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MODAL INTERPRETATIONS OF THREE VALUED LOGICS. I

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0 *Introduction* The present paper* extends a result of Peter Woodruff's reported in [1], to the effect that the three-valued logic Ł of Łukasiewicz may be interpreted as a modal system. Woodruff obtains his result by constructing a mapping from the wffs of Ł to those of the modal system S5. A definition is then produced which gives directions for the construction of interpretations of Ł from interpretations of S5, and it is further shown that no interpretation of Ł fails to be thus obtainable. The result is of interest especially because it has been argued that Ł cannot plausibly be viewed as a modal system, even though Łukasiewicz himself viewed it as one.[1] Here the question of the existence of modal interpretations of Ł via other mappings into S5 is explored. In order that the present paper be self-contained, no familiarity with [1] is presupposed; but the reader familiar with that work will appreciate this author's indebtedness to it.

In what follows, we use 'p's and 'q's as syntactic variables for wffs of both Ł and S5, trusting the context to signal which system is under discussion. It will be convenient to use the bracketless Polish notation, presumed to be familiar.

1 *The Systems* Ł *and* S5[2] We suppose Ł to be constructed from a denumerably infinite set of atoms, the set of wffs then being the least set that both contains the atoms and has Cpq and Np as members whenever p and q are members. An interpretation I for Ł is any function from the set of wffs to $\{1, \frac{1}{2}, 0\}$ such that $I(Np) = 1 - I(p)$ and $I(Cpq) = \min(1, 1 - (I(p) - I(q)))$. A wff p of Ł is valid (contravalid) if, for every I, $I(p) = 1(0)$; otherwise p

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is said to be indeterminate. We take S5 to be constructed from the same set of atoms as Ł, the set of wffs thus being the least set containing the atoms and such that Cpq, Np, and Mp are members whenever p and q are.

A K-interpretation I_K for S5 in a non-empty set K (of possible worlds, if you like) is any function from wffs of S5 to subsets of K such that $I_K(Np) = \overline{I_K(p)}$ (i.e., the complement of $I_K(p)$ with respect to K), $I_K(Mp) = {*I_K(p)}$ (where '$*$' is an operation defined on subsets of K as follows: $*\emptyset = \emptyset$; $*G = K$ for every other $G \subseteq K$), and $I_K(Cpq) = \overline{I_K(p)} \cup I_K(q)$. A wff p of S5 is said to be valid (contravalid) in K if for every K-interpretation I_K of S5 in K, $I_K(p) = K(\emptyset)$. A wff p of S5 is said to be valid (contravalid) simpliciter, if p is valid (contravalid) in every K. A wff p of S5 will be said to be indeterminate if it is neither valid nor contravalid. (To facilitate the exposition, we will henceforth speak of interpretations I, I', etc. for S5 in a set K, but the reader should note that the choice of interpretations for S5 is relative to a given K, as the more cumbersome notation suggests.) The following definitions are adopted, the first for Ł only,[3] the fifth and sixth for S5 only,[4] the rest for both Ł and S5:

(D1) $Mp =_{df} CNpp$	(D4) $Lp =_{df} NMNp$
(D2) $Apq =_{df} CCpqq$	(D5) $C_tp =_{df} KpMNp$
(D3) $Kpq =_{df} NANpNq$	(D6) $C_fp =_{df} KNpMp$.[5]

2 *Eight modal interpretations of Ł* In this section we develop eight mappings from the wffs of Ł into those of S5, and show of each that it yields an interpretation of Ł in modal terms. One of our eight mappings is in fact the one reported by Woodruff in [1], and we obtain the aforementioned result by generalizing arguments Woodruff produced in obtaining the result for his mapping. Each mapping will be denoted by a lower case 'f' with numerical superscripts and subscripts. Intuitively, the superscript indicates how to translate negations, the subscript, how to translate conditionals. We use 'n' as a variable for the integers 1 and 2, 'm', for the integers between 0 and 5 (exclusive). The mappings are defined as follows for all wffs p, q of Ł:

(1) where p is atomic, $f_m^n p = p$
(2) $f_m^1 Np = Nf_m^1 p$
(3) $f_m^2 Np = AC_t f_m^2 p\ LNf_m^2 p$
(4) $f_1^n Cpq = KCLf_1^n pf_1^n qCf_1^n pMf_1^n q$
(5) $f_2^n Cpq = KCLf_2^n pf_2^n qCMf_2^n pAf_2^n pMf_2^n q$
(6) $f_3^n Cpq = KCLf_3^n pALf_3^n qNf_3^n qCf_3^n pMf_3^n q$
(7) $f_4^n Cpq = KCLf_4^n pALf_4^n qC_f f_4^n qCMf_4^n pAf_4^n pMf_4^n q$

(f_1^1 is the mapping due to Woodruff.)

