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Mixed shock models

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Traditionally, shock models are of two kinds. The failure (of a system) is related either to the cumulative effect of a (large) number of shocks or it is caused by a shock which is larger than some critical level. The present paper is devoted to a mixed model, in which the system is supposed to break down either because of one (very) large shock, or as a result of many smaller ones.

Keywords: convergence; cumulative shock model; extreme shock model; first passage times; intershock time; mixed shock model; moments; renewal theory; stopped random walk; uniform integrability

1. Introduction

Shock models are systems that at random times are subject to shocks of random magnitude. One distinguishes between two major types: cumulative shock models and extreme shock models. Systems governed by the former type break down when the cumulative shock magnitude exceeds some given level, whereas systems modelled by the latter type break down as soon as an individual shock exceeds some given level. For some background and examples, see Shanthikumar and Sumita (1983), Sumita and Shanthikumar (1985), Anderson (1987; 1988), Gut (1990), Boshuizen and Gouweleeuw (1993), and Gut and Hüsler (1999).

In this paper we investigate a mixture of these models. We suppose that the system breaks down because of either a cumulative effect or a single large shock, depending on which attains its critical level first. This idea is not entirely new; as pointed out by a referee, Li and Shaked (1995; 1997) consider the same mixed kind of first passage times, but relative to certain damage processes and more general Markov processes, and with a somewhat different focus.

The paper is organized as follows. In Section 2 we describe the cumulative and the extreme shock models and present some basic results for later comparisons. We focus, in particular, on asymptotics for high levels, that is, when the level t increases to the upper end-point $t \to x_F := \sup\{x : F(x) < 1\} \leqslant \infty$ of the distribution. In Section 3 we describe the mixed shock model and state a result on distributional convergence, the proof of which is given in Section 4. Uniform integrability and moment convergence are treated in Section 5. Section 6 is devoted to applications, and Section 7 contains some further results and comments.

1350–7265

2. Background

In this section we describe the cumulative and extreme shock models and quote some basic results.

The general set-up in cumulative shock models (of what is called type I) is a family $\{(X_k, Y_k), k \geqslant 0\}$ (with $X_0 = Y_0 = 0$), of independent and identically distributed two-dimensional random vectors, with partial sums $T_n = \sum_{k=1}^n Y_k$ and $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$, respectively. The interpretation is that X_k represents the magnitude of the kth shock, and Y_k represents the time between the $(k-1)$th and the kth shock. The main object of interest is the lifetime or failure time of the system.

2.1. Cumulative shock models

Following Gut (1990), we define the first passage time process $\{\nu(t), t \geqslant 0\}$ by

$$\nu(t) = \min\{n : S_n > t\}. \tag{2.1}$$

The failure time then can be described by the random variable $T_{\nu(t)}$.

Note that $\nu(t)$ is the first passage time for the random walk $\{S_n, n \geqslant 1\}$. Since shocks in general are non-negative, 'everything' is known for the first passage times from classical renewal theory. The essential feature, however, is that $\mathrm{E}X_1 > 0$; see Gut (1988, Chapter III). Throughout this paper, therefore, we allow for arbitrarily signed shocks with positive mean in this model. (Some results in renewal theory remain valid when $\mu_x = \infty$ with $1/\infty = 0$, but this case will always be excluded in what follows.) Similarly for $\{Y_k, k \geqslant 1\}$ – although the sequence represents *times* in the present context, we do not (need to) presuppose any positivity in our results.

As a preliminary we recall from renewal theory for random walks – see, for example, Gut (1988, Theorem III.4.1) – that

$$\frac{\nu(t)}{t} \xrightarrow{\text{a.s.}} \frac{1}{\mu_x} \qquad \text{as } t \to \infty. \tag{2.2}$$

We now give two basic results from Gut (1990), which follow from Gut and Janson (1983); see also Gut (1988, Section IV.2).

Theorem 2.1. (i) *If $\mu_x = \mathrm{E}X_1 > 0$ and $\mu_y = \mathrm{E}Y_1$ exists and is finite, then*

$$\frac{T_{\nu(t)}}{t} \xrightarrow{\text{a.s.}} \frac{\mu_y}{\mu_x} \qquad \text{as } t \to \infty.$$

(ii) *If, in addition, $\sigma_x^2 = \mathrm{var}X_1 < \infty$, $\sigma_y^2 = \mathrm{var}Y_1 < \infty$, and $\gamma^2 = \mathrm{var}(\mu_x Y_1 - \mu_y X_1) > 0$, then*

$$\frac{T_{\nu(t)} - \dfrac{\mu_y}{\mu_x} t}{\sqrt{\mu_x^{-3}\gamma^2 t}} \xrightarrow{d} N(0, 1) \qquad \text{as } t \to \infty.$$

The proof of (i) uses strong laws for stopped random walks and renewal theory for random walks (for background, see Gut 1988).

