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MIXING AND MONODROMY OF ABSTRACT POLYTOPES

B. MONSON, DANIEL PELLICER, AND GORDON WILLIAMS

ABSTRACT. The monodromy group $\mathrm{Mon}(\mathcal{P})$ of an n-polytope $\mathcal{P}$ encodes the combinatorial information needed to construct $\mathcal{P}$. By applying tools such as mixing, a natural group-theoretic operation, we develop various criteria for $\mathrm{Mon}(\mathcal{P})$ itself to be the automorphism group of a regular n-polytope $\mathcal{R}$. We examine what this can say about regular covers of $\mathcal{P}$, study a peculiar example of a 4-polytope with infinitely many distinct, minimal regular covers, and then conclude with a brief application of our methods to chiral polytopes.

1. INTRODUCTION

This paper can be read as an attempt to correct our confused understanding of recent efforts to extend operations, such as mixing and covering, from the well-understood domain of maps to the domain of abstract polytopes of higher rank. In the case of maps, or 3-polytopes, the local topology is Euclidean, at least in finite cases, so that one can draw on a rich supply of topological techniques, such as the theory of covering surfaces, to make progress (see [16] for instance). But for higher ranks $n \geqslant 4$, it seems that we are forced to rely more on purely combinatorial methods, which become essentially group-theoretic for more symmetric polytopes.

Although we have several new results, and examples which may seem strange to the world of maps, it is true that much of our exposition consists in clarifying the properties of constructions and tools invented elsewhere. We hope that we have given proper credit for the most significant of such ideas. What is original to us is more or less the following.

In Section 2 we recall basic definitions for general n-polytopes $\mathcal{Q}$ and their morphisms. We exhibit in subsection 2.4 various quotients and covers with unexpected properties, including Example 2.13, which provides a modest counterexample to some published results.

In Section 3 we introduce the monodromy group $\mathrm{Mon}(\mathcal{Q})$ for frequent use later on. Theorem 3.9 and Propositions 3.11, 3.13 and 3.16 may be familiar in the world of maps, but we have been unable to track down much discussion of them for polytopes of higher rank.

Section 4 concerns the flag action, a well-established tool which is particularly suited to investigating 'better behaved' covers. We make a connection to quotients by sparse subgroups of string C-groups in Proposition 4.8.

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In Section 5 we introduce the mix of a family of groups and describe several new results, such as Theorems 5.11 and 5.12 concerning covers by regular polytopes and the mixing of their automorphism groups.

Section 6 contains our most interesting results and asks when $\mathrm{Mon}(\mathcal{Q})$ is isomorphic to the automorphism group of a regular polytope $\mathcal{P}$. This important question involves the uniqueness of minimal regular covers $\mathcal{P}$, a property which is forced for polyhedra (Proposition 6.1) but which can fail in higher ranks (Example 6.8). Theorems 6.4 and 6.7 provide guarantees of unique minimal regular covers.

Finally, in Section 7 we turn our attention to chiral polytopes $\mathcal{P}$. After reinterpreting $\mathrm{Mon}(\mathcal{P})$ in a natural way (Theorem 7.2), we discuss in Example 7.6 a recently discovered chiral 5-polytope with strange covering properties.

2. Covers and quotients of abstract polytopes

2.1. **Polytopes.** An abstract n-polytope $\mathcal{P}$ has some of the key combinatorial properties of the face lattice of a convex n-polytope; in general, however, $\mathcal{P}$ need not be a lattice, need not be finite, and need not have any familiar geometric realization. We refer to [17] for more details concerning the following overview of some basic ideas.

Definition 2.1. An *abstract n-polytope* $\mathcal{P}$ is a partially ordered set with properties **A**, **B** and **C** below. A *pre-polytope* need only satisfy properties **A** and **B** and a *flagged poset* just **A**:

A: $\mathcal{P}$ has a strictly monotone rank function with range $\{-1, 0, \dots, n\}$. Moreover, $\mathcal{P}$ has a unique least face F_{-1} of rank -1 and unique greatest face F_n of rank n.

An element $F \in \mathcal{P}$ with $\mathrm{rank}(F) = j$ is called a *j-face*; typically F_j will indicate a j-face. Naturally, faces of ranks 0, 1 and $n-1$ are called vertices, edges and facets, respectively. Faces of rank $n-2$ are called *ridges*. Notice that each maximal chain or *flag* in $\mathcal{P}$ contains $n+2$ faces. We let $\mathcal{F}(\mathcal{P})$ be the set of all flags in $\mathcal{P}$.

B: Whenever $F < G$ with $\mathrm{rank}(F) = j-1$ and $\mathrm{rank}(G) = j+1$, there are exactly two j-faces H with $F < H < G$.

For $0 \leqslant j \leqslant n-1$ and any flag Φ, there thus exists a unique *j-adjacent* flag Φ^j, differing from Φ in just the face of rank j . With this notion of adjacency, $\mathcal{F}(\mathcal{P})$ becomes the *flag graph* for $\mathcal{P}$. Whenever $F \leqslant G$ are incident faces in $\mathcal{P}$, the *section* G/F is defined by

$$G/F := \{H \in \mathcal{P} \mid F \leqslant H \leqslant G\} .$$

C: $\mathcal{P}$ is *strongly flag–connected*, that is, the flag graph for each section is connected.

It follows that G/F is a $(k-j-1)$-polytope in its own right whenever $F \leqslant G$ with $\mathrm{rank}(F) = j \leqslant k = \mathrm{rank}(G)$. For example, if F is a vertex, then the section F_n/F is called the *vertex-figure* over F. Similarly, any j-face F becomes a polytope in its own right if we give it the structure of F/F_{-1}.

Remark 2.2. Typically, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and so forth will denote *polytopes*, and occasionally pre-polytopes, for which notions like *flag adjacency* and *flag action* (defined in Section 4) remain meaningful. A pre-polytope $\mathcal{P}$ is said to be *flag-connected* if $\mathcal{F}(\mathcal{P})$ is a connected graph; it then fails to be a polytope if some proper section itself has a disconnected flag graph.

If $i_1, \dots, i_k \in \{0, \dots, n-1\}$ and $\Phi \in \mathcal{F}(\mathcal{P})$, it is natural to inductively define $\Phi^{i_1 \cdots i_k} := (\Phi^{i_1 \cdots i_{k-1}})^{i_k}$. Note how the sequence $i_1, \dots, i_k$ defines a walk in the flag graph $\mathcal{F}(\mathcal{P})$ [17, pp. 12-13].

2.2. **Morphisms.** A poset homomorphism $\eta : \mathcal{P} \to \mathcal{Q}$ need only preserve incidence and hence can be very general. For example, $\mathcal{P}$ and $\mathcal{Q}$ can have different ranks; η could collapse $\mathcal{P}$ onto a single face of $\mathcal{Q}$; or $\mathcal{P}$ could be a section of $\mathcal{Q}$ and η the natural inclusion. We have little need as yet for such generality. Here is the right sort of morphism for most of our purposes:

Definition 2.3 ([17, Sect. 2D])**.** Let $\mathcal{P}$ and $\mathcal{Q}$ be pre-polytopes, both of rank n. A *rap-map* is a rank and adjacency preserving homomorphism $\eta : \mathcal{P} \to \mathcal{Q}$. (This means that η induces a mapping $\mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{Q})$ which sends any j-adjacent pair of flags in $\mathcal{P}$ to another such pair in $\mathcal{Q}$.) A surjective rap-map is called a *covering*; we then say $\mathcal{P}$ is a *cover* of $\mathcal{Q}$ and write $\mathcal{P} \to \mathcal{Q}$.

If we restrict our considerations to covers $\mathcal{P}, \mathcal{R}$ from a particular class of polytopes (typically the regular polytopes), then a cover $\mathcal{R}$ is *minimal* over $\mathcal{Q}$ if $\mathcal{R} \to \mathcal{P} \to \mathcal{Q}$ implies $\mathcal{P} = \mathcal{R}$ or $\mathcal{P} = \mathcal{Q}$. Note that the latter is possible only if $\mathcal{Q}$ also belongs to the given class of polytopes.

We now establish some simple but useful properties of rap-maps.

Lemma 2.4. *Suppose $\eta : \mathcal{P} \to \mathcal{Q}$ is a rap-map of pre-polytopes with rank n. For any flag Φ of $\mathcal{P}$ and $i_1, \dots, i_k \in \{0, \dots, n-1\}$, we have*

$$((\Phi)\eta)^{i_1 \cdots i_k} = (\Phi^{i_1 \cdots i_k})\eta \,.$$

Proof. Since η is a rap-map we have $(\Psi^j)\eta = ((\Psi)\eta)^j$ for any flag Ψ and $0 \leqslant j \leqslant n-1$. Now use a simple induction on k. ☐

Lemma 2.5. *Suppose $\eta : \mathcal{P} \to \mathcal{Q}$ is a rap-map (of pre-polytopes). If $\mathcal{Q}$ is flag-connected, then η is surjective. (Indeed, $\mathcal{Q}$ is covered by a flag-component of $\mathcal{P}$.)*

Proof. Since the simple arguments used here reappear in so many situations, we give rather more detail than usual. First let us fix some 'base flags' Λ of $\mathcal{P}$. Since η is a rap-map, $\Phi := (\Lambda)\eta$ is a flag in $\mathcal{Q}$. Let Ψ be any other flag of $\mathcal{Q}$. By flag-connectedness, there exist consecutively adjacent flags

$$\Phi = \Phi_0, \Phi_1, \dots, \Phi_m = \Psi,$$

say with $\Phi_j = \Phi_{j-1}^{i_j}$, where $i_j \in \{0, \dots, n-1\}$ for $1 \leqslant j \leqslant m$. In brief, $\Psi = \Phi^{i_1 \cdots i_m}$.

But again since η is a rap-map, we must have $(\Lambda^{i_1 \cdots i_m})\eta = \Psi$ by Lemma 2.4. Thus η induces a surjection $\mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{Q})$, so η itself is certainly surjective. ☐

It follows at once that every rap-map of polytopes is a covering. In a similar way, we find that a covering of polytopes is determined by its effect on one particular base flag:

Lemma 2.6. *Let $\mathcal{P}, \mathcal{Q}$ be n-polytopes and suppose $\eta, \lambda : \mathcal{P} \to \mathcal{Q}$ are rap-maps. If $(\Phi)\eta = (\Phi)\lambda$ for some flag Φ of $\mathcal{P}$, then $\eta = \lambda$.*

Proof. Being rap-maps, η and λ must coincide on flags adjacent to Φ and so on all flags (by induction on the length of a chain of consecutively adjacent flags). Compare [17, Prop. 2A4]. ☐

Naturally, we define an *isomorphism* $\eta : \mathcal{P} \to \mathcal{Q}$ to be a bijection such that both η and η^{-1} are order preserving. Clearly isomorphisms are rap-maps. Notice that the condition that η^{-1} be order preserving is redundant, if $\mathcal{P}$ and $\mathcal{Q}$ are flag-connected pre-polytopes. This follows easily from the proof of Lemma 2.5, since incident faces in $\mathcal{Q}$ do lie in a flag, which in turn is covered by a flag of $\mathcal{P}$.

The group of all *automorphisms* $\eta : \mathcal{P} \to \mathcal{P}$ of $\mathcal{P}$ will be denoted $\Gamma(\mathcal{P})$, with identity 1. Each automorphism $\eta \in \Gamma(\mathcal{P})$ induces a bijection on the flag set:

$$\begin{aligned} \overline{\eta} : \mathcal{F}(\mathcal{P}) &\to \mathcal{F}(\mathcal{P}), \\ \Phi &\mapsto (\Phi)\eta. \end{aligned}$$

For any flagged poset $\mathcal{P}$, the resulting action of $\Gamma(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ is faithful, and the mapping $\eta \to \overline{\eta}$ embeds $\Gamma(\mathcal{P})$ in $\mathrm{Sym}(\mathcal{F}(\mathcal{P}))$, the symmetric group on flags of $\mathcal{P}$. When it suits us, we can therefore consider the automorphism group of $\mathcal{P}$ to be a subgroup of the symmetric group on flags.

Since we will soon have need of examples, let us now describe the most symmetric polytopes.

Definition 2.7. The n-polytope $\mathcal{P}$ is *regular* if $\Gamma(\mathcal{P})$ is transitive (hence sharply transitive) on the flag set $\mathcal{F}(\mathcal{P})$.

If $\mathcal{P}$ is regular, we may specify any one flag Φ as the *base flag*, then define ρ_j to be the (unique) automorphism mapping Φ to Φ^j, for $0 \leqslant j \leqslant n-1$. From [17, Sect. 2B] we recall that $\Gamma(\mathcal{P})$ is then a *string C-group*, meaning that it has the following properties **SC1** and **SC2**:

SC1: $\Gamma(\mathcal{P})$ is a *string group generated by involutions* (sggi), that is, it is generated by involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the commutativity relations typical of a Coxeter group with string diagram, namely

$$(\rho_j \rho_k)^{p_{jk}} = 1, \text{ for } 0 \leqslant j \leqslant k \leqslant n-1, \tag{2.1}$$

where $p_{jj} = 1$, $p_{jk} = p_{kj}$, for $0 \leqslant j, k \leqslant n-1$, and $p_{jk} = 2$ whenever $|j-k| > 1$. The periods $p_j := p_{j-1,j}$ in (2.1) are assembled into the *Schläfli* symbol $\{p_1, \ldots, p_{n-1}\}$ for the sggi.

SC2: $\Gamma(\mathcal{P})$ satisfies the *intersection condition*

$$\langle \rho_k : k \in I\rangle \cap \langle \rho_k : k \in J\rangle = \langle \rho_k : k \in I \cap J\rangle, \text{ for any } I, J \subseteq \{0, \ldots, n-1\}\ . \tag{2.2}$$

Naturally, we also say that the regular polytope $\mathcal{P}$ has Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$. The fact that one can reconstruct a regular polytope in a canonical way from any string C-group Γ is at the heart of the theory [17, Sect. 2E]. Later we introduce other sorts of symmetry conditions, such as *chirality*, which relax regularity in a natural way (see Section 7).

Example 2.8. For each $p \in \{2, \ldots, \infty\}$ there is, up to isomorphism, a unique 2-polytope or *polygon* $\{p\}$. In fact, $\{p\}$ happens to be (abstractly) regular, and its automorphism group is the *dihedral group*

$$\mathbb{D}_{2p} = \langle \rho_0, \rho_1 \,|\, \rho_0^2 = \rho_1^2 = (\rho_0\rho_1)^p = 1\rangle,$$

of order $2p$. It is convenient to introduce here a little terminology that we will first use in the proof of Proposition 5.15. We shall say that $\gamma \in \mathbb{D}_{2p}$ is *even* (resp. *odd*) if $\gamma = (\rho_0\rho_1)^k$ (resp. $\gamma = (\rho_0\rho_1)^k \rho_0$) for some integer k. Understood in this definition are the two involutory generators, so we might say (ρ_0, ρ_1)-even, for example, to be more explicit. Of course, the even elements in $\langle \rho_0, \rho_1\rangle$ are those that act like rotations in the usual action of the dihedral group on the (possibly infinite) polygon $\{p\}$; the odd elements are precisely those that act like reflections. No element is both even and odd.

We will encounter other kinds of sggi 's G, typically written $G = \langle r_0, \ldots, r_{n-1} \rangle$ to emphasize the specified list of involutory generators; we then say that G is an sggi of *rank* n. Usually we ask that homomorphisms of sggi 's respect these lists of generators. In Section 3 we will see that the monodromy group of $\mathcal{P}$ is always an sggi , though it might not be a string C-group.