Before turning to the proof that each mapping yields an interpretation of Ł in modal terms, we pause to note some features of the mappings, so that the ensuing arguments are simplified. It is well known that, given the values that an interpretation I for S5 (in a given K) assigns to the atomic wffs occurring in a wff p (of S5), one can calculate the value of $I(p)$. However, we require more than this of the wffs of S5 onto which we map those of Ł. Intuitively, we will associate below the value K with the value 1, the value $\emptyset$ with the value 0, and the remaining S5 values with the value $\frac{1}{2}$. As a result, our translations of the wffs of Ł must be such that we can determine their values to be K, $\emptyset$, or neither of these on a given interpretation I just from the information that the components of the translations are assigned K, $\emptyset$ or neither of these by I. More precisely, if we let 'p' and 'q' represent arbitrary wffs of Ł, and 'J' and 'H' represent arbitrary subsets of K other than K and $\emptyset$ then our translations are characterized by the following matrices[6,7]:

$f_m^n q$ \ $f_m^n p$	$f_m^1 Np$	$f_m^2 Np$	$f_1^n Cpq$			$f_2^n Cpq$			$f_3^n Cpq$			$f_4^n Cpq$		
			K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$
K	$\emptyset$	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	$\overline{H}$	$\emptyset$	K	$\overline{H}$	$\emptyset$
J	$\overline{J}$	J	K	K	$\overline{J}$	K	K	J	K	K	$\overline{J}$	K	K	J
Φ	K	K	K	K	K	K	K	K	K	K	K	K	K	K

We will appeal to these matrices freely below. We now justify each matrix, but the reader uninterested in these details may omit the materials within the asterisks without loss of continuity:

Let I be an arbitrary interpretation for S5 in a given K, and p, q be arbitrary wffs of Ł.

$$I(f_m^1 Np) = I(Nf_m^1 p),$$

so the first matrix is self-explanatory.

$$I(f_m^2 Np) = I(AC_t f_m^2 pLNf_m^2 p) = I(C_t f_m^2 p) \cup I(LNf_m^2 p).$$

Suppose first that $I(f_m^2 p) = K$. In this case, $I(C_t f_m^2 p) = \emptyset$ and $I(LNf_m^2 p) = \emptyset$, hence $I(f_m^2 Np) = \emptyset$. Suppose then that $I(f_m^2 p) = J$. Then $I(f_m^2 C_t p) = J$ but $I(LNf_m^2 p) = \emptyset$, so $I(f_m^2 Np) = J$. Suppose finally that $I(f_m^2 p) = \emptyset$. Then $I(f_m^2 Np) = K$ since $I(LNf_m^2 p) = K$. Thus the matrix for $I(f_m^2 Np)$.

$$\begin{aligned} I(f_1^n Cpq) &= I(KCLf_1^n pf_1^n qCf_1^n pMf_1^n q) \\ &= I(CLf_1^n pf_1^n q) \cap I(Cf_1^n pMf_1^n q) \\ &= [\overline{I(Lf_1^n p)} \cup I(f_1^n q)] \cap [\overline{I(f_1^n p)} \cup I(Mf_1^n q)] \\ &= [\overline{\overline{*\overline{I(f_1^n p)}}} \cup I(f_1^n q)] \cap [\overline{I(f_1^n p)} \cup * I(f_1^n q)] \\ &= [* I(f_1^n p) \cup I(f_1^n q)] \cap [\overline{I(f_1^n p)} \cup * I(f_1^n q)] \end{aligned}$$

It is easy to see that both sides of the last intersection (and so the whole intersection) equal K if either $I(f_1^n q) = K$ or $I(f_1^n p) = \emptyset$; it is also clear that if $I(f_1^n p) = K$ and $I(f_1^n q) = \emptyset$, $I(f_1^n Cpq) = \emptyset$. So suppose first that $I(f_1^n p) = J$ and $I(f_1^n q) = H$. In this case $I(f_1^n Cpq) = [* \overline{J} \cup H] \cap [\overline{J} \cup * H] = [K \cup H] \cap [\overline{J} \cup K] = K \cap K = K$. Suppose next that $I(f_1^n p) = K$ and $I(f_1^n q) = H$. Now $I(f_1^n Cpq) = [* \overline{K} \cup H] \cap [\overline{K} \cup H] = [* \emptyset \cup H] \cap [\emptyset \cup H] = H \cap H = H$. Finally, suppose $I(f_1^n p) = J$ and $I(f_1^n q) = \emptyset$. In this case $I(f_1^n Cpq) = [* \overline{J} \cup \emptyset] \cap [\overline{J} \cup \emptyset] = K \cap \overline{J} = \overline{J}$. Thus the matrix for $f_1^n Cpq$.

