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MÖBIUS ISOTROPIC SUBMANIFOLDS IN S^n

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Abstract. Let $x : \boldsymbol{M}^m \to \boldsymbol{S}^n$ be a submanifold in the n-dimensional sphere $\boldsymbol{S}^n$ without umbilics. Two basic invariants of x under the Möbius transformation group in $\boldsymbol{S}^n$ are a 1-form Φ called the Möbius form and a symmetric $(0, 2)$ tensor $\mathbf{A}$ called the Blaschke tensor. x is said to be Möbius isotropic in $\boldsymbol{S}^n$ if $\Phi \equiv 0$ and $\mathbf{A} = \lambda dx \cdot dx$ for some smooth function λ. An interesting property for a Möbius isotropic submanifold is that its conformal Gauss map is harmonic. The main result in this paper is the classification of Möbius isotropic submanifolds in $\boldsymbol{S}^n$. We show that (i) if $\lambda > 0$, then x is Möbius equivalent to a minimal submanifold with constant scalar curvature in $\boldsymbol{S}^n$; (ii) if $\lambda = 0$, then x is Möbius equivalent to the pre-image of a stereographic projection of a minimal submanifold with constant scalar curvature in the n-dimensional Euclidean space $\boldsymbol{R}^n$; (iii) if $\lambda < 0$, then x is Möbius equivalent to the image of the standard conformal map $\tau : \boldsymbol{H}^n \to \boldsymbol{S}^n_+$ of a minimal submanifold with constant scalar curvature in the n-dimensional hyperbolic space $\boldsymbol{H}^n$. This result shows that one can use Möbius differential geometry to unify the three different classes of minimal submanifolds with constant scalar curvature in $\boldsymbol{S}^n$, $\boldsymbol{R}^n$ and $\boldsymbol{H}^n$.

1. Introduction. Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be an m-dimensional submanifold in the n-dimensional sphere $\boldsymbol{S}^n$ without umbilics. Let $\{e_i\}$ be a local orthonormal basis for the first fundamental form $I = dx \cdot dx$ with dual basis $\{\theta_i\}$. Let $II = \sum_{ij\alpha} h^\alpha_{ij}\theta_i\theta_j e_\alpha$ be the second fundamental form of x and $H = \sum_\alpha H^\alpha e_\alpha$ the mean curvature vector of x, where $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of x. We define $\rho^2 = m/(m-1)\cdot(\|II\|^2 - m\|H\|^2)$, where $\| \ \|$ is the norm with respect to the induced metric $dx \cdot dx$ on $\boldsymbol{M}$. Then two basic Möbius invariants of x, the Möbius form $\Phi = \sum_i C^\alpha_i \theta_i e_\alpha$ and the Blaschke tensor $\mathbf{A} = \rho^2 \sum_{ij} A_{ij}\theta_i\theta_j$, are defined by (cf. [W])

$$C^\alpha_i = -\rho^{-2}\left(H^\alpha{}_{,i} + \sum_j (h^\alpha_{ij} - H^\alpha \delta_{ij})e_j(\log\rho)\right), \tag{1.1}$$

$$\begin{aligned} A_{ij} = &-\rho^{-2}\left(\mathrm{Hess}_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - \sum_\alpha H^\alpha h^\alpha_{ij}\right) \\ &- \frac{1}{2}\rho^{-2}(\|\nabla \log\rho\|^2 - 1 + \|H\|^2)\delta_{ij}\,, \end{aligned} \tag{1.2}$$

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where Hess_{ij} and ∇ are the Hessian-matrix and the gradient with respect to $dx \cdot dx$. A submanifold $x : \boldsymbol{M} \to \boldsymbol{S}^n$ is called Möbius isotropic if $\Phi \equiv 0$ and $\mathbf{A} = \lambda dx \cdot dx$ for some function λ.

Let $\boldsymbol{H}^n$ be the n-dimensional hyperbolic space defined by

$$\boldsymbol{H}^n = \{(y_0, y_1, \dots, y_n) \mid -y_0^2 + y_1^2 + \cdots y_n^2 = -1,\ y_0 > 0\}.$$

Let $\boldsymbol{S}^n_+$ be the hemisphere in $\boldsymbol{S}^n$ whose first coordinate is positive. Let $\sigma : \boldsymbol{R}^n \to \boldsymbol{S}^n \setminus \{(-1, 0)\}$ and $\tau : \boldsymbol{H}^n \to \boldsymbol{S}^n_+$ be the following conformal diffeomorphisms:

$$\sigma(u) = \left(\frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2}\right), \quad u \in \boldsymbol{R}^n, \tag{1.3}$$

$$\tau(y) = \left(\frac{1}{y_0}, \frac{y_1}{y_0}\right), \quad y_0 > 0, \quad -y_0^2 + y_1 \cdot y_1 = -1, \quad y_1 \in \boldsymbol{R}^n. \tag{1.4}$$

Then we can state our main result as follows:

CLASSIFICATION THEOREM. *Any Möbius isotropic submanifold in $\boldsymbol{S}^n$ is Möbius equivalent to one of the following Möbius isotropic submanifolds*:

(i) *minimal submanifolds with constant scalar curvature in $\boldsymbol{S}^n$*;

(ii) *the images of σ of minimal submanifolds with constant scalar curvature in $\boldsymbol{R}^n$*;

(iii) *the images of τ of minimal submanifolds with constant scalar curvature in $\boldsymbol{H}^n$*.

This paper is organized as follows. In Section 2 we give Möbius invariants and structure equations for submanifolds in $\boldsymbol{S}^n$. In Section 3 we show that the conformal Gauss map of an isotropic submanifold in $\boldsymbol{S}^n$ is harmonic. In Section 4 we give conformal invariants for submanifolds in $\boldsymbol{R}^n$ and $\boldsymbol{H}^n$ and relate them to the Möbius invariants of submanifolds in $\boldsymbol{S}^n$. Using these relations we show that all submanifolds in (i), (ii) and (iii) of the classification theorem are Möbius isotropic submanifolds . Then in Section 5 we prove the classification theorem for Möbius isotropic submanifolds.

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2. Möbius invariants for submanifolds in $\boldsymbol{S}^n$. In this section we define Möbius invariants and recall structure equations for submanifolds in $\boldsymbol{S}^n$. For more detail we refer to [W].

Let $\boldsymbol{R}^{n+2}_1$ be the Lorentzian space with inner product

$$\langle x, w\rangle = -x_0 w_0 + x_1 w_1 + \cdots + x_{n+1} w_{n+1}, \tag{2.1}$$

where $x = (x_0, x_1, \dots, x_{n+1})$ and $w = (w_0, w_1, \dots, w_{n+1})$. Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a m-dimensional submanifold of $\boldsymbol{S}^n$ without umbilics. We define the Möbius position vector $Y : \boldsymbol{M} \to \boldsymbol{R}^{n+2}_1$ of x by

$$Y = \rho(1, x) = (\rho, \rho x), \quad \rho^2 = m/(m-1) \cdot (\|II\|^2 - m\|H\|^2) > 0. \tag{2.2}$$

Then we have the following

THEOREM 2.1 ([W]). *Two submanifolds $x, \tilde{x} : \boldsymbol{M} \to \boldsymbol{S}^n$ are Möbius equivalent if and only if there exists T in the Lorentz group $O(n+1, 1)$ in $\boldsymbol{R}_1^{n+2}$ such that $Y = \tilde{Y}T$.*

As a matter of fact, the Möbius group in $\boldsymbol{S}^n$ is isomorphic to the subgroup $O^+(n+1, 1)$ of $O(n+1, 1)$ which preserves the positive part of the light cone in $\boldsymbol{R}_1^{n+2}$. It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY\rangle = \rho^2 dx \cdot dx \tag{2.3}$$

is a Möbius invariant (cf. [CH]). We call it the induced Möbius metric for x. Now let Δ be the Laplace operator of g. Then there is an identity given by

$$\langle \Delta Y, \Delta Y\rangle = 1 + m^2\kappa\ ,$$

where κ is the normalized scalar curvature of g (cf. [W]). We define

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}(1 + m^2\kappa)Y\ . \tag{2.4}$$

Then we have

$$\langle Y, Y\rangle = \langle N, N\rangle = 0\ , \quad \langle Y, N\rangle = 1\ . \tag{2.5}$$