2.3. **Quotients.** Let $\mathcal{P}$ be an n-polytope (or even any flagged poset), and let $\sim$ be any equivalence relation on the faces of $\mathcal{P}$ which is stratified by rank (so $F \sim G \Rightarrow \mathrm{rank}(F) = \mathrm{rank}(G)$). Then the set $\mathcal{Q} := \mathcal{P}/\!\sim$ of all classes $\widehat{F} := \{G : G \sim F\}$ can be ordered by agreeing that $\widehat{F} \leqslant \widehat{G}$ if and only if there exists a finite sequence of faces $F_1, \ldots, F_k, G_1, \ldots, G_k$ in $\mathcal{P}$ such that

$$F = F_1 \sim G_1 \leqslant F_2 \sim G_2 \leqslant \ldots \leqslant F_{k-1} \sim G_{k-1} \leqslant F_k \sim G_k = G\ .$$

Certainly '$\leqslant$' is a reflexive and transitive relation on $\mathcal{Q}$. Moreover, if $\mathrm{rank}(F) = \mathrm{rank}(G)$ in such a chain of faces, then all faces must have the same rank and so lie in the same class. Hence the relation is also antisymmetric, and $\mathcal{Q}$ is partially ordered. (We have defined the transitive closure of the more obvious order relation.) In fact, $\mathcal{Q}$ is a flagged poset and

$$\begin{array}{rcl} \eta : \mathcal{P} & \to & \mathcal{Q}, \\ F & \mapsto & \widehat{F} \end{array}$$

is a rank preserving, surjective poset homomorphism.

Definition 2.9. $\mathcal{Q}$ is the *quotient* of $\mathcal{P}$ induced by $\sim$, with *natural map* η.

This notion of a quotient is very general. We will mainly need two more specialized versions, the first of which is described in

Lemma 2.10. *Let $\lambda : \mathcal{P} \to \mathcal{Q}$ be a rap-map of pre-polytopes, taking $\mathcal{Q}$ to be flag-connected. Then λ induces a quotient $\mathcal{P}_\lambda$ isomorphic to $\mathcal{Q}$, where the faces of $\mathcal{P}_\lambda$ are just the fibres $(H)\lambda^{-1}, H \in \mathcal{Q}$.*

Proof. Note that λ is surjective by Lemma 2.5. It is easy to see that any chain of mutually incident faces in $\mathcal{Q}$ lifts under λ to a similar such chain in $\mathcal{P}$. One can then check that $\widehat{F} \leqslant \widehat{G}$ (in the quotient order on $\mathcal{P}_\lambda$) if and only if $(F)\lambda = (F')\lambda$ and $(G)\lambda = (G')\lambda$ for certain $F' \leqslant G'$ in $\mathcal{P}$. Now it is easy to verify that

$$\begin{array}{rcl} \mathcal{Q} & \to & \mathcal{P}_\lambda, \\ H & \mapsto & (H)\lambda^{-1} \end{array}$$

defines an isomorphism of posets. ∎

In fact, as is the case elsewhere in the literature, quotients for us will usually be induced by group actions, although we shall see a few examples of a more peculiar nature below. Thus, if $\mathcal{P}$ is a polytope, or even any flagged poset, we begin with a subgroup N of the automorphism group $\Gamma(\mathcal{P})$. We then define $F \sim G$ if and only if $G = F\tau$ for some $\tau \in N$. Clearly, this defines on the faces of $\mathcal{P}$ an equivalence relation stratified by rank. The equivalence classes are just the N-orbits on $\mathcal{P}$, and we customarily write

$$\mathcal{Q} = \mathcal{P}/N \; .$$

Notice that the order induced on $\mathcal{Q}$ is now guaranteed to be transitive, and we have $\widehat{F} \leqslant \widehat{G}$ in $\mathcal{Q}$ if and only if $F_1 \leqslant G_1$ in $\mathcal{P}$ for faces $F_1 \in \widehat{F}, G_1 \in \widehat{G}$.

Before resuming our general discussion we present some examples. These illustrate that it is very easy to be led astray when thinking of the sort of combinatorial coverings and quotients described above.

2.4. **Examples.** Our first example illustrates that the image of a polytope under a rap-map might not be a polytope.

Example 2.11. In an ordinary 3-cube $\mathcal{P}$ identify two opposite squares $F, -F$, without identifying any of their vertices or edges. In the resulting pre-polytope $\mathcal{Q}$, the new facet $\widehat{F}$, or more precisely the section $\widehat{F}/\mathcal{Q}_{-1}$, is no longer connected; it consists of two disjoint 4-gons. However, the natural map $\eta : \mathcal{P} \to \mathcal{Q}$ is clearly a covering (surjective rap-map).

The next example shows that the inverse image of a flag under a certain rap-map need not necessarily contain a flag.

Example 2.12. Consider two opposite 2-faces in a 5-cube $\mathcal{P}$. Observe that if we merely identify a single pair $F, -F$ of opposite 2-faces, again without identifying their edges or vertices, then the resulting quotient is not actually a pre-polytope. Indeed, if in $\mathcal{P}$ we choose an edge E of F and a 3-cube C on $-F$, then $\widehat{F} = \{F, -F\}$ is the only 2-face in this quotient with $E < \widehat{F} < C$.

However, if we identify *all* pairs of opposite 2-faces in $\mathcal{P}$, then the resulting quotient $\mathcal{Q}$ is a pre-polytope. To see this inspect the various sections $G < H$ in $\mathcal{Q}$, where $\mathrm{rank}(G) = \mathrm{rank}(H) - 2 = -1, 0, 1, 2, 3$. The natural map $\eta : \mathcal{P} \to \mathcal{Q}$ is still a covering; but notice that the section defined by a 2-face once more consists of two disjoint 4-gons, so $\mathcal{Q}$ is not a polytope. Now let F_0, F_1 be an incident vertex and edge in a particular 2-face F of $\mathcal{P}$; and let F_3, F_4 be an incident ridge and facet of $\mathcal{P}$ containing the face $-F$ opposite to F. Then

$$\Psi = [\widehat{F_0}, \widehat{F_1}, \widehat{F}, \widehat{F_3}, \widehat{F_4}]$$

is a flag of $\mathcal{Q}$ which is covered under η by no flag of $\mathcal{P}$. ($\widehat{F_0}$ is covered only by F_0, $\widehat{F_4}$ is covered only by F_4 and these faces lie in no flag of $\mathcal{R}$.) In fact, $\mathcal{Q}$ has 7680 flags (twice as many as does $\mathcal{P}$) and so cannot be flag-connected. Note that we customarily suppress the improper faces in a flag; for Ψ these are $\mathcal{Q}_{-1}$ and $\mathcal{Q}_5$.

We show next that quotients by group actions need not arise from rap-maps; likewise, the quotient induced by a rap-map need not arise from a group action.

Example 2.13. The regular toroidal polyhedron $\mathcal{P} = \{4, 4\}_{(2,0)}$ has group $\Gamma = \langle \rho_0, \rho_1, \rho_2 \rangle$ of order 32. The action on the 8 edges is faithful, and we may write

$$\rho_0 = (2\,3)(4\,6) \; , \; \rho_1 = (1\,2)(3\,5)(4\,7)(6\,8) \; , \; \rho_2 = (2\,4)(3\,6) \; .$$

A typical base flag Φ is shaded in Figure 1.

A 'northeast' translation is given by $\tau = (\rho_1\rho_2)^2(\rho_0\rho_1)^2$. We let $\Sigma = \langle \tau, \rho_1 \rangle$, a subgroup of order 4. Then $\mathcal{Q} = \mathcal{P}/\Sigma \simeq \{2, 2\}$ is a regular 3-polytope. But since $\rho_1 \in \Sigma$, the base flag Φ is identified with Φ^1 and the natural map $\mathcal{P} \to \mathcal{Q}$ cannot be a rap-map.

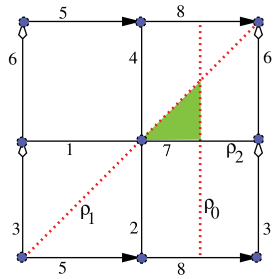


FIGURE 1. The toroid $\{4,4\}_{(2,0)}$ has the dihedron $\{2,2\}$ as a quotient.

Remark 2.14. Example 2.13 provides a counterexample to the 'only if' implications of Lemma 2D5(b) and Proposition 2D11(b) in [17]. However, these seem to be minor blemishes and have little consequence elsewhere. Either the 'if' parts of the results are used, or a natural sharpening of hypotheses will rule out the above sort of peculiarities; see [9, Cor. 2.3].

Example 2.15. Let $\mathcal{P}$ and $\mathcal{Q}$ be the two maps of type $\{4,4\}$ on the left and right copies of the Klein bottle shown in Figure 2.

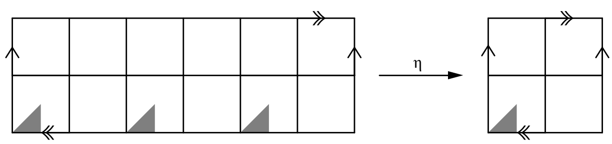


FIGURE 2. One Klein bottle covers another.

Then there is a rap-map $\eta : \mathcal{P} \to \mathcal{Q}$ under which the three gray flags in $\mathcal{P}$ are mapped to the gray flag in $\mathcal{Q}$. However, $\Gamma(\mathcal{P})$ is a group of order 16, hence with 6 flag orbits in $\mathcal{P}$. Since $\mathcal{P}$ is a 3-fold cover of $\mathcal{Q}$, the latter polyhedron cannot possibly arise from the action of a subgroup of $\Gamma(\mathcal{P})$. The maps $\mathcal{P}$ and $\mathcal{Q}$ can be described as $\{4,4\}_{|2,6|}$ and $\{4,4\}_{|2,2|}$, respectively, in the notation of [31].

2.5. **Quotients and sggi's.** Suppose $G = \langle r_0, \ldots, r_{n-1}\rangle$ is an sggi of rank n. We require some notation for standard subgroups of G. For $-1 \leqslant j \leqslant n$, let

$$G_j := \langle r_i \,:\, i \neq j\rangle,$$
$$G_{<j} := \langle r_i \,:\, i < j\rangle,$$
$$G_{>j} := \langle r_i \,:\, i > j\rangle.$$

Notice that this convenient notation is a little inconsistent. As natural conventions we set $G_{<0} = G_{>n-1} = \{1\}$ and $G_{<n} = G_{>-1} = G$. It is easy to check that

$$G_j = G_{<j}G_{>j} = G_{>j}G_{<j}$$

for $0 \leqslant j \leqslant n-1$ (the essence of stringiness).

We have already observed that the automorphism group $\Gamma(\mathcal{P})$ for a regular n-polytope $\mathcal{P}$ is an sggi of rank n. To emphasize the crucial role of the specified generators consider the symmetry group $[4,3] = \langle \rho_0, \rho_1, \rho_2\rangle$ of the cube. Let $r_0 = \rho_0, r_1 = \rho_1, r_2 = \rho_2, r_3 = (\rho_0\rho_1\rho_2)^3$. Then $G = \langle r_0, r_1, r_2, r_3\rangle$ is a rank 4 sggi; but note that $r_3 \in G_3$. The intersection condition (2.2) clearly fails.

We record some results on double cosets in G.

Lemma 2.16 ([8, Lemmas 2.1, 2.2])**.** *Let $G = \langle r_0, \dots, r_{n-1}\rangle$ be an sggi and N any subgroup of G.*

(a) *If $G_i\, uN = G_j\, vN$ for $u, v \in G$ and $i \neq j$, then $G_i\, uN = G(= G_j\, vN)$.*

(b) *Let $0 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n-1$. Suppose $u_{i_1}, \dots, u_{i_m} \in G$ so that*

$$G_{i_j} u_{i_j} N \cap G_{i_{j+1}} u_{i_{j+1}} N \neq \emptyset$$

for $1 \leqslant j < m$. Then there exists some $u \in G$ such that $G_{i_j} u_{i_j} N = G_{i_j} uN$for all j.

Proof (adapted from [8]*).* Part (a) actually holds for any group with specified generators r_j, not necessarily involutions and not necessarily arranged in a stringy way, so long as we define G_j as before. Thus $G_i uN = G_j uN$ for $i \neq j$ implies

$$G_i G_j (G_i uN) = G_i G_j (G_j uN) = G_i (G_j uN) = G_i (G_i uN) = G_i uN\ .$$

One then proves by induction on $k \geqslant 1$ that $G_i (G_j G_i)^k uN = G_i uN$. Now note that any element of G is a product of terms selected alternately from G_i and $\langle r_i\rangle$.

Part (b) does depend on the r_i's being arranged in a stringy way; but again these generators need not be involutions. The case $m = 1$ is trivial. For $m = 2$ we need only note that $u \in G_{i_j} u_{i_j} N$ implies $G_{i_j} u_{i_j} N = G_{i_j} uN$; so suppose $m \geqslant 3$. By induction we have u' such that $G_{i_j} u' N = G_{i_j} u_{i_j} N$ for $j = 2, \dots, m$; and we have v such that $G_{i_j} vN = G_{i_j} u_{i_j} N$ for $j = 1, 2$. From the overlap at i_2 we get $g' \in G_{i_2}$ and $n' \in N$ such that $v = g' u' n'$. But $g' = ab = ba$ for some $a \in G_{<i_2} \subseteq G_{i_j}$, when $j = 2, \dots, m$, and some $b \in G_{>i_2} \subseteq G_{i_j}$, when $j = 1, 2$. Then $u = b^{-1} v = a u' n'$ will work. □

Following [8, §2], we now define a flagged poset $\mathcal{Q}$ for any pair G, N:

Definition 2.17. Suppose N is a subgroup of an sggi G of rank n. Let the j-faces of $\mathcal{Q}$ be the double cosets $G_j uN$, $u \in G$. Define $G_i uN \leqslant G_j vN$ if and only if $i \leqslant j$ and $G_i uN \cap G_j vN \neq \emptyset$.

By Lemma 2.16(b), the proper faces in a typical flag of $\mathcal{Q}$ look like

$$[G_0 uN, \dots, G_j uN, \dots, G_{n-1} uN]$$

(with a common double coset representative u). If N produces the sort of degeneracy suggested in Lemma 2.16(a), then one rebuilds the ordering by indexing double cosets with $j \in \{-1, \dots, n\}$. In any case, from Lemma 2.16 we get

Proposition 2.18 ([8, Th. 2.3])**.** *Let N be a subgroup of the sggi $G = \langle r_0, \dots, r_{n-1}\rangle$. Then $\mathcal{Q}$ as defined above is a flagged poset of rank n.*

As a special case we have

Proposition 2.19. *Suppose $\mathcal{P}$ is a regular n-polytope with automorphism group $\Gamma = \langle \rho_0, \dots, \rho_{n-1}\rangle$. For any subgroup N of Γ, let $\mathcal{Q}$ be the flagged poset with faces $\Gamma_j \tau N$, $\tau \in \Gamma$, as defined just above. Then*

$$\mathcal{P}/N \simeq Q\ .$$

Proof. Let $\Phi = [F_0, \dots, F_{d-1}]$ be the base flag of $\mathcal{P}$ corresponding to the given generators ρ_j. A typical j-face of $\mathcal{P}/N$ is an orbit $\widehat{F} = \{F\alpha : \alpha \in N\}$, where F is a j-face of $\mathcal{P}$. If $F = F_j \tau$, where $\tau \in \Gamma$, we define $\eta : \mathcal{P}/N \to \mathcal{Q}$ by $\widehat{F}\eta := \Gamma_j \tau N$. It is a routine matter to show that η is a bijection and that both η and its inverse are order preserving. □

Clearly we must impose some special conditions on the groups in the previous propositions in order to guarantee that $\mathcal{Q}$ be a polytope; see Proposition 4.8 below, for example, or consult [10]. Notice that when $N = \{1\}$, we recover a description of $\mathcal{P}$ itself in which a typical flag becomes

$$[\Gamma_0\tau, \ldots, \Gamma_j\tau, \ldots, \Gamma_{n-1}\tau],\ \tau \in \Gamma. \tag{2.3}$$

Note that the corresponding j-adjacent flag has j-face $\Gamma_j\rho_j\tau$ [17, Sect. 2E].

3. The monodromy group of a polytope

Next we describe some tools for working with covers $\mathcal{R} \to \mathcal{Q}$. In order to understand how $\mathcal{Q}$ arises by identifications in $\mathcal{R}$, we modify Hartley's approach in [8] and instead exploit the monodromy group. For the moment, our pre-polytopes need not have any special symmetry properties.

Definition 3.1. Let $\mathcal{P}$ be a (pre-)polytope of rank $n \geqslant 1$. For $0 \leqslant j \leqslant n-1$, let r_j be the bijection on $\mathcal{F}(\mathcal{P})$ which maps each flag Φ to the j-adjacent flag Φ^j. Then the *monodromy group* for $\mathcal{P}$ is

$$\mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1}\rangle$$

(a subgroup of the symmetric group on $\mathcal{F}(\mathcal{P})$).