$$\begin{aligned} I(f_2^nCpq) &= I(KCLf_2^npf_2^nqCMf_2^npAf_2^npMf_2^nq) \\ &= I(CLf_2^npf_2^nq) \cap I(CMf_2^npAf_2^npMf_2^nq) \\ &= [\overline{I(Lf_2^np)} \cup I(f_2^nq)] \cap [\overline{I(Mf_2^np)} \cup (I(f_2^np) \cup I(Mf_2^nq))] \\ &= [\overline{\overline{*\,\overline{I(f_2^np)}}} \cup I(f_2^nq)] \cap [\overline{*\,I(f_2^np)} \cup (I(f_2^np) \cup *\,I(f_2^nq))] \\ &= [*\,\overline{I(f_2^np)} \cup I(f_2^nq)] \cap [\overline{*\,I(f_2^np)} \cup (I(f_2^np) \cup *\,I(f_2^nq))] \end{aligned}$$

The left side of this intersection has value $\emptyset$ if $I(f_2^np) = K$ and $I(f_2^nq) = \emptyset$, so $I(f_2^nCpq) = \emptyset$ in this case. It is also fairly clear that both sides of the intersection equal K if either $I(f_2^np) = \emptyset$ or $I(f_2^nq) = K$, and so $I(f_2^nCpq) = K$ in these cases. So suppose first that $I(f_2^np) = J$ and $I(f_2^nq) = H$. In this case $I(f_2^nCpq) = [*\,\bar{J} \cup H] \cap [\overline{*\,J} \cup (J \cup *\,H)] = (K \cup \ldots] \cap [\ldots \cup K] = K$. Suppose then that $I(f_2^np) = K$ and $I(f_2^nq) = H$. Then $I(f_2^nCpq) = [*\,\bar{K} \cup H] \cap [\overline{*\,K} \cup (K \cup H)] = [\emptyset \cup H] \cap K = H$. Finally, suppose $I(f_2^np) = J$ and $I(f_2^nq) = \emptyset$. In this case, $I(f_2^nCpq) = [*\,\bar{J} \cup \emptyset] \cap [\overline{*\,J} \cup (J \cup \emptyset)] = K \cap J = J$. Thus the matrix for f_2^nCpq.

$$\begin{aligned} I(f_3^nCpq) &= I(KCLf_3^npALf_3^nq\mathbf{N}f_3^nqCf_3^npMf_3^nq) \\ &= I(CLf_3^npALf_3^nqNf_3^nq) \cap I(Cf_3^npMf_3^nq) \\ &= [\overline{I(Lf_3^np)} \cup I(ALf_3^nqNf_3^nq)] \cap [\overline{I(f_3^np)} \cup I(Mf_3^nq)] \\ &= [\overline{\overline{*\,\overline{I(f_3^np)}}} \cup (I(Lf_3^nq) \cup I(Nf_3^nq))] \cap [\overline{I(f_3^np)} \cup *\,I(f_3^nq)] \\ &= [*\,\overline{I(f_3^np)} \cup (\overline{*\,\overline{I(f_3^nq)}} \cup \overline{I(f_3^nq)})] \cap [\overline{I(f_3^np)} \cup *\,I(f_3^nq)] \end{aligned}$$

The right side of this intersection equals $\emptyset$ when $I(f_3^np) = K$ and $I(f_3^nq) = \emptyset$, and so $I(f_3^nCpq) = \emptyset$ in this case. Again it is readily verified that $I(f_3^nCpq) = K$ if either $I(f_3^np) = \emptyset$ or $I(f_3^nq) = K$. So suppose first that $I(f_3^np) = J$ and $I(f_3^nq) = H$. In this case $I(f_3^nCpq) = [*\,\bar{J} \cup \ldots] \cap [\ldots \cup *\,H] = K \cap K = K$. Suppose then that $I(f_3^np) = K$ and $I(f_3^nq) = H$. Then $I(f_3^nCpq) = [*\,\bar{K} \cup (*\,\bar{\bar{H}} \cup \bar{H})] \cap [\bar{K} \cup *\,H] = [\emptyset \cup (\emptyset \cup \bar{H})] \cap [\emptyset \cup K] = \bar{H} \cap K = \bar{H}$. Finally, suppose $I(f_3^np) = J$ and $I(f_3^nq) = \emptyset$. In this case $I(f_3^nCpq) = [*\,\bar{J} \cup \ldots] \cap [\bar{J} \cup *\,\emptyset] = K \cap \bar{J} = \bar{J}$. Thus the matrix for f_3^nCpq.