Moreover, if we take a local orthonormal basis $\{E_i\}$ for the Möbius metric g with dual basis $\{\omega_i\}$, then we have

$$\langle E_i(Y), E_j(Y)\rangle = \delta_{ij}\ , \quad \langle E_i(Y), Y\rangle = \langle E_i(Y), N\rangle = 0\ , \quad 1 \le i, j \le m\ . \tag{2.6}$$

Let $\mathbf{V}$ be the orthogonal complement to the subspace in $\boldsymbol{R}_1^{n+2}$ spanned by $\{Y, N, E_i(Y)\}$. Then we have the following orthogonal decomposition:

$$\boldsymbol{R}_1^{n+2} = \mathrm{span}\{Y, N\} \oplus \mathrm{span}\{E_1(Y), \dots, E_m(Y)\} \oplus \mathbf{V}\ . \tag{2.7}$$

$\mathbf{V}$ is called the Möbius normal bundle of x. A local orthonormal basis $\{E_\alpha\}$ for $\mathbf{V}$ can be written as

$$E_\alpha = (H^\alpha, H^\alpha x + e_\alpha), \quad m+1 \le \alpha \le n\ . \tag{2.8}$$

Now, let $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}_1^{n+2})$ be the Grassmannian manifold consisting of all positive definite oriented $(n-m)$-planes in the Lorentz space $\boldsymbol{R}_1^{n+2}$. The conformal Gauss map $f : \boldsymbol{M} \to \boldsymbol{G}^+_{n-m}(\boldsymbol{R}_1^{n+2}) \subset \bigwedge^{n-m}(\boldsymbol{R}_1^{n+2})$ is then defined by

$$f = E_{m+1} \wedge E_{m+2} \wedge \cdots \wedge E_n\ . \tag{2.9}$$

Since $\{Y, N, E_1(Y), \dots, E_m(Y), E_{m+1}, \dots, E_n\}$ are Möbius invariant moving frame in $\boldsymbol{R}_1^{n+2}$ along $\boldsymbol{M}$, we can write the structure equations as

$$E_i(N) = \sum_j A_{ij}E_j(Y) + \sum_\alpha C_i^\alpha E_\alpha\ , \tag{2.10}$$

$$E_j(E_i(Y)) = -A_{ij}Y - \delta_{ij}N + \sum_k \Gamma_{ij}^k E_k(Y) + \sum_\alpha B_{ij}^\alpha E_\alpha\ , \tag{2.11}$$

$$E_i(E_\alpha) = -C_i^\alpha Y - \sum_j B_{ij}^\alpha E_j(Y) + \sum_\beta \Gamma_{\alpha i}^\beta E_\beta\ , \tag{2.12}$$

where $\{\Gamma^k_{ij}\}$ is the Levi-Civita connection of the Möbius metric g; $\{\Gamma^\beta_{\alpha i}\}$ is the normal connection for $x : \boldsymbol{M} \to \boldsymbol{S}^n$, which is a Möbius invariant; $\mathbf{A} = \sum_{ij} A_{ij}\omega_i \otimes \omega_j$ and $\Phi = \sum_{i\alpha} C^\alpha_i \omega_i(\rho^{-1}e_\alpha)$ are called the Blaschke tensor and the Möbius form, respectively; and $\mathbf{B} = \sum_{ij\alpha} B^\alpha_{ij}\omega_i\omega_j(\rho^{-1}e_\alpha)$ is called the Möbius second fundamental form of x. The relations between $\mathbf{A}$, Φ, $\mathbf{B}$ and the Euclidean invariants of x are given by (1.1), (1.2) and

$$B^\alpha_{ij} = \rho^{-1}(h^\alpha_{ij} - H^\alpha\delta_{ij})\,. \tag{2.13}$$

The integrability conditions for the structure equations (2.10) through (2.12) are given by (cf. [W])

$$A_{ij,k} - A_{ik,j} = \sum_\alpha (B^\alpha_{ik}C^\alpha_j - B^\alpha_{ij}C^\alpha_k)\,, \tag{2.14}$$

$$C^\alpha_{i,j} - C^\alpha_{j,i} = \sum_k (B^\alpha_{ik}A_{kj} - B^\alpha_{kj}A_{ki})\,, \tag{2.15}$$

$$B^\alpha_{ij,k} - B^\alpha_{ik,j} = \delta_{ij}C^\alpha_k - \delta_{ik}C^\alpha_j\,, \tag{2.16}$$

$$R_{ijkl} = \sum_\alpha (B^\alpha_{ik}B^\alpha_{jl} - B^\alpha_{il}B^\alpha_{jk}) + (\delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il})\,, \tag{2.17}$$

$$R_{\alpha\beta ij} = \sum_k (B^\alpha_{ik}B^\beta_{kj} - B^\beta_{ik}B^\alpha_{kj})\,, \tag{2.18}$$

$$\sum_i B^\alpha_{ii} = 0, \quad \sum_{ij\alpha}(B^\alpha_{ij})^2 = \frac{m-1}{m}, \quad \operatorname{tr}\mathbf{A} = \sum_i A_{ii} = \frac{1}{2m}(1 + m^2\kappa)\,, \tag{2.19}$$

where κ is the normalized scalar curvature of g. From (2.16) and (2.19) we get

$$\sum_i B^\alpha_{ij,i} = (1-m)C^\alpha_j\,. \tag{2.20}$$

DEFINITION 2.2. Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a submanifold in $\boldsymbol{S}^n$ without umbilics. We call x a Möbius isotropic submanifold in $\boldsymbol{S}^n$ if $\Phi \equiv 0$ and there exists a function $\lambda : \boldsymbol{M} \to \boldsymbol{R}$ such that $\mathbf{A} = \lambda g$.

PROPOSITION 2.3. *Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a Möbius isotropic submanifold in $\boldsymbol{S}^n$. Then the function λ in Definition* 2.2 *has to be constant.*

PROOF. Since $\Phi \equiv 0$ and $\mathbf{A} = \lambda g$, we can write (2.10) as $dN = \lambda dY$, which implies that $d\lambda \wedge dY = 0$. Since $\{E_1(Y), \dots, E_m(Y)\}$ are linearly independent, we get $\lambda =$ constant.

3. Conformal Gauss map of submanifolds in $\boldsymbol{S}^n$. Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a submanifold in $\boldsymbol{S}^n$. We assume that $\boldsymbol{M}$ is oriented. Then we can give the normal bundle $\boldsymbol{N}(\boldsymbol{M})$ of x an orientation. Let $\{e_\alpha\}$ be a local orthonormal basis for $\boldsymbol{N}(\boldsymbol{M})$ which gives the orientation. Using the bundle isometry $\tau : \boldsymbol{N}(\boldsymbol{M}) \to \mathbf{V}$ defined by $e_\alpha \to (H^\alpha, H^\alpha x + e_\alpha)$, we can give $\mathbf{V}$ an orientation. We define the conformal Gauss map $f : \boldsymbol{M} \to \boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1) \subset \bigwedge^{n-m}(\boldsymbol{R}^{n+2}_1)$ by

$$f = E_{m+1} \wedge E_{m+2} \wedge \cdots \wedge E_n\,, \tag{3.1}$$

where $\{E_\alpha\}$ is an oriented orthonormal basis for $\mathbf{V}$. We denote by I_G the induced metric of the standard embedding of $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ in $\bigwedge^{n-m}(\boldsymbol{R}^{n+2}_1)$. Our goal in this section is to prove the following

THEOREM 3.1. *Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a Möbius isotropic submanifold in $\boldsymbol{S}^n$. Then its conformal Gauss map $f : (\boldsymbol{M}, g) \to (\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1), I_G)$ is harmonic.*

Let $(\boldsymbol{M}, g)$ and $(\boldsymbol{N}, h)$ be two semi-Riemannian manifolds. We assume that g is positive definite and h is a metric of type (r, s). Then locally we can write

$$g = \sum_{i=1}^{m} \theta_i^2, \quad h = -\sum_{\alpha=1}^{r} \theta_\alpha^2 + \sum_{\lambda=r+1}^{r+s} \theta_\lambda^2 . \tag{3.2}$$

