Modal logic without contraction in a metatheory without contraction

Patrick Girard and Zach Weber

Abstract

Standard reasoning about Kripke semantics for modal logic is almost always based on a background framework of classical logic. Can proofs for familiar definability theorems be carried out using a *non-classical substructural logic* as the metatheory? This paper presents a semantics for positive substructural modal logic and studies the connection between frame conditions and formulas, via definability theorems. The novelty is that all the proofs are carried out with a *non-contractive logic* in the background. This sheds light on which modal principles are invariant under changes of meta-logic, and provides (further) evidence for the general viability of non-classical mathematics.

1 Introduction: Repeating assumptions about possibility

Logics without the rule of *contraction* have been of interest for some time. Informally, contraction is the principle that if ψ follows from φ and φ, then ψ follows from φ. One reason for sustained interest in non-contractive logics is that they evade Curry's paradox, and so may be suitable for unrestricted set/property abstraction [4, 9, 10, 12, 21, 24]. And while substructural logics do yield consistent (or non-trivial) naive set theories, this is only a necessary, not sufficient, condition for being 'interesting'. The essential question—of just how much 'ordinary reasoning' can be carried out in a non-contractive metatheoretic framework[1]—has remained open.

In fact, there has been serious doubt about the viability of a non-contractive metatheory. Without contraction, after all, the following argument is *invalid*:

If p, then q

p

Therefore, p and q

[1] [18] has a detailed discussion of the interplay between classical and non-classical foundational frameworks for mathematics and beyond.

The conclusion fails because p was 'used up' already to get q; unless we assume contraction, it is no longer around. Similarly, the inference from p to $p \,\&\, p$ is invalid in a non-contractive setting. Without these seemingly basic inferences available, even proponents of such systems have expressed pessimism: Terui writes at the conclusion of his paper [24, p.38] that the system, light affine set theory (LAST),

> ...is hardly considered as a *working* system of mathematics, because the reasoning allowed by LAST is too poor to formalize proofs of mathematically interesting theorems.

This would appear to support a claim by Feferman, that "nothing like sustained ordinary reasoning can be carried out" in such logics [7].

In this paper we show by 'honest toil' that logics without contraction *are* able to support ordinary reasoning about *modality*, in the form of familiar proofs about Kripke frames. In particular we establish familiar frame correspondents for normal logics up to S5, as in [26], using non-contractive set theory as the modelling clay. Van Benthem says in [26, p.331]:

> Three pillars of wisdom support the edifice of Modal Logic. There is the ubiquitous Completeness Theory, the present Correspondence, or, more generally, Definability Theory, and finally, the Duality Theory between Kripke frames and 'modal algebras' ....

Completeness and Duality are out of reach for the time being. But we show that a non-contractive modal logician can get the first pillar up: Definability.

The main task of this paper, then, amounts to revisiting some elementary results about Kripke semantics, but newly respecting how many times assumptions are repeated in the course of a proof, e.g. how many times we need to reiterate a premise like 'world x is possible relative to world y'. On a practical level, we suggest a handful of useful tricks for working around contraction. The K axiom requires special attention, since it seems to encode a sort of contraction. The general theme is that familiar results are recoverable, if more attention is paid to the fine details; we observe how non-substructural reasoning is very blunt, often assuming much more than is required (for instance, that $a = b$, when it would do to assume that a and b have just one specific property in common). The logic we use is very minimal indeed, and can support expansions in several different directions—for example, adding a paraconsistent negation and reasoning about impossible worlds. But for now, we want to make this basic toolkit available for any purposes in which it matters how often assumptions are repeated.[2]

Section 2 presents the background logic and set theory. Section 3 defines the semantics. Section 4 is a study of the K axiom, and which of its forms

[2]This work is orthogonal to work giving *classical* structures as models for modal substructural logics, e.g. [15, 19, 27], or [1, 16]. It rather continues efforts to use non-classical structures, as floated in [11] and [28], and more broadly relates to the so-called 'classical recapture' problem for non-classical mathematics.

are obtainable without contraction from the frame conditions alone. Section 5 produces some standard definability theorems, and section 6 concludes. Section §4.2 includes some discussion of what can *not* be derived without contraction, but the emphasis throughout is proof theoretic and positive—on what *can* be shown.

2 Background Logic

The logic we use is a fragment of the substructural logic BCK, as studied in [20]; cf. [14, §7.25]. For simplicity, we keep the logic stripped down—in particular, with no negation.[3] Also for simplicity, similarly to [24], we only use one kind of conjunction (multiplicative) and one kind of disjunction (additive).[4] The language is then that of positive first order logic, with connectives $\&, \vee, \Rightarrow, \exists, \forall$. Letters $p, q, \ldots$ denote propositional atoms, $\varphi, \psi, \chi, \ldots$ denote well formed formulas built up in the usual way,[5] and big $\Gamma, \Delta, \ldots$ denote collections of formulas. Since, in the absence of contraction, the number of occurrences of a formula matters, these are *multisets*, collections that respect how many times a member occurs (so e.g. the multiset $[p, q]$ is not the same multiset as $[p, p, q]$).

Here is the logic presented as a Hilbert system. The logic as a Gentzen system is given in the appendix; that the two presentations correspond is a standard classical result (cf. [25]).

Axioms	
A1 (B)	$(\psi \Rightarrow \chi) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$
A2 (C)	$(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \chi))$
A3 (K)	$\varphi \Rightarrow (\psi \Rightarrow \varphi)$
A4	$\varphi \Rightarrow \varphi \vee \psi \qquad \psi \Rightarrow \varphi \vee \psi$
A5	$(\varphi \Rightarrow \chi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \vee \psi \Rightarrow \chi))$
A6	$\varphi \Rightarrow (\psi \Rightarrow \varphi \,\&\, \psi)$
A7	$(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi \,\&\, \psi \Rightarrow \chi)$
A8	$\forall x\varphi \Rightarrow \varphi^x_t$
A9	$\varphi^x_t \Rightarrow \exists x\varphi$
	t is any term substitutable for x in φ.
A10	$\forall x(\varphi \Rightarrow \psi) \Rightarrow (\exists y\varphi^x_y \Rightarrow \psi)$
A11	$\forall x(\psi \Rightarrow \varphi) \Rightarrow (\psi \Rightarrow \forall y\varphi^x_y)$
	x is not free in ψ, $x = y$, or y is not free in φ.

[3] Once some set theory is introduced, we can define an absurdity constant $\perp$ and a corresponding negation $\varphi \Rightarrow \perp$, but no use is made of this.

[4] The standard presentation of BCK has the full suite of connectives—implication, additive conjunction and disjunction, multiplicative conjunction and disjunction (as in [20]).

[5] Proof that induction works in this setting is in [2]. That said, there are many details we are skimming over here, since our aim in this note is to carry out one small ‘proof of concept’ exercise (on Kripke frames) in this system, *not* to develop the ‘foundations’ for all mathematics and logic in substructural set theory; the latter is a project-sized open problem, some small steps into which may be found in [24] and relatedly [22].

Rules		
MP	$\varphi, \varphi \Rightarrow \psi \vdash \psi$	
Generalisation	$\varphi^x_y \vdash \forall x \varphi$	($x = y$ or y not free in φ)

We say that φ is *deducible* from Γ, and we write $\Gamma \vdash \varphi$, when there is a sequence of formulas $\langle \varphi_0, ..., \varphi_n \rangle$ such that φ is φ_n, and for every φ_i, either $\varphi_i \in \Gamma$, φ_i is an instance of an axiom, or φ_i or results from application of a rule on previous lines that haven't been used in other applications of rules.[6]

The table below lists some derived theorems. Some particularly useful derived theorems are designated as *tricks*—recurring inferential moves that help navigate through the substructural proofs. They call for special attention where classically they would pass unnoticed; proofs in Appendix 2.