$$\begin{aligned} I(f_4^nCpq) &= I(KCLf_4^npALf_4^nqC_ff_4^nqCMf_4^npAf_4^npMf_4^nq) \\ &= I(CLf_4^npALf_4^nqC_ff_4^nq) \cap I(CMf_4^npAf_4^npMf_4^nq) \\ &= [\overline{I(Lf_4^np)} \cup I(ALf_4^nqC_ff_4^nq)] \cap [\overline{I(Mf_4^np)} \cup I(Af_4^npMf_4^nq)] \\ &= [\overline{\overline{*\,\overline{I(f_4^np)}}} \cup (I(Lf_4^nq) \cup I(C_ff_4^nq))] \cap \\ &\qquad [*\,I(f_4^np) \cup (I(f_4^np) \cup I(Mf_4^nq))] \\ &= [*\,\overline{I(f_4^np)} \cup (\overline{\overline{*\,\overline{I(f_4^nq)}}} \cup (I(f_4^nq) \cap *\,\overline{I(f_4^nq)}))] \cap \\ &\qquad [\overline{*\,I(f_4^np)} \cup (I(f_4^np) \cup *\,I(f_4^nq))] \end{aligned}$$

If $I(f_4^np) = K$ and $I(f_4^nq) = \emptyset$, the left side of this intersection equals $\emptyset$, so $I(f_4^nCpq) = \emptyset$ in that case. Again it is easily verified that if $I(f_4^np) = \emptyset$ or $I(f_4^nq) = K$, both sides of the intersection are equal to K and hence $I(f_4^nCpq) = K$ in these cases. So suppose first that $I(f_4^np) = J$ and $I(f_4^nq) = H$. In this case, $I(f_4^nCpq) = [*\,\bar{J} \cup \ldots] \cap [\ldots \cup *\,H] = K \cap K = K$. Suppose next that $I(f_4^np) = K$ and $I(f_4^nq) = H$. Now $I(f_4^nCpq) = [*\,\bar{K} \cup (*\,\bar{\bar{H}} \cup (\bar{H} \cap *\,H))] \cap [\ldots \cup *\,H] = [\emptyset \cup (\emptyset \cup \bar{H})] \cap K = \bar{H}$. Finally, suppose $I(f_4^np) = J$ and

$I(f_4^n q) = \emptyset$. In this case $I(f_4^n Cpq) = [*\bar{J} \cup \ldots] \cap [\overline{*J} \cup (J \cup *\emptyset)] = K \cap [\emptyset \cup J] = K \cap J = J$. Thus the matrix for $f_4^n Cpq$.

By means of the following definitions and theorems (holding for every $n (1 \leqslant n \leqslant 2)$ and $m (1 \leqslant m \leqslant 4)$), we now show that each mapping yields an interpretation of Ł in modal terms. For any interpretation I of S5 (in a given set K), let If_m^n be the function from wffs of Ł to $\{1, \frac{1}{2}, 0\}$ defined thus:

$$If_m^n(p) = \begin{cases} 1, \text{ if } I(f_m^n p) = K \\ 0, \text{ if } I(f_m^n p) = \emptyset \\ \frac{1}{2} \text{ otherwise} \end{cases}$$

Theorem 1 *If_m^n is an interpretation of* Ł.

Proof: It suffices to show:

(a) $If_m^n(Np) = 1 - If_m^n(p)$;
(b) $If_m^n(Cpq) = \min(1, 1 - (If_m^n(p) - If_m^n(q)))$.

For proof of (a): $If_m^n(Np) = 1$ iff $K = I(f_m^n Np)$. But as the matrices show, $K = I(f_m^n Np)$ iff $I(f_m^n p) = \emptyset$, and $I(f_m^n p) = \emptyset$ iff $0 = If_m^n(p)$. Hence $If_m^n(Np) = 1$ iff $1 = 1 - If_m^n(p)$. Similarly, $If_m^n(Np) = 0$ iff $\emptyset = I(f_m^n Np)$, and again the matrices show that $\emptyset = I(f_m^n Np)$ iff $I(f_m^n p) = K$. But $I(f_m^n p) = K$ iff $1 = If_m^n(p)$. Hence $If_m^n(Np) = 0$ iff $0 = 1 - If_m^n(p)$. This suffices for (a).

For (b): $If_m^n(Cpq) = 1$ iff $K = I(f_m^n Cpq)$. But as the matrices indicate, $K = I(f_m^n Cpq)$ iff at least one of the following holds: (i) $I(f_m^n p) = \emptyset$; (ii) $I(f_m^n q) = K$; or (iii) both $I(f_m^n q) \neq \emptyset$ and $I(f_m^n p) \neq K$. Note that (i)-(iii) are respectively equivalent to: (i′) $If_m^n(p) = 0$; (ii′) $If_m^n(q) = 1$; and (iii′) both $If_m^n(q) \neq 0$ and $If_m^n(p) \neq 1$. Since $\min(1, 1 - (If_m^n(p) - If_m^n(q))) = 1$ iff at least one of (i′)-(iii′) holds, $If_m^n(Cpq) = 1$ iff $1 = \min(1, 1 - (If_m^n(p) - If_m^n(q)))$. Likewise $If_m^n(Cpq) = 0$ iff $\emptyset = I(f_m^n Cpq)$, and the matrices indicate that $\emptyset = I(f_m^n Cpq)$ iff both $I(f_m^n p) = K$ and $I(f_m^n q) = \emptyset$. But $I(f_m^n p) = K$ and $I(f_m^n q) = \emptyset$ iff $If_m^n(p) = 1$ and $If_m^n(q) = 0$, the last holding just in case $\min(1, 1 - (If_m^n(p) - If_m^n(q))) = 0$. Thus $If_m^n(Cpq) = 0$ iff $0 = \min(1, 1 - (If_m^n(p) - If_m^n(q)))$. This suffices for (b) and the theorem is proved.