Remark 3.2. It is easy to check that $r_i^2 = 1$ and that $(r_i r_j)^2 = 1$, for $|j-i| > 1$, so that $\mathrm{Mon}(\mathcal{P})$ is an sggi. Thus $\mathrm{Mon}(\mathcal{P})$ is a string C-group if and only if it satisfies the intersection condition (2.2).

If $w = r_{j_1} \cdots r_{j_m}$, then $\Phi^w = \Phi^{j_1\cdots j_m}$ for any flag Φ. Therefore, a relation $r_{j_1} \cdots r_{j_m} = 1$ in $\mathrm{Mon}(\mathcal{P})$ forces the corresponding type of flag-walk to close, regardless of the initial flag. This in turn suggests how $\mathcal{P}$ arises by identifications in some cover; we refer to [8] for more details and to [12] for an application.

The monodromy group is a well-established tool in the theory of maps. We refer, for example, to [3, pp. 20-31], or to [30, p. 540], which uses the term 'connection group'.

Lemma 3.3. *Suppose $\mathcal{P}$ is a flag-connected pre-polytope. Then* $\mathrm{Mon}(\mathcal{P})$ *is transitive on $\mathcal{F}(\mathcal{P})$, and all flag-stabilizers are conjugate.*

Proof. This follows at once from the flag-connectedness of $\mathcal{P}$. □

If $\eta \in \Gamma(\mathcal{P})$, we have from Lemma 2.4 that $(\Phi^i)\eta = (\Phi\eta)^i$, for $0 \leqslant i \leqslant n-1$. We conclude that $r_i\overline{\eta} = \overline{\eta}r_i$, where $\overline{\eta}$ is the bijection induced on $\mathcal{F}(\mathcal{P})$ by η. In other words, the actions of $\Gamma(\mathcal{P})$ and $\mathrm{Mon}(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ must commute.

More generally, a rap-map $\xi : \mathcal{R} \to \mathcal{P}$ might induce a surjection $\overline{\xi} : \mathcal{F}(\mathcal{R}) \to \mathcal{F}(\mathcal{P})$ on flag sets. (This will certainly be so when $\mathcal{P}$ is a polytope, or more generally, a flag-connected pre-polytope, in which case ξ must be a covering.) In such cases, $\overline{\xi}$ will commute with the actions of the respective monodromy groups $\mathrm{Mon}(\mathcal{R})$ and $\mathrm{Mon}(\mathcal{P})$. Here are some useful consequences of these observations:

Lemma 3.4. *Let $\mathcal{Q}$ be an n-polytope with monodromy group* $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, \ldots, r_{n-1}\rangle$. *Suppose F and G are incident faces of $\mathcal{Q}$ with* $\mathrm{rank}(G) = j < k = \mathrm{rank}(F)$. *Let $\mathcal{S}$ be the section F/G, a $(k-j-1)$-polytope.*

Let $\tilde{\Phi}$ and $\tilde{\Lambda}$ be two flags of $\mathcal{Q}$ containing both F and G and such that the restricted flags Φ and Λ of $\mathcal{S}$ are in the same $\Gamma(\mathcal{S})$-orbit.

Then for $w \in \langle r_{j+1}, \ldots, r_{k-1}\rangle$ we have $\tilde{\Phi}^w = \tilde{\Phi}$ if and only if $\tilde{\Lambda}^w = \tilde{\Lambda}$.

Proof. Note that each $w = r_{t_1} \cdots r_{t_l} \in \langle r_{j+1}, \ldots, r_{k-1} \rangle$ restricts in a natural way to an element $\overline{w} = \overline{r}_{t_1} \cdots \overline{r}_{t_l} \in \mathrm{Mon}(\mathcal{S})$. Moreover, $(\Phi^{\overline{w}})\overline{\eta} = (\Phi\overline{\eta})^{\overline{w}}$ for all $\overline{\eta} \in \Gamma(\mathcal{S})$. If $\Phi\overline{\eta} = \Lambda$ we conclude that $\Phi^{\overline{w}} = \Phi$ if and only if $\Lambda^{\overline{w}} = \Lambda$. The result follows from observing that w fixes all faces of $\tilde{\Phi}, \tilde{\Lambda}$ which lie outside the section $\mathcal{S}$. □

The following results are just special cases of the previous lemma.

Lemma 3.5. *Let Φ and Λ be two flags of polytope $\mathcal{Q}$ which lie in the same $\Gamma(\mathcal{Q})$-orbit, and let $w \in \mathrm{Mon}(\mathcal{Q})$. Then $\Phi^w = \Phi$ if and only if $\Lambda^w = \Lambda$.*

Lemma 3.6. *Let Φ and Λ be two flags of an n-polytope $\mathcal{Q}$ which lie on a common facet F, and suppose there is an automorphism of F which maps Φ to Λ. Let $w \in \langle r_0, \ldots, r_{n-2} \rangle \subset \mathrm{Mon}(\mathcal{Q})$. Then $\Phi^w = \Phi$ if and only if $\Lambda^w = \Lambda$.*

Next we describe the interplay between the actions of the monodromy group and the automorphism group on the flag set.

Lemma 3.7. *Let $\mathcal{P}$ be any n-polytope, with some specified flag Ψ; and suppose $\tau_1, \ldots, \tau_k \in \Gamma(\mathcal{P})$. For $1 \leqslant j \leqslant k$ choose $w_j \in \mathrm{Mon}(\mathcal{P})$ such that $\Psi^{w_j} = (\Psi)\tau_j$. Then*

$$\Psi^{w_k \cdots w_1} = (\Psi)\tau_1 \cdots \tau_k.$$

Proof. We use induction on k. The case $k = 1$ follows from our choice of the w_j's, which in turn is enabled by Lemma 3.3. Assume that $\Psi^{w_{k-1} \cdots w_1} = (\Psi)\tau_1 \cdots \tau_{k-1}$. Since any automorphism τ_k commutes with the action of $\mathrm{Mon}(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$, we obtain

$$(\Psi)\tau_1 \cdots \tau_{k-1}\tau_k = (\Psi^{w_{k-1} \cdots w_1})\tau_k = (\Psi\tau_k)^{w_{k-1} \cdots w_1} = (\Psi^{w_k})^{w_{k-1} \cdots w_1} = \Psi^{w_k \cdots w_1}.$$

□

Corollary 3.8. *Let $\mathcal{P}$ be a regular n-polytope with base flag Φ, automorphism group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and monodromy group $\mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$. Let $\tau = \rho_{i_1} \cdots \rho_{i_k} \in \Gamma(\mathcal{P})$. Then*

(a) *the flag $\Psi := \Phi\tau^{-1}$ satisfies $\Psi^{\tau^*} = \Phi$, for $\tau^* = r_{i_k} \cdots r_{i_1} \in \mathrm{Mon}(\mathcal{P})$.*

(b) *$\Psi = \Phi\rho_{i_k} \ldots \rho_{i_1}$ if and only if $\Psi = \Phi^{r_{i_1} \cdots r_{i_k}}$.*

Proof. In Lemma 3.7 choose $w_j = r_j$ for $\tau_j = \rho_j$. □

Our next result is verging on folklore, at least in the polyhedral case (but see [32, §7]).

Theorem 3.9. *Let $\mathcal{P}$ be a regular n-polytope with base flag Φ, automorphism group $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and monodromy group $\mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$. Then there is an isomorphism $\Gamma(\mathcal{P}) \simeq \mathrm{Mon}(\mathcal{P})$ mapping each ρ_j to r_j.*

Proof. We attempt to define

$$\begin{aligned} \varphi : \Gamma(\mathcal{P}) &\;\to\; \mathrm{Mon}(\mathcal{P}), \\ \rho_{i_1} \cdots \rho_{i_k} &\;\mapsto\; r_{i_1} \cdots r_{i_k}. \end{aligned}$$

Now suppose $\rho_{i_1} \cdots \rho_{i_k} = 1$ is a relation in $\Gamma(\mathcal{P})$. Applying Corollary 3.8 to the flag $\Psi = (\Phi)\rho_{i_1} \cdots \rho_{i_k} = \Phi$, we get $\Phi^{r_{i_k} \cdots r_{i_1}} = \Phi$, hence also $\Phi^{r_{i_1} \cdots r_{i_k}} = \Phi$. But since $\mathcal{P}$ is regular, an arbitrary flag $\Lambda \in \mathcal{F}(\mathcal{P})$ can be written as $\Lambda = (\Phi)\mu$ for suitable $\mu \in \Gamma(\mathcal{P})$. Thus

$$\Lambda = (\Phi^{r_{i_1} \cdots r_{i_k}})\mu = ((\Phi)\mu)^{r_{i_1} \cdots r_{i_k}} = \Lambda^{r_{i_1} \cdots r_{i_k}}\ ,$$

since μ is an automorphism. Thus $r_{i_1} \cdots r_{i_k} = 1$ in $\mathrm{Mon}(\mathcal{P})$. It follows that φ is a well-defined homomorphism. It is even easier to show that φ is injective; and clearly φ is surjective. $\square$

Corollary 3.10. *Let $\mathcal{P}$ be a regular polytope and Λ, Ψ be two flags of $\mathcal{P}$. Then there is a unique element in* $\mathrm{Mon}(\mathcal{P})$ *that maps Λ to Ψ.*

Proof. Corollary 3.8 asserts for $\tau = \rho_{i_1} \cdots \rho_{i_k} \in \Gamma(\mathcal{P})$ that $\Phi = (\Psi)\tau$ if and only if $\Psi = \Phi^{(\tau)\varphi}$. The proof follows easily from this observation and from the proof of Theorem 3.9. $\square$

Now we can use the monodromy group to rephrase questions about coverings.

Proposition 3.11. *Suppose $\kappa : \mathcal{R} \to \mathcal{P}$ is a covering of n-polytopes (or even flag-connected pre-polytopes). Then there is an epimorphism*

$$\overline{\kappa} : \mathrm{Mon}(\mathcal{R}) \to \mathrm{Mon}(\mathcal{P})$$

(of sggi's, i.e. mapping standard generators to standard generators).

Suppose also that κ maps the flag Λ' in $\mathcal{R}$ to the flag Λ in $\mathcal{P}$. Then

$$(\mathrm{Stab}_{\mathrm{Mon}(\mathcal{R})}\Lambda')\,\overline{\kappa} \subseteq \mathrm{Stab}_{\mathrm{Mon}(\mathcal{P})}\Lambda\,. \tag{3.1}$$

Proof. We have $\mathrm{Mon}(\mathcal{R}) = \langle r'_0, \ldots, r'_{n-1}\rangle$ and $\mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1}\rangle$. All we can do is attempt the obvious definition: if $w = r'_{j_1} \cdots r'_{j_k} \in \mathrm{Mon}(\mathcal{R})$, then we let $(w)\overline{\kappa} := r_{j_1} \cdots r_{j_k}$. Clearly we need only show that this mapping is well defined. So let $r'_{j_1} \cdots r'_{j_k} = 1$ be a relation in $\mathrm{Mon}(\mathcal{R})$, and set $u = r_{j_1} \cdots r_{j_k} \in \mathrm{Mon}(\mathcal{P})$. Let $\Lambda \in \mathcal{F}(\mathcal{P})$ and note that Λ is covered by some flag $\Lambda' \in \mathcal{F}(\mathcal{R})$. (It is here that we want $\mathcal{P}$ to be flag-connected.) Thus $\Lambda = (\Lambda')\kappa$, and with the aid of Lemma 2.4 we get

$$\Lambda^u = ((\Lambda')\kappa)^{r_{j_1}\cdots r_{j_k}} = ((\Lambda')^{r'_{j_1}\cdots r'_{j_k}})\kappa = (\Lambda')\kappa = \Lambda.$$

Since Λ is arbitrary, we have $u = 1$. A similar calculation gives the second part. $\square$

The following result is easy, but nevertheless useful when working with non-regular polytopes.

Corollary 3.12. *Suppose $\kappa : \mathcal{Q} \to \tilde{\mathcal{Q}}$ is an isomorphism of polytopes, and let Ψ be a flag of $\mathcal{Q}$. Then* $\mathrm{Mon}(\mathcal{Q})$ *and* $\mathrm{Mon}(\tilde{\mathcal{Q}})$ *are isomorphic as sggi's and*

$$(\mathrm{Stab}_{\mathrm{Mon}(\mathcal{Q})}\Psi)\,\overline{\kappa} = \mathrm{Stab}_{\mathrm{Mon}(\tilde{\mathcal{Q}})}(\Psi)\kappa\,.$$

We also have a converse to Proposition 3.11.

Proposition 3.13. *Suppose that $\mathcal{R}$ and $\mathcal{P}$ are n-polytopes and that*

$$\overline{\kappa} : \mathrm{Mon}(\mathcal{R}) \to \mathrm{Mon}(\mathcal{P})$$

is an epimorphism of sggi's. Suppose also that there are flags Λ' of $\mathcal{R}$ and Λ of $\mathcal{P}$ such that

$$(\mathrm{Stab}_{\mathrm{Mon}(\mathcal{R})}\Lambda')\,\overline{\kappa} \subseteq \mathrm{Stab}_{\mathrm{Mon}(\mathcal{P})}\Lambda\,. \tag{3.2}$$

Then there is a covering $\kappa : \mathcal{R} \to \mathcal{P}$, which induces $\overline{\kappa}$ as in Proposition 3.11.

Proof. If $\Lambda' = [F'_0, \ldots, F'_{n-1}]$ and $\Lambda = [F_0, \ldots, F_{n-1}]$, we set $(F'_j)\kappa := F_j$, for $0 \leqslant j \leqslant n-1$. Now suppose F' is any j-face of $\mathcal{R}$, say in flag Φ'. Then $\Phi' = (\Lambda')^w$ for some $w \in \mathrm{Mon}(\mathcal{R})$. We define $(F')\kappa$ to be the j-face of $\Lambda^{(w)\overline{\kappa}}$. Now the strong flag-connectedness of $\mathcal{R}$ ensures that $\mathrm{Mon}(\mathcal{R})_j = \langle r_i \,|\, i \neq j\rangle$ acts transitively on flags of $\mathcal{R}$ which contain F'. Using this and (3.2) we see that the j-face in $\Lambda^{(w)\overline{\kappa}}$ is independent of our choice of w.

Now it is clear that κ is a rap-map. In particular, κ is adjacency preserving, since we do assume that $\overline{\kappa} : r'_j \mapsto r_j$, for $0 \leqslant j \leqslant n-1$. □

Remark 3.14. If $\mathcal{R}$ is regular, then condition (3.1) or (3.2) is fulfilled automatically, since all flags Λ' are equivalent, with trivial stabilizer, in $\Gamma(\mathcal{R}) \simeq \mathrm{Mon}(\mathcal{R})$ (see Theorem 3.9). In such cases, a covering $\kappa : \mathcal{R} \to \mathcal{P}$ induces an epimorphism

$$\Gamma(\mathcal{R}) = \langle \rho_0, \ldots, \rho_{n-1}\rangle \to \mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1}\rangle$$

sending ρ_i to r_i, for $0 \leqslant i \leqslant n-1$ (see Section 4).

Proposition 3.13 holds more generally when $\mathcal{P}$ is a flag-connected pre-polytope. However, it is clear from the proof that $\mathcal{R}$ should be a polytope.

Corollary 3.15. *Suppose $\eta : \mathcal{R} \to \mathcal{P}$ is a cover of regular n-polytopes, which maps the base flag Ψ for $\mathcal{R}$ to the base flag Φ for $\mathcal{P}$. Let $\Gamma(\mathcal{R}) = \langle \sigma_0, \ldots, \sigma_{n-1}\rangle$ and $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1}\rangle$ be the corresponding string C-groups. Then there is an epimorphism*

$$\eta_* : \Gamma(\mathcal{R}) \to \Gamma(\mathcal{P})$$

of string C-groups mapping σ_j to ρ_j, for $0 \leqslant j \leqslant n-1$.

Proof. Compose isomorphisms from Theorem 3.9 with the epimorphism from Proposition 3.11. □

We conclude the general results of this section with a look at minimal regular covers.