Remark 2.1. Traditionally one studies the counting process $\{N(t),\ t \geqslant 0\}$ (that is, $N(t) = \max\{n : S_n \leqslant t\}$) rather than the first passage time process. Mathematically first passage times are more convenient because they are stopping times, whereas the counting variables are not. However, their asymptotics are, generally, the same. □

2.2. Extreme shock models

In this model we introduce, instead of (2.1), the first passage time process $\{\tau(t),\ t \geqslant 0\}$, defined by

$$\tau(t) = \min\{n : X_n > t\}, \qquad t \geqslant 0. \tag{2.3}$$

The failure time now is described by $T_{\tau(t)}$. As remarked in Gut and Hüsler (1999), there are some important differences in this setting. First of all, whereas 'high level' for the cumulative shock model means 'as $t \to \infty$', it now is to be interpreted as 'as t increases to the upper end-point of the distribution', that is, as $t \to x_F := \sup\{x : F(x) < 1\}$, finite or infinite. We also do not assume that the size of the shocks has a finite mean. That is to say, the crucial assumption is that

$$p_t = P(X_1 > t) = 1 - F_X(t) \to 0 \qquad \text{as } t \to x_F. \tag{2.4}$$

Secondly, we recall the well-known, and easily established, facts that $\tau(t)$ is geometric with mean $1/p_t$, and that

$$p_t \tau(t) \xrightarrow{d} \mathrm{Exp}(1) \qquad \text{as } t \to x_F. \tag{2.5}$$

In particular, this shows that the stopping times behave very differently from the previous ones in that there is no law of large numbers available for $\tau(t)$, and, hence, no Anscombe theorem to exploit as before. In spite of this the failure time, $T_{\tau(t)}$, is still a *stopped random walk.*

We now give the counterpart of Theorem 2.1 related to the extreme case taken from Gut and Hüsler (1999) (see also Shanthikumar and Sumita 1983, Corollary 1.A.5).

Theorem 2.2. *If* (2.4) *holds and* $\mu_y = \mathrm{E}Y_1$ *exists and is finite, then*

$$p_t T_{\tau(t)} \xrightarrow{d} \mu_y\, \mathrm{Exp}(1) \stackrel{d}{=} \text{sign}\,(\mu_y)\mathrm{Exp}(|\mu_y|) \qquad \text{as } t \to x_F.$$

The proof in Gut and Hüsler (1999) follows immediately via (2.5) and the strong law of large numbers for stopped random walks – see Gut (1988, Theorem I.2.3); note that $p_t T_{\tau(t)} = p_t\tau(t)\cdot T_{\tau(t)}/\tau(t)$.

The analogue of Remark 2.1 also applies to this model.

3. Mixed shock models

In this model the system breaks down when the cumulative shocks reach some 'high' level or when a single 'large' shock appears, whichever comes first. The number of shocks at the time of failure thus equals $\min\{\nu(t), \tau(t)\}$. However, an inspection of Theorems 2.1 and 2.2 shows that in order to obtain a non-trivial result (that is, in order to avoid one of the stopping times dominating the other and failure only being able (almost surely) to occur for one of the reasons), the levels to attain must be chosen in such a fashion that the normalizations in the theorems, that is, p_t and t, are of the same order of magnitude. Moreover, we must join some of the assumptions from the two models. With respect to the cumulative model, we must assume that $\mathrm{E}X_1 > 0$, and, since 'high level' in that model means 'as $t \to \infty$', it follows that we must also have $x_F = +\infty$.

Let $\nu(t) = \min\{n: S_n > t\}$, $t \geqslant 0$ as before. In view of (2.2) and (2.5), in order for $\nu(t)$ and $\tau(t)$ to be of the same order of magnitude asymptotically, we let λ_t denote the θ/t quantile of the distribution of X_1 for some $\theta > 0$; that is, for $t > \theta > 0$, we define λ_t via the relation

$$P(X_1 > \lambda_t) = 1 - F_{X_1}(\lambda_t) = \theta/t. \tag{3.1}$$

Since $x_F = +\infty$, it follows that (2.4) holds automatically, and that $\lambda_t \to +\infty$ as $t \to \infty$. Moreover,

$$\lambda_t = o(t) \qquad \text{as } t \to \infty, \tag{3.2}$$

since the mean shock size is finite. With

$$\tau_\lambda(t) = \min\{n: X_n > \lambda_t\}, \qquad t \geqslant 0, \tag{3.3}$$

the number of shocks until failure equals

$$\kappa(t) = \min\{\nu(t), \tau_\lambda(t)\}. \tag{3.4}$$

Having noted that

$$\kappa(t) \xrightarrow{\text{a.s.}} +\infty, \qquad \text{as } t \to \infty, \tag{3.5}$$

we are ready to state our first result, the proof of which is given in the next section.