Theorem 2 *For any interpretation I of* Ł *there is an interpretation $\mathcal{I}$ of* S5 *such that $I = \mathcal{I}f_m^n$.*

Proof: Let I be an arbitrary interpretation of Ł. We define $\mathcal{I}$ in $\mathbf{K}(=\{1, 0\})$ as follows for atomic p: if $I(p) = 1$, $\mathcal{I}(p) = \mathbf{K}$; if $I(p) = 0$, $\mathcal{I}(p) = \emptyset$; $\mathcal{I}(p) = \{0\}$ otherwise. It is well known of both Ł and S5 that any interpretation of the atoms determines a unique interpretation for the system. Hence we have characterized an interpretation of S5 by our definition of $\mathcal{I}$; moreover, it is clear by construction that I and $\mathcal{I}f_m^n$ agree on the atoms of Ł, and hence are the same interpretation of Ł.

Theorem 3 *For every wff p of* Ł:

(a) *p is valid (in* Ł*) iff $f_m^n p$ is valid (in* S5*).*
(b) *p is contravalid (in* Ł*) iff $f_m^n p$ is contravalid (in* S5*).*
(c) *p is indeterminate (in* Ł*) iff $f_m^n p$ is indeterminate (in* S5*).*

Proof: (a) and (b). Suppose first that $f_m^n p$ is not (contra-)valid in S5. Then by Theorem 1 p is not (contra-)valid in Ł either. So suppose on the other hand that p is not (contra-)valid in Ł. Then by Theorem 2 there is an interpretation for S5 in **K** that does not assign $f_m^n p(\emptyset)$ **K**, and hence $f_m^n(p)$ is not (contra-)valid in S5. Hence (a) and (b).

(c) Proof immediate from (a) and (b).

3 *A sense in which the mappings are exhaustive* In this section we will sketch a proof to the effect that, within certain constraints, the list of mappings from Ł to S5 we have provided is exhaustive. We require some additional terminology. The wffs p, q of S5 will be said to be strictly equivalent on a given interpretation I of S5 if $I(p) = I(q)$; we indicate this in symbols by writing $p \underset{I}{\leftrightarrow} q$. And p, q will be said to be semantically equivalent (in S5) if $p \underset{I}{\leftrightarrow} q$ for every I (in symbols: $p \equiv q$). Finally, the mappings m and m' from the wffs of Ł to those of S5 will be said to be equivalent if $mp \equiv m'p$ (in S5) for every wff p of Ł. We will also employ the following notational device: where $p, \ldots, q$ are wffs of Ł (S5), $\mathcal{A}(p, \ldots, q)$ is to be taken as denoting a wff of Ł (S5) compounded from just $p, \ldots, q$ by means of the usual formation rules. Other script Roman capitals will appear in place of $\mathcal{A}$ when clarity so demands, their role being exactly similar.

Now let @ be a mapping from wffs of Ł into those of S5, with @ presumed to satisfy both of the following conditions.

Condition 1: @ has a definition of the following form: For all wffs p, q of Ł:

(i) if p is atomic, $@p = p$.
(ii) $@Np = \mathcal{A}(@p)$
(iii) $@Cpq = \mathcal{B}(@p@q)$.[8]

Condition 2: @ is such that the following definition, in which p is a variable for wffs of Ł and I is an arbitrary interpretation of S5 in an arbitrary K, guarantees that the appropriate analogues of Theorems 1-3 hold true of $I@$[9]:

$$I@(p) = \begin{cases} 1, \text{ if } I(@p) = K \\ 0, \text{ if } I(@p) = \emptyset \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

We now show that @ is equivalent to some f_m^n. In the following lemmas and theorem, K is an arbitrarily chosen set and I an arbitrary interpretation for S5 in K. As before, J and H are arbitrary non-empty proper subsets of K.

Lemma 1 *Let $I(p) = J$. Then one of the following is sure to hold for $\mathcal{A}(p)$:*

(i) $I(\mathcal{A}(p)) = K$
(ii) $I(\mathcal{A}(p)) = \emptyset$
(iii) $I(\mathcal{A}(p)) = J$
(iv) $I(\mathcal{A}(p)) = \bar{J}$.