	Derivable
Axiom modus ponens	$\varphi \,\&\, (\varphi \Rightarrow \psi) \Rightarrow \psi$
B0	$(\varphi \Rightarrow \psi) \,\&\, (\chi \Rightarrow \xi) \Rightarrow (\varphi \,\&\, \chi \Rightarrow \psi \,\&\, \xi)$
B1	$\varphi \Rightarrow \varphi$
B2	$(\varphi \,\&\, \psi) \vee (\varphi \,\&\, \chi) \Rightarrow \varphi \,\&\, (\psi \vee \chi)$
B3	$\varphi \,\&\, (\psi \vee \chi) \Rightarrow (\varphi \,\&\, \psi) \vee (\varphi \,\&\, \chi)$
B4	$(\varphi \,\&\, \psi \Rightarrow \chi) \Leftrightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))$
Trick 1	$(\varphi \Rightarrow (\psi \Rightarrow \xi)) \Rightarrow ((\xi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi)))$
Trick 2	$(\varphi \,\&\, \psi) \,\&\, (\psi \Rightarrow \xi) \Rightarrow \varphi \,\&\, \xi$
Trick 3	$(\varphi \Rightarrow (\psi \Rightarrow \xi)) \,\&\, \psi \Rightarrow (\varphi \Rightarrow \xi)$.

where a biconditional is defined:

$$\varphi \Leftrightarrow \psi := (\varphi \Rightarrow \psi) \,\&\, (\psi \Rightarrow \varphi)$$

As flagged in the introduction, the appeal of this system is that one can add, without incoherence, the highly intuitive *naive set comprehension principle*. Extending the language with a binary relation $\in$, and a term-forming operator $\{x : \varphi\}$, then

Comprehension $x \in \{z : \varphi\} \Leftrightarrow \varphi$

[6] Contraction is avoided by taking due care with the implicit meaning of the clause "results from", to ensure that repeated assumptions are accounted for. In the course of a derivation from multiset Γ to φ, we associate with each member φ_k of the sequence of formulas a multiset Γ_k made up of exactly the previous members of the sequence that φ_k followed from (so Γ_k is empty if φ_k is an axiom or member of Γ). When all these 'premise sets' Γ_k are combined together in a (multiset) union, a derivation is valid only if every member of the union is already a member of Γ. In this way, for example, the valid $\psi, \psi \vdash \psi \;\&\; \psi$ does not reduce to $\psi \vdash \psi \;\&\; \psi$, because multiset $[\psi, \psi]$ is not included in the multiset $[\psi]$. Cf. the definition of a valid derivation in [2]. Any valid derivation will respect this 'tracking' of occurrences, we claim, because contraction is not valid in the language—this one—in which we are stating the definition of the Hilbert system; for elaboration of the idea that an 'object language' logic reflects the assumptions made in the 'metalogic', see [28]. For some problems with contraction and consequence, see [23], and for replies, and more on multisets, see [6].

is true for all formulas. Even in the positive implicational fragment of the logic the comprehension principle gives rise to fixed points, e.g. in the form of Curry's set $\{z : z \in z \Rightarrow \varphi\}$ for arbitrary φ; nevertheless, the logic above doesn't trivialise upon addition of this axiom, due to the absence of contraction.

Set-theoretic definitions are as usual:

Subset	$X \subseteq Y$	$:=$	$\forall z(z \in X \Rightarrow z \in Y)$
Union	$X \cup Y$	$:=$	$\{z : z \in X \vee z \in Y\}$
Binary intersection	$X \cap Y$	$:=$	$\{z : z \in X \ \& \ z \in Y\}$
Leibniz identity	$X = Y$	$:=$	$\forall z(X \in z \Leftrightarrow Y \in z)$

Leibniz identity supports substitution—if $X = Y$ then X and Y may be interchanged *salva veritate*.

However, with identity, proceed with caution. Property terms are quite *intensional*. That X and Y have the same extension (i.e., $\forall z(z \in X \Leftrightarrow z \in Y)$) does *not* imply that $X = Y$; extensional coincidence of X and Y will *not* validate substitution of X for Y, on pain of Grišin's paradox.[7] That's because identity *does* contract, in the sense that

$$X = Y \Rightarrow (X = Y \,\&\, X = Y)$$

since $(X = X) \,\&\, (X = X)$ is always true. Then if comprehension provides a term for every sentence of the language, contraction becomes valid on $\in$-sentences, too, and the system collapses. (See [4, cor. 3.21].) One can show that two sets are co-extensive, but this is not enough for identity. So while the identity relation $=$ obeys substitution, in practice one can almost never show that two sets are so related. For this reason, in this paper we effectively work entirely without substitution.

Extensionality notwithstanding, a useful feature of this set theory is that it validates the *law of ordered pairs*,

$$\langle x, y\rangle = \langle u, v\rangle \Leftrightarrow (x = u) \,\&\, (y = v)$$

as in [4, p.356]. This in turn is sufficient to define a *relation* as a set of ordered pairs, a subset of some cartesian product:

$$X \times Y = \{\langle x, y\rangle : x \in X \ \& \ y \in Y\}$$

The particular relation we have in mind is Kripke's notion of *relative possibility*.

3 Modal Logic: Syntax and Semantics

Reasoning constrained by relative possibility can be described in a positive modal language. The language $\mathcal{L}$ is based on a set of propositional variables

[7] The theory should really be called *property theory*, then, rather than set theory, since sets are extensions. Having now flagged this, we'll follow the tradition from [24] and keep saying 'set theory'.

PROP, and constructed in the usual way with connectives $\&$, $\vee$, $\Rightarrow$ and modalities $\Box, \Diamond$. We take $\mathcal{L}$ to be an 'object language', which we define from within our set theory, that happens to use the same symbols with the same intended meaning as the 'meta-language' of the theory, as is usual in standard model theory for first order languages.

The modal language is interpreted over frames and models. The definitions are the familiar Kripke definitions, but built on a naive set theory:

Definition 1 (Kripke Frames and models)**.** A *frame* $F = \langle \mathcal{W}, R \rangle$ is a structure with $\mathcal{W}$ a non-empty set (thought of as a containing possible worlds), and $R \subseteq \mathcal{W} \times \mathcal{W}$ a relation on $\mathcal{W}$. A model $M = \langle F, V \rangle$ is a frame together with a valuation $V : \mathtt{PROP} \longrightarrow \mathscr{P}(W)$ taking propositional atoms to sets of worlds.

A truth definition for formulas of the language uses pointed models (cf., [3, p.107]), written as $M, w \models \varphi$, and read as "φ is true in model M at world w." It is regimented by semantic principles that follow the construction of formulas:

Definition 2 (Semantics)**.** A semantics for the modal language is a relation $\models$ between pointed models M, w and formulas that satisfy the following *positive* constraints:

$M, w \models p$	$\Leftrightarrow$	$w \in V(p)$
$M, w \models \varphi \,\&\, \psi$	$\Leftrightarrow$	$(M, w \models \varphi) \,\&\, (M, w \models \psi)$
$M, w \models \varphi \vee \psi$	$\Leftrightarrow$	$(M, w \models \varphi) \vee (M, w \models \psi)$
$M, w \models \varphi \Rightarrow \psi$	$\Leftrightarrow$	$(M, w \models \varphi) \Rightarrow (M, w \models \psi)$
$M, w \models \Box\varphi$	$\Leftrightarrow$	$\forall v(Rwv \Rightarrow M, v \models \varphi)$
$M, w \models \Diamond\varphi$	$\Leftrightarrow$	$\exists v(Rwv \,\&\, M, v \models \varphi)$

We will simplify the notation and write $w \models \varphi$ on most occasions. On discussion of the meaning of these conditions, see [11].