We denote by $\{\theta_{ij}\}$ the connection forms of g with respect to $\{\theta_i\}$ and denote by $\{\theta_{\alpha\beta}, \theta_{\alpha\lambda}, \theta_{\lambda\mu}\}$ the connection forms of h with respect to $\{\theta_\alpha, \theta_\lambda\}$. Here we use the following ranges of the indices:

$$1 \le i, j \le m, \quad 1 \le \alpha, \beta \le r, \quad r+1 \le \lambda, \mu \le r+s . \tag{3.3}$$

Then we have

$$d\theta_i = \sum_j \theta_{ij} \wedge \theta_j , \tag{3.4}$$

$$d\theta_\alpha = -\sum_\beta \theta_{\alpha\beta} \wedge \theta_\beta + \sum_\lambda \theta_{\alpha\lambda} \wedge \theta_\lambda , \quad d\theta_\lambda = -\sum_\beta \theta_{\lambda\beta} \wedge \theta_\beta + \sum_\mu \theta_{\lambda\mu} \wedge \theta_\mu . \tag{3.5}$$

Now, let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a smooth map. We define $\{f_{\alpha i}, f_{\lambda i}\}$ by

$$f^*\theta_\alpha = \sum_i f_{\alpha i}\theta_i , \quad f^*\theta_\lambda = \sum_i f_{\lambda i}\theta_i . \tag{3.6}$$

The second fundamental form $\{f_{\alpha i,j}, f_{\lambda i,j}\}$ of $f : \boldsymbol{M} \to \boldsymbol{N}$ is defined by

$$df_{\alpha i} + \sum_j f_{\alpha j}\theta_{ji} - \sum_\beta f_{\beta i} f^*\theta_{\beta\alpha} + \sum_\lambda f_{\lambda i} f^*\theta_{\lambda\alpha} = \sum_j f_{\alpha i,j}\theta_j , \tag{3.7}$$

$$df_{\lambda i} + \sum_j f_{\lambda j}\theta_{ji} - \sum_\alpha f_{\alpha i} f^*\theta_{\alpha\lambda} + \sum_\mu f_{\mu i} f^*\theta_{\mu\lambda} = \sum_j f_{\lambda i,j}\theta_j . \tag{3.8}$$

Then $f : \boldsymbol{M} \to \boldsymbol{N}$ is harmonic if and only if

$$\sum_i f_{\alpha i,i} = 0 , \quad \sum_i f_{\lambda i,i} = 0 , \quad 1 \le \alpha \le r , \quad r+1 \le \lambda \le r+s . \tag{3.9}$$

To prove Theorem 3.1 we study first the geometry of the Grassmannian manifold $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ as a submanifold in the pseudo-Euclidean space $\bigwedge^{n-m}(\boldsymbol{R}^{n+2}_1)$ with the inner product induced by $(\boldsymbol{R}^{n+2}_1, \langle\, , \rangle)$. Let $\tilde{O}(n+1, 1)$ be the manifold defined by

$$\tilde{O}(n+1, 1) = \{T \in GL(n+2, \boldsymbol{R}) \mid {}^tT I_1 T = J\} , \tag{3.10}$$

where $I_1 = \mathrm{diag}\{-1, 1, \dots, 1\}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathrm{diag}\{1, \dots, 1\}$. Then

$$T = (\xi_{-1}, \xi_0, \xi_1, \dots, \xi_n) \in \tilde{O}(n+1, 1)$$

if and only if

$$\langle \xi_{-1}, \xi_{-1}\rangle = \langle \xi_0, \xi_0\rangle = 0\,, \quad \langle \xi_{-1}, \xi_0\rangle = 1\,, \tag{3.11}$$

$$\langle \xi_a, \xi_{-1}\rangle = \langle \xi_a, \xi_0\rangle = 0\,, \quad \langle \xi_a, \xi_b\rangle = \delta_{ab}\,, \quad 1 \le a, b \le n\,. \tag{3.12}$$

Let $\pi : \tilde{O}(n+1,1) \to \boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ be the fibre bundle defined by

$$\pi(T) = \xi_{m+1} \wedge \cdots \wedge \xi_n\,. \tag{3.13}$$

Then around each point in $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ there exists an open set $U \subset \boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ such that we have a local section

$$T = (\xi_{-1}, \xi_0, \xi_1, \ldots, \xi_n) : U \to \tilde{O}(n+1,1)\,. \tag{3.14}$$

Thus the embedding of $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ in $\bigwedge^{n-m}(\boldsymbol{R}^{n+2}_1)$ can be written locally by the position vector

$$\xi = \xi_{m+1} \wedge \cdots \wedge \xi_n : U \to \bigwedge\nolimits^{n-m}(\boldsymbol{R}^{n+2}_1)\,. \tag{3.15}$$

Since $\{\xi_{-1}, \xi_0, \xi_1, \ldots, \xi_n\}$ is a moving frame in $\boldsymbol{R}^{n+2}_1$ along $U \subset \boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$, we can write the structure equations as

$$d\xi_A = \sum_B \theta_{AB}\xi_B\,, \quad -1 \le A, B \le n\,, \tag{3.16}$$

where d stands for the differential operator on $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ and $\{\theta_{AB}\}$ are local 1-forms on $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$. The integrability conditions for (3.16) are given by

$$d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB}\,, \quad -1 \le A, B, C \le n\,. \tag{3.17}$$

Since (3.11) and (3.12) hold on U, we get from (3.16) that

$$\theta_{0(-1)} = \theta_{(-1)0} = 0, \quad \theta_{00} = -\theta_{(-1)(-1)}\,, \tag{3.18}$$

$$\theta_{0a} = -\theta_{a(-1)}\,, \quad \theta_{(-1)a} = -\theta_{a0}\,, \quad \theta_{ab} = -\theta_{ba}\,, \quad 1 \le a, b \le n\,. \tag{3.19}$$

We make the following convention on the range of indices:

$$1 \le i, j, k \le m\,, \quad m+1 \le \alpha, \beta, \gamma \le n\,, \quad -1 \le A, B, C \le n\,.$$

Then from (3.15) we get

$$\begin{aligned} d\xi &= \sum_\alpha \xi_{m+1} \wedge \cdots \wedge d\xi_\alpha \wedge \cdots \wedge \xi_n \\ &= \sum_\alpha (-1)^{\alpha-m-1}\theta_{\alpha(-1)}\xi_{-1} \wedge \xi_{m+1} \wedge \cdots \wedge \widehat{\xi_\alpha} \wedge \cdots \wedge \xi_n \\ &\quad + \sum_\alpha (-1)^{\alpha-m-1}\theta_{\alpha 0}\xi_0 \wedge \xi_{m+1} \wedge \cdots \wedge \widehat{\xi_\alpha} \wedge \cdots \wedge \xi_n \\ &\quad + \sum_{\alpha,i} (-1)^{\alpha-m-1}\theta_{\alpha i}\xi_i \wedge \xi_{m+1} \wedge \cdots \wedge \widehat{\xi_\alpha} \wedge \cdots \wedge \xi_n\,. \end{aligned} \tag{3.20}$$

Thus the induced metric I_G of $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ in $\bigwedge^{n-m}(\boldsymbol{R}^{n+2}_1)$ is given by

$$I_G = \langle d\xi, d\xi\rangle = \sum_{\alpha}(\theta_{\alpha(-1)} \otimes \theta_{\alpha 0} + \theta_{\alpha 0} \otimes \theta_{\alpha(-1)}) + \sum_{\alpha i}\theta^2_{\alpha i}\,. \tag{3.21}$$

If we define

$$\phi_{\alpha(-1)} = \frac{1}{\sqrt{2}}(\theta_{\alpha(-1)} - \theta_{\alpha 0}), \quad \phi_{\alpha 0} = \frac{1}{\sqrt{2}}(\theta_{\alpha(-1)} + \theta_{\alpha 0})\,, \tag{3.22}$$

then we can write

$$I_G = -\sum_{\alpha}\phi^2_{\alpha(-1)} + \sum_{\alpha}\phi^2_{\alpha 0} + \sum_{\alpha i}\theta^2_{\alpha i}\,. \tag{3.23}$$

