}_{\delta(x_0,t_0)}$ so that it satisfies (1)$\sim$(5) in $\bar{R}_+^n \times R^1$, and we denote it by the same notation $\{\tilde{A}, \tilde{B}_i\}$. Since

$$\tilde{A}^* v = \tilde{w} \text{ in } R_+^n \times (-\delta^2, 0), \quad \tilde{B}_i' v = 0 \text{ on } R^{n-1} \times (-\delta^2, 0)\ (i=1, \cdots, m'),$$

$$D_t^j v|_{t=0} = 0\ (j=0, \cdots, m-1) \qquad (\{\tilde{A}^*, \tilde{B}_i'\} : \text{ adjoint system of } \{\tilde{A}, \tilde{B}_i\})$$

is solvable, we obtain from Green's formula, $(u, w)_{D_{\delta(x_0,t_0)}} = 0$ for $w \in L^2(D_{\delta(x_0,t_0)})$. Hence $u=0$ in $D_{\delta(x_0,t_0)}$.

By making use of the well known method of sweeping out, we have

Theorem 2. *If* $u \in \mathfrak{H}_T^{m-1+r}$ *satisfies* $[A, \{B_i\}, \{D_t^j\}; 0, \{0\}, \{0\}]$ *in* $D'_{\delta(x_0,t_0)}$, *then* $u=0$ *in* $D'_{\delta(x_0,t_0)}$.

Corollary. *Let* $u \in \mathfrak{H}_T^{m-1+r}$ *be the solution of* (P).
If supports of datas $\subset D_K = \bigcup_{x_0 \in K} \left\{(x,t) \in \bar{\Omega}_T ; |x-x_0| < \frac{1}{\delta_0} t\right\}$ *where* K *is a compact set in* $\bar{\Omega}$, *then* $\mathrm{supp}[u] \subset D_K$.

§ 3. Existence theorems in a general domain.

Let S be a closed C^∞ hypersurface in R^n and Ω be the interior or exterior domain of S. There exists a finite open covering $\{U_p\}$ of S in R^n and there exist C^∞ injections $\{\phi_p ; \bar{\Omega} \cap U_p \to \bar{R}^n_+, (S \cap U_p \to R^{n-1})\}$ such that assumptions (1)～(4) for $\{A, B_i\}$ are stable for $\{\phi_p\}$.

By making use of the method of partition of unity, we have from Theorem 1, Theorem 2 and its corollary,

Theorem 3. *For*

$$(f, g_i, u_j) \in H^r(\Omega_T) \times \prod_{i=1}^{m'} H^{m-1-m_i+r}(S_T) \times \prod_{j=0}^{m-1} H^{m-1-j+r}(\Omega)$$

satisfying the compatibility condition of order $m+r$, *there exists unique solution* $u \in \mathfrak{H}_T^{m-1+r}$ *of* (P). *It is valid for this* u,

$$\|u\|^2_{m-1+r,(t_0)} + \sum_{j=0}^{m-1+r} \langle D_N^j u\rangle^2_{m-1-j+r,(t_0)} + \sum_{j=0}^{m-1+r} [D_t^j u(t_0)]^2_{m-1-j+r} \leqslant C\|u\|^2_{\mathfrak{H}_{t_0}^{m-1+r}}$$

$$(t_0 \in [0, T]).$$

Theorem 4. *If* $u \in \mathfrak{H}_T^{m-1+r}$ *satisfies* $[A, \{B_i\}, \{D_t^j\} : 0, \{0\}, \{0\}]$ *in* $D'_{\delta(x_0,t_0)}$, *then* $u=0$ *in* $D'_{\delta(x_0,t_0)}$ $((x_0, t_0) \in \bar{\Omega}_T, 0 < \delta \leqslant \delta_0)$.

Corollary. *Let* $u \in \mathfrak{H}_T^{m-1+r}$ *be the solution of* (P).

If supports of datas $\subset D_K = \bigcup_{x_0 \in K} \left\{(x,t) \in \bar{\Omega}_T ; |x-x_0| < \frac{1}{\delta_0} t\right\}$ *where* K *is a compact set in* $\bar{\Omega}$, *then* $\mathrm{supp}[u] \subset D_K$.

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