Proposition 3.16. *Suppose $\mathcal{Q}$ is an n-polytope whose monodromy group is a string C-group, and let $\mathcal{R}$ be the regular n-polytope with $\Gamma(\mathcal{R}) \simeq \mathrm{Mon}(\mathcal{Q})$ (as sggi's). Then $\mathcal{R}$ is a minimal regular cover of $\mathcal{Q}$ and is the only minimal regular cover up to isomorphism.*

Proof. Let $\mathcal{P}$ be any regular cover of $\mathcal{Q}$. By Proposition 3.11 and Theorem 3.9, there is an epimorphism $\mathrm{Mon}(\mathcal{P}) \to \mathrm{Mon}(\mathcal{Q}) \simeq \Gamma(\mathcal{R}) \simeq \mathrm{Mon}(\mathcal{R})$. By Proposition 3.13 we obtain a cover $\kappa : \mathcal{P} \to \mathcal{R}$. Clearly $\mathcal{R}$ is a minimal regular cover, unique to isomorphism. □

Example 3.17. In [12] Hartley and Williams determine $\mathrm{Mon}(\mathcal{P}) = \langle r_0, r_1, r_2\rangle$ for each classical (convex) Archimedean solid $\mathcal{P}$ in $\mathbb{E}^3$. For example, the regular toroidal map $\mathcal{R} = \{6,3\}_{(2,2)}$ covers the truncated tetrahedron $\mathcal{P}$ and

$$\mathrm{Mon}(\mathcal{P}) \simeq \Gamma(\mathcal{R}) = \Gamma(\{6,3\}_{(2,2)}),$$

a string C-group of order 144; see Figure 3 and compare [17, p. 19]. In this case, the map $\overline{\kappa}$ in Proposition 3.11 is an isomorphism.

Here is another way to look at this. Since $\mathcal{R}$ is regular, each flag stabilizer in $\mathrm{Mon}(\mathcal{R}) \simeq \Gamma(\mathcal{R})$ is trivial, by Theorem 3.9. In $\mathrm{Mon}(\mathcal{P})$, the flag stabilizer of a 'triangular' flag is $\langle (r_0r_1)^3\rangle$; for a 'hexagonal' flag it is $\langle x^{-1}(r_0r_1)^3x\rangle$, where $x = r_2$ or r_2r_1. There are 24 flags in each of three types in $\mathcal{P}$. Now that we

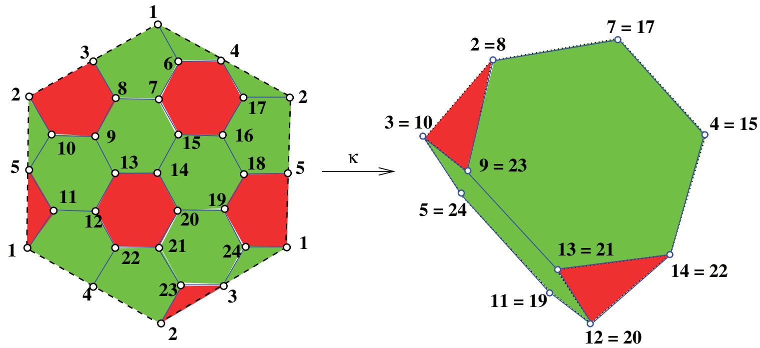


FIGURE 3. The toroid $\{6,3\}_{(2,2)}$ covers the truncated tetrahedron.

understand these flag-stabilizers, we further conclude from Proposition 3.13 that the truncated tetrahedron itself covers the tetrahedron. Looking ahead a bit, let us finally consider the subgroup $N = \langle(\rho_0\rho_1)^3\rangle$ of $\Gamma(\mathcal{R})$. It follows from Theorem 4.3 in the next section that $\mathcal{R}/N \simeq \mathcal{P}$.

4. Flag actions

Definition 4.1. Suppose that $\Gamma = \langle \rho_0, \ldots, \rho_{n-1}\rangle$ is a string C-group of rank n and let $\mathcal{Q}$ be an n-polytope with monodromy group $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, \ldots, r_{n-1}\rangle$. Then we say that $\mathcal{Q}$ admits the *flag action* by Γ if there exists an epimorphism

$$\varphi : \Gamma \to \mathrm{Mon}(\mathcal{Q})$$

which maps ρ_j to r_j for $0 \leqslant j \leqslant n-1$. To indicate this action we write

$$\Psi^\tau := \Psi^{(\tau)\varphi}$$

for $\Psi \in \mathcal{F}(\mathcal{Q})$ and $\tau \in \Gamma$.

Remark 4.2. It is easy to see that this definition is equivalent to that of Hartley in [8]. Notice that every n-polytope $\mathcal{Q}$ admits the flag action by the universal string Coxeter group

$$U := [\infty, \ldots, \infty]$$

of rank n. We could in fact ask whether $\mathcal{Q}$ admits the flag action under a more general sggi W of rank n; for example, W could be the monodromy group of a seemingly unrelated n-polytope $\mathcal{K}$. As yet, we have no use for this level of generality.

Theorem 4.3 ([8, Theorem 5.2])**.** *Suppose $\mathcal{P}$ is a regular n-polytope with base flag $\Phi = [F_0, \ldots, F_{n-1}]$ and corresponding specified generators for the automorphism group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1}\rangle$. Also suppose that $\mathcal{Q}$ is any n-polytope admitting the flag action by Γ. Fix a flag Ψ for $\mathcal{Q}$ and let*

$$N := \{\tau \in \Gamma : \Psi^\tau = \Psi\} =: \mathrm{Stab}_\Gamma(\Psi)$$

be the Γ-stabilizer of Ψ. Then there is a polytope isomorphism

$$\eta : \mathcal{P}/N \to \mathcal{Q}$$

which maps $\Phi N = [F_0N, \ldots, F_{n-1}N]$ to Ψ.

Proof. We may identify the j-faces of $\mathcal{P}$ with the right cosets $\Gamma_j\tau$, for $\tau \in \Gamma$; in particular, the base face F_j becomes Γ_j. After a look at Proposition 2.19, we attempt to define $\eta : \mathcal{P}/N \to \mathcal{Q}$ by letting $(\Gamma_j\tau N)\eta := [\Psi^{\tau^{-1}}]_j$ (the j-face in the flag $\Psi^{\tau^{-1}}$). Our arguments will imply that $\mathcal{P}/N$ actually is an n-polytope.

Now if $\tilde{\tau} = \gamma\tau\alpha$ with $\alpha \in N$ and $\gamma \in \Gamma_j$, then

$$\Psi^{\tilde{\tau}^{-1}} = \Psi^{\alpha^{-1}\tau^{-1}\gamma^{-1}} = \Psi^{\tau^{-1}\gamma^{-1}}$$

has the same j-face as $\Psi^{\tau^{-1}}$, since $\gamma^{-1} \in \Gamma_j$. Thus η is well defined.

Clearly η is onto since $\mathcal{Q}$ is flag-connected; and η is a rank preserving poset homomorphism because of Lemma 2.16(b).

Suppose $\Psi^{\tau^{-1}}$ and $\Psi^{\lambda^{-1}}$ have the same j-face in $\mathcal{Q}$. By strong flag-connectivity in $\mathcal{Q}$, there exists $\gamma \in \Gamma_j$ mapping the first of these flags to the second via the flag action. Thus $\tau^{-1}\gamma\lambda = n \in N$. Then in $\mathcal{P}/N$ we must have $\Gamma_j\tau N = \Gamma_j\lambda N$. Thus η is $1-1$.

Finally, we note that two incident faces in $\mathcal{Q}$ lie in a flag, which we may write as $\Psi^{\tau^{-1}}$ for some $\tau \in \Gamma$. Thus η^{-1} is also order preserving. $\square$

Corollary 4.4. *In the previous theorem, the polytope $\mathcal{P}$ is a regular cover of $\mathcal{Q}$.*

Proof. Regarding the covering, just compose maps as follows:

$$\mathcal{P} \overset{\text{natural}}{\to} \mathcal{P}/N \overset{\eta}{\to} \mathcal{Q}\,.$$

We need only show that the natural map sends adjacent flags in $\mathcal{P}$ to adjacent flags in $\mathcal{P}/N$. If not, then N must identify the two distinct j-faces in some rank 1 section of $\mathcal{P}$. Thus, for some $u \in \Gamma$ we would have $\Gamma_j\rho_j uN = \Gamma_j uN$, whence $\rho_j u = gux$ for some $g \in \Gamma_j$ and $x \in N$. It follows that the flag $\Psi^{u^{-1}}$ in $\mathcal{Q}$ is fixed under flag action by $g^{-1}\rho_j$. This is a contradiction since g^{-1} fixes the j-faces in all flags, whereas ρ_j does not. Alternatively, the covering is implied by Proposition 3.13 and Theorem 3.9. $\square$

There is a converse of sorts:

Proposition 4.5. *Let $\mathcal{Q}$ be an n-polytope and let $\mathcal{P}$ be a regular cover of $\mathcal{Q}$. Then $\mathcal{Q}$ admits the flag action by the string C-group $\Gamma(\mathcal{P})$.*

Proof. By Proposition 3.11 there exists an epimorphism $\overline{\kappa} : \mathrm{Mon}(\mathcal{P}) \to \mathrm{Mon}(\mathcal{Q})$ of sggi's. But by Theorem 3.9, $\Gamma(\mathcal{P}) \simeq \mathrm{Mon}(\mathcal{P})$. $\square$

Remark 4.6. The epimorphism $\overline{\kappa} : \mathrm{Mon}(\mathcal{P}) \to \mathrm{Mon}(\mathcal{Q})$ provides a useful reminder that the covering $\mathcal{P} \to \mathcal{Q}$ is a rap-map. Perhaps, then, it would be a little more natural to replace $\Gamma = \Gamma(\mathcal{P})$ by $\mathrm{Mon}(\mathcal{P})$ in Definition 4.1.

Next we consider the quotients of regular polytopes by 'sparse' subgroups of the automorphism group. (We refer to [17, p. 58] for a history of this term.)

Definition 4.7. Let $\Gamma = \langle\rho_0, \ldots, \rho_{n-1}\rangle$ be the automorphism group of the regular n-polytope $\mathcal{P}$. A subgroup $N \leqslant \Gamma$ is said to be *sparse* if

$$\tau^{-1}N\tau \cap \Gamma_0\Gamma_{n-1} = \{1\}\,, \tag{4.1}$$

for all $\tau \in \Gamma$.

Proposition 4.8. *Let $\mathcal{P}$ be a regular n-polytope with automorphism group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and corresponding base flag $\Phi = [F_0, \ldots, F_{n-1}]$. Suppose that N is a sparse subgroup of Γ. Then $Q := \mathcal{P}/N$ is an n-polytope which admits the flag-action by the string C-group Γ.*

Proof. By [17, Prop. 2E23] we know that Q is an n-polytope whose facets and vertex-figures are respectively isomorphic to those of $\mathcal{P}$. (Thus, in this special case, facets and vertex-figures of $\mathcal{P}$ are preserved; note that $\mathcal{Q}$ is equivelar.)

By Lemma 2.16(b), a typical flag of $\mathcal{Q}$ can be written as

$$\Psi = [\Gamma_0 \tau N, \ldots, \Gamma_j \tau N, \ldots, \Gamma_{n-1} \tau N] \, ,$$

that is, with one and the same $\tau \in \Gamma$ representing each double coset. Note that the j-adjacent flag Ψ^j is represented by $\rho_j \tau$ and so has j-face $\Gamma_j \rho_j \tau N$. (The fact that $\Gamma_j \rho_j \tau N \neq \Gamma_j \tau N$ follows from the sparseness of N and the intersection condition on Γ.)

For any $\alpha \in \Gamma$, we therefore want to define Ψ^α to be the flag in $\mathcal{Q}$ with representative $\alpha^{-1}\tau$. We need only show that this action is well defined. Suppose then that the flag Ψ is also represented by $\mu \in \Gamma$, so that $\mu = \gamma_j \tau \lambda_j$, for $0 \leqslant j \leqslant n-1$, with $\lambda_j \in N$ and $\gamma_j \in \Gamma_j$. Since Γ is an sggi, we have $\gamma_j = \alpha_j \beta_j = \beta_j \alpha_j$ for certain $\alpha_j \in \Gamma_{<j} \subseteq \Gamma_{n-1}$ and $\beta_j \in \Gamma_{>j} \subseteq \Gamma_0$. For any $j < i$ we have $\gamma_i \tau \lambda_i = \mu = \gamma_j \tau \lambda_j$, so that

$$\begin{aligned} \tau \lambda_j \lambda_i^{-1} \tau^{-1} &= \gamma_j^{-1} \gamma_i \\ &= \beta_j^{-1} \alpha_j^{-1} \beta_i \alpha_i \\ &= (\beta_j^{-1} \beta_i)\,(\alpha_j^{-1} \alpha_i) \in \Gamma_0 \Gamma_{n-1} \, . \end{aligned}$$

(Since $j < i$, α_j^{-1} commutes with β_i.) By (4.1), we have $\gamma_j = \gamma_i$. Indeed, $\gamma_0 = \ldots = \gamma_{n-1} \in \bigcap_{i=0}^{n-1} \Gamma_i = \{1\}$, by the intersection condition. Hence $\mu = \tau\lambda$ for some $\lambda \in N$. But the flags with representatives τ and $\mu = \tau\lambda$ are clearly identical. $\square$

Remark 4.9. Clearly we need some sort of condition on N in order that the quotient $\mathcal{P}/N$ be a polytope. Perhaps the most familar instance of Proposition 4.8 is when $\mathcal{P} = \{4, 4\}$ is the familiar regular tessellation of the plane by squares, with N some subgroup of $\Gamma(\mathcal{P})$ generated by translations. A proper quotient will then be a toroidal, or cylindrical, polyhedron with the same Schläfli type.

Corollary 4.10. *The Γ-stabilizer of the base flag $[\Gamma_0 N, \ldots, \Gamma_{n-1} N]$ in $\mathcal{Q}$ is N.*

Proof. If $\tau = 1$ in the proof of Proposition 4.8, then $\mu = \tau\lambda = \lambda \in N$. $\square$

Definition 4.11. Let N be a subgroup of G. Then the core of N in G, or core(G, N), is the intersection of all G-conjugates of N:

$$\mathrm{core}(G, N) := \bigcap_{g \in G} g^{-1} N g \, .$$

We note that core(G, N) is the largest normal subgroup of G which is contained in N. We easily verify the following

Lemma 4.12. *Suppose $\mathcal{Q}$ is an n-polytope which admits the flag action by the string C-group Γ, through the epimorphism*

$$\begin{aligned} \varphi : \Gamma &\to \mathrm{Mon}(\mathcal{Q}), \\ \rho_j &\mapsto r_j . \end{aligned}$$

Let N be the Γ-stabilizer of a fixed flag Ψ in $\mathcal{Q}$. Then

$$\ker \varphi = \operatorname{core}(\Gamma, N).$$

Remark 4.13. We will see in Example 6.8 that $\Gamma/\operatorname{core}(\Gamma, N) \simeq \operatorname{Mon}(\mathcal{Q})$ is not always a string C-group. However, if $\operatorname{Mon}(\mathcal{Q})$ is a string C-group, then clearly $\mathcal{Q}$ admits a flag action by $\operatorname{Mon}(\mathcal{Q})$ itself via the identity map $\operatorname{Mon}(\mathcal{Q}) \to \operatorname{Mon}(\mathcal{Q})$.

We can now gather together several useful and related results from the literature:

Theorem 4.14. *Let $\mathcal{P}$ be a regular n-polytope with automorphism group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and suppose that $\mathcal{Q}$ is an n-polytope which admits the flag action by Γ. Let $N := \operatorname{Stab}_\Gamma(\Psi)$ be the Γ-stabilizer of a base flag Ψ for $\mathcal{Q}$.*

(a) *Let $\omega, \mu \in \Gamma$. An automorphism $\alpha_\omega \in \Gamma(\mathcal{Q})$ mapping Ψ to Ψ^ω exists if and only if $\omega \in \operatorname{Norm}_\Gamma(N)$. In such cases, $\alpha_\omega = \alpha_\mu$ if and only if $\omega\mu^{-1} \in N$.*

(b) $\Gamma(\mathcal{Q}) \simeq \operatorname{Norm}_\Gamma(N)/N$.