Definition 3 (Model Validity)**.** A formula φ is *valid in a model* M, written $M \models \varphi$, if:

$$\forall w(w \in W \Rightarrow M, w \models \varphi)$$

Definition 4 (Frame Validity)**.** A formula φ is *valid in a frame* F, written $F \models \varphi$, if $M \models \varphi$ for all models M based on F. A formula φ is *valid in a set of frames* $\mathtt{F}$, written $\mathtt{F} \models \varphi$, if φ is valid in every frame $F \in \mathtt{F}$. We write $\models \varphi$ when φ is valid in all frames.

4 What is a normal modal logic?

In orthodox approaches to modal logic, frame semantics generate some logical behavior 'for free'. Most notably, the K axiom (showing that modal operators distribute over connectives) is inherent in the Kripke semantics. (Not so in more general semantics, like Chellas' neighborhood semantics [5].) If one reads Definition 2 with a classical eye, what is referred to as 'the axiom K' can be formulated in various logically equivalent ways, depending perhaps only on an author's connective inclination (cf., [13]). But only a handful are valid in frames when studied in a non-contractive set theory.

The following table summarizes what we are about to show; the signs to mark validity follow *pollice verso* (from ancient Roman gladiatorial contests).

$\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$	👎
$\Box(\varphi \,\&\, \psi) \Rightarrow (\Box\varphi \,\&\, \Box\psi)$	👎
$(\Box\varphi \,\&\, \Box\psi) \Rightarrow \Box(\varphi \,\&\, \psi)$	👎
$(\Box\varphi \vee \Box\psi) \Rightarrow \Box(\varphi \vee \psi)$	👍
$\Diamond(\varphi \vee \psi) \Rightarrow (\Diamond\varphi \vee \Diamond\psi)$	👍
$(\Diamond\varphi \vee \Diamond\psi) \Rightarrow \Diamond(\varphi \vee \psi)$	👍
$\Diamond(\varphi \,\&\, \psi) \Rightarrow (\Diamond\varphi \,\&\, \Diamond\psi)$	👎

A 'thumbs up' indicates that there is a proof, for which we give examples below. 'Thumbs down' indicates an essential appeal to contraction, which we will discuss in section §4.2.

4.1 Three theorems

Theorem 1. $\models (\Box\varphi \vee \Box\psi) \Rightarrow \Box(\varphi \vee \psi)$.

Proof. We give a detailed proof:

1.	$(w \models \Box\varphi \vee \Box\psi) \Rightarrow (w \models \Box\varphi \vee w \models \Box\psi)$	(Def. 2)
2.	$(w \models \Box\varphi) \Rightarrow \forall x(Rwx \Rightarrow x \models \varphi)$	(Def. 2)
3.	$(w \models \Box\varphi) \Rightarrow (Rwv \Rightarrow v \models \varphi)$	(2, A1 + A8)
4.	$(v \models \varphi) \Rightarrow ((v \models \varphi) \vee (v \models \psi))$	(A4)
5.	$(v \models \varphi) \Rightarrow v \models \varphi \vee \psi$	(4, Def. 2 + A1)
6.	$(w \models \Box\varphi \Rightarrow (Rwv \Rightarrow v \models \varphi))$	
	$\Rightarrow ((v \models \varphi \Rightarrow (v \models \varphi \vee \psi))$	
	$\Rightarrow (w \models \Box\varphi \Rightarrow (Rwv \Rightarrow v \models \varphi \vee \psi)))$	(Trick 1)
7.	$w \models \Box\varphi \Rightarrow (Rwv \Rightarrow v \models \varphi \vee \psi)$	(4-6, MP)
8.	$\forall x(w \models \Box\varphi \Rightarrow (Rwx \Rightarrow x \models \varphi \vee \psi))$	(7, Generalisation)
9.	$w \models \Box\varphi \Rightarrow \forall x(Rwx \Rightarrow x \models \varphi \vee \psi)$	(8, A11)
10.	$w \models \Box\varphi \Rightarrow w \models \Box(\varphi \vee \psi)$	(9, Def. 2 + A1)
...	Repeat steps 1-12 with ψ	...
11.	$w \models \Box\psi \Rightarrow w \models \Box(\varphi \vee \psi)$	
12.	$(w \models \Box\varphi \Rightarrow w \models \Box(\varphi \vee \psi))$	
	$\Rightarrow ((w \models \Box\psi \Rightarrow w \models \Box(\varphi \vee \psi))$	
	$\Rightarrow ((w \models \Box\varphi \vee w \models \Box\psi) \Rightarrow w \models \Box(\varphi \vee \psi)))$	(A5)
13.	$(w \models \Box\varphi \vee w \models \Box\psi) \Rightarrow w \models \Box(\varphi \vee \psi)$	(10-12)
14.	$(w \models \Box\varphi \vee \Box\psi) \Rightarrow w \models \Box(\varphi \vee \psi)$	(1, 13, A1)
15.	$w \models (\Box\varphi \vee \Box\psi) \Rightarrow \Box(\varphi \vee \psi)$	(14, Def. 2)

This completes the proof. $\therefore$

Theorem 2. $\models (\Diamond\varphi \vee \Diamond\psi) \Rightarrow \Diamond(\varphi \vee \psi)$.

Proof.

1.	$w \models \Diamond\varphi \Rightarrow \exists x(Rwx \,\&\, x \models \varphi)$	(Def. 2)
2.	$(\forall x(Rwx \,\&\, x \models \varphi) \Rightarrow (Rwv \,\&\, v \models \varphi))$ $\Rightarrow ((\exists x(Rwx \,\&\, x \models \varphi) \Rightarrow (Rwv \,\&\, v \models \varphi))$	(A10)
3.	$\forall x(Rwx \,\&\, x \models \varphi) \Rightarrow (Rwv \,\&\, v \models \varphi)$	(A8)
4.	$\exists x(Rwx \,\&\, x \models \varphi) \Rightarrow (Rwv \,\&\, v \models \varphi)$	(2-3)
5.	$w \models \Diamond\varphi \Rightarrow (Rwv \,\&\, v \models \varphi)$	(1, 4 + A1)
6.	$v \models \varphi \Rightarrow (v \models \varphi \vee v \models \psi)$	(A4)
7.	$v \models \varphi \Rightarrow v \models \varphi \vee \psi$	(6, Def. 2 + A1)
8.	$(Rwv \,\&\, v \models \varphi) \Rightarrow$ $((v \models \varphi \Rightarrow v \models \varphi \vee \psi) \Rightarrow (Rwv \,\&\, v \models \varphi \vee \psi))$	(Trick 2)
9.	$(Rwv \,\&\, v \models \varphi) \Rightarrow (Rwv \,\&\, v \models \varphi \vee \psi)$	(7-8 + Trick 3)
10.	$w \models \Diamond\varphi \Rightarrow (Rwv \,\&\, v \models \varphi \vee \psi)$	(5, 9 + A1)
11.	$(Rwv \,\&\, v \models \varphi \vee \psi) \Rightarrow \exists x(Rwx \,\&\, x \models \varphi \vee \psi)$	(A10)
12.	$w \models \Diamond\varphi \Rightarrow w \models \Diamond(\varphi \vee \psi)$	(9-11, Def. 2 + A1)
...	Repeat steps 1-12 with ψ	...
13.	$w \models \Diamond\psi \Rightarrow w \models \Diamond(\varphi \vee \psi)$	
14.	$(w \models \Diamond\varphi \Rightarrow w \models \Diamond(\varphi \vee \psi))$ $\Rightarrow ((w \models \Diamond\psi \Rightarrow w \models \Diamond(\varphi \vee \psi))$ $\Rightarrow ((w \models \Diamond\varphi \vee w \models \Diamond\psi) \Rightarrow w \models \Diamond(\varphi \vee \psi)))$	(A5)
15.	$(w \models \Diamond\varphi \vee w \models \Diamond\psi) \Rightarrow w \models \Diamond(\varphi \vee \psi)$	(12-14)
16.	$w \models \Diamond\varphi \vee \Diamond\psi \Rightarrow w \models \Diamond(\varphi \vee \psi)$	(15, Def. 2 + A1)
17.	$w \models (\Diamond\varphi \vee \Diamond\psi) \Rightarrow \Diamond(\varphi \vee \psi)$	(16, Def. 2)

This completes the proof. $\therefore$

Theorem 3. $\models \Diamond(\varphi \vee \psi) \Rightarrow (\Diamond\varphi \vee \Diamond\psi)$.