Theorem 3.1. *If $\mu_x = \mathrm{E}X_1 > 0$, and $\mu_y = \mathrm{E}Y_1$ exists and is finite, then:*

(i)

$$\frac{\kappa(t)}{t} \xrightarrow{d} Z \qquad \text{as } t \to \infty,$$

where

$$f_Z(y) = \theta \mathrm{e}^{-\theta y}, \quad 0 < y < 1/\mu_x, \quad \text{and} \quad P(Z = 1/\mu_x) = \mathrm{e}^{-\theta/\mu_x},$$

or, equivalently,

$$F_Z(y) = \begin{cases} 1 - e^{-\theta y}, & \textit{for } 0 < y < 1/\mu_x, \\ 1, & \textit{for } y \geqslant 1/\mu_x; \end{cases}$$

(ii)

$$\frac{T_{\kappa(t)}}{t} \xrightarrow{d} \mu_y Z \qquad \textit{as } t \to \infty;$$

(iii)

$$\frac{S_{\kappa(t)}}{t} \xrightarrow{d} \mu_x Z \qquad \textit{as } t \to \infty;$$

(iv)

$$\frac{X_{\kappa(t)}}{t} \xrightarrow{p} 0 \quad \textit{and} \quad \frac{\max_{1 \leqslant k \leqslant \kappa(t)} X_k}{t} \xrightarrow{p} 0 \qquad \textit{as } t \to \infty.$$

Remark 3.1. With respect to (iv), we remark that $X_{\kappa(t)}$ is the last shock (which may or may not have caused failure), whereas $\max_{1 \leqslant \kappa \leqslant \kappa(t)} X_k$ is the largest shock so far at the time of failure. In the event that failure is caused by an extreme shock, these quantities coincide.

Remark 3.2. Since $X_{\kappa(t)} = X_{\nu(t)} I\{\kappa(t) = \nu(t)\} + X_{\tau_\lambda(t)} I\{\kappa(t) = \tau_\lambda(t)\} > 0$, the last shock is always positive, irrespective of the cause of failure.

Remark 3.3. If the system fails due to an extreme shock, the level that has been surpassed is $\lambda_t = o(t)$ (recall (3.2)), and if it is surpassed because of the cumulative effect, the maximal shock at that time point is smaller than λ_t, with the possible exception of the last one, that is, in the particular case when failure is due to both causes, in which case the last shock is $o(t)$ almost surely; see Gut, (1988, Section I.8). Conclusion (iv), which may seem contradictory at first sight, is thus in order.

4. Proof of Theorem 3.1

4.1. An auxiliary result

An essential ingredient in the proof to follow is (the first half of) the following more general result.

Proposition 4.1. *Suppose that* $\{U_t,\ t \geqslant 0\}$ *and* $\{V_t,\ t \geqslant 0\}$ *are families of random variables such that*

$$U_t \xrightarrow{p} a \quad \textit{and} \quad V_t \xrightarrow{d} V \qquad \textit{as } t \to \infty,$$

for some finite constant a and some random variable V. Then

$$P(\min\{U_t, V_t\} > y) \to \begin{cases} P(V > y), & \text{if } y < a, \\ 0, & \text{if } y > a, \end{cases} \quad \textit{as } t \to \infty,$$

and

$$P(\max\{U_t, V_t\} \leqslant y) \to \begin{cases} 0, & \text{if } y < a, \\ P(V \leqslant y), & \text{if } y > a, \end{cases} \quad \textit{as } t \to \infty.$$

Proof. Since the proofs of the two relations are fairly routine and, furthermore, essentially the same, we confine ourselves to giving an outline of the first one.

Let $\varepsilon > 0$ be given. Then

$$\begin{aligned} &P(\min\{U_t, V_t\} > y) \\ &\quad = P(\{\min\{U_t, V_t\} > y\} \cap \{|U_t - a| \leqslant \varepsilon\}) + P(\{\min\{U_t, V_t\} > y\} \cap \{|U_t - a| > \varepsilon\}) \\ &\quad = P_1 + P_2, \end{aligned}$$

and the conclusion follows upon observing that

$$P_1 = \begin{cases} P(U_t > y), & \text{if } y < a - \varepsilon, \\ P(\{U_t > y\} \cap \{y < V_t < a + \varepsilon\}), & \text{if } |y - a| \leqslant \varepsilon, \\ 0, & \text{if } y > a + \varepsilon, \end{cases}$$

and that

$$P_2 \leqslant P(|U_t - a| > \varepsilon) \to 0 \qquad \text{as } t \to \infty. \qquad \square$$

Remark 4.1. The proposition implies, in particular, that the limiting random variables have point masses at $y = a$.