Thus $\{\phi_{\alpha(-1)}, \phi_{\alpha 0}, \theta_{\alpha i}\}$ is a local orthonormal basis of $T^*\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$, which implies that I_G is a semi-Riemannian metric on $\boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ of type $((n-m), (n-m)(m+1))$. From (3.22), (3.17), (3.18) and (3.19) we get

$$d\phi_{\alpha(-1)} = \sum_{\beta}\theta_{\alpha\beta} \wedge \phi_{\beta(-1)} + \theta_{00} \wedge \phi_{\alpha 0} + \sum_{k}\frac{1}{\sqrt{2}}(\theta_{k0} - \theta_{k(-1)}) \wedge \theta_{\alpha k}\,, \tag{3.24}$$

$$d\phi_{\alpha 0} = \theta_{00} \wedge \phi_{\alpha(-1)} + \sum_{\beta}\theta_{\alpha\beta} \wedge \phi_{\beta 0} - \sum_{k}\frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0}) \wedge \theta_{\alpha k}\,, \tag{3.25}$$

$$\begin{aligned} d\theta_{\alpha k} =& \frac{1}{\sqrt{2}}(\theta_{k0} - \theta_{k(-1)}) \wedge \phi_{\alpha(-1)} + \frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0}) \wedge \phi_{\alpha 0} \\ &+ \sum_{j\beta}(-\theta_{jk}\delta_{\alpha\beta} + \theta_{\alpha\beta}\delta_{jk})\theta_{\beta j}\,. \end{aligned} \tag{3.26}$$

By (3.5) we obtain the following connection forms of I_G with respect to the orthonormal basis $\{\phi_{\alpha(-1)}, \phi_{\alpha 0}, \theta_{\alpha i}\}$:

$$\Omega_{\alpha(-1)\beta(-1)} = -\theta_{\alpha\beta}\,, \ \Omega_{\alpha(-1)\beta 0} = \theta_{00}\delta_{\alpha\beta}\,, \ \Omega_{\alpha(-1)\beta k} = \frac{1}{\sqrt{2}}(\theta_{k0} - \theta_{k(-1)})\delta_{\alpha\beta}\,, \tag{3.27}$$

$$\Omega_{\alpha 0\beta(-1)} = -\theta_{00}\delta_{\alpha\beta}\,, \quad \Omega_{\alpha 0\beta 0} = \theta_{\alpha\beta}\,, \quad \Omega_{\alpha 0\beta k} = -\frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0})\delta_{\alpha\beta}\,, \tag{3.28}$$

$$\begin{aligned} \Omega_{\alpha k\beta(-1)} =& \frac{1}{\sqrt{2}}(\theta_{k(-1)} - \theta_{k0})\delta_{\alpha\beta}\,, \quad \Omega_{\alpha k\beta 0} = \frac{1}{\sqrt{2}}(\theta_{k(-1)} + \theta_{k0})\delta_{\alpha\beta}\,, \\ \Omega_{\alpha k\beta j} =& -\theta_{jk}\delta_{\alpha\beta} + \theta_{\alpha\beta}\delta_{jk}\,. \end{aligned} \tag{3.29}$$

Now, let $f : \boldsymbol{M} \to \boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ be the conformal Gauss map of a submanifold $x : \boldsymbol{M} \to \boldsymbol{S}^n$. Let $\{Y, N, E_1(Y), \ldots, E_m(Y), E_{m+1}, \ldots, E_n\}$ be the Möbius moving frame in $\boldsymbol{R}^{n+2}_1$ along $\boldsymbol{M}$. Then we can find a local section T of $\pi : \tilde{O}(n+1, 1) \to \boldsymbol{G}^+_{n-m}(\boldsymbol{R}^{n+2}_1)$ given by (3.14) such that

$$(Y, N, E_1(Y), \ldots, E_m(Y), E_{m+1}, \ldots, E_n) = T \circ f = (f^*\xi_{-1}, \ldots, f^*\xi_n)\,. \tag{3.30}$$

It follows from (2.10), (2.11), (2.12) and (3.16) that

$$f^*\theta_{00} = 0\,, \quad f^*\theta_{k(-1)} = -\sum_j A_{kj}\omega_j\,, \quad f^*\theta_{k0} = -\omega_k\,, \tag{3.31}$$

$$f^*\theta_{ij} = \omega_{ij} := \sum_k \Gamma^j_{ik}\omega_k\,, \quad f^*\theta_{\alpha\beta} = \omega_{\alpha\beta} := \sum_i \Gamma^\beta_{\alpha i}\omega_i\,, \tag{3.32}$$

$$f^*\theta_{\alpha(-1)} = -\sum_i C^\alpha_i\omega_i\,, \quad f^*\theta_{\alpha0} = 0\,, \quad f^*\theta_{\alpha k} = -\sum_j B^\alpha_{kj}\omega_j\,. \tag{3.33}$$

If we define $\{f_{\alpha(-1)i}, f_{\alpha0i}, f_{\alpha ki}\}$ by

$$f^*\phi_{\alpha(-1)} = \sum_i f_{\alpha(-1)i}\omega_i\,, \quad f^*\phi_{\alpha0} = \sum_i f_{\alpha0i}\omega_i\,, \quad f^*\theta_{\alpha k} = \sum_i f_{\alpha ki}\omega_i\,. \tag{3.34}$$

Then by (3.22) and (3.33) we have

$$f_{\alpha(-1)i} = -\frac{1}{\sqrt{2}}C^\alpha_i\,, \quad f_{\alpha0i} = -\frac{1}{\sqrt{2}}C^\alpha_i\,, \quad f_{\alpha ki} = -B^\alpha_{ki}\,. \tag{3.35}$$

By definition (cf. (3.7) and (3.8)) the second fundamental form $\{f_{\alpha(-1)i,j}, f_{\alpha0i,j}, f_{\alpha ki,j}\}$ are defined by the following formulas

$$\begin{aligned} df_{\alpha(-1)i} &+ \sum_j f_{\alpha(-1)j}\omega_{ji} - \sum_\beta f_{\beta(-1)i} f^*\Omega_{\beta(-1)\alpha(-1)} + \sum_\beta f_{\beta0i} f^*\Omega_{\beta0\alpha(-1)} \\ &+ \sum_{\beta k} f_{\beta ki} f^*\Omega_{\beta k\alpha(-1)} = \sum_j f_{\alpha(-1)i,j}\omega_j\,, \end{aligned} \tag{3.36}$$

$$\begin{aligned} df_{\alpha0i} &+ \sum_j f_{\alpha0j}\omega_{ji} - \sum_\beta f_{\beta(-1)i} f^*\Omega_{\beta(-1)\alpha0} + \sum_\beta f_{\beta0i} f^*\Omega_{\beta0\alpha0} \\ &+ \sum_{\beta k} f_{\beta ki} f^*\Omega_{\beta k\alpha0} = \sum_j f_{\alpha0i,j}\omega_j\,, \end{aligned} \tag{3.37}$$

$$\begin{aligned} df_{\alpha ki} &+ \sum_j f_{\alpha kj}\omega_{ji} - \sum_\beta f_{\beta(-1)i} f^*\Omega_{\beta(-1)\alpha k} + \sum_\beta f_{\beta0i} f^*\Omega_{\beta0\alpha k} \\ &+ \sum_{\beta j} f_{\beta ji} f^*\Omega_{\beta j\alpha k} = \sum_j f_{\alpha ki,j}\omega_j\,. \end{aligned} \tag{3.38}$$

It follows from (3.27) through (3.29) and (3.31) through (3.35) that

$$f_{\alpha(-1)i,j} = -\frac{1}{\sqrt{2}}\left(C^\alpha_{i,j} - \sum_k B^\alpha_{ik}A_{kj} + B^\alpha_{ij}\right), \tag{3.39}$$

$$f_{\alpha0i,j} = -\frac{1}{\sqrt{2}}\left(C^\alpha_{i,j} - \sum_k B^\alpha_{ik}A_{kj} - B^\alpha_{ij}\right), \tag{3.40}$$

$$f_{\alpha ki,j} = -(B^\alpha_{ki,j} + C^\alpha_i\delta_{kj})\,. \tag{3.41}$$

Thus we know from (2.19) and (2.20) that the conformal Gauss map $f : \boldsymbol{M} \to \boldsymbol{G}^{+}_{n-m}(\boldsymbol{R}^{n+2}_{1})$ is harmonic if and only if

$$\sum_i C^{\alpha}_{i,i} - \sum_{i,j} B^{\alpha}_{ij} A_{ij} = 0\,, \quad (m-2)C^{\alpha}_{k} = 0\,, \quad 1 \le k \le m\,,\ \ 1 \le \alpha \le n\,. \tag{3.42}$$