Proof.

1.	$w \models \Diamond(\varphi \vee \psi) \Rightarrow \exists x(Rwx \,\&\, x \models \varphi \vee \psi)$	(Def. 2)
...	Repeat lines 2-4 from the proof of Theorem 2	...
2.	$w \models \Diamond(\varphi \vee \psi) \Rightarrow (Rwv \,\&\, v \models \varphi \vee \psi)$	(1)
3.	$v \models \varphi \vee \psi \Rightarrow (v \models \varphi \vee v \models \psi)$	(Def. 2)
...	Use tricks 2 and 3 as in the proof of Theorem 2	...
4.	$w \models \Diamond(\varphi \vee \psi) \Rightarrow (Rwv \,\&\, (v \models \varphi \vee v \models \psi))$	(2-3)
5.	$(Rwv \,\&\, (v \models \varphi \vee v \models \psi))$ $\Rightarrow ((Rwv \,\&\, v \models \varphi) \vee (Rwv \,\&\, v \models \varphi))$	(B2)
6.	$w \models \Diamond(\varphi \vee \psi) \Rightarrow ((Rwv \,\&\, v \models \varphi) \vee (Rwv \,\&\, v \models \varphi))$	(4-5 + A1)

7.	$(Rwv \,\&\, v \models \varphi) \Rightarrow \exists x(Rwx \,\&\, x \models \varphi)$	(A9)
8.	$\exists x(Rwx \,\&\, x \models \varphi) \Rightarrow w \models \Diamond\varphi$	(Def. 2)
9.	$w \models \Diamond\varphi \Rightarrow (w \models \Diamond\varphi \vee \Diamond\psi)$	(A4)
10.	$Rwv \,\&\, v \models \varphi \Rightarrow w \models \Diamond\varphi \vee \Diamond\psi$	(7-9 + A1)
...	Repeat steps 7-9 with ψ	...
11.	$Rwv \,\&\, v \models \psi \Rightarrow w \models \Diamond\varphi \vee \Diamond\psi$	
12.	$w \models \Diamond(\varphi \vee \psi) \Rightarrow (\Diamond\varphi \vee \Diamond\psi)$	(6, 10, 11, A5 + A1)

This completes the proof. $\therefore$

4.2 K contracts

Consider the following attempt to prove that K is valid in all frames.

1.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow \forall x(Rwx \Rightarrow x \models \varphi \Rightarrow \psi)$	(Definition 2)
2.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow (Rwv \Rightarrow v \models \varphi \Rightarrow \psi)$	(1, A1 + A8)
3.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow (Rwv \Rightarrow (v \models \varphi \Rightarrow v \models \psi))$	(2, Definition 2 + A1)

And now, contraction is required to distribute Rwv over $v \models \varphi \Rightarrow v \models \psi$ to obtain $(Rwv \Rightarrow v \models \varphi) \Rightarrow (Rwv \Rightarrow v \models \psi)$. But we only have one instance of Rwv available, so we run out of assumptions. That contraction is essential for the derivation can be demonstrated by an easy checking of the failure of any proof search, carried out by inspection of all available Gentzen cut-free proofs (cf., Appendix 1). For instance, with $\varphi := Rwv, \psi := v \models \varphi$ and $\xi := v \models \psi$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \qquad \dfrac{\dfrac{\times}{\vdash \varphi} \qquad \dfrac{\psi \vdash \psi \quad \xi \vdash \xi}{\psi, \psi \Rightarrow \xi \vdash \xi}}{\varphi \Rightarrow \psi, \psi \Rightarrow \xi \vdash \xi}}{\varphi \Rightarrow (\psi \Rightarrow \xi), \varphi \Rightarrow \psi, \varphi \vdash \xi}}{\varphi \Rightarrow (\psi \Rightarrow \xi), \varphi \Rightarrow \psi \vdash (\varphi \Rightarrow \xi)}}{\varphi \Rightarrow (\psi \Rightarrow \xi) \vdash (\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \xi)}}{\vdash (\varphi \Rightarrow (\psi \Rightarrow \xi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \xi))}
$$

In light of this, the 'thumbs down' from our table becomes a theorem—provided the reader accepts the following classical chain of reasoning. Because the logic satisfies cut-elimination (cf. [20, Theorem 2.3]), it satisfies the subformula property. In the absence of contraction, the number of occurrences of each formula in identity axioms is then bounded by the number of occurrences of the formula to be proved. In our example case, the formula φ can occur at most three times in axioms. Two occurrences are secured in the axiom $\varphi \vdash \varphi$. The remaining occurrence hangs unsupported, with weakening as the only potential, but unsuccessful origin. Our non-provability claims, which substantiate the thumbs-down verdicts above, all follow similar arguments from inspection of available Gentzen proofs:

Formula to prove	Principle required
$\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$	$(\varphi \Rightarrow (\psi \Rightarrow \xi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \xi))$
$\Box(\varphi \,\&\, \psi) \Rightarrow (\Box\varphi \,\&\, \Box\psi)$	$(\varphi \Rightarrow (\psi \,\&\, \xi)) \Rightarrow ((\varphi \Rightarrow \psi) \,\&\, (\varphi \Rightarrow \xi))$
$(\Box\varphi \,\&\, \Box\psi) \Rightarrow \Box(\varphi \,\&\, \psi)$	$((\varphi \Rightarrow \psi) \,\&\, (\varphi \Rightarrow \xi)) \Rightarrow (\varphi \Rightarrow (\psi \,\&\, \xi))$
$\Diamond(\varphi \,\&\, \psi) \Rightarrow (\Diamond\varphi \,\&\, \Diamond\psi)$	$(\varphi \,\&\, (\psi \,\&\, \xi)) \Rightarrow ((\varphi \,\&\, \psi) \,\&\, (\varphi \,\&\, \xi))$

Cementing these negative results further, as a classical proof-theoretic exercise, is here omitted. Obtaining similar unprovability results in our *non-classical* framework—which is philosophically preferable, given the project we've taken on—would require the development of either general proof theory or general model theory in the substrucutual metatheory, which is out of range for this paper. As our goal is to focus on what *can* be proved, and how to do it, we proceed with positive results.

4.3 Recovering K

Though not all versions of K are valid on all frames, we can recover them by imposing restrictions on frames. These are "contractive properties," loosely speaking. Contractive properties are hard-wired by brute force whereas they obtain for free in contractive logics. The restrictions can be computed quite straightforwardly by considering the standard translation of modal logic into second-order logic ([3, §3.2]), for instance:[8]

$K^{\Diamond}$	$\Diamond(p \,\&\, q) \Rightarrow (\Diamond p \,\&\, \Diamond q)$	
	Second-order translation:	$\forall P \forall Q \forall x(\exists y(Rxy \,\&\, P(y) \,\&\, Q(y)) \Rightarrow$ $(\exists y(Rxy \,\&\, P(y)) \,\&\, \exists y(Rxy \,\&\, Q(y)))$
	Frame restriction:	$\forall X \forall Y \forall x(\exists y(Rxy \,\&\, y \in X \cap Y)$ $\Rightarrow (\exists y(Rxy \,\&\, y \in X) \,\&\, \exists y(Rxy \,\&\, y \in Y)))$

Theorem 4. *The following table displays a list of frame conditions under which various K theses are valid:*

[8] This is 'second-order' (also at the end of §5 below), but we note that a nice feature of using naive set theory is all quantification is simply quantification over sets (of any order), so the distinction is inessential.