(c) *All orbits of $\Gamma(\mathcal{Q})$ acting on $\mathcal{F}(\mathcal{Q})$ have cardinality $|\Gamma(\mathcal{Q})|$. The number of such flag orbits is*

$$[\Gamma : \operatorname{Norm}_\Gamma(N)] = [\operatorname{Mon}(\mathcal{Q}) : \operatorname{Norm}_{\operatorname{Mon}(\mathcal{Q})}(\operatorname{Stab}_{\operatorname{Mon}(\mathcal{Q})}(\Psi))]\ .$$

(d) *Let $\mathcal{R}$ be a regular n-polytope which also admits the flag action by Γ, and let $K = \operatorname{Stab}_\Gamma(\Lambda)$ be the Γ-stabilizer of a base flag Λ for $\mathcal{R}$. Then $K \triangleleft \Gamma$, $\Gamma(\mathcal{R}) \simeq \Gamma/K$ and $\mathcal{P}/K \simeq \mathcal{R} \simeq \mathcal{P}(\Gamma/K)$.*

(e) *Suppose that $\operatorname{Mon}(\mathcal{Q})$ is a string C-group. Let $\mathcal{R}$ be any regular n-polytope such that $\mathcal{Q}$ also admits the flag action by $\Gamma(\mathcal{R})$. Then $\mathcal{R}$ covers $\mathcal{P}(\operatorname{Mon}(\mathcal{Q}))$, which in turn is a regular cover of $\mathcal{Q}$.*

(f) *Let $\tilde{\mathcal{Q}}$ be an n-polytope; suppose Λ is a flag of $\tilde{\mathcal{Q}}$. If $\eta : \mathcal{Q} \to \tilde{Q}$ is an isomorphism of polytopes, then $\tilde{Q}$ admits the flag action by Γ; and $\operatorname{Stab}_\Gamma(\Lambda) = \tau N \tau^{-1}$, for some $\tau \in \Gamma$ satisfying $\Lambda^\tau = (\Psi)\eta$.*

Conversely, suppose that $\tilde{Q}$ admits flag action by Γ and that $\operatorname{Stab}_\Gamma(\Lambda) = \tau N \tau^{-1}$, for some $\tau \in \Gamma$. Then there exists an isomorphism $\eta : \mathcal{Q} \to \tilde{Q}$ which maps Ψ to Λ^τ.

Proof. For fixed $\omega \in \operatorname{Norm}_\Gamma(N)$, we define $\alpha_\omega \in \Gamma(\mathcal{Q})$ as follows. If F is a j-face of $\mathcal{Q}$, then $F \in \Psi^\beta$ for some $\beta \in \Gamma$. If also $F \in \Psi^\gamma$, then $\Psi^\gamma = \Psi^{\beta\mu}$, for some $\mu \in \Gamma_j$, by the strong flag-connectivity of $\mathcal{Q}$. Thus, $\beta\mu\gamma^{-1} \in N$, and indeed $\omega(\beta\mu\gamma^{-1})\omega^{-1} \in N$. It follows that

$$\Psi^{\omega\beta} = \Psi^{\omega\gamma\mu^{-1}},$$

so $\Psi^{\omega\beta}$ and $\Psi^{\omega\gamma}$ share the same j-face, which we define to be $(F)\alpha_\omega$. Notice that a (general) flag Ψ^β of $\mathcal{Q}$ is mapped to another flag $\Psi^{\omega\beta}$. It is now easy to check that α_ω is an automorphism of $\mathcal{Q}$ and that all automorphisms arise in this way. Furthermore, the mapping

$$\begin{array}{ccc} \operatorname{Norm}_\Gamma(N) & \to & \Gamma(\mathcal{Q}), \\ \omega & \mapsto & \alpha_{\omega^{-1}} \end{array}$$

is an epimorphism with kernel N. This addresses parts (a) and (b).

For part (c) compare [17, Prop. 2A5]. It is easy to check that as β runs through Γ, the $\Gamma(\mathcal{Q})$-orbit of the flag Ψ^β corresponds naturally to the coset $\operatorname{Norm}_\Gamma(N)\beta$. In light of this correspondence, the given equality makes sense when both sides are infinite.

For (d) just apply (a) and (b) with $\mathcal{R}$ in place of $\mathcal{Q}$ and K in place of N. Regularity of $\mathcal{R}$ forces $\mathrm{Norm}_\Gamma(K) = \Gamma$, by part (c).

In part (e) we have $\mathrm{Mon}(\mathcal{Q}) \simeq \Gamma(\mathcal{P}(\mathrm{Mon}(\mathcal{Q})))$ by Theorem 3.9. Clearly, the identity map $\mathrm{Mon}(\mathcal{Q}) \to \mathrm{Mon}(\mathcal{Q})$ induces the regular cover $\mathcal{P}(\mathrm{Mon}(\mathcal{Q})) \to \mathcal{Q}$. Since

$$\mathrm{Mon}(\mathcal{P}(\mathrm{Mon}(\mathcal{Q}))) \simeq \Gamma(\mathcal{P}(\mathrm{Mon}(\mathcal{Q})) \simeq \mathrm{Mon}(\mathcal{Q}),$$

the map $\Gamma(\mathcal{R}) \to \mathrm{Mon}(\mathcal{Q})$ induces a covering $\mathcal{R} \to \mathcal{P}(\mathrm{Mon}(\mathcal{Q}))$, by Proposition 3.13.

For the first part of (f), suppose that $\varphi : \Gamma \to \mathrm{Mon}(\mathcal{Q})$ defines the flag action on $\mathcal{Q}$. From Corollary 3.12 we conclude that $\varphi\overline{\eta}$ defines the flag action of Γ on $\tilde{Q}$. The remaining details use the fact that Γ acts transitively on $\mathcal{F}(\tilde{\mathcal{Q}})$.

Conversely, given the conjugate stabilizers $\tilde{N} = \mathrm{Stab}_\Gamma(\Lambda) = \tau N \tau^{-1}$, the isomorphism η is readily constructed using the description of $\mathcal{Q}$ (and of $\tilde{Q}$) provided by Theorem 4.3. □

Remark 4.15. In some form or another, part (a) of Theorem 4.14 appears as [14, Prop. 7]; part (b) as [9, Th. 3.6] (but see also [17, Prop. 2D8]); part (c) in [12, Th. 2.2] or [14, Prop. 9]; part (e) in [12, Th. 2.3]; and part (f) in [8, Th. 5.3]. Several subgroup relations are summarized in

$$\begin{array}{ccc}
\Gamma & \longrightarrow & \mathrm{Mon}(\mathcal{Q}) \\
| & & | \\
\mathrm{Norm}_\Gamma(N) & \longrightarrow & \mathrm{Norm}_{\mathrm{Mon}(\mathcal{Q})}(\mathrm{Stab}_{\mathrm{Mon}(\mathcal{Q})}(\Psi)) \\
| & & | \\
N = \mathrm{Stab}_\Gamma(\Psi) & \longrightarrow & \mathrm{Stab}_{\mathrm{Mon}(\mathcal{Q})}(\Psi) \\
| & & | \\
\mathrm{core}(\Gamma, N) & \longrightarrow & 1
\end{array}$$

5. The mix or parallel product

The 'parallel product' of a pair of groups is a natural construction with many applications in the theory of maps and polytopes. The idea is described in [30], where it is used, for instance, to investigate chiral maps and their regular covers; see also [22, 23]. Here we use instead the term 'mix', which is defined in [17, Ch. 7] as part of a more general discussion of mixing operations on sggi's. We begin with an extension of the idea to quite general families of groups with specified generators.

Definition 5.1. Let n be a positive integer, and suppose that $\{G(m) : m \in M\}$ is a family of groups, each generated by a specified list $g_{0,m}, \ldots, g_{n-1,m}$ of n elements. (We allow redundant generators, such as some $g_{j,m} = 1$.) The *mix*

$$\diamondsuit_{m\in M} G(m)$$

of this family of groups is the subgroup of the (strong) direct product $\bigotimes_{m\in M} G(m)$ generated by $g_0, \ldots, g_{n-1}$, where $g_j := (g_{j,m})_{m\in M}$.

Remark 5.2. Note that the mix also has n specified generators, the jth of which has $g_{j,m}$ as its mth component. A typical element of the mix is a sequence (x_m), not necessarily of finite support if M is infinite, with $x_m \in G(m)$ for all m. In our work, the indexing set M will always be at most countable, and usually finite, in which case we will use notation such as $G\diamondsuit H$ or $G\diamondsuit H\diamondsuit\cdots\diamondsuit K$ without any fuss.

We naturally say that the groups $G(m)$ have *rank* n. Usually we tacitly assume that an epimorphism $\varphi : \tilde{G} \to G$ from one group of rank n to another respects specified generators, so that $\tilde{g}_j \mapsto g_j$, for $0 \leqslant j \leqslant n-1$. On the other hand, each natural projection $\pi_k : \diamondsuit_{m\in M} G(m) \to G(k)$ is clearly onto, so that the mix is a kind of subdirect product.

It is easy to check that the mix is a commutative and associative operation, up to isomorphism in the class of groups of rank n.

We now establish a few important but readily verified properties of the mix.

Lemma 5.3. *Suppose $\tilde{G}$ is a group of rank n such that for each $m \in M$ there exists an epimorphism $\varphi_m : \tilde{G} \to G(m)$, where one such map, say φ_1, is in fact an isomorphism. Then*

$$\diamondsuit_{m\in M} G(m) \simeq \tilde{G}\ .$$

Proof. Just check that

$$\begin{array}{rcl} \eta : \tilde{G} & \to & \diamondsuit_{m\in M} G(m), \\ g & \mapsto & (g\varphi_1, \ldots, g\varphi_m, \ldots) \end{array}$$

is an isomorphism respecting specified generators. □

Of course, we will mainly be concerned with the mix of sggi's. It is easy to check the next result.

Lemma 5.4. *Suppose that each $G(m)$ is an sggi of rank n, say with Schläfli symbol $\{p_{1,m}, \ldots, p_{n-1,m}\}$. Then the mix $\diamondsuit_{m\in M} G(m)$ is also an sggi, with Schläfli symbol $\{\overline{p}_1, \ldots, \overline{p}_{n-1}\}$, where $\overline{p}_j = \mathrm{lcm}\{p_{j,m} : m \in M\}$, for $1 \leqslant j \leqslant n-1$.*

Naturally, for a given j we take $\overline{p}_j = \infty$ when there are infinitely many different polygonal sizes $p_{j,m}$ for $m \in M$. We also observe that if $G, H, \ldots, K$ are isomorphic sggi's, then $G\diamondsuit H\diamondsuit\cdots\diamondsuit K \simeq G$.

We can already use mixing to help understand the standard subgroups of the monodromy group of any polytope:

Theorem 5.5. *Let $\mathcal{Q}$ be an n-polytope with monodromy group* $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, \ldots, r_{n-1}\rangle$. *For fixed $-1 \leqslant j \leqslant k \leqslant n$, let $\{\mathcal{S}(m) : m \in M\}$ be the set of all sections F/G in $\mathcal{Q}$, where $G < F$,* $\mathrm{rank}(G) = j$ *and* $\mathrm{rank}(F) = k$. (*Thus each $\mathcal{S}(m)$ is a $(k-j-1)$-polytope; and the indexing set M can be infinite if $\mathcal{Q}$ is an infinite polytope.*) *Then*

$$\langle r_{j+1}, \ldots, r_{k-1}\rangle \simeq \diamondsuit_{m\in M} \mathrm{Mon}(\mathcal{S}(m))\ .$$

Proof. Any flag $\Phi = [G, \ldots, F]$ of the section $\mathcal{S}(m) = F/G$ certainly extends to a flag $\tilde{\Phi}$ for $\mathcal{Q}$, perhaps in several ways. For each $g \in \langle r_{j+1}, \ldots, r_{k-1}\rangle$, $\tilde{\Phi}^g$ differs from $\tilde{\Phi}$ only in proper faces of $\mathcal{S}(m)$, so that g induces a permutation $g(m)$ of the flags in $\mathcal{F}(\mathcal{S}(m))$. In particular, $r_i(m)$ maps any flag Φ of $\mathcal{S}(m)$ to Φ^i, for $j+1 \leqslant i \leqslant k-1$, and so we may write

$$\mathrm{Mon}(\mathcal{S}(m)) = \langle r_{j+1}(m), \ldots, r_{k-1}(m)\rangle$$

for all $m \in M$. Now we may define an epimorphism

$$\begin{array}{rcl} \varphi : \langle r_{j+1}, \ldots, r_{k-1}\rangle & \to & \diamondsuit_{m\in M} \mathrm{Mon}(\mathcal{S}(m)), \\ g & \mapsto & (g(m))_{m\in M}. \end{array}$$

If $g(m) = 1$ for all $m \in M$, then g fixes (elementwise) all sections F/G with $\mathrm{rank}(G) = j$ and $\mathrm{rank}(F) = k$; and in any case, since $g \in \langle r_{j+1}, \ldots, r_{k-1} \rangle$, g must also fix all sections of the form Q_n/F or G/Q_{-1}, where Q_{-1} and Q_n are the minimal and maximal faces of $\mathcal{Q}$. Thus $g = 1$ and φ is an isomorphism of sggi's. □

Corollary 5.6. *Suppose $\mathcal{Q}$ is an n-polytope with monodromy group* $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, \ldots, r_{n-1} \rangle$ *and facet set* $\{F(m) : m \in M\}$. *Then*

$$\langle r_0, \ldots, r_{n-2} \rangle \simeq \diamondsuit_{m \in M} \mathrm{Mon}(F(m)) \ .$$

Corollary 5.7. *Suppose $\mathcal{Q}$ is an n-polytope with facet set $\{F(m) : m \in M\}$, where each $F(m)$ is a regular $(n-1)$-polytope. Then the subgroup $\langle r_0, \ldots, r_{n-2} \rangle$ of* $\mathrm{Mon}(\mathcal{Q})$ *is isomorphic to the mix*

$$\diamondsuit_{m \in M} \Gamma(F(m)) \ .$$

Proof. This follows at once from Theorems 3.9 and 5.5. □

Let us now examine the mix of the automorphism groups of a family of regular n-polytopes.

Definition 5.8. Let $\mathcal{P}$ and $\mathcal{Q}$ be regular n-polytopes with automorphism groups $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ and $\Gamma(\mathcal{Q}) = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$, and let $\tau_j := (\rho_j, \sigma_j) \in \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$, $0 \leqslant j \leqslant n-1$. Whenever the sggi $\Gamma(\mathcal{P}) \diamondsuit \Gamma(\mathcal{Q}) = \langle \tau_0, \ldots, \tau_{n-1} \rangle$ is a string C-group, we shall say that the corresponding regular polytope is the *mix* of $\mathcal{P}$ and $\mathcal{Q}$ and denote it by $\mathcal{P} \diamondsuit \mathcal{Q}$. On the other hand, if $\Gamma(\mathcal{P}) \diamondsuit \Gamma(\mathcal{Q})$ is not a string C-group, we say that the corresponding flagged poset $\mathcal{P} \diamondsuit \mathcal{Q}$ is *non-polytopal.* Similar definitions apply to any family $\{\mathcal{P}(m) : m \in M\}$ of regular n-polytopes.

Example 5.9. By Proposition 5.15 below, non-polytopal mixes can occur only in ranks $n \geqslant 4$; compare [31, Th. 2] and the example in [17, p. 185]. Perhaps the simplest of many non-polytopal mixes is $\{2, 3, 3\} \diamondsuit \{3, 3, 2\}$. Here with a little help from GAP [7], or easily enough by hand, we find that

$$(\tau_2 \tau_1)^{((\tau_1 \tau_0)^3)} = (\tau_1 \tau_2)^{((\tau_2 \tau_3)^3)} \notin \langle \tau_1, \tau_2 \rangle \ .$$

In [6] Cunningham has determined all polytopal mixes of regular *convex* polytopes. See also [5] for a wide-ranging exploration of related problems, including a study of *semi-polytopes*, a class of pre-polytopes which generalize abstract polytopes.

Sometimes then the mix of regular polytopes is polytopal. In these cases, it is useful to set up a little machinery:

Lemma 5.10. *Suppose $\{\mathcal{P}(m) : m \in M\}$ is a family of regular n-polytopes whose mix is the regular n-polytope $\mathcal{R}$. For each $m \in M$ choose a base flag $\Phi(m)$ for $\mathcal{P}(m)$ and let $\rho_{0,m}, \ldots, \rho_{n-1,m}$ be the corresponding involutory generators of $\Gamma(\mathcal{P}(m))$. Then the automorphism group for $\mathcal{R}$ is*

$$\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle := \diamondsuit_{m \in M} \Gamma(\mathcal{P}(m)) \ ,$$

with $\rho_j = (\rho_{j,m})_{m \in M}$, for $0 \leqslant j \leqslant n-1$. Let Φ be the base flag of $\mathcal{R}$ associated to the ρ_j's.