Proof: By strong induction on the length of $\mathcal{A}(p)$. Details left to the reader. In the basis, $\mathcal{A}(p) = p$. In the inductive step, there are three cases: $\mathcal{A}(p) = N\mathcal{B}(p)$; $\mathcal{A}(p) = M\mathcal{B}(p)$; $\mathcal{A}(p) = C\mathcal{B}(p)\mathcal{C}(p)$. The hypothesis is of course that (i)-(iv) hold true of any wff $\mathcal{B}(p)$ less complex than $\mathcal{A}(p)$.

Lemma 2 *If* $p \underset{I}{\leftrightarrow} q$, *then* $\mathcal{A}(\ldots, p, \ldots) \underset{I}{\leftrightarrow} \mathcal{A}(\ldots, q, \ldots)$.

Proof left to the reader. (The lemma records an obvious consequence of the familiar fact that strict equivalence is preserved in S5 under substitution of strict equivalents).

Lemma 3 *Let* $I(p) = K$ *and* $I(q) = H$. *Then one of the following is sure to hold for* $\mathcal{A}(p, q)$:

(i) $I(\mathcal{A}(p, q)) = K$
(ii) $I(\mathcal{A}(p, q)) = \emptyset$
(iii) $I(\mathcal{A}(p, q)) = H$
(iv) $I(\mathcal{A}(p, q)) = \overline{H}$

Proof: Since $I(p) = K$ and $I(q) = H$, $I(Mq)$ $(= * H = K) = I(p)$. Hence, $p \underset{I}{\leftrightarrow} Mq$, so by Lemma 2, $\mathcal{A}(p, q) \underset{I}{\leftrightarrow} \mathcal{A}(Mq, q)$. But $\mathcal{A}(Mq, q)$ qualifies as a compound $\mathcal{B}(q)$ of q, hence Lemma 3 by Lemma 1.

Lemma 4 *Let* $I(p) = J$ *and* $I(q) = \emptyset$. *Then one of the following is sure to hold for* $\mathcal{A}(p, q)$:

(i) $I(\mathcal{A}(p, q)) = K$
(ii) $I(\mathcal{A}(p, q)) = \emptyset$
(iii) $I(\mathcal{A}(p, q)) = J$
(iv) $I(\mathcal{A}(p, q)) = \overline{J}$

Proof: Like that for Lemma 3.

Lemma 5

(a) *If* $I(@\, p) = \emptyset$, *then* $I(@\, Np) = K$.
(b) *If* $I(@\, p) = K$, *then* $I(@\, Np) = \emptyset$
(c) *If* $I(@\, p) = J$, *then either* $I(@\, Np) = J$ *or* $I(@\, Np) = \overline{J}$.

Proof: (a) Suppose $I(@\, p) = \emptyset$. Then $I@\,(p) = 0$, in which case $I@\,(Np) = 1$ (since $I@$ is an interpretation of Ł by Theorem 1). But $I@\,(Np) = 1$ iff $I(@\, Np) = K$. Hence, (a). (b) Proof like that of (a). (c) Suppose $I(@\, p) = J$. Then $I@\,(p) = \frac{1}{2}$, in which case $I@\,(Np) = \frac{1}{2}$, this last holding iff $K \neq I(@\, Np) \neq \emptyset$. But by Lemma 1, together with our supposition that $I(@\, p) = J$, one of the following holds:

(i) $I(@\, Np) = K$
(ii) $I(@\, Np) = \emptyset$
(iii) $I(@\, Np) = J$
(iv) $I(@\, Np) = \overline{J}$.

Hence, since neither (i) nor (ii), (c).

Lemma 6

(a) *If* $I(@p) = K$ *and* $I(@q) = \emptyset$, *then* $I(@Cpq) = \emptyset$

(b) *If any of the following obtains:*

(i) $I(@p) = \emptyset$

(ii) $I(@q) = K$

(iii) $I(@p) \neq K$ and $I(@q) \neq \emptyset$,

then $I(@Cpq) = K$.

(c) *If both* $I(@p) = K$ *and* $I(@q) = H$, *then either* $I(@Cpq) = H$ *or* $I(@Cpq) = \overline{H}$.

(d) *If both* $I(@p) = J$ *and* $I(@q) = \emptyset$, *then either* $I(@Cpq) = J$ *or* $I(@Cpq) = \overline{J}$.

Proof: Proof of (a) and (b) like proof of Lemma 5 (a) and Lemma 5 (b). Proof of (c) like proof of Lemma 5(c), using Lemma 3 in place of Lemma 1. Proof of (d) like proof of (c), using Lemma 4 in place of Lemma 3.

Theorem 4 *@ is equivalent to some* f_m^n.