	Frame Restriction	
K	*None*	$\Box((\varphi \Rightarrow \psi) \,\&\, \varphi) \Rightarrow \Box\psi$ $(\Box\varphi \vee \Box\psi) \Rightarrow \Box(\varphi \vee \psi)$ $\Diamond(\varphi \vee \psi) \Rightarrow (\Diamond\varphi \vee \Diamond\psi)$ $(\Diamond\varphi \vee \Diamond\psi) \Rightarrow \Diamond(\varphi \vee \psi)$
$K^{\Rightarrow}$	$\forall X \forall Y \forall x \forall y((Rxy \Rightarrow (y \in X \Rightarrow y \in Y))$ $\Rightarrow ((Rxy \Rightarrow y \in X) \Rightarrow (Rxy \Rightarrow y \in Y)))$	$\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$
$K^{\&}$	$\forall X \forall Y \forall x \forall y((Rxy \Rightarrow y \in X \cap Y)$ $\Leftrightarrow ((Rxy \Rightarrow y \in X) \,\&\, (Rxy \Rightarrow y \in Y)))$	$\Box(\varphi \,\&\, \psi) \Leftrightarrow (\Box\varphi \,\&\, \Box\psi)$
$K^{\Diamond}$	$\forall X \forall Y \forall x(\exists y(Rxy \,\&\, y \in X \cap Y)$ $\Rightarrow (\exists y(Rxy \,\&\, y \in X) \,\&\, \exists y(Rxy \,\&\, y \in Y)))$	$\Diamond(\varphi \,\&\, \psi) \Rightarrow (\Diamond\varphi \,\&\, \Diamond\psi)$

We need to introduce the notion of a truth-set to prove Theorem 4.

Definition 5 (Truth-set)**.** The *truth-set* of φ in M is the set of worlds at which it is true:

$$x \in [\![\varphi]\!]^M \Leftrightarrow M, x \models \varphi$$

We will simplify notation and write $[\![\varphi]\!]$. Notice that we do *not* have $[\![\varphi]\!]^M = \{x \in W : M, x \models \varphi\}$ with the identity $=$. So we will need to use truth sets without appealing to substitution.

The following extra tricks will come in handy; see Appendix 2.

	Derivable
Trick 4	$(\varphi \Rightarrow \psi) \Rightarrow (((\varphi \Leftrightarrow \chi) \,\&\, (\psi \Leftrightarrow \xi)) \Rightarrow (\chi \Rightarrow \xi))$.
Trick 5	$((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)) \Rightarrow (((\psi \Leftrightarrow \xi) \,\&\, (\chi \Leftrightarrow \gamma)) \Rightarrow ((\varphi \Rightarrow \xi) \Rightarrow (\varphi \Rightarrow \gamma)))$
Trick 6	$(\varphi^x_t \Rightarrow \psi^x_t) \Rightarrow (\forall x\varphi \Rightarrow \forall x\psi)$

Proof of Theorem 4. We exemplify with a proof for $K^{\Rightarrow}$.

1.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow \forall x(Rwx \Rightarrow w \models \varphi \Rightarrow \psi)$	(Def. 2)
2.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow (Rwv \Rightarrow (v \models \varphi \Rightarrow v \models \psi))$	(1, Def. 2 + A1 + A8)
3.	$(v \models \varphi \Rightarrow v \models \psi)$ $\Rightarrow (((v \models \varphi \Leftrightarrow v \in [\![\varphi]\!]) \,\&\, (v \models \psi \Leftrightarrow v \in [\![\psi]\!]))$ $\Rightarrow (v \in [\![\varphi]\!] \Rightarrow v \in [\![\psi]\!]))$	(Defition 5 + Trick 4)
4.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow (Rwv \Rightarrow (v \in [\![\varphi]\!] \Rightarrow v \in [\![\psi]\!]))$	(2-3, Def. 5 + A1)
5.	$(Rwv \Rightarrow (v \in [\![\varphi]\!] \Rightarrow v \in [\![\psi]\!]))$ $\Rightarrow ((Rwv \Rightarrow v \in [\![\varphi]\!]) \Rightarrow (Rwv \Rightarrow v \in [\![\psi]\!]))$	($K^{\Rightarrow}$)
6.	$((Rwv \Rightarrow v \in [\![\varphi]\!]) \Rightarrow (Rwv \Rightarrow v \in [\![\psi]\!]))$ $\Rightarrow (((v \in [\![\varphi]\!] \Leftrightarrow v \models \varphi) \,\&\, (v \in [\![\psi]\!] \Leftrightarrow v \models \psi))$ $\Rightarrow ((Rwv \Rightarrow v \models \varphi) \Rightarrow (Rwv \Rightarrow v \models \psi))$	(Trick 5)

7.	$((Rwv \Rightarrow v \models \varphi) \Rightarrow (Rwv \Rightarrow v \models \psi)) \Rightarrow$ $((\forall x(Rwx \Rightarrow x \models \varphi)) \Rightarrow (\forall x(Rwx \Rightarrow x \models \psi))$	(Trick 6)
8.	$((\forall x(Rwx \Rightarrow x \models \varphi)) \Rightarrow (\forall x(Rwx \Rightarrow x \models \psi))$ $\Rightarrow (w \models \Box\varphi \Rightarrow \Box\psi)$	(7, Def. 2 + Trick 4)
9.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow (w \models \Box\varphi \Rightarrow \Box\psi)$	(4-8, A1)
10.	$w \models \Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$	(9, Def. 2)

This completes the proof. $\therefore$

4.4 Extensions of K

It is now only an exercise to show that standard frame conditions validate modal formulas. Transitivity validates the 4 formulas $\Box\varphi \Rightarrow \Box\Box\varphi$ and (its contrapositive) $\Diamond\Diamond\varphi \Rightarrow \Diamond\varphi$. Like with K above, however, some classical frame conditions appear to be inherently contractive, in which case we build in more in the frame conditions to relieve the logic. That's why, for instance, the D conditions below are more complex than the classical $\forall x \exists y Rxy$.

	Frame Condition		
T	$\forall x Rxx$	$\Box\varphi \Rightarrow \varphi$ $\varphi \Rightarrow \Diamond\varphi$	
4	$\forall x \forall y \forall z(Rxy \Rightarrow (Ryz \Rightarrow Rxz)$	$\Box\varphi \Rightarrow \Box\Box\varphi$ $\Diamond\Diamond\varphi \Rightarrow \Diamond\varphi$	
B	$\forall x \forall y(Rxy \Rightarrow Ryx)$	$\varphi \Rightarrow \Box\Diamond\varphi$ $\Diamond\Box\varphi \Rightarrow \varphi$	
5	$\forall x \forall y \forall z(Rxy \Rightarrow (Rxz \Rightarrow Ryz))$	$\Diamond\varphi \Rightarrow \Box\Diamond\varphi$ $\Diamond\Box\varphi \Rightarrow \Box\varphi$	
D	$\forall x \exists y Rxy$ $\forall Y \forall x \forall y((Rxy \Rightarrow y \in Y) \Rightarrow \exists z(Rxz \,\&\, z \in Y))$	$\Box\varphi \Rightarrow \Diamond\varphi$ $\Box\varphi \Rightarrow \Diamond\varphi$	👎 👍

The lesson is this: Because our logic is weaker than the usual classical meta-logic, we cannot always prove the same validity of formulas under standard frame conditions. This is because, in general, a weaker logic allows for more frames. But we can recover standard validities by adopting stronger frame conditions, which are logically equivalent only in the stronger (classical) logic.[9]

5 Definability

Non-contractive meta-theory can go beyond merely showing that certain formulas are valid in sets of frames with given restrictions.

[9] There is some similarity here with relevant (substructural) modal logics studied under a classical metatheory, and how the K principle behaves. See [17].