In the case $m = 2$, the first equation of (3.42) is exactly the Euler-Lagrange equation for the Willmore functional (which is the Möbius volume functional, cf. [W]). The surfaces in $\boldsymbol{S}^n$ satisfying this equation are known as Willmore surfaces in $\boldsymbol{S}^n$. The conformal Gauss map of a surface in $\boldsymbol{S}^n$ has been studied by Bryant ([BR]) for $n = 3$ and Rigoli ([R]) for $n > 3$ by using complex coordinate on the surface. It follows immediately from (3.42) that

THEOREM 3.2 ([BR], [R]). *A surface $x : \boldsymbol{M} \to \boldsymbol{S}^n$ is Willmore if and only if its conformal Gauss map is harmonic.*

In the case $m > 2$, we know that the conformal Gauss map of $x : \boldsymbol{M} \to \boldsymbol{S}^n$ is harmonic if and only if it satisfies

$$C^{\alpha}_{k} \equiv 0\,, \quad \sum_{i,j} B^{\alpha}_{ij} A_{ij} \equiv 0\,, \quad 1 \le k \le m,\ \ m+1 \le \alpha \le n\,. \tag{3.43}$$

Since for any Möbius isotropic submanifold we have $C^{\alpha}_{k} \equiv 0$ and $A_{ki} \equiv \lambda\delta_{ki}$ for some λ, which implies (3.42). Thus we complete the proof of Theorem 3.1.

4. Conformal invariants for submanifolds in $\boldsymbol{R}^n$ and $\boldsymbol{H}^n$. Let $\sigma : \boldsymbol{R}^n \to \boldsymbol{S}^n$ and $\tau : \boldsymbol{H}^n \to \boldsymbol{S}^n_{+}$ be the conformal maps definded by (1.3) and (1.4). Using σ and τ, we can regard submanifolds in $\boldsymbol{R}^n$ and $\boldsymbol{H}^n$ as submanifolds in $\boldsymbol{S}^n$. In this section we give the conformal invariants for submanifolds in $\boldsymbol{R}^n$ and $\boldsymbol{H}^n$, and relate them to the Möbius invariants for submanifolds in $\boldsymbol{S}^n$. By using these relations, we show that any minimal submanifolds with constant scalar curvature in $\boldsymbol{R}^n$, $\boldsymbol{H}^n$ and $\boldsymbol{S}^n$ are Möbius isotropic.

Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a minimal submanifold with constant scalar curvature in $\boldsymbol{S}^n$. Then by the Gauss equation we know that $\rho^2 = m/(m-1)\cdot(\|II\|^2 - m\|H\|^2)$ is a constant. Thus from (1.1) and (1.2) we get

$$C^{\alpha}_{i} = 0\,, \quad A_{ij} = \frac{1}{2}\rho^{-2}\delta_{ij}\,.$$

By definition x is a Möbius isotropic submanifold in $\boldsymbol{S}^n$.

Let $u : \boldsymbol{M} \to \boldsymbol{R}^n$ be a submanifold without umbilics in $\boldsymbol{R}^n$. Let $\{\tilde{e}_i\}$ be a local orthonormal basis for the first fundamental form $\tilde{I} = du \cdot du$ with the dual basis $\{\tilde{\theta}_i\}$. Let $\widetilde{II} = \sum_{ij\alpha} \tilde{h}^{\alpha}_{ij}\tilde{\theta}_i\tilde{\theta}_j\tilde{e}_{\alpha}$ be the second fundamental form of u and $\tilde{H} = \sum_{\alpha} \tilde{H}^{\alpha}\tilde{e}_{\alpha}$ be the mean curvature vector of u, where $\{\tilde{e}_{\alpha}\}$ is a local orthonormal basis for the normal bundle of u. We define

$$\tilde g = \tilde\rho^2 du\cdot du\,,\quad \tilde\rho^2 = m/(m-1)\cdot(||\widetilde{II}||^2 - m||\tilde H||^2)\,, \tag{4.1}$$

$$\tilde B^\alpha_{ij} = \tilde\rho^{-1}(\tilde h^\alpha_{ij} - \tilde H^\alpha\delta_{ij})\,, \tag{4.2}$$

$$\tilde C^\alpha_i = -\,\tilde\rho^{-2}\left(\tilde H^\alpha{}_{,i} + \sum_j(\tilde h^\alpha_{ij} - \tilde H^\alpha\delta_{ij})\tilde e_j(\log\tilde\rho)\right)\,, \tag{4.3}$$

$$\begin{aligned}\tilde A_{ij} = &-\,\tilde\rho^{-2}\left(\mathrm{Hess}_{ij}(\log\tilde\rho) - \tilde e_i(\log\tilde\rho)\tilde e_j(\log\tilde\rho) - \sum_\alpha \tilde H^\alpha\tilde h^\alpha_{ij}\right)\\ &-\frac12\tilde\rho^{-2}\left(\|\nabla\log\tilde\rho\|^2 + \sum_\alpha(\tilde H^\alpha)^2\right)\delta_{ij}\,.\end{aligned} \tag{4.4}$$

We call the globally defined tensors $\tilde g$, $\tilde\Phi = \sum_{i\alpha}\tilde C^\alpha_i\tilde\theta_i\tilde e_\alpha$, $\tilde{\mathbf{A}} := \tilde\rho^2\sum_{ij}\tilde A_{ij}\tilde\theta_i\tilde\theta_j$ and $\tilde{\mathbf{B}} = \tilde\rho\sum_{ij\alpha}\tilde B^\alpha_{ij}\tilde\theta_i\tilde\theta_j\tilde e_\alpha$ the Möbius metric, the Möbius form, the Blaschke tensor and the Möbius second fundamental form of $u : \boldsymbol{M}\to\boldsymbol{R}^n$, respectively.

Now, let $\sigma : \boldsymbol{R}^n\to\boldsymbol{S}^n$ be the conformal map given by (1.3). We define $x := \sigma\circ u : \boldsymbol{M}\to\boldsymbol{S}^n$. Then x is a submanifold in $\boldsymbol{S}^n$ without umbilics. We denote by Φ and $\mathbf{A}$ the Möbius form and the Blaschke tensor of x defined by (1.1) and (1.2), and denote by g and $\mathbf{B}$ the Möbius metric and the Möbius second fundamental form defined by (2.3) and (2.13) for $x = \sigma\circ u$, respectively. Our goal in this section is to prove the following

THEOREM 4.1. *$g = \tilde g$, $\mathbf{B} = d\sigma(\tilde{\mathbf{B}})$, $\Phi = d\sigma(\tilde\Phi)$ and $\mathbf{A} = \tilde{\mathbf{A}}$. In particular, $\{\tilde g, \tilde{\mathbf{B}}, \tilde\Phi, \tilde{\mathbf{A}}\}$ are conformal invariants for submanifolds in $\boldsymbol{R}^n$.*

Let $\sigma : \boldsymbol{R}^n\to\boldsymbol{S}^n$ be the conformal map given by

$$x = \sigma(u) = \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right),\quad u\in\boldsymbol{R}^n\,. \tag{4.5}$$

Then for any vector $V\in T_u\boldsymbol{R}^n$ we have

$$d\sigma(V) = \frac{2}{1+|u|^2}\{-(u\cdot V)x + (-u\cdot V, V)\}\,. \tag{4.6}$$

Thus we get

$$dx\cdot dx = \frac{4}{(1+|u|^2)^2}du\cdot du\,. \tag{4.7}$$

Now, let $u : \boldsymbol{M}\to\boldsymbol{R}^n$ be a submanifold in $\boldsymbol{R}^n$ and $x = \sigma\circ u : \boldsymbol{M}\to\boldsymbol{S}^n$. We denote by $\{\tilde e_i\}$ and $\{\tilde e_\alpha\}$ local orthonormal basis for $du\cdot du$ and the normal bundle of u respectively, and define

$$e_i = \frac{1+|u|^2}{2}\tilde e_i\,,\quad e_\alpha = \frac{1+|u|^2}{2}d\sigma(\tilde e_\alpha)\,. \tag{4.8}$$