(a) *Then for each $m \in M$, the natural projection*

$$\begin{aligned} \pi_m : \Gamma \quad &\to \quad \Gamma(\mathcal{P}(m)), \\ \rho_j \quad &\mapsto \quad \rho_{j,m} \end{aligned}$$

induces a covering $\kappa_m : \mathcal{R} \to \mathcal{P}(m)$.

(b) *For* $\tau \in \Gamma$, $(\Phi\tau)\kappa_m = \Phi(m)$ *if and only if* $\tau \in \ker \pi_m$.

(c) $(\Phi\tau)\kappa_m = \Phi(m)$ *for all* $m \in M$ *if and only if* $\tau = 1$. *Thus,* Φ *is the unique flag of* $\mathcal{R}$ *which satisfies* $\Phi\kappa_m = \Phi(m)$, *for all* $m \in M$.

Proof. This is a routine application of Theorem 3.9 and Proposition 3.13. From the proof of the latter we note that κ_m maps the general flag $[\Gamma_0\tau, \ldots, \Gamma_{n-1}\tau]$ of $\mathcal{R}$ to the flag $[\Gamma(m)_0(\tau\pi_m), \ldots, \Gamma(m)_{n-1}(\tau\pi_m)]$ of $\mathcal{P}(m)$. (See equation (2.3) which describes a flag of a regular polytope qua coset geometry.) $\square$

Theorem 5.11. *Let* $\{\mathcal{P}(m) : m \in M\}$ *be a family of regular* n*-polytopes whose mix*

$$\mathcal{R} := \diamondsuit_{m\in M}\mathcal{P}(m)$$

is polytopal. Then $\mathcal{R}$ *covers each* $\mathcal{P}(m)$. *Furthermore, if* $\mathcal{S}$ *is any regular* n*-polytope covering all the* $\mathcal{P}(m)$*'s, then* $\mathcal{S}$ *covers* $\mathcal{R}$.

Proof. Our first claim follows at once from Lemma 5.10. Now suppose for each m there exists a covering $\gamma(m) : \mathcal{S} \to \mathcal{P}(m)$. Let Ψ be a fixed base flag of $\mathcal{S}$ and choose $\Phi(m) := (\Psi)\gamma(m)$ before proceeding to construct the mixed polytope $\mathcal{R}$. By Corollary 3.15, we have induced epimorphisms

$$\gamma(m)_* : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{P}(m))$$

of string C-groups. Clearly, we now have an epimorphism

$$\begin{array}{rcl} \gamma : \Gamma(\mathcal{S}) & \to & \Gamma(\mathcal{R}), \\ \alpha & \mapsto & (\,(\alpha)\gamma(m)_*\,)_{m\in M}\ . \end{array}$$

This map induces the desired covering of $\mathcal{R}$ by $\mathcal{S}$. $\square$

In stating the next result it is convenient to abuse language a little. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are regular n-polytopes. Then all facets of $\mathcal{P}$ are isomorphic to some regular $(n-1)$-polytope, say $\mathcal{K}$; likewise all facets of $\mathcal{Q}$ are isomorphic to some $\mathcal{L}$. We say that the *facets of* $\mathcal{P}$ *cover those of* $\mathcal{Q}$ if $\mathcal{K}$ covers $\mathcal{L}$.

Theorem 5.12. *Let* $\{\mathcal{P}(m) : m \in M\}$ *be a family of regular* n*-polytopes. Suppose that there is a particular polytope, say* $\mathcal{P}(1)$, *whose facets cover the facets of every polytope* $\mathcal{P}(m)$ *in the family. Then the mix*

$$\diamondsuit_{m\in M}\mathcal{P}(m)$$

is a regular n*-polytope.*

Proof. For each $m \in M$ choose a base flag $\Phi(m)$ for $\mathcal{P}(m)$; as in Lemma 5.10, let $\Gamma(\mathcal{P}(m)) = \langle \rho_{0,m}, \ldots, \rho_{n-1,m}\rangle$, and let $\Gamma := \diamondsuit_{m\in M}\Gamma(\mathcal{P}(m)) = \langle \rho_0, \ldots, \rho_{n-1}\rangle$, where $\rho_j := (\rho_{j,m})_{m\in M}$. We must show that Γ satisfies the intersection condition.

From our hypothesis and Corollary 3.15 we obtain epimorphisms

$$\varphi_m : \langle \rho_{0,1}, \ldots, \rho_{n-2,1}\rangle \to \langle \rho_{0,m}, \ldots, \rho_{n-2,m}\rangle\ .$$

Notice that $\pi_1\varphi_m = \pi_m$ on the subgroup $\langle \rho_0, \ldots, \rho_{n-2}\rangle$ of Γ.

Now let $\mathcal{K}$ be a facet of $\mathcal{P}(1)$. It follows from Lemma 5.3 that $\langle \rho_0, \ldots, \rho_{n-2}\rangle \simeq \Gamma(\mathcal{K})$ is a string C-subgroup of Γ. By [17, Prop. 2E16(b)] we need only prove for $1 \leqslant k \leqslant n-1$ that

$$\langle \rho_0, \ldots, \rho_{n-2}\rangle \cap \langle \rho_k, \ldots, \rho_{n-1}\rangle = \langle \rho_k, \ldots, \rho_{n-2}\rangle\ .$$

So assume $\tau \in \langle \rho_0, \ldots, \rho_{n-2} \rangle \cap \langle \rho_k, \ldots, \rho_{n-1} \rangle$. Clearly,

$$(\tau)\pi_m \in \langle \rho_{0,m}, \ldots, \rho_{n-2,m} \rangle \cap \langle \rho_{k,m}, \ldots, \rho_{n-1,m} \rangle = \langle \rho_{k,m}, \ldots, \rho_{n-2,m} \rangle,$$

since $\Gamma(\mathcal{P}(m))$ is a string C-group. Taking $m = 1$, we get $(\tau)\pi_1 = \rho_{i_1,1} \cdots \rho_{i_l,1}$, for suitable $i_j \in \{k, \ldots, n-2\}$. Let $\lambda = \rho_{i_1} \cdots \rho_{i_l} \in \langle \rho_k, \ldots, \rho_{n-2} \rangle$. Then for each $m \in M$ we have

$$\begin{aligned} (\lambda)\pi_m &= \rho_{i_1,m} \cdots \rho_{i_l,m} \\ &= (\rho_{i_1,1} \cdots \rho_{i_l,1})\varphi_m \\ &= (\tau)\pi_1\varphi_m \\ &= (\tau)\pi_m \,. \end{aligned}$$

Thus $\tau = \lambda \in \langle \rho_k, \ldots, \rho_{n-2} \rangle$. □

Remark 5.13. Compare [2, Lemma 3.3] or [5, Proposition 2.41]. For ease of reading, we have omitted from Theorem 5.12 equally valid dual conditions concerning vertex-figures. Clearly, the conclusion holds when, for example, all the $\mathcal{P}(m)$ share isomorphic facets.

It is clear on geometric grounds that the mix of any family of polygons is another polygon. This fact also follows immediately from Theorem 5.12, since all 1-polytopes are isomorphic. More precisely, we have

Corollary 5.14. *Suppose $\{\,\{p_m\} : m \in M\}$ is a finite or countably infinite family of polygons, and let $\overline{p} := \operatorname{lcm}\{p_m : m \in M\}$. Then*

$$\diamondsuit_{m\in M}\{p_m\} \simeq \{\overline{p}\}$$

(and $\diamondsuit_{m\in M}\mathbb{D}_{2p_m} \simeq \mathbb{D}_{2\overline{p}}$).

The proof is routine. With just a bit more effort we obtain a similar result in rank 3. (Compare [5, Corollary 2.46].)

Proposition 5.15. *Let $\{\,\mathcal{P}(m) : m \in M\}$ be any family of regular polyhedra; and suppose $\mathcal{P}(m)$ has Schläfli type $\{p_m, q_m\}$. Then $\mathcal{Q} = \diamondsuit_{m\in M}\mathcal{P}(m)$ is also a regular polyhedron, with type $\{\overline{p}, \overline{q}\}$, where $\overline{p} := \operatorname{lcm}\{p_m : m \in M\}$ and $\overline{q} := \operatorname{lcm}\{q_m : m \in M\}$.*

Proof. As in the proof of Theorem 5.12, let $\Gamma(\mathcal{P}(m)) = \langle \rho_{0,m}, \rho_{1,m}, \rho_{2,m} \rangle$, and let $\Gamma := \diamondsuit_{m\in M}\Gamma(\mathcal{P}(m)) = \langle \rho_0, \rho_1, \rho_2 \rangle$, where $\rho_j := (\rho_{j,m})_{m\in M}$. By Corollary 5.14, both $\langle \rho_0, \rho_1 \rangle$ and $\langle \rho_1, \rho_2 \rangle$ are (dihedral) string C-subgroups of Γ. We may therefore apply [17, Prop. 2E16], so let $\alpha \in \langle \rho_0, \rho_1 \rangle \cap \langle \rho_1, \rho_2 \rangle$; we need to show $\alpha \in \langle \rho_1 \rangle$. We can assume that α is (ρ_0, ρ_1)-even; otherwise replace α by $\alpha\rho_1$ (see Example 2.8). In other words, $\alpha = (\rho_0\rho_1)^k$ for some integer k. Projecting to $\Gamma(\mathcal{P}(m))$, we conclude for each m that

$$(\rho_{0,m}\rho_{1,m})^k \in \langle \rho_{0,m}, \rho_{1,m} \rangle \cap \langle \rho_{1,m}, \rho_{2,m} \rangle = \langle \rho_{1,m} \rangle,$$

by the intersection condition in $\Gamma(\mathcal{P}(m))$. This forces $(\rho_{0,m}\rho_{1,m})^k = 1$, for each m, so that $\alpha = 1$. The values for $\overline{p}, \overline{q}$ follow from Lemma 5.4. □

We conclude this section with a look at mixing the regular facets of a polytope.

Lemma 5.16. *Suppose $\mathcal{Q}$ is an n-polytope whose facets are all regular. Then all ridges in $\mathcal{Q}$ (faces of rank $n-2$) are isomorphic regular polytopes.*

Proof. Suppose F_{n-2}, G_{n-2} are two faces of rank $n-2$ in $\mathcal{Q}$. By 'ridges' we more accurately mean the corresponding sections F_{n-2}/Q_{-1} and G_{n-2}/Q_{-1}. Flags in such sections can be extended to flags of $\mathcal{Q}$, say $\Phi = [F_0, \ldots, F_{n-2}, F_{n-1}]$ and $\Psi = [G_0, \ldots, G_{n-2}, G_{n-1}]$, again suppressing improper faces. Let $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_{k-1}, \Phi_k = \Psi$ be a chain of consecutively adjacent flags. Now suppose $\Phi_{j+1} = \Phi_j^t$. If $0 \leqslant t \leqslant n-2$, then Φ_{j+1} and Φ_j share the same (regular!) face, say H_{n-1} of rank $n-1$. Thus the two faces of rank $n-2$ in these flags are isomorphic, being facets of H_{n-1}. On the other hand, if $t = n-1$, then the two faces of rank $n-2$ are identical, being common to the two flags. It follows at once by induction that F_{n-2} and G_{n-2} are isomorphic. $\square$

Corollary 5.17. *Let $\mathcal{Q}$ be an n-polytope with the property that all its facets (or dually, its vertex-figures) are regular. Then the mix of all regular polytopes occurring as facets (or vertex-figures) of $\mathcal{Q}$ is a regular $(n-1)$-polytope $\mathcal{P}$.*

Proof. Consider the case that $\mathcal{Q}$ has regular facets. By Lemma 5.16, these regular facets in turn have their own isomorphic facets (namely, the ridges of $\mathcal{Q}$). Now we can apply Theorem 5.12 (with $n-1$ instead of n). $\square$

Remark 5.18. Given an at most countable family of regular $(n-1)$-polytopes with isomorphic facets, we can conversely ask whether these polytopes are, up to isomorphism, precisely the facets of some n-polytope. See [20] for some results in this direction.

A somewhat related problem is to determine the mix of the monodromy groups of *all* finite polyhedra (taken over isomorphism classes, of course). Obviously we get a group of type $\{\infty, \infty\}$, but must this be the universal Coxeter group $U := [\infty, \infty]$? In a conversation with one of us, Steve Wilson has explained that U must indeed arise, even if one mixes just the regular polyhedral maps. On the other hand, we may still ask whether U results if we mix, say, over all *convex* polyhedra.

6. Polytopality of monodromy groups

We have observed in Proposition 3.16 that an n-polytope $\mathcal{Q}$ whose monodromy group $\mathrm{Mon}(\mathcal{Q})$ is a string C-group must have an essentially unique minimal regular cover $\mathcal{R}$. Since $\Gamma(\mathcal{R}) \simeq \mathrm{Mon}(\mathcal{Q})$, this regular cover is certainly finite when $\mathcal{Q}$ itself is finite. It is therefore interesting to establish criteria which guarantee that $\mathrm{Mon}(\mathcal{Q})$ is a string C-group. Or could this always be the case? Well, 'no' in fact; but because of our next result, the search for counterexamples must begin in rank 4.

Proposition 6.1. *The monodromy group of any polyhedron $\mathcal{Q}$ (not necessarily regular) is a string C-group of type $\{\overline{p}, \overline{q}\}$, where $\overline{p}$ is the least common multiple of all facet sizes in $\mathcal{Q}$ and $\overline{q}$ is the least common multiple of all vertex degrees.*

Proof. Let $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, r_1, r_2 \rangle$, and suppose that $\{\,\{p_m\} : m \in M\}$ is the set of (polygonal) facets of $\mathcal{Q}$. Of course, each $\{p_m\}$ is automatically regular; and $\mathrm{Mon}(\{p_m\}) \simeq \mathbb{D}_{2p_m} = \langle \rho_{0,m}, \rho_{1,m} \rangle$, say, by Theorem 3.9. It follows from Theorem 5.5 and Corollary 5.14 that $\langle r_0, r_1 \rangle$ is a string C-group, namely the dihedral group $\mathbb{D}_{2\overline{p}} \simeq \diamondsuit_{m \in M} \mathbb{D}_{2p_m}$ obtained by mixing over the facets of $\mathcal{Q}$. By mixing over vertex-figures of $\mathcal{Q}$ we similarly get $\langle r_1, r_2 \rangle \simeq \mathbb{D}_{2\overline{q}}$.

Appealing again to [17, Prop. 2E16], we now let $g \in \langle r_0, r_1 \rangle \cap \langle r_1, r_2 \rangle$. We want to show that $g = 1$ or r_1. Consider any flag $\Phi = [V, E, F]$ of $\mathcal{Q}$. Since $g \in \langle r_0, r_1 \rangle$, Φ^g and Φ share the same facet F; similarly, Φ^g and Φ share vertex V. Thus $\Phi^g = \Phi$

or Φ^1. But a look at the polygon $\{\overline{p}\}$ shows that the former case occurs if and only if g is r_0r_1-even, in which case $\Lambda^g = \Lambda$ for *all* flags Λ. Thus $g = 1$ or r_1. ☐

Corollary 6.2. *Any* 3*-polytope* $\mathcal{Q}$ *has an essentially unique minimal, regular cover* $\mathcal{R}$*. Moreover,* $\mathcal{R}$ *is finite if and only if* $\mathcal{Q}$ *is finite.*

Remark 6.3. The monodromy groups and associated regular covers of general maps have appeared frequently in the literature; see [15, 16, 30], for instance.

We now try to extend the above result to polytopes of higher rank.