Definition 6 (Frame Definability)**.** Let $\mathtt{F}$ be a set of frames. We say that φ *defines* $\mathtt{F}$ if for all frames $F \in \mathtt{F}$:

$$F \in \mathtt{F} \Leftrightarrow F \models \varphi$$

We illustrate how to establish definability results with a simple example.

Theorem 5. *The formula $\Box p \Rightarrow p$ defines the set of reflexive frames, namely the frames in which the relation Rxx holds for all x.*

Proof. We prove each direction separately.

$\Rightarrow$) Take a frame $F = \langle \mathcal{W}, R \rangle$ with the property Rxx for any $x \in \mathcal{W}$.

1.	$(w \models \Box p) \Rightarrow \forall x(Rwx \Rightarrow x \models p)$	(Definition 2)
2.	$\forall x(Rwx \Rightarrow x \models p) \Rightarrow (Rww \Rightarrow w \models p)$	(A8)
3.	$(w \models \Box p) \Rightarrow (Rww \Rightarrow w \models p)$	(1-2, A1)
4.	$\forall x Rxx$	(Assumption)
5.	Rww	(4, A8, MP)
6.	$(w \models \Box p) \Rightarrow (w \models p)$	(3, 5, Trick 3)
7.	$w \models \Box p \Rightarrow p$	(6, Definition 2)

$\Leftarrow$) Assumptions:

(a) Take a frame $F = \langle \mathcal{W}, R \rangle$ such that $F \models \Box p \Rightarrow p$.

(b) Take a valuation V such that (see Trick 7):

i. $\forall x(Rwx \Rightarrow x \in V(p))$.

ii. $\forall x(x \in V(p) \Rightarrow Rwx)$.

1.	$w \models \Box p \Rightarrow p$	(Assumptions (a))
2.	$w \models \Box p \Rightarrow w \models p$	(1, Definition 2)
3.	$Rwv \Rightarrow v \models p$	(Assumption (b-i) + Definition 2)
4.	$\forall x(Rwx \Rightarrow x \models p)$	(3, Generalisation)
5.	$w \models \Box p$	(4, Definition 2)
6.	$w \models p$	(2, 5, MP)
7.	$w \in V(p)$	(6, Definition 2)
8.	$w \in V(p) \Rightarrow Rww$	(Assumption (b-ii))
9.	Rww	(7-8, MP)
10.	$\forall x Rxx$	(9, Generalisation)

This completes the proof. $\therefore$

Trick 7 In contractive settings, one would use $V(p) = \{y : Rwy\}$ and substitution of identicals (more than once!). But this identity is too strong to be really contraction free. Instead, we isolate the assumptions that are needed for the proof, without having to repeat them, both of which are unpacked from the classical identity.

Theorem 6. *The formula $p \Rightarrow \Diamond p$ also defines the set of reflexive frames, namely the frames in which the relation R is reflexive.*

Proof. We leave the first direction to the reader.

$\Leftarrow$) Assumptions:

(a) Take a frame $F = \langle \mathcal{W}, R\rangle$ such that $F \models p \Rightarrow \Diamond p$.

(b) Take a valuation V such that (see Trick 8):

i. $\forall x(x \in \{y : Rwy \Rightarrow Rww\} \Rightarrow x \in V(p))$

ii. $\forall x(x \in V(p) \Rightarrow x \in \{y : Rwy \Rightarrow Rww\})$
Notice that this valuation is well-defined, since at least w satisfies both conditions.

1.	$w \models p \Rightarrow \Diamond p$	(Assumptions (a))
2.	$w \models p \Rightarrow w \models \Diamond p$	(1, Definition 2)
3.	$w \in \{y : Rwy \Rightarrow Rww\} \Rightarrow w \in V(p)$	(Assumption (b-i))
4.	$Rww \Rightarrow Rww$	(B1)
5.	$w \in \{y : Rwy \Rightarrow Rww\}$	(Comprehension)
6.	$w \in V(p)$	(3,5,MP)
7.	$w \models p$	(6 + Definition 2)
8.	$w \models \Diamond p$	(2,7, MP)
9.	$\exists x(Rwx \,\&\, x \models p)$	(8, Definition 2)
10.	$Rwv \,\&\, v \models p$	(9, A10)
11.	$Rwv \,\&\, v \in V(p)$	(10, Defintion 2 + Trick 2)
12.	$v \in V(p) \Rightarrow v \in \{y : Rwy \Rightarrow Rww\}$	(Assumption (b-ii))
13.	$v \in \{y : Rwy \Rightarrow Rww\} \Rightarrow (Rwv \Rightarrow Rww)$	(12, Comprehension)
14.	$Rwv \,\&\, (Rwv \Rightarrow Rww)$	(11-13, A1 + Trick 2)
15.	Rww	(14, MP)
16.	$\forall x Rxx$	(15, Generalisation)

This completes the proof. $\therefore$

Trick 8 As in the proof of Theorem 5, we only assume what is required to get the proof going, instead of the sledgehammer-to-crack-a-nut assumption that $V(p) = \{w\}$.

Having now seen a few examples, it is clear that cases involving & and $\Box$ are trickier than $\vee$ or $\Diamond$. We cover these cases now, moving a bit more speedily through the arguments.

Theorem 7. *The formula* $\Box p \Rightarrow \Box\Box p$ *defines frames where* $\forall x \forall y \forall z((Rxy \,\&\, Ryz) \Rightarrow Rxz)$.

Proof. We leave the first direction to the reader.

$\Leftarrow$) Assumptions:

(a) Take a frame $F = \langle W, R\rangle$ such that $\models \Box p \Rightarrow \Box\Box p$.

(b) Take a valuation V such that (see Trick 8):

i. $\forall x(Rwx \Rightarrow x \in V(p))$.

ii. $\forall x(x \in V(p) \Rightarrow Rwx)$.

1.	$w \models \Box p \Rightarrow \Box\Box p$	(Assumptions (a))
2.	$w \models \Box p \Rightarrow w \models \Box\Box p$	(1, Definition 2)
3.	$w \models \Box p$	(Assumption (b-i) + Definition 2)
4.	$w \models \Box\Box p$	(2-3, MP)
5.	$\forall x \forall y(Rwx \Rightarrow (Rxy \Rightarrow y \models p))$	(4, Definition 2)
6.	$Rwv \Rightarrow (Rvu \Rightarrow u \models p)$	(5, A8)
7.	$Rwv \Rightarrow (Rvu \Rightarrow u \in V(p))$	(6, Definition 6, A1)
8.	$Rwv \Rightarrow (Rvu \Rightarrow Rwu)$	(7, Assumption (b-ii))
9.	$\forall x \forall y \forall z(Rxy \Rightarrow (Ryz \Rightarrow Rxz))$	(8, Generalisation)

This completes the proof. $\therefore$

Theorem 8. *The formula* $\Diamond\Diamond p \Rightarrow \Diamond p$ *also defines the set of transitive frames, namely the frames in which the relation* R *is transitive.*

Proof. We leave the first direction to the reader.

$\Leftarrow$) Assumptions:

(a) Take a frame $F = \langle \mathcal{W}, R\rangle$ such that $F \models \Diamond\Diamond p \Rightarrow \Diamond p$.

(b) Take a valuation V such that (see Trick 8):

i. $u \in V(p)$

ii. $\forall x(x \in V(p) \Rightarrow x \in \{y : Rwy \Rightarrow Rwu\})$
Notice that this valuation is well-defined, since at least u satisfies (ii).