Then $\{e_i\}$ is a local orthonormal basis for $dx \cdot dx$ with dual basis $\{\theta_i\}$ and $\{e_\alpha\}$ is a local orthonormal basis for the normal bundle of x in $\boldsymbol{S}^n$. It follows from (4.6) that

$$e_i(x) = \frac{1+|u|^2}{2} d\sigma(\tilde{e}_i(u)) = -(u \cdot \tilde{e}_i(u))x + (-u \cdot \tilde{e}_i(u), \tilde{e}_i(u)) \,, \tag{4.9}$$

$$\begin{aligned} e_\alpha &= \frac{1+|u|^2}{2} d\sigma(\tilde{e}_\alpha) = -\frac{2u \cdot \tilde{e}_\alpha}{1+|u|^2}(1,u) + (0, \tilde{e}_\alpha) \\ &= -(u \cdot \tilde{e}_\alpha)x + (-u \cdot \tilde{e}_\alpha, \tilde{e}_\alpha) \,. \end{aligned} \tag{4.10}$$

By (4.9) we get

$$e_i e_j(x) = \frac{1+|u|^2}{2}((-\delta_{ij}, 0) + (-u \cdot \tilde{e}_j \tilde{e}_i(u), \tilde{e}_j \tilde{e}_i(u))) \quad \mathrm{mod}\,(x, e_i(x)) \,. \tag{4.11}$$

Thus (4.10) and (4.11) yield

$$h_{ij}^\alpha = \frac{1+|u|^2}{2} \tilde{h}_{ij}^\alpha + \tilde{e}_\alpha \cdot u \delta_{ij} \,, \quad H^\alpha = \frac{1+|u|^2}{2} \tilde{H}^\alpha + \tilde{e}_\alpha \cdot u \,. \tag{4.12}$$

It follows from (4.12) and (4.7) that

$$\rho^2 = \frac{(1+|u|^2)^2}{4} \tilde{\rho}^2 \,, \tag{4.13}$$

$$g = \rho^2 dx \cdot dx = \tilde{\rho}^2 du \cdot du = \tilde{g} \,. \tag{4.14}$$

It is clear that $\tilde{g}$ is a conformal invariant. By (4.12) and (4.13) we get

$$B_{ij}^\alpha = \rho^{-1}(h_{ij}^\alpha - H^\alpha \delta_{ij}) = \tilde{\rho}^{-1}(\tilde{h}_{ij}^\alpha - \tilde{H}^\alpha \delta_{ij}) = \tilde{B}_{ij}^\alpha \,. \tag{4.15}$$

By (4.10) we get

$$de_\alpha = (-u \cdot d\tilde{e}_\alpha, d\tilde{e}_\alpha) \quad \mathrm{mod}\,(x, dx) \,,$$

which implies that

$$\theta_{\alpha\beta} = de_\alpha \cdot e_\beta = d\tilde{e}_\alpha \cdot \tilde{e}_\beta = \tilde{\theta}_{\alpha\beta} \,. \tag{4.16}$$

Let $\{H^\alpha{}_{,i}\}$ and $\{\tilde{H}^\alpha{}_{,i}\}$ be the covariant derivatives of the mean curvature vector in the normal bundle of $x = \sigma \circ u : \boldsymbol{M} \to \boldsymbol{S}^n$ and $u : \boldsymbol{M} \to \boldsymbol{R}^n$, respectively. By definition we have

$$dH^\alpha + \sum_\beta H^\beta \theta_{\beta\alpha} = \sum_i H^\alpha{}_{,i}\, \theta_i \,, \quad d\tilde{H}^\alpha + \sum_\beta \tilde{H}^\beta \tilde{\theta}_{\beta\alpha} = \sum_i \tilde{H}^\alpha{}_{,i}\, \tilde{\theta}_i \,.$$

Since $\tilde{\theta}_i = ((1+|u|^2)/2)\theta_i$, from (4.12) and (4.16) we get

$$H^\alpha{}_{,i} = \left(\frac{1+|u|^2}{2}\right)^2 \tilde{H}^\alpha{}_{,i} - \frac{1+|u|^2}{2} \sum_j (\tilde{h}_{ij}^\alpha - \tilde{H}^\alpha \delta_{ij})(\tilde{e}_j(u) \cdot u) \,. \tag{4.17}$$

By (4.13) we get

$$e_j(\log \rho) = \frac{1+|u|^2}{2} \tilde{e}_j(\log \tilde{\rho}) + \tilde{e}_j(u) \cdot u \,. \tag{4.18}$$

We define $\{C_i^\alpha\}$ and $\{\tilde{C}_i^\alpha\}$ by (1.1) and (4.3), respectively. It follows from (4.17) and (4.18) that

$$C_i^\alpha = \tilde{C}_i^\alpha \,. \tag{4.19}$$

Let $\{\theta_{ij}\}$ and $\{\tilde{\theta}_{ij}\}$ be the Levi-Civita connections of $dx \cdot dx$ and $du \cdot du$ with respect to the basis $\{e_i\}$ and $\{\tilde{e}_i\}$, respectively. Then by (4.7) we have

$$\theta_{ij} = \tilde{\theta}_{ij} + \frac{2u \cdot \tilde{e}_j(u)}{1+|u|^2}\tilde{\theta}_i - \frac{2u \cdot \tilde{e}_i(u)}{1+|u|^2}\tilde{\theta}_j \,. \tag{4.20}$$

We define the $\mathrm{Hess}_{ij}(\log\rho)$ and $\mathrm{Hess}_{ij}(\log\tilde{\rho})$ by

$$\begin{aligned} d\,(e_i(\log\rho)) + \sum_j e_j(\log\rho)\theta_{ji} &= \sum_j \mathrm{Hess}_{ij}(\log\rho)\theta_j \,,\\ d\,(\tilde{e}_i(\log\tilde{\rho})) + \sum_j \tilde{e}_j(\log\tilde{\rho})\tilde{\theta}_{ji} &= \sum_j \mathrm{Hess}_{ij}(\log\tilde{\rho})\tilde{\theta}_j \,. \end{aligned}$$

Using (4.18) and (4.20), we get

$$\begin{aligned} \mathrm{Hess}_{ij}(\log\rho) &= \left(\frac{1+|u|^2}{2}\right)^2 \mathrm{Hess}_{ij}(\log\tilde{\rho}) + (u\cdot\tilde{e}_i(u))(u\cdot\tilde{e}_j(u))\\ &+ \frac{1+|u|^2}{2}\left(\sum_\alpha \tilde{h}_{ij}^\alpha(\tilde{e}_\alpha\cdot u) + (u\cdot\tilde{e}_j(u))\tilde{e}_i(\log\tilde{\rho}) + (u\cdot\tilde{e}_i(u))\tilde{e}_j(\log\tilde{\rho})\right)\\ &+ \left(\frac{1+|u|^2}{2} - \frac{1+|u|^2}{2}\sum_k(u\cdot\tilde{e}_k(u))\tilde{e}_k(\log\tilde{\rho}) - \sum_k(u\cdot\tilde{e}_k(u))^2\right)\delta_{ij} \,. \end{aligned} \tag{4.21}$$

Using (4.12) and (4.18), we also get

$$\begin{aligned} & e_i(\log\rho)e_j(\log\rho) + \sum_\alpha H^\alpha h_{ij}^\alpha\\ &\quad = \left(\frac{1+|u|^2}{2}\right)^2\left(\tilde{e}_i(\log\tilde{\rho})\tilde{e}_j(\log\tilde{\rho}) + \sum_\alpha \tilde{H}^\alpha\tilde{h}_{ij}^\alpha\right)\\ &\quad + \frac{1+|u|^2}{2}(\tilde{e}_i(\log\tilde{\rho})(\tilde{e}_j(u)\cdot u) + \tilde{e}_j(\log\tilde{\rho})(\tilde{e}_i(u)\cdot u))\\ &\quad + (\tilde{e}_i(u)\cdot u)(\tilde{e}_j(u)\cdot u) + \frac{1+|u|^2}{2}\tilde{h}_{ij}^\alpha(\tilde{e}_\alpha\cdot u)\\ &\quad + \left(\sum_\alpha(\tilde{e}_\alpha\cdot u)^2 + \frac{1+|u|^2}{2}\sum_\alpha(\tilde{e}_\alpha\cdot u)\tilde{H}^\alpha\right)\delta_{ij} \,, \end{aligned} \tag{4.22}$$