Remark 4.2. Although the present proof is simple and immediate, we mention that an alternative argument (which, however, is essentially the same) would be to invoke Billingsley (1968, Theorem 4.4) according to which $(U_t, V_t) \xrightarrow{d} (U, V)$, where $P(U = a) = 1$, as $t \to \infty$, and then apply the continuous mapping theorem to conclude that $\min\{U_t, V_t\} \xrightarrow{d} \min\{U, V\}$, and $\max\{U_t, V_t\} \xrightarrow{d} \max\{U, V\}$, respectively, as $t \to \infty$, which is equivalent to the formulation of the proposition.

4.2. Proof of the theorem

4.2.1. Proof of (i)

We first recall from (2.2) that $\nu(t)/t \xrightarrow{\text{a.s.}} 1/\mu_x$ as $t \to \infty$. As for $\tau_\lambda(t)$, the nonlinearity of the boundary is no complication: $\tau_\lambda(t)$ is geometric with mean θ/t, and, hence, in analogy with Gut and Hüsler (1999) (cf. also (2.5)),

$$\frac{\tau_\lambda(t)}{t} \xrightarrow{d} \text{Exp}(1/\theta) \qquad \text{as } t \to \infty. \tag{4.1}$$

An application of Proposition 4.1 finishes the proof; the limiting random variable Z is exponential with mean $1/\theta$ in $[0, 1/\mu_x)$, and has point mass $\exp\{-\theta/\mu_x\}$ at $1/\mu_x$.

4.2.2. Proof of (ii)

Since

$$\frac{T_{\kappa(t)}}{t} = \frac{T_{\kappa(t)}}{\kappa(t)} \cdot \frac{\kappa(t)}{t},$$

the conclusion is immediate from the strong law of large numbers for stopped random walks (recall (3.5)), (i), and Proposition 4.1 – see also Gut and Hüsler (1999) and/or Theorem 2.2.

4.2.3. Proof of (iii)

The proof is the same as that of (ii), with $T_{\kappa(t)}$ replaced by $S_{\kappa(t)}$ (and μ_y by μ_x).

4.2.4. Proof of (iv)

Since the mean shock size is finite, we know that $X_n/n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$. The first conclusion then follows from Gut (1988, Theorem I.2.3(i)), (3.5), (i) and Proposition 4.1.

As for the second one, we invoke the fact that $\max_{1 \leqslant k \leqslant n} X_k/n \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, which, by (3.5) and Gut (1988, Theorem I.2.1), implies that $\max_{1 \leqslant k \leqslant \kappa(t)} X_k/\kappa(t) \xrightarrow{\text{a.s.}} 0$ as $t \to \infty$. An application of Proposition 4.1, together with (i), finishes the proof.

5. Uniform integrability and moment convergence

For cumulative shock models, $\mathrm{E}(\nu(t))^r < \infty$ if and only if $\mathrm{E}|\min\{0, X_1\}|^r < \infty$, and $\mathrm{E}(S_{\nu(t)})^r < \infty$ if and only if $\mathrm{E}(\max\{0, X_1\})^r < \infty$; see Gut (1988, Theorem III.3.1). Also, by Gut (1988, Theorem IV.2.1), $\mathrm{E}|T_{\nu(t)}|^r < \infty$ provided $\mathrm{E}|X_1|^r < \infty$ and $\mathrm{E}|Y_1|^r < \infty$. The family of first passage times and the family of stopped sums, suitably normalized, are uniformly integrable under their respective conditions (Gut 1988, Section III.7). As for extreme shock models, $\tau(t)$ being geometric obviously possesses moments of all orders. Moreover, $\{(p_t\tau(t))^r,\ t \geqslant 1\}$ is uniformly integrable for all $r > 0$; see Gut and Hüsler (1999, formula (2.8)). In this section we establish analogous results for the mixed shock model.

Theorem 5.1. (i) *For all* $r > 0$,

$$\left\{ \left(\frac{\kappa(t)}{t} \right)^r,\ t \geqslant 1 \right\} \textit{ is uniformly integrable},$$

and $\mathrm{E}(\kappa(t)/t)^r \to \mathrm{E}Z^r$ *as* $t \to \infty$.

(ii) *If* $\mathrm{E}|Y_1|^r < \infty$ *for some* $r \geqslant 1$, *then*

$$\left\{\left|\frac{T_{\kappa(t)}}{t}\right|^r,\ t \geqslant 1\right\} \text{ is uniformly integrable,}$$

and $\mathrm{E}|T_{\kappa(t)}/t|^r \to |\mu_y|^r\,\mathrm{E}Z^r$ *as* $t \to \infty$.