1.	$Rwv \,\&\, Rvu \Rightarrow w \models \Diamond\Diamond p$	(Assumption (b-i))
2.	$w \models \Diamond\Diamond p \Rightarrow w \models \Diamond p$	(Assumption (a)+Definition 2)
3.	$Rwv \,\&\, Rvu \Rightarrow w \models \Diamond p$	(1-2, A1)
4.	$w \models \Diamond p \Rightarrow \exists x(Rwx \,\&\, x \models p)$	(Definition 2)
5.	$\exists x(Rwx \,\&\, x \models p) \Rightarrow (Rwx \,\&\, x \models p)$	(A10)
6.	$Rwx \,\&\, x \models p \Rightarrow (Rwx \,\&\, x \in V(p))$	(Definition 6)
7.	$Rwv \,\&\, Rvu \Rightarrow (Rwx \,\&\, x \in V(p))$	(3-6, A1)
8.	$x \in V(p) \Rightarrow x \in \{y : Rwy \Rightarrow Rwu\}$	(Assumption (b-ii), A11)
9.	$x \in \{y : Rwy \Rightarrow Rwu\} \Rightarrow (Rwx \Rightarrow Rwu)$	(Comprehension)
10	$x \in V(p) \Rightarrow (Rwx \Rightarrow Rwu)$	(8-9, A1)

11.	$Rwx \,\&\, x \in V(p) \Rightarrow Rwx \,\&\, (Rwx \Rightarrow Rwu)$	(10, Trick 2)
12.	$(Rwx \,\&\, (Rwx \Rightarrow Rwu)) \Rightarrow Rwu$	(11, Axiom modus ponens)
13.	$Rwv \,\&\, Rvu \Rightarrow Rwu$	(7, 11, 12, A1)
14.	$\forall x \forall y \forall z(Rxy \,\&\, Ryz \Rightarrow Rxz)$	(13, A11)

This completes the proof. ∴

Theorem 9. *Frames where $\forall x \forall y(Rxy \Rightarrow Ryx)$ are defined by the formula $p \Rightarrow \Box\Diamond p$.*

Proof. The proof technique should now become familiar. To show that frames in which $p \Rightarrow \Box\Diamond p$ are symmetric, use a valuation V such that $w \in V(p)$ and $\forall x(x \in V(p) \Rightarrow x \in \{y : Rvy \Rightarrow Rvw\})$. As above, notice that this valuation is well-defined, as it is satisfied by w. ∴

Theorem 10. *Frames where $\forall x \forall y \forall z(Rxy \Rightarrow (Rxz \Rightarrow Ryz))$ are defined by $\Diamond p \Rightarrow \Box\Diamond p$*

Proof. To show that frames in which $\Diamond p \Rightarrow \Box\Diamond p$ are euclidean, take three worlds such that Rwu and Rwv, and use a valuation V such that $v \in V(p)$ and $\forall x(x \in V(p) \Rightarrow x \in \{y : Ruy \Rightarrow Ruv\})$. As above, notice that this valuation is well-defined, as it is satisfied by v. ∴

As the above proofs illustrate, showing definability of frame conditions only requires minimal non-contractive and positive logical tools. The following table summarises the standard conditions that can be defined by various first-order formulas.

	First-Order Frame Condition	Definition
T	$\forall x Rxx$	$\Box p \Rightarrow p$ $p \Rightarrow \Diamond p$
4	$\forall x \forall y \forall z((Rxy \,\&\, Ryz) \Rightarrow Rxz)$	$\Box p \Rightarrow \Box\Box p$ $\Diamond\Diamond p \Rightarrow \Diamond p$
B	$\forall x \forall y(Rxy \Rightarrow Ryx)$	$p \Rightarrow \Box\Diamond p$ $\Diamond\Box p \Rightarrow p$
5	$\forall x \forall y \forall z(Rxy \Rightarrow (Rxz \Rightarrow Ryz))$	$\Diamond p \Rightarrow \Box\Diamond p$ $\Diamond\Box p \Rightarrow \Box p$

Nevertheless, not all standard first-order definable conditions are recoverable. We have seen above that various versions of K are not valid over all frames, but can be recovered from restrictions on frames. As we noted there, we can use the standard translation of modal formulas into second-order logic to compute the second-order frame correspondents defined by standard modal formulas:

	Second-Order Frame Condition	Definition
$K^{\Rightarrow}$	$\forall X \forall Y \forall x \forall y((Rxy \Rightarrow (y \in X \Rightarrow y \in Y))$ $\Rightarrow ((Rxy \Rightarrow y \in X) \Rightarrow (Rxy \Rightarrow y \in Y)))$	$\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$
$K^{\&}$	$\forall X \forall Y \forall x \forall y((Rxy \Rightarrow y \in X \cap Y)$ $\Leftrightarrow ((Rxy \Rightarrow y \in X) \,\&\, (Rxy \Rightarrow y \in Y)))$	$\Box(p \,\&\, q) \Leftrightarrow (\Box p \,\&\, \Box q)$
$K^{\Diamond}$	$\forall X \forall Y \forall x(\exists y(Rxy \,\&\, y \in X \cap Y)$ $\Rightarrow (\exists y(Rxy \,\&\, y \in X) \,\&\, \exists y(Rxy \,\&\, y \in Y)))$	$\Diamond(p \,\&\, q) \Rightarrow (\Diamond p \,\&\, \Diamond q)$
D	$\forall X \forall Y \forall x \forall y((Rxy \Rightarrow y \in Y)$ $\Rightarrow \exists z(Rxz \,\&\, z \in Y))$	$\Box p \Rightarrow \Diamond p$

One last item. In classical metatheory, frames in which accessibility is *universal* (every world accesses every other) have exactly the same validities as frames in which accessibility is an *equivalence* relation (reflexive, symmetric, transitive). Without more model theory, we can only conjecture (but with some confidence) that this elision will fail in a substructural framework; cf. [29].

6 Conclusion

We have shown by direct demonstration that definability theory—the tight relationship between the structure of frames and validity of specific modal formulas—is not dependent on classical logic. A metatheory without contraction can demonstrate these same relationships, suitably phrased, for a modal logic without contraction. What else is possible, and what else is possible, for a contraction-free approach yet awaits.

Appendix 1: Sequent calculus

The logic `BCK` is presented as a Gentzen system. The Hilbert system and the Gentzen system are equivalent [20, 2]; cf. [8, cor.2.21].

A *sequent* is of the form $\Gamma \vdash \varphi$, and are obtained in the following ways.

$$\frac{}{\varphi \vdash \varphi}\, id$$

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\, weakening \qquad \frac{\Gamma \vdash \psi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\, cut \qquad \frac{\Gamma, \varphi, \psi \vdash \chi}{\Gamma, \psi, \varphi \vdash \chi}\, exchange$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

$$\frac{\Gamma \vdash \psi \qquad \Delta \vdash \chi}{\Gamma, \Delta \vdash \psi \,\&\, \chi} \qquad \frac{\Gamma, \varphi, \psi \vdash \chi}{\Gamma, \varphi \,\&\, \psi \vdash \chi}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta, \varphi \Rightarrow \psi \vdash \chi}$$

$$\frac{\Gamma, \varphi^x_t \vdash \psi}{\Gamma, \forall x \varphi \vdash \psi} \ t \text{ any term} \qquad \frac{\Gamma \vdash \varphi^x_y}{\Gamma \vdash \forall x \varphi} \ \text{y not free in } \Gamma$$

$$\frac{\Gamma \vdash \varphi^x_t}{\Gamma \vdash \exists x \varphi} \ t \text{ any term} \qquad \frac{\Gamma, \varphi^x_y \vdash \psi}{\Gamma, \exists x \varphi \vdash \psi} \ \text{y not free in } \Gamma$$

Cut is eliminable [20, thm 2.3]. Thus we can appeal to the sub-formula property.

Without contraction, the two rules for property abstraction

$$\frac{\Gamma \vdash \varphi(t)}{\Gamma \vdash t \in \{x : \varphi(x)\}} \qquad \frac{\Gamma, \varphi(t) \vdash \psi}{\Gamma, t \in \{x : \varphi(x)\} \vdash \psi}$$

may be added, and cut is still eliminable [24, thm 2.2]. Dropping contraction to save naive comprehension is essentially a proof-theoretic idea.