Proof: @ and each f_m^n have the same values for atomic arguments. Consider then the possible matrices for $@Np$. By Lemma 5, these will be identical to the matrices for f_m^1Np and f_m^2Np, so suppose the actual matrix for $@Np$ is identical to that for f_m^jNp. Next consider the possible matrices for $@Cpq$. By Lemma 6, these will be identical to the matrices for $f_1^nCpq, \ldots, f_4^nCpq$, and suppose here that the actual matrix for $@Cpq$ is identical to that for f_k^nCpq. A straightforward induction (omitted here) shows that $@p \equiv f_k^jp$ for every wff p of Ł, and thus proves @ to be equivalent to f_k^j.

4 *Peculiarities of the mappings* The reader familiar with [1] may recall that the mapping devised by Woodruff (f_1^1, in the present paper) suffers from a certain defect: it does not preserve semantic equivalence. Given that $p \equiv q$ in Ł, Woodruff has shown that it is not always the case that $f_1^1p \equiv f_1^1q$ in S5.[10] In this section we show that none of the mappings constructed thus far preserves semantic equivalence. This result is of all the more significance since the preceding section shows that we have exhausted the ways of translating '*N*' and '*C*' into S5. Of course, if one is willing to modify the syntax for S5, further translations may become possible.[11]

We will show that none of the four ways of translating '*C*' preserves equivalence by means of a test case. It is well known that in Ł $Apq \equiv Aqp$ (in primitives, that $CCpqq \equiv CCqpp$); but we now show that in S5 $f_m^nApq \not\equiv f_m^nAqp$ ($f_m^nCCpqq \not\equiv f_m^nCCqpp$). We proceed by constructing matrices, taking it that two wffs are semantically equivalent iff they have identical matrices. To construct the matrix for f_1^nCCpqq, recall first the matrix for f_1^nCpq:

f_1^np \ f_1^nq	K	H	$\emptyset$
K	K	H	$\emptyset$
J	K	K	$\overline{J}$
$\emptyset$	K	K	K

It is useful to note that this matrix simply defines a function from pairs of subsets of K to subsets of K. If we denote this function by 'F_1' and think of '$f_1^1 p$' and '$f_1^n q$' as variables for subsets of K, the matrix may be thought of as shorthand for the following definition:

$$F_1(f_1^n p, f_1^n q) = \begin{cases} f_1^n q, \text{ if } f_1^n p = K \\ K, \text{ if } f_1^n p = \emptyset \\ * f_1^n q \cup f_1^n p \text{ otherwise.} \end{cases}$$

The matrix for $f_1^n CCpqq$ will similarly be a shorthand description of the compound function $F_1(F_1(f_1^n p, f_1^n q), f_1^n q)$, and the following is readily verified to be that matrix:

$f_1^n p$ \ $f_1^n q$	K	H	$\emptyset$
K	K	K	K
J	K	H	J
$\emptyset$	K	H	$\emptyset$

By analogous reasoning, the matrix for $f_1^n CCqpp$ may be gotten from the matrix for $f_1^n Cqp$:

$f_1^n p$ \ $f_1^n q$	$f_1^n Cqp$			$f_1^n CCqpp$		
	K	H	$\emptyset$	K	H	$\emptyset$
K	K	K	K	K	K	K
J	J	K	K	K	J	J
$\emptyset$	$\emptyset$	$\overline{H}$	K	K	H	$\emptyset$

And similarly by reflecting on these matrices,

$f_m^n p$ \ $f_m^n q$	$f_2^n Cpq$			$f_2^n Cqp$			$f_3^n Cpq$			$f_3^n Cqp$			$f_4^n Cpq$			$f_4^n Cqp$		
	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$
K	K	H	$\emptyset$	K	K	K	K	$\overline{H}$	$\emptyset$	K	K	K	K	$\overline{H}$	$\emptyset$	K	K	K
J	K	K	J	J	K	K	K	K	$\overline{J}$	$\overline{J}$	K	K	K	K	J	$\overline{J}$	K	K
$\emptyset$	K	K	K	$\emptyset$	H	K	K	K	K	$\emptyset$	$\overline{H}$	K	K	K	K	$\emptyset$	H	K

we get these:

$f_m^n p$ \ $f_m^n q$	$f_2^n CCpqq$			$f_2^n CCqpp$			$f_3^n CCpqq$			$f_3^n CCqpp$			$f_4^n CCpqq$			$f_4^n CCqpp$		
	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$
K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K
J	K	H	J	K	J	J	K	$\overline{H}$	J	K	$\overline{J}$	$\overline{J}$	K	$\overline{H}$	J	K	$\overline{J}$	$\overline{J}$
$\emptyset$	K	H	$\emptyset$	K	H	$\emptyset$	K	$\overline{H}$	$\emptyset$	K	H	$\emptyset$	K	$\overline{H}$	$\emptyset$	K	H	$\emptyset$

The matrices thus show that $f_m^n Apq \not\equiv f_m^n Aqp$.