(iii) *If* $\mathrm{E}|X_1|^r < \infty$ *for some* $r \geqslant 1$, *then*

$$\left\{\left|\frac{S_{\kappa(t)}}{t}\right|^r,\ t \geqslant 1\right\} \text{ is uniformly integrable,}$$

and $\mathrm{E}|S_{\kappa(t)}/t|^r \to \mu_x^r\,\mathrm{E}Z^r$ *as* $t \to \infty$.

(iv) *If* $\mathrm{E}(\max\{0, X_1\})^r < \infty$ *for some* $r \geqslant 1$, *then*

$$\left\{\frac{(X_{\kappa(t)})^r}{t},\ t \geqslant 1\right\} \text{ is uniformly integrable,}$$

and $\mathrm{E}(X_{\kappa(t)})^r/t \to 0$ *as* $t \to \infty$.

(v) *If* $\mathrm{E}(\max\{0, X_1\})^r < \infty$ *for some* $r \geqslant 1$, *then*

$$\left\{\left(\frac{\max_{1\leqslant k\leqslant\kappa(t)} X_k}{t}\right)^r,\ t \geqslant 1\right\} \text{ is uniformly integrable,}$$

and $\mathrm{E}(\max_{1\leqslant k\leqslant\kappa(t)} X_k/t)^r \to 0$ *as* $t \to \infty$.

Proof. We mimic Gut and Hüsler (1999). Since $\tau_\lambda(t)$ is geometric with mean θ/t, it follows, for any $k \geqslant 1$, that $\mathrm{E}[(\tau_\lambda(t)-1)(\tau_\lambda(t)-2)\ldots(\tau_\lambda(t)-k)] = k!((t-\theta)/\theta)^k$ and, hence, that

$$\mathrm{E}\left(\frac{\tau_\lambda(t)}{t}\right)^k \to k!\theta^{-k} = \mathrm{E}(\mathrm{Exp}(1/\theta))^k \qquad \text{as } t \to \infty, \tag{5.1}$$

which, together with (4.1) and Billingsley (1968, Theorem 5.4), proves that

$$\left\{\left(\frac{\tau_\lambda(t)}{t}\right)^r,\ \mathrm{t} \geqslant 1\right\} \text{ is uniformly integrable for all } r > 0. \tag{5.2}$$

Since (obviously) $0 < \kappa(t) \leqslant \tau_\lambda(t)$, uniform integrability as claimed has been verified, from which moment convergence follows via an application of Theorem 3.1(i).

Parts (ii) and (iii) follow from Gut (1988, Theorem I.6.1) and Theorem 3.1(ii) and 3.1(iii), respectively; part (iv) from Gut (1988, Theorem I.8.1) (see also Gut 1988, Theorem III.7.2), and Theorem 3.1(iv) (recall Remark 3.2); and part (v), finally, via the fact that $0 \leqslant \max_{1\leqslant k\leqslant\kappa(t)} X_k \leqslant \sum_{k=1}^{\kappa(t)} \max\{0, X_k\}$, and (iii). □

Remark 5.1. Note that it may, in fact, happen that $S_{\kappa(t)}$ is negative.

Remark 5.2. Since the distribution of Z is explicitly defined, it is possible to compute moments of any order. In particular,

$$\mathrm{E}Z = \frac{1}{\theta}(1 - \mathrm{e}^{-\theta/\mu_x}) \quad \text{and} \quad \mathrm{var}\,Z = \frac{1}{\theta^2}\left(1 - \frac{2\theta}{\mu_x}\,\mathrm{e}^{-\theta/\mu_x} - \mathrm{e}^{-2\theta/\mu_x}\right).$$

A case of particular interest is $\theta = \mu_x$, because then $\mathrm{E}\{\nu(t)\}/t$ and $\mathrm{E}\{\tau_\lambda(t)\}/t$ both converge to $1/\mu_x$ as $t \to \infty$. In this case the above relations become

$$\mathrm{E}Z = \frac{1}{\mu_x}(1 - \mathrm{e}^{-1}) \approx \frac{0.632}{\mu_x} \quad \text{and} \quad \mathrm{var}\,Z = \frac{1}{\mu_x^2}(1 - 2\,\mathrm{e}^{-1} - \mathrm{e}^{-2}) \approx \frac{0.129}{\mu_x^2}.$$

6. Applications

Example 6.1. A somewhat drastic, but rather illuminating, example that comes to mind is boxing. In a fight a knockout may be caused by either a series of small (moderate) punches or by a single very big one.

Example 6.2. *Rainfall*.
On 17 August 1997, the city of Uppsala experienced extremely heavy rain for about an hour, and the basement in my home was flooded. A year later it rained fairly heavily on and off for a few days, which led to the basement being flooded again. The first instance obviously corresponds to an extreme shock causing failure, the second one to the cumulative situation.
A more general example of the same kind is flooding in rivers or dams.