Appendix 2: Derivable theorems of the logic

Axiom modus ponens: it is a theorem that

$$(\varphi \,\&\, (\varphi \Rightarrow \psi)) \Rightarrow \psi$$

This is because $(\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \psi)$ by B1, and then $\varphi \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow \psi)$ by A2 (permutation), and finally the desired modus ponens form by A7.

- B0: Use the modus ponens axiom and Trick 2 to derive

  $$(((\varphi \Rightarrow \psi) \,\&\, \varphi) \,\&\, ((\chi \Rightarrow \xi) \,\&\, \chi)) \Rightarrow (\varphi \,\&\, \chi)$$

  then use permutation and B4.

- The proof for B1 is to plug axiom (A3), weakening, into axiom (A2), permutation. This yields $\psi \Rightarrow (\varphi \Rightarrow \varphi)$. Then pick any axiom for ψ.

- The proof for B2 is as follows. By B1 and A4, since $\varphi \Rightarrow \varphi$ and $\psi \Rightarrow \psi \vee \chi$, then

  $$\varphi \,\&\, \psi \Rightarrow \varphi \,\&\, (\psi \vee \chi)$$

  using B0. Analogously,

  $$\varphi \,\&\, \chi \Rightarrow \varphi \,\&\, (\psi \vee \chi)$$

  Therefore, using argument by cases (A5) completes the derivation.

- To show B3 (distribution):

1	$\varphi\&\psi \Rightarrow (\varphi\&\psi) \vee (\varphi\&\chi)$	(A4)
2	$\psi \Rightarrow (\varphi \Rightarrow (\varphi\&\psi) \vee (\varphi\&\chi))$	(1, B4)
3	$\varphi\&\chi \Rightarrow (\varphi\&\psi) \vee (\varphi\&\chi)$	(A4)
4	$\chi \Rightarrow (\varphi \Rightarrow (\varphi\&\psi) \vee (\varphi\&\chi))$	(3, B4)
5	$\psi \vee \chi \Rightarrow (\varphi \Rightarrow (\varphi\&\psi) \vee (\varphi\&\chi))$	(2, 4, A5)
6	$(\psi \vee \chi)\&\varphi \Rightarrow (\varphi\&\psi) \vee (\varphi\&\chi)$	(5, B4)

- For B4, right to left is (A7). Then for left to right, use Trick 1.

- Trick 1: By (A1), if $\xi \Rightarrow \chi$ then $(\psi \Rightarrow \xi) \Rightarrow (\psi \Rightarrow \chi)$. By (A2), then,

$$(\psi \Rightarrow \xi) \Rightarrow ((\xi \Rightarrow \chi) \Rightarrow (\psi \Rightarrow \chi))$$

  So now let $\varphi \Rightarrow (\psi \Rightarrow \xi)$. Then by (A1) again the theorem follows.

- Trick 2: Putting together axiom modus ponens $\psi\&(\psi \Rightarrow \xi) \Rightarrow \xi$ with (A6) $\xi \Rightarrow (\varphi \Rightarrow \varphi\&\xi)$, then

$$\psi\&(\psi \Rightarrow \xi) \Rightarrow (\psi \Rightarrow \varphi\&\xi)$$

  by transitivity. Then $(\varphi\&\psi)\&(\psi \Rightarrow \xi) \Rightarrow (\varphi\&\xi)$ from (B4) as required.

- Hints for other tricks: Trick 3 uses A2 and then A7. Trick 4 is a rearrangement of A1 using A2. The same for Trick 5. And Trick 6 is by A8.

A check of the dependencies of the Bi and Tricks show that: B1 and T1 follow only from axioms, and everything else follows non-circularly from these and axioms.

Acknowledgement

Thanks to Andrew Withy, Shawn Standefer, Guillermo Badia, members of the the Pukeko Logic Group, members of the Melbourne Logic Group, and an anonymous referee for very helpful comments. Part of this research was supported by the Marsden Grant, Royal Society of New Zealand.

References

[1] Gerard Allwein and J. Michael Dunn. Kripke semantics for linear logic. *Journal of Symbolic Logic*, 58(02):514–545, 1993.

[2] Guillermo Badía and Zach Weber. A substructural logic for inconsistent mathematics. In A Reiger and G Young, editors, *Dialetheism and its applications*. Springer (Trends in Logic), to appear.

[3] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*. Cambridge University Press 2001, New York, 2001.

[4] Andrea Cantini. The undecidability of Grisin's set theory. *Studia Logica*, 74(3):345–368, 2003.

[5] Brian F. Chellas. *Modal logic: an introduction*. Cambridge University Press, 1980.

[6] Petr Cintula and Francesco Paoli. Is mutliset consequence trivial? *Synthese*, x(x):1–25, 201x.

[7] Solomon Feferman. Toward useful type-free theories, i. *Journal of Symbolic Logic*, 49(1):75–111, 1984.

[8] N Galatos, P Jipsen, T Kowalski, and H Ono. *Residuated Lattices*. Elsevier, 2007.

[9] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, (50), 1987.

[10] Jean-Yves Girard. Light linear logic. *Information and Computation*, 143:175–204, 1998.

[11] Patrick Girard and Zach Weber. Bad worlds. *Thought*, 4:93–101, 2015.

[12] Viacheslav Nikolaevich Grišin. Predicate and set theoretic calculi based on logic without contraction rules. *Math. USSR Izvestija*, 18(1):41–59, 1982.

[13] G. E. Hughes and M.J. Cresswell. *A new introduction to modal logic*. Routledge, New York, 1996.

[14] Lloyd Humberstone. *The Connectives*. MIT, 2011.

[15] Norihiro Kamide. Kripke semantics for modal substructural logics. *Journal of Logic, Language and Information*, 11:453–470, 2002.

[16] Wendy MacCaull. Kripke semantics for logics with bck implication. *Bulletin of the Section of Logic*, 25:41–51, 1996.

[17] Edwin D. Mares and Robert K. Meyer. The semantics of R4. *Journal of Philosophical Logic*, 22(1):95–110, 1993.

[18] Toby Meadows and Zach Weber. Computation in non-classical foundations? *Philosophers' Imprint*, 16:1–17, 2016.

[19] Hiroakira Ono. Semantics for substructural logics. In Peter Schroeder-Heister and Kosta Došen, editors, *Substructural Logics*, pages 259–291. Oxford University Press, 1993.

[20] Hiroakira Ono and Yuichi Komori. Logic without the contraction rule. *Journal of Symbolic Logic*, 50:169–201, 1985.

[21] Uwe Petersen. Logic without contraction as based on inclusion and unrestriced abstraction. *Studia Logica*, 64:365–403, 2000.

[22] Uwe Petersen. $l^i d^Z_\lambda$ as a basis for pra. *Archive for mathematical logic*, 42:665–694, 2003.

[23] David Ripley. Contraction and closure. *Thought*, 4:131–138, 2015.

[24] Kazushige Terui. Light affine set theory: A naive set theory of polynomial time. *Studia Logica*, 77:9–40, 2004.

[25] Anne S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*, volume 43 of *Cambridge tracts in theoretical computer science*. Cambridge University Press, 1996.

[26] Johan van Benthem. Correspondence theory. In Dov M. Gabbay and Franz Günthner, editors, *Handbook of Philosophical Logic*, volume 2, pages 167–247. Reidel, Dordrecht, 1984.

[27] Heinrich Wansing. *The Logic of Information Structures*. Number 681 in Lecture Notes in Artificial Intelligence. Springer-Verlag, 1993.

[28] Zach Weber, Guillermo Badia, and Patrick Girard. What is an inconsistent truth table? *Australasian Journal of Philosophy*, 94(3), 2016.

[29] Zach Weber and Hitoshi Omori. Observations on the trivial world. *Erkenntnis*, 201x. to appear.