Theorem 6.4. *Let* $\mathcal{Q}$ *be an* n*-polytope with the property that all its facets (or dually, its vertex-figures) are covered by one particular facet (or vertex-figure) which happens to be regular. Then* $\mathrm{Mon}(\mathcal{Q}) = \langle r_0, \ldots, r_{n-1}\rangle$ *is a string C-group.*

Proof. We adapt the proof of Theorem 5.12. Let $\{F(m) : m \in M\}$ be the facet set of $\mathcal{Q}$, and say the regular facet $F(1)$ covers all facets. (Recall that for simplicity we often write $F(m)$ in place of the section $F(m)/\mathcal{Q}_{-1}$.) Suppose $\Gamma(F(1)) = \langle \rho_0, \ldots, \rho_{n-2}\rangle$; and let $\kappa_m : F(1) \to F(m)$ be a cover. From Theorem 3.9 and Proposition 3.11 we get epimorphisms

$$\overline{\kappa}_m : \langle \rho_0, \ldots, \rho_{n-2}\rangle \to \mathrm{Mon}(F(m)) \; .$$

By Theorem 5.5 and Lemma 5.3 we find that

$$\langle r_0, \ldots, r_{n-2}\rangle \simeq \diamondsuit_{m\in M}\mathrm{Mon}(F(m)) \simeq \langle \rho_0, \ldots, \rho_{n-2}\rangle \; ,$$

which is a string C-group of rank $n-1$. Again we apply [17, Prop. 2E16(b)], so suppose $1 \leqslant k \leqslant n-1$, and let $g \in \langle r_0, \ldots, r_{n-2}\rangle \cap \langle r_k, \ldots, r_{n-1}\rangle$. We must show $g \in \langle r_k, \ldots, r_{n-2}\rangle$. Notice that if Λ is any flag of $\mathcal{Q}$, then Λ and Λ^g share the same j-face, for $j = 0, 1, \ldots, k-1$ or $n-1$.

In particular, suppose Φ is some flag of $\mathcal{Q}$ which contains the special regular facet $F(1)$. Since $\mathcal{Q}$ is strongly flag-connected, there is a sequence of flags $\Phi = \Psi_0, \Psi_1, \ldots, \Psi_s = \Phi^g$ such that $\Psi_j = \Psi_{j-1}^{i_j}$, with $i_j \in \{k, \ldots, n-2\}$ for $j = 1, \ldots, s$. Thus $h = r_{i_1} \cdots r_{i_s} \in \langle r_k, \ldots, r_{n-2}\rangle$ and $\Phi^g = \Phi^h$. Clearly $\Phi^u = \Phi$, where $u = gh^{-1} \in \langle r_0, \ldots, r_{n-2}\rangle$.

Now for any flag Λ of $\mathcal{Q}$, let Λ_* be the flag induced in the facet of Λ. Since $u \in \langle r_0, \ldots, r_{n-2}\rangle$ we have $(\Lambda_*)^u = (\Lambda^u)_*$. Now suppose Λ has facet $F(m)$. Since $F(1)$ is regular, we can assume $(\Phi_*)\kappa_m = \Lambda_*$. Then

$$(\Lambda_*)^u = ((\Phi_*)\kappa_m)^u = (\Phi_*^u)\kappa_m = (\Phi_*)\kappa_m = \Lambda_* \; .$$

(Recall that any cover κ_m commutes with monodromy actions.) On the other hand, u certainly fixes the facet in each flag Λ; we conclude that $u = 1$ and $g = h \in \langle r_k, \ldots, r_{n-2}\rangle$. ☐

Corollary 6.5. *Let* $\mathcal{Q}$ *be an* n*-polytope with the property that all its facets (or dually, its vertex-figures) are isomorphic to a particular regular* $(n-1)$*-polytope. Then* $\mathrm{Mon}(\mathcal{Q})$ *is a string C-group, and* $\mathcal{Q}$ *has an essentially unique minimal regular cover.*

Example 6.6. Any simple or simplicial convex polytope has a unique (and finite) minimal regular cover.

For instance, let $\mathcal{Q}$ be the cyclic 4-polytope with 6 vertices. Then $\mathrm{Mon}(\mathcal{Q})$ is a string C-group of order $2^6 \cdot 3^7$ and the corresponding regular polytope has Schläfli type $\{3, 3, 12\}$. The vertex-figure of this regular cover is a map $\mathcal{M}$ of type $\{3, 12\}$

and listed as $\{3, 12\} * 1296$(b) in Hartley's Atlas [11]. Each (convex) vertex-figure in $\mathcal{Q}$ itself is a triangular bipyramid; thus some edges of $\mathcal{Q}$ are surrounded by 3 tetrahedra and others by 4, thereby accounting for the 12 in the Schläfli symbol.

Various other geometric conditions guarantee that $\mathrm{Mon}(\mathcal{Q})$ is a string C-group. Here is one last result of this sort:

Theorem 6.7. *Let $\mathcal{Q}$ be an n-polytope with regular facets such that all vertex-figures are facet-transitive. Then* $\mathrm{Mon}(\mathcal{Q})$ *is a string C-group.*

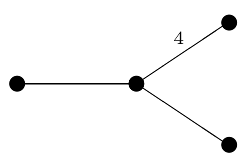
Proof. We use much the same notation and approach as in the proof of Theorem 6.4. First, from Corollaries 5.6 and 5.17 we conclude that $\langle r_0, \ldots, r_{n-2}\rangle$ is a string C-group. Next we fix some flag Φ of $\mathcal{Q}$, and in similar fashion obtain an element $u = gh^{-1} \in \langle r_0, \ldots, r_{n-2}\rangle$ satisfying $\Phi^u = \Phi$.

Since facets are regular, we conclude from Lemma 3.4 that $\Lambda^u = \Lambda$ for any flag Λ having the same $(n-1)$-face as Φ. On the other hand, $u \in \langle r_k, \ldots, r_{n-1}\rangle$ must (globally) fix every vertex-figure of $\mathcal{Q}$. Again by Lemma 3.4 we conclude that u fixes any flag containing the vertex in Φ. A standard inductive argument on the length of flag chains allows us to conclude that $u = 1$. $\square$

The several results above certainly direct our search for a polytope $\mathcal{Q}$ for which $\mathrm{Mon}(\mathcal{Q})$ is not a string C-group: the rank must be at least 4, the facets (or vertex-figures) cannot be isomorphic and regular, etc. Eventually, we did find such a polytope, which we have called the *Tomotope* $\mathcal{T}$, as a small nod to the contributions of our colleague Tomaž Pisanski. We describe the properties of $\mathcal{T}$ in considerable detail in [18]. A brief summary follows.

Example 6.8. Let $\mathcal{U}$ be the familiar face-to-face tiling of Euclidean space $\mathbb{R}^3$ by regular tetrahedra and octahedra, two of each arranged alternately around every edge of the tiling. (We may take the centres of the octahedra to be points of $\mathbb{Z}^3$ with odd coordinate sum.) In fact, $\mathcal{U}$ is an infinite semiregular 4-polytope, and $\Gamma(\mathcal{U}) \simeq \tilde{B}_3$, the affine Coxeter group with diagram

(6.1)



(By definition, a *semiregular* polytope has regular facets and a vertex-transitive automorphism group; see [20].)

Now slice out a $2 \times 2 \times 2$-cube containing eight tetrahedra, a core octahedron and three other octahedra, each split into four identical but non-regular tetrahedra. The latter pieces fit into the twelve 'dimples' on the surface of the *stella octangula* shown in Figure 4.

Next identify opposite square faces of the $2 \times 2 \times 2$-cube in toroidal fashion, so that the eight original vertices of the cube become one. Finally, we reflect in the centre of the core octahedron and so identify antipodal faces of all ranks. The resulting 4-polytope $\mathcal{T}$ is clearly a quotient of $\mathcal{U}$. One readily counts 4 vertices (labelled $1, 2, 3, 4$); $12 = 4 \cdot 6/2$ edges; $16 = 4 \cdot 4$ triangular 2-faces; 4 tetrahedral facets; and 4 *hemioctahedral* facets. Thus $\mathcal{T}$ has $192 = 4 \cdot (24 + 24)$ flags. Each vertex-figure is a hemicuboctahedron.

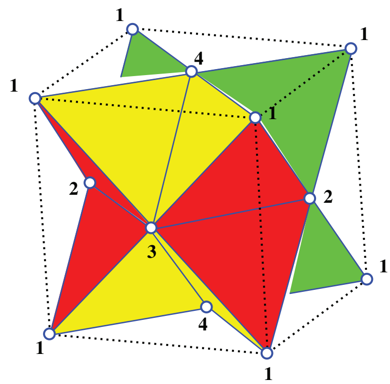


FIGURE 4. The Tomotope.

In fact, $\mathcal{T}$ remains semiregular. Its automorphism group $\Gamma(\mathcal{T})$ has order 96 and is a quotient of $\Gamma(\mathcal{U})$. On the other hand, a calculation in GAP shows that $\mathrm{Mon}(\mathcal{T})$ is an sggi of order 18432 and type $\{3, 12, 4\}$, which *fails the intersection condition.* Moreover, the conclusion of Proposition 3.16 also fails for $\mathcal{T}$; in [18, Th. 5.9] we demonstrate that $\mathcal{T}$ has infinitely many minimal, mutually non-isomorphic regular covers.

Although polytopes such as $\mathcal{T}$ may seem very peculiar, it is likely that $\mathrm{Mon}(\mathcal{Q})$ fails to be a string C-group for almost all abstract polytopes $\mathcal{Q}$ of higher rank.

7. Chiral polytopes

From one intuitive point of view, regular polytopes have maximal 'reflection' symmetry. Chiral polytopes, on the other hand, have maximal 'rotational' symmetry, while failing to be regular. To start thinking about chirality, recall that for any regular n-polytope $\mathcal{P}$, the *rotations* $\sigma_j := \rho_{j-1}\rho_j$, $1 \leqslant j \leqslant n-1$, generate a subgroup $\Gamma(\mathcal{P})^+$ having index 1 or 2 in $\Gamma(\mathcal{P})$. In the latter case, $\mathcal{P}$ is said to be *directly regular*, and certain properties of the σ_j's lead, in a natural way, to the theory of chiral polytopes. We refer to [26, 27] for a deeper look at this very interesting class of symmetric polytopes.

Definition 7.1. Suppose that $\mathcal{P}$ has rank $n \geqslant 3$. Then $\mathcal{P}$ is *chiral* if it is not regular, but if for some *base flag* $\Phi := [F_{-1}, F_0, \ldots, F_n]$ there exist automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ of $\mathcal{P}$ such that σ_j fixes all faces in $\Phi \backslash \{F_{j-1}, F_j\}$ and cyclically permutes consecutive j-faces of $\mathcal{P}$ in the rank 2 section F_{j+1}/F_{j-2} of $\mathcal{P}$.

The automorphism group of $\mathcal{P}$ now has two flag orbits, with adjacent flags always in different orbits, so that the flag graph $\mathcal{F}(\mathcal{P})$ is bipartite. Let us say that the orbit of Φ contains white flags, while those in the orbit of Φ^0 are black.

Actually, the σ_j's can be chosen so that if F'_j denotes the j-face of $\mathcal{P}$ with $F_{j-1} < F'_j < F_{j+1}$ and $F'_j \neq F_j$, then $F_j\sigma_j = F'_j$ for $j = 1, \ldots, n-1$. These σ_j's generate $\Gamma(\mathcal{P})$ and furthermore satisfy at least relations of the following form:

$$
\begin{gathered}
\sigma_1^{p_1} = \sigma_2^{p_2} = \cdots = \sigma_{n-1}^{p_{n-1}} = 1, \\
(\sigma_i \cdots \sigma_j)^2 = 1,\ 1 \leqslant i < j \leqslant n-1,
\end{gathered}
\tag{7.1}
$$

for certain $2 \leqslant p_1, \ldots, p_{n-1} \leqslant \infty$. Once more $\mathcal{P}$ has Schläfli type $\{p_1, p_2, \ldots, p_{n-1}\}$. The specified generators also satisfy an intersection condition analogous to but a little more technical than that for the regular case in (2.2).

It is conversely possible to reconstruct a chiral polytope from a suitable group [26]. If $\Lambda = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \rangle$ satisfies (7.1) and the appropriate intersection condition, then there exists a chiral (or directly regular) n-polytope $\mathcal{P}(\Lambda)$ of type $\{p_1, p_2, \ldots, p_{n-1}\}$ with $\Gamma(\mathcal{P}(\Lambda)) \simeq \Lambda$ (or $\Gamma^+(\mathcal{P}(\Lambda)) \simeq \Lambda$). The directly regular case occurs if and only if Λ admits an involutory automorphism ρ such that $(\sigma_1)\rho = \sigma_1^{-1}$, $(\sigma_2)\rho = \sigma_1^2\sigma_2$, and $(\sigma_j)\rho = \sigma_j$ for $3 \leqslant j \leqslant n-1$.

Each (isomorphism type of) chiral polytope gives rise to two *enantiomorphic* chiral polytopes: if one of these, say $\mathcal{P}$, is associated with the base flag Φ, then its enantiomorphic 'twin' $\mathcal{P}^-$ is associated to an adjacent flag, say $\Phi^0 := (\Phi \setminus \{F_0\}) \cup \{F_0'\}$. As a result of this change, $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ must be replaced by new generators $\overline{\sigma}_1 := \sigma_1^{-1}, \overline{\sigma}_2 := \sigma_1^2\sigma_2$ and $\overline{\sigma}_j := \sigma_j$, for $3 \leqslant j \leqslant n-1$ (cf. [27, Section 3]). Here we borrow an idea from [2]: if ω is any word in the σ_j's and their inverses, then $\overline{\omega}$ is the new word obtained from ω by replacing each σ_k by $\overline{\sigma}_k$. For example, if $\omega = \sigma_2\sigma_3$, then

$$\overline{\omega} = \sigma_1^2\sigma_2\sigma_3 = \sigma_1(\sigma_1\sigma_2\sigma_3)^{-1} = \sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_1\sigma_2\sigma_3\sigma_1^{-1}, \tag{7.2}$$

where we have invoked (7.1) a few times. Hence this particular $\overline{\omega}$ has period 2, being a conjugate of ω. (We will never have to worry whether 'barring' is well defined under refactorization of a group element in $\Gamma(\mathcal{P})$.) We emphasize that $\mathcal{P}$ and $\mathcal{P}^-$ are isomorphic polytopes, rather as a left hand and right hand are isometric; likewise $\Gamma(\mathcal{P})$ and $\Gamma(\mathcal{P}^-)$ are differently generated versions of the same group.

One way to better understand all this is to employ the monodromy group (but compare the approach in [27, Sect. 3]). The first task in our approach is to mix the automorphism groups of enantiomorphic twins. Suppose then that $\mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$ is the monodromy group of a chiral n-polytope $\mathcal{P}$; and let $s_j = r_{j-1}r_j$, for $1 \leqslant j \leqslant n-1$. Since $\mathrm{Mon}(\mathcal{P})$ is an sggi, it is easy to see that $\mathrm{Mon}^+(\mathcal{P}) := \langle s_1, \ldots, s_{n-1} \rangle$ has index at most 2 in $\mathrm{Mon}(\mathcal{P})$.

Now check directly that $(\Phi)\sigma_j = \Phi^{s_j^{-1}}$. Since the actions on Φ of σ_j and s_j commute, we also get $(\Phi)\sigma_j^{-1} = \Phi^{s_j}$, so altogether we have $(\Phi)\sigma_j{}^{\epsilon} = \Phi^{s_j^{-\epsilon}}$, $1 \leqslant j \leqslant n-1$, $\epsilon = \pm 1$. Any $z \in \mathrm{Mon}^+(\mathcal{P})$ can now be written as, say, $z = t_1 \cdots t_m$, where all $t_j \in \{s_1^{\epsilon}, \ldots, s_{n-1}^{\epsilon}\}$. Let $(z)\varphi := \tau_1 \cdots \tau_m$ be the corresponding element of $\Gamma(\mathcal{P})$, for appropriate $\tau_j \in \{\sigma_1^{\epsilon}, \ldots, \sigma_{n-1}^{\epsilon}\}$. It follows at once from Lemma 3.7 that

$$\Phi^{z^{-1}} = (\Phi)(z)\varphi\,. \tag{7.3}$$

(We will soon see that φ is well defined.)