Example 6.3. *Fatigue*.
A material, for example a rope or a wire, can break either because of the cumulative effect of 'normal' loads after a certain time period or because of a sudden very big load.

Example 6.4. *Environmental damage*.
A factory may leak poisonous waste products into a river. After some time the vegetation and the fish in the river may die as a result of the cumulative effect. Or they may die because of some catastrophy in the factory that instantaneously pours a huge amount of waste into the river.

Example 6.5. *Radioactivity.*
A variation on the previous example is the radioactivity emitted by an atomic power station. At 'normal' levels this may, after some (long?) time, cause a higher rate of cancer in the nearby population. The same result may also be brought about somewhat more rapidly in the case of a sudden meltdown.

For some similar and further examples, see Li and Shaked (1997).

7. Further results and remarks

7.1. Comparing stopping times

Once failure has occurred it is of interest to find out whether it was caused by a single large shock or by the cumulative effect. Mathematically this means that we would like to find out how $\nu(t)$, $\tau_\lambda(t)$, and $\kappa(t)$ relate to each other.

7.1.1. Mean and variance

A first, simple, indication is provided by checking the expected value and variance of the respective stopping times. In Remark 5.2 we computed the mean and variance of the limiting random variable Z. A comparison between the three models shows the following:

$$\frac{\mathrm{E}\nu(t)}{t} \to \frac{1}{\mu_x} \qquad \text{as } t \to \infty,$$

$$\frac{\mathrm{E}\tau_\lambda(t)}{t} \to \frac{1}{\theta} \qquad \text{as } t \to \infty,$$

$$\frac{\mathrm{E}\kappa(t)}{t} \to \frac{1}{\theta}(1 - \mathrm{e}^{-\theta/\mu_x}) \qquad \text{as } t \to \infty.$$

Since $\mathrm{e}^{-x} \geqslant 1 - x$ for $x > 0$, it follows that the limiting expected value for the mixed model is always the smallest of the three (which, of course, is no surprise).

7.1.2. $\kappa(t)$ and $\nu(t)$

Another approach is to compare the stopping times themselves. We confine ourselves to comparing $\kappa(t)$ and $\nu(t)$ in the special case when $\theta = \mu_x$, since then, as we have just found, $\mathrm{E}\nu(t)$ and $\mathrm{E}\tau_\lambda(t)$ both are asymptotically equal to t/μ_x as $t \to \infty$.

By (2.2) and Theorem 3.1(i),

$$\frac{\kappa(t)}{\nu(t)} \xrightarrow{d} \mu_x Z \qquad \text{as } t \to \infty. \tag{7.1}$$

Since, moreover $\kappa(t)/\nu(t) \leqslant 1$ (note also that $0 \leqslant \mu_x Z \leqslant 1$), it also follows that

$$\left\{ \left(\frac{\kappa(t)}{\nu(t)} \right)^r, \ t \geqslant 1 \right\} \text{ is uniformly integrable for all } r > 0, \tag{7.2}$$

and that $\mathrm{E}(\kappa(t)/\nu(t))^r \to \mu_x^r \, \mathrm{E}(Z)^r$ as $t \to \infty$; for the last statement we rely on Billingsley (1968, Theorem 5.4). In particular,

$$\mathrm{E}\frac{\kappa(t)}{\nu(t)} \to \mu_x \, \mathrm{E}Z = 1 - \mathrm{e}^{-1} \approx 0.632.$$

7.1.3. $\kappa(t)$ and $\tau_\lambda(t)$

It seems harder to find an analogue of (7.1) for the ratio $\kappa(t)/\tau_\lambda(t)$, since numerator and denominator, divided by t, both converge in distribution only. However, the analogue of (7.2),

$$\left\{\left(\frac{\kappa(t)}{\tau_\lambda(t)}\right)^r, t \geqslant 1\right\} \text{ is uniformly integrable for all } r > 0, \tag{7.3}$$

is trivial (since $\kappa(t)/\tau_\lambda(t) \leqslant 1$).