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\nabla\log\rho\|^2-1+\sum_\alpha(H^\alpha)^2\Big)=\frac{1}{2}\Big(\frac{1+|u|^2}{2}\Big)^2\Big(\|\nabla\log\tilde\rho\|^2+\sum_\alpha(\tilde H^\alpha)^2\Big)\\
&\quad+\frac{1+|u|^2}{2}\Big(\sum_k\tilde e_k(\log\tilde\rho)(\tilde e_k(u)\cdot u)+\sum_\alpha\tilde H^\alpha(\tilde e_\alpha\cdot u)\Big)\\
&\quad+\frac{1}{2}\sum_k(u\cdot\tilde e_k(u))^2+\frac{1}{2}\sum_\alpha(u\cdot\tilde e_\alpha(u))^2-\frac{1}{2}\,.
\end{aligned}
\tag{4.23}
$$

Let $\{A_{ij}\}$ and $\{\tilde A_{ij}\}$ be the tensor defined by (1.2) and (4.4), respectively. Then we get from (4.13), (4.21), (4.22) and (4.23) that

$$A_{ij}=\tilde A_{ij}\,. \tag{4.24}$$

Now, we come to the proof of Theorem 4.1. It follows from (4.14) that $g=\tilde g$. We take $\omega_i=\rho\theta_i=\tilde\rho\tilde\theta_i$. Then by (4.24) we get $\mathbf{A}=\tilde{\mathbf{A}}$. From (4.8) and (4.13) we get $d\sigma(\tilde\rho^{-1}\tilde e_\alpha)=\rho^{-1}e_\alpha$. Thus we get from (4.15) and (4.19) that $d\sigma(\tilde{\mathbf{B}})=\mathbf{B}$ and $d\sigma(\tilde\Phi)=\Phi$. This completes the proof of Theorem 4.1.

It follows from (4.3) and (4.4) that

THEOREM 4.2. *The images of σ of minimal submanifolds with constant scalar curvature in $\boldsymbol{R}^n$ are Möbius isotropic submanifolds in $\boldsymbol{S}^n$.*

Let $\boldsymbol{R}_1^{n+1}$ be the Lorentzian space with inner product

$$\langle y,w\rangle=-y_0w_0+y_1w_1+\cdots+y_nw_n,\quad y=(y_0,\dots,y_n),\,w=(w_0,\dots,w_n).$$

Let $\boldsymbol{H}^n=\{y\in R_1^{n+1}\mid\langle y,y\rangle=-1,\,y_0>0\}$ be the n-dimensional hyperbolic space. We define now the conformal invariants for the submanifolds in $\boldsymbol{H}^n$. Let $y:\boldsymbol{M}\to\boldsymbol{H}^n$ be a submanifold in $\boldsymbol{H}^n$ without umbilics. Let $\{\hat e_i\}$ be a local orthonormal basis for $\langle dy,dy\rangle$ with dual basis $\{\hat\theta_i\}$. Let $\widehat{II}=\sum_{\alpha ij}\hat h_{ij}^\alpha\hat\theta_i\hat\theta_j\hat e_\alpha$ be the second fundamental form of y and $\hat H=\sum_\alpha\hat H^\alpha\hat e_\alpha$ the mean curvature vector of y, where $\{\hat e_\alpha\}$ is a local orthonormal basis for the normal bundle of y. We define

$$\hat g=\hat\rho^2\langle dy,dy\rangle,\quad \hat\rho^2=m/(m-1)\cdot(\|\widehat{II}\|^2-m\|\hat H\|^2)\,, \tag{4.25}$$

$$\hat B_{ij}^\alpha=\hat\rho^{-1}(\hat h_{ij}^\alpha-\hat H^\alpha\delta_{ij})\,, \tag{4.26}$$

$$\hat C_i^\alpha=-\hat\rho^{-2}\Big(\hat H^\alpha{}_{,i}+\sum_j(\hat h_{ij}^\alpha-\hat H^\alpha\delta_{ij})\hat e_j(\log\hat\rho)\Big)\,, \tag{4.27}$$

$$
\begin{aligned}
\hat A_{ij}=&-\hat\rho^{-2}\Big(\mathrm{Hess}_{ij}(\log\hat\rho)-\hat e_i(\log\hat\rho)\hat e_j(\log\hat\rho)-\sum_\alpha\hat H^\alpha\hat h_{ij}^\alpha\Big)\\
&-\frac{1}{2}\hat\rho^{-2}\Big(\|\nabla\log\hat\rho\|^2+1+\sum_\alpha(\hat H^\alpha)^2\Big)\delta_{ij}\,.
\end{aligned}
\tag{4.28}
$$

We call $\hat{g}$ the Möbius metric of y, $\hat{\mathbf{B}} = \hat{\rho}\sum_{ij\alpha}\hat{B}^{\alpha}_{ij}\hat{\theta}_i\hat{\theta}_j\hat{e}_\alpha$ the Möbius second fundamental form of y, $\hat{\Phi} = \sum_{i\alpha}\hat{C}^{\alpha}_i\hat{\theta}_i\hat{e}_\alpha$ the Möbius form of y and $\hat{\mathbf{A}} = \sum_{ij}\hat{\rho}^2\hat{A}_{ij}\hat{\theta}_i\hat{\theta}_j$ the Blaschke tensor of y, respectively.

Set $D^n = \{u \in \boldsymbol{R}^n \mid |u|^2 < 1\}$. Let $\mu : D^n \to \boldsymbol{H}^n$ be the conformal diffeomorphism given by

$$\mu(u) = \left(\frac{1+|u|^2}{1-|u|^2}, \frac{2u}{1-|u|^2}\right). \tag{4.29}$$

Then $u = \mu^{-1}\circ y : \boldsymbol{M} \to D^n$ is a submanifold in D^n without umbilics. We denote by $\{\tilde{g}, \tilde{\mathbf{B}}, \tilde{\Phi}, \tilde{\mathbf{A}}\}$ the basic Möbius invariants for $u = \mu^{-1}\circ y : \boldsymbol{M} \to D^n \subset \boldsymbol{R}^n$. Using the same method as in the proof of Theorem 4.1, we can prove that

THEOREM 4.3. *$\hat{g} = \tilde{g}$, $\hat{\mathbf{B}} = d\mu(\tilde{\mathbf{B}})$, $\hat{\Phi} = d\mu(\tilde{\Phi})$ and $\hat{\mathbf{A}} = \tilde{\mathbf{A}}$. In particular, $\{\hat{g}, \hat{\mathbf{B}}, \hat{\Phi}, \hat{\mathbf{A}}\}$ are conformal invariants for submanifolds in $\boldsymbol{H}^n$.*

Let $\tau : \boldsymbol{H}^n \to \boldsymbol{S}^n_+$ be the conformal diffeomorphism defined by (1.4). Then we have $\tau = \sigma\circ\mu^{-1}$. Thus from Theorem 4.1 and Theorem 4.3 we get

THEOREM 4.4. *Let $y : \boldsymbol{M} \to \boldsymbol{H}^n$ be a submanifold in $\boldsymbol{H}^n$ without umbilics. Let $x = \tau\circ y : \boldsymbol{M} \to \boldsymbol{S}^n_+$. Then we have*

$$g = \hat{g}, \quad \mathbf{B} = d\tau(\hat{\mathbf{B}}), \quad \Phi = d\tau(\hat{\Phi}), \quad \mathbf{A} = \hat{\mathbf{A}}.$$

In particular, $\{\hat{g}, \hat{\mathbf{B}}, \hat{\Phi}, \hat{\mathbf{A}}\}$ are conformal invariants for submanifolds in $\boldsymbol{H}^n$.

It follows immediately from (4.27) and (4.28) that

THEOREM 4.5. *The images of τ of minimal submanifolds with constant scalar curvature in $\boldsymbol{H}^n$ are Möbius isotropic submanifolds in $\boldsymbol{S}^n$.*

5. The classification of Möbius isotropic submanifolds in $\boldsymbol{S}^n$. In this section we prove the classification theorem mentioned in Section 1.