In similar fashion we define $\overline{s}_j := r_0 s_j r_0$, so that $\overline{s}_1 = s_1^{-1}, \overline{s}_2 = s_1^2 s_2, \overline{s}_k = s_k$, for $3 \leqslant k \leqslant n-1$. Then all $(\Phi)\overline{\sigma}_j{}^{\epsilon} = \Phi^{\overline{s}_j^{-\epsilon}}$. But we can also write $z \in \mathrm{Mon}^+(\mathcal{P})$ as $z = \overline{t}_1 \cdots \overline{t}_m$, with $\overline{t}_j \in \{\overline{s}_1{}^{\epsilon}, \ldots, \overline{s}_{n-1}{}^{\epsilon}\}$, and then set $(z)\psi := \overline{\tau}_1 \cdots \overline{\tau}_m$. Now we obtain

$$\Phi^{z^{-1}} = (\Phi)(z)\psi\,. \tag{7.4}$$

It is now clear that $\mathrm{Mon}^+(\mathcal{P})$ has index exactly 2 in $\mathrm{Mon}(\mathcal{P})$. For if $r_0 \in \mathrm{Mon}^+(\mathcal{P})$, then writing $r_0 = t_1 \cdots t_m$, we get from (7.3) that $\Phi^0 = \Phi^{r_0} = (\Phi)\tau_1 \cdots \tau_m$, contradicting the fact that Φ and Φ^0 lie in different $\Gamma(\mathcal{P})$-orbits.

Theorem 7.2. *Suppose the chiral n-polytope $\mathcal{P}$ has automorphism group*

$$\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{n-1} \rangle,$$

so that the enantiomorphic twin $\mathcal{P}^-$ has group

$$\Gamma(\mathcal{P}^-) = \langle \overline{\sigma}_1, \overline{\sigma}_2, \overline{\sigma}_3, \ldots, \overline{\sigma}_{n-1} \rangle = \langle \sigma_1^{-1}, \sigma_1^2\sigma_2, \sigma_3, \ldots, \sigma_{n-1} \rangle.$$

Then

$$\mathrm{Mon}(\mathcal{P}) \simeq (\Gamma(\mathcal{P}) \diamondsuit \Gamma(\mathcal{P}^-)) \rtimes C_2 \simeq \mathrm{Mon}(\mathcal{P}^-)\ . \tag{7.5}$$

Proof. Recall that the generators σ_j for $\Gamma(\mathcal{P})$ correspond to the white base flag Φ, whereas the generators $\overline{\sigma}_j$ for $\Gamma(\mathcal{P}^-)$ correspond to the black base flag Φ^0.

Since $\mathcal{P}$ and $\mathcal{P}^-$ are isomorphic, we need only consider $\mathrm{Mon}(\mathcal{P})$. For brevity write $H := \Gamma(\mathcal{P}) \diamondsuit \Gamma(\mathcal{P}^-)$, with generators $\beta_j := (\sigma_j, \overline{\sigma}_j)$, $1 \leqslant j \leqslant n-1$. First, referring to the notation just above, we set

$$\begin{array}{rcl} \eta : \mathrm{Mon}^+(\mathcal{P}) & \to & H, \\ x & \mapsto & ((x)\varphi, (r_0 x r_0)\psi)\ . \end{array} \tag{7.6}$$

This mapping is well defined: for if $x = t_1 \cdots t_m = 1$, then also $r_0 x r_0 = \overline{t}_1 \cdots \overline{t}_m = 1$; and $(x)\eta = (1,1)$ by (7.3) and (7.4). Since $(s_j)\eta = \beta_j$, it is clear that η is an epimorphism.

Suppose then that $x \in \ker(\eta)$. For any white flag Ψ, there exists $g \in \mathrm{Mon}^+(\mathcal{P})$ such that $\Psi = \Phi^g$. Since $gxg^{-1} \in \ker(\eta)$, we conclude from (7.3) that $\Phi = (\Phi)(gxg^{-1})\varphi = \Phi^{gx^{-1}g^{-1}}$. Thus $\Psi^{x^{-1}} = \Psi$, so x fixes all white flags. Similarly any black flag Ψ can be written as $\Psi = \Phi^{r_0 g}$, for suitable $g \in \mathrm{Mon}^+(\mathcal{P})$. Since $r_0 gxg^{-1} r_0 \in \ker(\eta)$, we further conclude that $\Psi^{x^{-1}} = \Psi$. Thus x fixes *all* flags and η is an ismorphism.

The map

$$\begin{array}{rcl} \rho : H & \to & H, \\ (\alpha, \gamma) & \mapsto & (\gamma, \alpha) \end{array}$$

is clearly an automorphism of period 2, so long as it is well defined. But this follows from the fact that $(\beta_1)\rho = \beta_1^{-1}, (\beta_2)\rho = \beta_1^2\beta_2, (\beta_k)\rho = \beta_k$, for $3 \leqslant k \leqslant n-1$. Furthermore, since r_0 transforms generator s_j of $\mathrm{Mon}^+(\mathcal{P})$ in the same way that ρ transforms β_j, we can extend η to an isomorphism

$$\widehat{\eta} : \mathrm{Mon}(\mathcal{P}) \to (\Gamma(\mathcal{P}) \diamondsuit \Gamma(\mathcal{P}^-)) \rtimes C_2\ ,$$

where the factor $C_2 = \langle \rho \rangle$. □

Remark 7.3. The *chirality group* $X(\mathcal{P})$ for a chiral or directly regular polytope $\mathcal{P}$ provides some measure of how $\mathcal{P}$ deviates from regularity. We refer to [1, 2] for details and applications, as well as to [5, Ch. 3] for extensions of the idea to wider classes of polytopes. It is also clear from [5] that $X(\mathcal{P})$ is isomorphic to the kernel of the natural projection from $\Gamma(\mathcal{P}) \diamondsuit \Gamma(\mathcal{P}^-)$ to either component. Referring to (7.6) and (7.3), we may therefore set

$$X(\mathcal{P}) = \{x \in \mathrm{Mon}^+(\mathcal{P}) : (x)\varphi = 1\} = \{x \in \mathrm{Mon}^+(\mathcal{P}) : \Phi^{x^{-1}} = \Phi\}.$$

On the other hand we also conclude from (7.6) that $X(\mathcal{P})$ is isomorphic to

$$\{(r_0 x r_0)\psi\ :\ x \in \mathrm{Mon}^+(\mathcal{P}), (x)\varphi = 1\},$$

which is a normal subgroup of $\Gamma(\mathcal{P}^-)$. (The corresponding quotient of $\Gamma(\mathcal{P}^-)$ by this normal subgroup is the *comix* $\Gamma(\mathcal{P}) \,\square\, \Gamma(\mathcal{P}^-)$ described in [5].)

We now have

Corollary 7.4. *Let $\mathcal{P}$ be a chiral or directly regular n-polytope. Then the chirality group $X(\mathcal{P})$ is isomorphic to the stabilizer of, say, the white flag Φ under the action of* $\mathrm{Mon}^+(\mathcal{P})$. *$X(\mathcal{P})$ is also isomorphic to a normal subgroup of* $\Gamma(\mathcal{P}^-)$. *Analogous interpretations of $X(\mathcal{P})$ hold after swapping the roles of $\mathcal{P}$ and $\mathcal{P}^-$.*

If, as for the Tomotope, the monodromy group of a chiral polytope fails to be a string C-group, then such peculiar behaviour already occurs within the natural mix described in (7.5). Now recall from [26, Props. 4 and 9] that all facets (likewise all vertex-figures) of a chiral n-polytope are isomorphic and are either themselves chiral or directly regular. All faces of rank $n-2$ and all cofaces of edges are, in fact, regular. Certainly, we may conclude at once from Proposition 6.1 and Corollary 6.5 that many chiral polytopes do behave 'nicely':

Corollary 7.5. *The monodromy group of a chiral polytope $\mathcal{P}$ with regular facets (or with regular vertex-figures) is a string C-group. In this case $\mathcal{P}$ and $\mathcal{P}^-$ have a common and essentially unique minimal regular cover, as constructed from the mix described in* Theorem 7.2. *In particular, any chiral polyhedron has such a unique minimal regular cover.*

Our search for a chiral n-polytope $\mathcal{P}$ for which $\mathrm{Mon}(\mathcal{P})$ is not a string C-group is constrained in much the same way as our search for the Tomotope: the rank $n \geqslant 4$, and the facets and vertex-figures must themselves be chiral $(n-1)$-polytopes. This already rules out many instances on the grounds of the Schläfli symbol alone. We did find an example, but we didn't have to invent it! (This polytope also provides an affirmative answer to Problem 3 in [25].)

Example 7.6. The long-standing search for the first finite chiral polytopes of rank $n > 4$ ended in [4]. Several new examples were described there and were discovered with the aid of efficient new algorithms for assessing 'low index subgroups' of the rotation subgroups of suitable infinite Coxeter groups. In Section 5 of that paper the authors describe in detail an interesting chiral 5-polytope $\mathcal{P}$, with $\Gamma(\mathcal{P}) \simeq S_6$ generated by

$$\begin{array}{rcl} \sigma_1 & = & (1,2,3), \\ \sigma_2 & = & (1,3,2,4), \\ \sigma_3 & = & (1,5,4,3), \\ \sigma_4 & = & (1,2,3)(4,6,5). \end{array}$$

This example also appears in a recent atlas of chiral polytopes arising from almost simple groups [13]. See also [24] for a contrasting approach to the existence of chiral polytopes of higher rank.

Evidently $\mathcal{P}$ has Schläfli type $\{3,4,4,3\}$. It is *improperly* self-dual with 6 vertices, 15 edges, 40 triangles, 15 octahedral 3-faces and 6 chiral facets, each in fact a copy of the universal 4-polytope $\mathcal{K}$ of type $\{\,\{3,4\}\,,\,\{4,4\}_{(1,2)}\,\}$. (The vertex-figures are dual to these.) The middle section is the simplest chiral toroidal polyhedron $\mathcal{M} = \{4,4\}_{(1,2)}$.

This suggests that we can erect $\mathcal{P}$ on the complete graph K_6. However, we must fill in two triangles on each triple of points and allocate the octahedra and chiral facets in careful fashion. (Of course, such details are implicit in the selection of generators σ_j above.)

Our contribution is brief, though we think interesting. The monodromy group $\mathrm{Mon}(\mathcal{P}) = \langle r_0, r_1, r_2, r_3, r_4 \rangle$ has order $518400 = 720^2$, and also has type $\{3, 4, 4, 3\}$, since $\mathcal{P}$ is equivelar. However, the intersection condition (2.2) does fail. To see this we begin with the epimorphism

$$\begin{array}{rcl}\Gamma(\mathcal{P}) \times \Gamma(\mathcal{P}^-) & \to & \{\pm 1\} \times \{\pm 1\}, \\ (\alpha, \beta) & \mapsto & (\mathrm{sgn}(\alpha), \mathrm{sgn}(\beta))\ .\end{array}$$

For convenience let $\Gamma(\mathcal{P}) \perp \Gamma(\mathcal{P}^-)$ denote the pre-image of $\{\pm(1, 1)\}$. This group has index 2 in the direct product and consists of all pairs (α, β) for which $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)$. Since the permutations σ_j and $\overline{\sigma}_j$ have the same cycle structure, we get

$$\Gamma(\mathcal{P}) \perp \Gamma(\mathcal{P}^-) \leqslant \Gamma(\mathcal{P}) \times \Gamma(\mathcal{P}^-).$$

Similar statements hold for $\Gamma(\mathcal{K})$ and $\Gamma(\mathcal{M})$, each of which contains odd permutations.

Next we examine the chiral map $\mathcal{M}$; it is the quotient of the tessellation $\{4, 4\}$ by the translation group T generated by vectors $\langle 1, 2 \rangle$ and $\langle -2, 1 \rangle$. Thus

$$\Gamma(\mathcal{M}) \simeq \Gamma^+(\{4, 4\})/T$$

has order $20 = 4(1^2 + 2^2)$. From Theorems 5.5 and 7.2 we have an isomorphism

$$\begin{array}{rcl}\mathrm{Mon}^+(\mathcal{M}) & \overset{\eta}{\to} & \Gamma(\mathcal{M}) \diamondsuit \Gamma(\mathcal{M}^-), \\ s_j & \mapsto & (\sigma_j, \overline{\sigma}_j), \quad j = 2, 3.\end{array}$$

Now a careful look at $\{4, 4\}$ shows that the translation with vector $\langle 1, 2 \rangle$ acts trivially on $\mathcal{M}$ but in the same way as $t = s_2^2 s_3^2 s_2^{-1} s_3 \in \mathrm{Mon}^+(\mathcal{M})$. Since $\tau := (t)\eta = (\,(\,)\,,\,(1, 4, 5, 2, 3)\,)$, the chirality groups $X(\mathcal{M})$, $X(\mathcal{F})$ and $X(\mathcal{P})$ all contain the element τ of order 5. It is now easy to see that $\Gamma(\mathcal{M}) \diamondsuit \Gamma(\mathcal{M}^-)$ has order 100 and is isomorphic to the rotation group of the regular toroidal map $\{4, 4\}_{(5,0)}$, which covers both $\mathcal{M}$ and $\mathcal{M}^-$. Thus $\Gamma(\mathcal{M}) \diamondsuit \Gamma(\mathcal{M}^-)$ actually has index 2 in $\Gamma(\mathcal{M}) \perp \Gamma(\mathcal{M}^-)$, so there exists

$$\lambda = (\alpha, \beta) \in \Gamma(\mathcal{M}) \perp \Gamma(\mathcal{M}^-) \subseteq \Gamma(\mathcal{K}) \perp \Gamma(\mathcal{K}^-),$$

but with $\lambda \notin \Gamma(\mathcal{M}) \diamondsuit \Gamma(\mathcal{M}^-)$.

Now consider the facet $\mathcal{K}$. The chirality group $X(\mathcal{K})$ contains τ, but by Corollary 7.4 is isomorphic to a normal subgroup of $\Gamma(\mathcal{K}) \simeq S_5$, so $X(\mathcal{K}) \simeq S_5$ or A_5. This forces the left-most inclusion in

$$A_5 \times A_5 \subseteq \Gamma(\mathcal{K}) \diamondsuit \Gamma(\mathcal{K}^-) \subseteq \Gamma(\mathcal{K}) \perp \Gamma(\mathcal{K}^-).$$

But $\Gamma(\mathcal{K})$, $\Gamma(\mathcal{K}^-)$ contain odd permutations, so the right-most inclusion is equality and $\lambda \in \Gamma(\mathcal{K}) \diamondsuit \Gamma(\mathcal{K}^-)$. A similar look at the vertex-figure $\mathcal{K}^*$ gives $\lambda \in \Gamma(\mathcal{K}^*) \diamondsuit \Gamma(\mathcal{K}^{*-})$. The element λ therefore forces the intersection condition (2.2) to fail for $\mathrm{Mon}(\mathcal{P})$.

As a byproduct of these calculations and Remark 7.3, we note that the chirality group $X(\mathcal{P}) \simeq A_6$ (see [2, p. 1280]).

At this time we do not understand the regular polytopal covers of $\mathcal{P}$ (let alone those for chiral polytopes of higher rank in general). However, Egon Schulte has pointed out that $\mathcal{P}$ at least has some finite regular cover $\mathcal{R}$. We begin with the crystallographic string Coxeter group $W = [3, 4, 4, 3]$. Following [19] we reduce W

modulo 5 to obtain $W^5 \simeq O(5,5,0)$, the general orthogonal group of dimension 5 over $GF(5)$. Using [19, Th. 3.3(b)], we find that W^5 is a string C-group of order $2^8 \cdot 3^2 \cdot 5^4 \cdot 13$ and Schläfli type $\{3,4,4,3\}$. Finally, we let $G = \mathrm{Mon}(\mathcal{P})\diamondsuit W^5$. It follows from an argument detailed in [21] that G is the automorphism group of a regular 5-polytope $\mathcal{R}$, still of type $\{3,4,4,3\}$, which covers the chiral polytope $\mathcal{P}$ (and its twin $\mathcal{P}^-$). It is amusing to note that $\mathcal{R}$ has $2426112000000 = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 13$ flags, compared to a measly 1440 for $\mathcal{P}$ itself!

We intend to pursue these ideas in a sequel to this paper.

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Department of Mathematics, University of New Brunswick, Fredericton, New Brunswick, Canada

E-mail address: bmonson@unb.ca

Instituto de Matematicas, UNAM Morelia, Morelia, Michoacán, México

E-mail address: pellicer@matmor.unam.mx

Department of Mathematics, University of Alaska Fairbanks, Fairbanks, Alaska 99775

E-mail address: giwilliams@alaska.edu