7.1.4. $\nu(t)$ and $\tau_\lambda(t)$

By combining (2.2) and (2.5) we find that

$$\frac{\tau_\lambda(t)}{\nu(t)} \xrightarrow{d} \text{Exp}(\mu_x/\theta) \qquad \text{as } t \to \infty, \tag{7.4}$$

which, in particular, shows that

$$P(\tau_\lambda(t) \leqslant \nu(t)) \to 1 - \mathrm{e}^{-\theta/\mu_x} \qquad \text{as } t \to \infty, \tag{7.5}$$

and, in the special case that $\theta = \mu_x$ again, that

$$P(\tau_\lambda(t) \leqslant \nu(t)) \to 1 - \mathrm{e}^{-1} \qquad \text{as } t \to \infty. \tag{7.6}$$

Remark 7.1. The result gives the impression that the only quantities that matter here are μ_x and θ. At first sight this seems strange, since one would imagine different conclusions depending on the heaviness of the tails of the shock size distribution. However, since λ_t is defined via quantiles of that distribution, the rate of decrease of the tails is implicitly there. For example, for the standard exponential distribution we have $\lambda_t = \log t$, and for the Pareto distribution with finite mean, $\lambda_t = ct^{1/\alpha}$, $\alpha > 1$. □

7.2. The last/largest shock(s)

In Theorem 3.1(iv) we proved that the last shock and the largest shock, respectively (they sometimes coincide), were $o_P(t)$ as $t \to \infty$ at the time of failure. This may be compared with the relations

$$\frac{X_{\tau_\lambda(t)}}{\lambda_t} > 1 \qquad \text{for all } t,$$

$$\frac{X_{\tau_\lambda(t)}}{t} \xrightarrow{p} 0 \qquad \text{as } t \to \infty,$$

$$\frac{X_{\nu(t)}}{t} \xrightarrow{\text{a.s.}} 0 \qquad \text{as } t \to \infty.$$

The first relation holds by definition, the second one follows analogously to Theorem 3.1(iv), and the third one is immediate from Gut (1988, Theorem I.2.3(i)). This illustrates the different asymptotics of the ‘last shock’, relative to the different stopping times. As for the first two we refer, once again, back to formula (3.2).

Remark 7.2. The different modes of convergence are a consequence of the different convergence modes for $\tau_\lambda(t)$ and $\nu(t)$ (properly normalized).

7.3. Some stopped sums

By completely analogous arguments it is also possible to obtain results for $S_{\tau_\lambda(t)}$, $X_{\nu(t)}$, $S_{\kappa(t)}$, and so on, that is, for the status of various quantities at the moment of stopping for either reason. We just mention without any comment that, for the cumulative shock sizes, we have

$$\frac{S_{\nu(t)}}{t} > 1 \qquad \text{for all } t,$$

$$\frac{S_{\tau_\lambda(t)}}{t} \xrightarrow{d} \text{Exp}(\mu_x/\theta) \qquad \text{as } t \to \infty,$$

$$\frac{S_{\kappa(t)}}{t} \xrightarrow{d} \mu_x Z \qquad \text{as } t \to \infty,$$

and for the different failure times,

$$\frac{T_{\nu(t)}}{t} \xrightarrow{\text{a.s.}} \frac{\mu_y}{\mu_x} \qquad \text{as } t \to \infty,$$

$$\frac{T_{\tau_\lambda(t)}}{t} \xrightarrow{d} \mu_y \text{Exp}(1/\theta) = \text{sign}(\mu_y)\,\text{Exp}(|\mu_y|/\theta) \qquad \text{as } t \to \infty,$$

$$\frac{T_{\kappa(t)}}{t} \xrightarrow{d} \mu_y Z \qquad \text{as } t \to \infty,$$

where, of course, the last conclusions in both cases are nothing but Theorem 3.1(iii) and 3.1(ii), respectively.

As for the means, a comparison shows that

$$\frac{\mathrm{E}T_{\nu(t)}}{t} \to \frac{\mu_y}{\mu_x} \qquad \text{as} \quad t \to \infty,$$

$$\frac{\mathrm{E}T_{\tau_\lambda(t)}}{t} \to \frac{\mu_y}{\theta} \qquad \text{as } t \to \infty,$$

$$\frac{\mathrm{E}T_{\kappa(t)}}{t} \to \frac{\mu_y}{\theta}(1 - \mathrm{e}^{-\theta/\mu_x}) \qquad \text{as } t \to \infty.$$

In particular, if $\theta = \mu_x$, the first two limits are equal to μ_y/μ_x, and the last one equals $(1 - \mathrm{e}^{-1})\mu_y/\mu_x \approx 0.632\mu_y/\mu_x$.

7.4. Stopping at a given time

By switching the roles of the components, that is, by stopping time and checking the stopped sum or maximum, together with the same kind of arguments as before, we can find the size of the largest shock so far or the cumulative shock size at some specific time point. Technically, our starting point is to let

$$\eta(s) = \min\{n\colon T_n > s\}, \qquad s \geqslant 0. \tag{7.7}$$

Note that, just as we needed μ_x to be positive before, whereas only existence was needed for μ_y, now it is the other way around. In the present subsection we assume, for simplicity, that both means are positive. We may thus deduce that, for example,

$$\frac{\eta(s)}{s} \stackrel{\text{a.s.}}{\to} \frac{1}{\mu_y} \qquad \text{as } s \to \infty,$$

$$\frac{S_{\eta(s)}}{s} \stackrel{\text{a.s.}}{\to} \frac{\mu_x}{\mu_y} \qquad \text{as } s \to \infty,$$

$$\frac{\max_{1\leqslant k\leqslant \eta(s)} X_k}{s} \stackrel{\text{a.s.}}{\to} 0 \qquad \text{as } s \to \infty.$$