Let $x : \boldsymbol{M} \to \boldsymbol{S}^n$ be a Möbius isotropic submanifold in $\boldsymbol{S}^n$. By definition we have

$$A_{ij} = \lambda\delta_{ij}, \quad C^{\alpha}_i \equiv 0. \tag{5.1}$$

It follows from (2.10) and Proposition 2.3 that

$$dN = \lambda dY \tag{5.2}$$

for some constant λ. Using (5.1) and the last equation in (2.19), we get

$$A_{ij} = \frac{1}{2m^2}(1+m^2\kappa)\delta_{ij}, \quad \kappa = \text{constant}, \tag{5.3}$$

where κ is the normalized scalar curvature of the Möbius metric. By (5.2) we can find a constant vector $\boldsymbol{c} \in \boldsymbol{R}^{n+2}_1$ such that

$$N = \frac{1}{2m^2}(1+m^2\kappa)Y + \boldsymbol{c}. \tag{5.4}$$

It follows from (5.4) and (2.5) that

$$\langle \boldsymbol{c}, \boldsymbol{c} \rangle = -\frac{1}{m^2}(1 + m^2\kappa)\,, \quad \langle Y, \boldsymbol{c} \rangle = 1\,. \tag{5.5}$$

Then we consider the following three cases: (i) $\boldsymbol{c}$ is timelike; (ii) $\boldsymbol{c}$ is lightlike; (iii) $\boldsymbol{c}$ is spacelike.

First, we consider the case (i) that $\langle \boldsymbol{c}, \boldsymbol{c} \rangle = -r^2$ with $r = \sqrt{1 + m^2\kappa}/m > 0$. By (2.2) and $\langle Y, N \rangle = 1$ we know that the first coordinate of Y is positive and of N is negative. Thus by (5.4) we know that the first coordinate of $\boldsymbol{c}$ is negative. So there exists a $T \in O^+(n+1, 1)$ such that

$$(-r, 0) = \boldsymbol{c}T = NT - \frac{r^2}{2}YT\,. \tag{5.6}$$

Let $\tilde{x} : \boldsymbol{M} \to \boldsymbol{S}^n$ be the submanifold which is Möbius equivalent to x such that $\tilde{Y} = YT$ (cf. Theorem 2.1). Then we have $\tilde{N} = NT$. Since

$$\boldsymbol{c}T = (-r, 0)\,, \quad \langle \tilde{Y}, \boldsymbol{c}T \rangle = 1\,, \quad \tilde{Y} = \tilde{\rho}(1, \tilde{x})\,, \tag{5.7}$$

we get

$$\tilde{\rho} = r^{-1} = \text{constant}\,. \tag{5.8}$$

It follows from (5.6) and (2.4) that

$$(-r, 0) = \tilde{N} - \frac{r^2}{2}\tilde{Y}, \quad \tilde{N} = -\frac{1}{m}\tilde{\Delta}\tilde{Y} - \frac{1}{2}r^2\tilde{Y}\,. \tag{5.9}$$

Since $\tilde{\rho} = r^{-1}$, we know from $\tilde{g} = \tilde{\rho}^2 d\tilde{x} \cdot d\tilde{x}$ that the Laplace operator $\Delta_{\boldsymbol{M}}$ of $d\tilde{x} \cdot d\tilde{x}$ is given by $\Delta_{\boldsymbol{M}} = \tilde{\rho}^2\tilde{\Delta}$. Thus by (5.9) we get

$$\Delta_{\boldsymbol{M}}\tilde{x} + m\tilde{x} = 0\,. \tag{5.10}$$

By Takahashi's theorem ([T]) we know that $\tilde{x} : \boldsymbol{M} \to \boldsymbol{S}^n$ is a minimal submanifold. The normalized scalar curvature $\tilde{\kappa}$ of $d\tilde{x} \cdot d\tilde{x}$ is a constant given by

$$\tilde{\kappa} = \tilde{\rho}^2\kappa = \frac{m^2\kappa}{1 + m^2\kappa}\,. \tag{5.11}$$

Next, we consider the case (ii) that $\langle \boldsymbol{c}, \boldsymbol{c} \rangle = 0$. By making use of a Möbius transformation if necessary, we may assume that $\boldsymbol{c} = (-1, 1, 0)$. Thus by (5.4) and (2.4) we have

$$\boldsymbol{c} = (-1, 1, 0) = N = -\frac{1}{m}\Delta Y\,. \tag{5.12}$$

We write $x = (x_0, x_1)$. Then $Y = (\rho, \rho x_0, \rho x_1)$. By (5.5) and (5.12) we get $\langle Y, \boldsymbol{c} \rangle = \rho(1 + x_0) = 1$, which implies that $x_0 \neq -1$ and $x(\boldsymbol{M}) \subset \boldsymbol{S}^n \backslash \{(-1, 0)\}$.

Now, let $\sigma^{-1} : \boldsymbol{S}^n \backslash \{(-1, 0)\} \to \boldsymbol{R}^n$ be the stereographic projection from the point $(-1, 0) \in \boldsymbol{S}^n$. We define $u = \sigma^{-1} \circ x : \boldsymbol{M} \to \boldsymbol{R}^n$. Then by (1.3) we have

$$Y = \rho(1, x) = \left(\rho, \frac{\rho(1 - |u|^2)}{1 + |u|^2}, \frac{2\rho u}{1 + |u|^2}\right)\,. \tag{5.13}$$

From $\langle Y, \boldsymbol{c}\rangle = 1$ we get $\rho = (1 + |u|^2)/2$. Thus we get from (5.13) that

$$Y = \left(\frac{1+|u|^2}{2}, \frac{1-|u|^2}{2}, u\right).$$

The Möbius metric of x is given by

$$g = \langle dY, dY\rangle = du \cdot du\,, \tag{5.14}$$

which is exactly the first fundamental form of $u = \sigma^{-1} \circ x : \boldsymbol{M} \to \boldsymbol{R}^n$. In particular, the Laplace operator Δ of g coincides with the Laplace operator of $du \cdot du$. Comparing the last coordinate in (5.12), we get $\Delta u = 0$. Thus $u = \sigma^{-1} \circ x : \boldsymbol{M} \to \boldsymbol{R}^n$ is a minimal submanidold. By (5.14) and (5.4) we know that the normalized scalar curvature of u is exactly the scalar curvature κ of g. Since $\langle \boldsymbol{c}, \boldsymbol{c}\rangle = -(1 + m^2\kappa)/m^2 = 0$, we get $\kappa = -1/m^2$.

Finally, we consider the case that $\langle \boldsymbol{c}, \boldsymbol{c}\rangle = r^2$ with $r = \sqrt{-(1 + m^2\kappa)}/m > 0$. By making use of a Möbius transformation if necessary, we may assume that $\boldsymbol{c} = (0, r, 0)$. We write $x = (x_0, x_1)$. Then $Y = (\rho, \rho x_0, \rho x_1)$. It follows from (5.5) that $\langle Y, \boldsymbol{c}\rangle = \rho r x_0 = 1$, which implies that $x_0 > 0$ and $x(\boldsymbol{M}) \subset \boldsymbol{S}^n_+$.

Now, let $\tau : \boldsymbol{H}^n \to \boldsymbol{S}^n_+$ be the conformal diffeomorphsim defined by (1.4) and $y = \tau^{-1} \circ x : \boldsymbol{M} \to \boldsymbol{H}^n \subset \boldsymbol{R}^{n+1}_1$. Since $\langle Y, \boldsymbol{c}\rangle = \rho r x_0 = 1$, we get $x_0 = 1/r\rho$. By (1.4) we get $y_0 = 1/x_0 = r\rho$ and

$$Y = (\rho, \rho x_0, \rho x_1) = \left(\frac{y_0}{r}, \frac{1}{r}, \frac{y_1}{r}\right). \tag{5.15}$$

It follows that

$$g = \langle dY, dY\rangle = r^{-2}\langle dy, dy\rangle\,. \tag{5.16}$$

The Laplace operator $\Delta_{\boldsymbol{M}}$ of $\langle dy, dy\rangle$ is given by $\Delta_{\boldsymbol{M}} = r^{-2}\Delta$. By (5.4) and (2.4) we have

$$-\frac{1}{m}\Delta Y + \frac{r^2}{2}Y = -\frac{r^2}{2}Y + (0, r, 0)\,, \tag{5.17}$$

which is equivalent to the equation

$$\Delta_{\boldsymbol{M}} y - my = 0\,. \tag{5.18}$$

Thus $y = \tau^{-1} \circ x : \boldsymbol{M} \to \boldsymbol{H}^n$ is a minimal submanifold. Since the Möbius metric g has constant scalar curvature, we know from (5.16) that $y : \boldsymbol{M} \to \boldsymbol{H}^n$ has constant scalar curvature.

Thus we complete the proof of the classification theorem.

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