5 *Conclusions and further issues* Those who are not disconcerted by the result of the immediately preceding section may be interested to know that the eight mappings may be multiplied by treating the defined connectives of Ł as primitive and compounding mappings from our original eight.

For example, consider the following definition of a mapping m, like f_3^2 except in clause (iv), which is like $f_1^n CCpqq$:

(i) where p is atomic, $mp = p$.
(ii) $mNp = Ac_t mp\ LNmp$
(iii) $mCpq = KCLmpALmqNmqCmpMmq$
(iv) $mApq = KCLKCLmpmqCmpMmqmqCKCLmpmqCmpMmqMmq$.

Theorem 1-3 will still hold of such mappings. The same, however, cannot be expected of Theorem 4 since these mappings do not have a definition in the requisite form. Of course, no such compound mappings will preserve semantic equivalence.

Those of us of a more conservative bent, on the other hand, may find the peculiarities of the mappings discussed here somewhat objectionable. A sequel to the present paper will show that if S5 is modified slightly, the number of mappings from Ł to S5 is increased, and some of the new ones preserve semantic equivalence. The functionally complete version of Ł developed by Słupecki in [4] will be discussed in detail, so we do not dwell on the problem of giving a modal interpretation to that system here. But we do note in closing that the results for our mappings extend to the functionally complete case along the lines discussed by Woodruff in [1].

NOTES

1. As Woodruff notes in [1], Łukasiewicz advances the view that Ł is to be understood modally in [2], and this view receives criticism in Rescher [3] (see p. 98). It is interesting to note, though, that Woodruff's vindication of Łukasiewicz's view is foreshadowed by remarks of Turquette in [6] (see esp. p. 267).

2. Much of the material here follows Woodruff's account closely and is included only so that the present paper be independent of Woodruff's. However the account of an interpretation for S5 is an adaptation of material presented in Chapter XVII of [5].

3. Łukasiewicz attributes this definition to Tarski.

4. Intuitively, D5 and D6 define contingent truth and falsity. We hesitate to adopt these definitions for Ł since the definiens are semantically equivalent in Ł. Thus we find it philosophically preferable to say that the notions of contingent truth and falsity cannot be adequately represented in Ł.

5. At this point, we calculate from the definitions how the defined symbols of S5 are interpreted. This material will prove useful below. $I(Apq) = I(p) \cup I(q)$. $I(Kpq) = I(p) \cap I(q)$. $I(Lp) = {}^*\overline{\overline{I(p)}}$. $I(C_t p) = \Phi$ if either $I(p) = K$ or $I(p) = \Phi$; otherwise $I(C_t p) = I(p)$. $I(C_f p) = \Phi$ if either $I(p) = K$ or $I(p) = \Phi$; $I(C_f p) = \overline{I(p)}$ otherwise. It is also useful to note that $I(Lp) = K$ if $I(p) = K$, $I(Lp) = \Phi$ otherwise.

6. Notice that the arrays of Φ's and K's do not change from table to table. The remaining entries, however, guarantee that no two of the mappings are equivalent in the sense of section 3 (below).

7. In the arguments to follow, the fact that the entries of the matrices unambiguously denote either K, Φ, or neither of these plays an important role. Not all wffs of S5 have matrices that are unambiguous in this respect: e.g., the matrix for Kpq contains the entry '$J \cap H$' for the

case where p is assigned J and q is assigned H. But that entry is ambiguous, in a sense we cannot allow, since in some cases (i.e., for some values of J and H) it denotes $\emptyset$, and in others, it does not denote $\emptyset$. The reader familiar with [1] will find that these remarks compare interestingly with those on p. 436 of [1].

8. The constraint that where p is an atom, $@p = p$ might at first appear overly stringent. (The other constraints should, I hope, appear to be quite natural.) But there is nothing of great theoretical interest in allowing atoms of Ł to be mapped onto more complex wffs of S5, given that we require Th2 to hold of @ (see condition 2 below). In order to obtain Th2 for @, different atoms of Ł must be mapped onto wffs of S5 whose values can vary independently, because the values of the atoms of Ł vary independently. Suppose that @ does not map atoms into atoms, and let S^A be the set of values for @ at atomic arguments. Let S include S^A plus all the wffs compounded out of the members of S^A by means of the usual formation rules. The fragment of S5 made up of members of S is shown to be synonymous with the whole of S5 by mapping the members of S^A onto the atoms of S5 and reducing the other members of S accordingly.

9. By "appropriate analogues" is meant the result of replacing occurrences of 'f_m^n' by '@'.

10. The problem, intuitively, is that when p and q evaluate to 1/2 in Ł, they are not automatically assured of being mapped onto wffs of S5 that get assigned to the same proper non-empty subset of K. Woodruff comments that nonetheless the S5 translations are equivalent in the sense of both being contingent. But this seems rather strained.

11. In fact, further mappings do become possible. But discussion of this must be saved for a future time.

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