7.5. Further extensions

In the context of reliability one has the concept of 'replacement based on age', which means that a component is replaced at failure or at some given prespecified time, whichever comes first. An analogous shock model would be to consider the state of the system either at failure for one of the two reasons or at some given time. In order to model that situation we suppose that $0 < \mu_x,\ \mu_y < \infty$, and consider the simplest case, namely

$$\begin{aligned}\gamma(t) &= \min\{n\colon S_n > t \text{ or } X_n > \lambda_t \text{ or } T_n > t\}\\ &= \min\{\nu(t),\ \tau_\lambda(t),\ \eta(t)\} = \min\{\kappa(t),\ \eta(t)\}, \qquad t > 0.\end{aligned}$$

Then

$$\min\{\nu(t),\ \eta(t)\} \stackrel{\text{a.s.}}{\to} \min\left\{\frac{1}{\mu_x}, \frac{1}{\mu_y}\right\} = \frac{1}{\max\{\mu_x,\ \mu_y\}} \qquad \text{as } t \to \infty,$$

which, together with Proposition 4.1 and the arguments from Section 4, proves the following result.

Theorem 7.1. *Let* $\gamma(t)$ *be defined as above, set* $\mu = \max\{\mu_x,\ \mu_y\}$, *where* $\mu_x = \mathrm{E}X_1 > 0$ *and* $\mu_y = \mathrm{E}Y_1 > 0$. *Then:*

(i)

$$\frac{\gamma(t)}{t} \stackrel{d}{\to} W \qquad \text{as } t \to \infty,$$

where

$$f_W(y) = \theta e^{-\theta y}, \quad 0 < y < 1/\mu, \quad \text{and} \quad P(W = 1/\mu) = e^{-\theta/\mu},$$

or, equivalently,

$$F_W(y) = \begin{cases} 1 - e^{-\theta y}, & \text{for } 0 < y < /\mu, \\ 1, & \text{for } y \geqslant 1/\mu; \end{cases}$$

(ii)

$$\frac{T_{\gamma(t)}}{t} \xrightarrow{d} \mu_y W \qquad \text{as } t \to \infty;$$

(iii)

$$\frac{S_{\gamma(t)}}{t} \xrightarrow{d} \mu_x W \qquad \text{as } t \to \infty;$$

(iv)

$$\frac{X_{\gamma(t)}}{t} \xrightarrow{p} 0 \quad \text{and} \quad \frac{\max_{1 \leqslant k \leqslant \gamma(t)} X_k}{t} \xrightarrow{p} 0 \qquad \text{as } t \to \infty.$$

Remark 7.3. It is also possible to formulate a result on uniform integrability and moment convergence *à la* Theorem 5.1. We omit the details.

Another alternative is the following more natural stopping time, which, in addition, leaves room for choosing certain parameters according to some given rule or need, namely,

$$\gamma(t, s) = \min\{n : S_n > at \text{ or } X_n > b\lambda_t \text{ or } T_n > cs\}$$

$$= \min\{\nu(at), \tau_\lambda(bt), \eta(cs)\}, \qquad t, s \geqslant 0,$$

for a, b, $c > 0$, with s typically being some function of t, and so on.

We close by mentioning a sample of possible generalizations of the model. For example, the cumulative component could be $S_n = \sum_{i=1}^{n} \alpha^k X_k$, $n \geqslant 1$, where $0 < \alpha < 1$, that is, in the accumulation of the shocks there is a kind of 'discount' as past shocks become more distant in time. Another possibility would be a moving average type of sum, $S_n = \sum_{k=n-m}^{n} a_k X_k$, $n > m$, where $\{a_k, k \geqslant 1\}$ are constants, and m some given integer. One might also consider the case when failure occurs as soon as the cumulative sum exceeds some given level or, the sum of, say, the last three shocks exceed a(nother) level:

$$\min\{n : S_n > t \text{ or } \{X_{n-2} + X_{n-1} + X_n > \lambda_t\}\}, \qquad t \geqslant 0.$$

A final variation is when failure occurs as soon as the cumulative sum exceeds some given level, or not one but some given fixed number of consecutive 'large' shocks all exceed some other level. One simple example of such a stopping time would be

$$\min\{n : S_n > t \text{ or } \{X_{n-1} > \lambda_t \text{ and } X_n > \lambda_t\}\}, \qquad t \geqslant 0.$$

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