Modal logics for region-based theories of space

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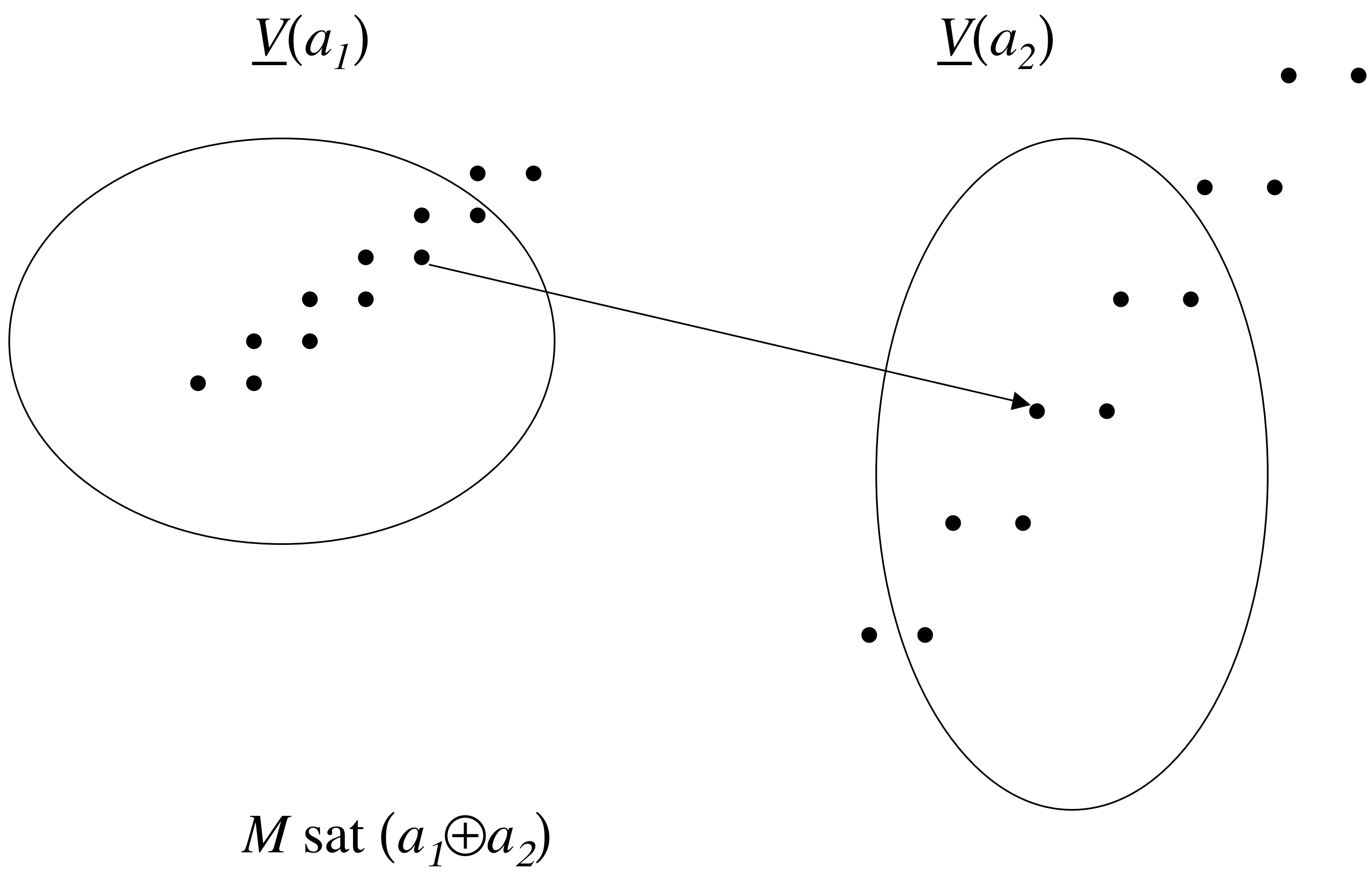
Syntax

- **Boolean terms**
  - $a ::= p \mid 0 \mid \neg a \mid (a_1 \vee a_2)$
- **Modal formulas**
  - $\phi ::= (\boldsymbol{a_1 \oplus a_2}) \mid 0 \mid \neg \phi \mid (\phi_1 \vee \phi_2) \mid (\boldsymbol{a_1 = a_2})$
- Abbreviations (Boolean terms)
  - $1 ::= \neg 0$, $(a_1 \wedge a_2) ::= \neg(\neg a_1 \vee \neg a_2)$
- Abbreviations (modal formulas)
  - $1 ::= \neg 0$, $(\phi_1 \wedge \phi_2) ::= \neg(\neg \phi_1 \vee \neg \phi_2)$
- $(\boldsymbol{a_1 \oplus a_2})$ is equivalent to $\langle U \rangle (a_1 \wedge \langle R \rangle a_2)$ and $(\boldsymbol{a_1 = a_2})$ is equivalent to $[U](a_1 \leftrightarrow a_2)$

Semantics

- A **model** is a structure of the form $M = <W,R,V>$ where
  - W is a nonempty set
  - R is a binary relation on W
  - V associates a subset $V(p)$ of W to each Boolean variable p
- $\underline{V}$ associates a subset $\underline{V}(a)$ of W to each Boolean term a
  - $\underline{V}(p) = V(p)$
  - $\underline{V}(0) = \emptyset$, $\underline{V}(\neg a) = W / \underline{V}(a)$, $\underline{V}(a_1 \vee a_2) = \underline{V}(a_1) \cup \underline{V}(a_2)$
- Remark that
  - $\underline{V}(1) = W$, $\underline{V}(a_1 \wedge a_2) = \underline{V}(a_1) \cap \underline{V}(a_2)$


$\underline{V}(a_1)$
$\underline{V}(a_2)$
M sat $(a_1 \oplus a_2)$

Semantics

- **Satisfiability** of ϕ in M = <W,R,V> is defined by:
  – M sat $(\boldsymbol{a_1 \oplus a_2})$ iff there exists $w \in \underline{V}(a_1)$ such that for some $w' \in R(w)$, $w' \in \underline{V}(a_2)$
  – Not M sat 0, M sat $\neg\phi$ iff not M sat ϕ, M sat $(\phi_1 \vee \phi_2)$ iff M sat ϕ_1 or M sat ϕ_2
  – M sat $(\boldsymbol{a_1 = a_2})$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
- Remark that
  – M sat 1, M sat $(\phi_1 \wedge \phi_2)$ iff M sat ϕ_1 and M sat ϕ_2
- **Validity** of modal formula ϕ in frame F = <W,R> is defined by:
  – F val ϕ iff for all models M = <W,R,V> based on F, M sat ϕ

Semantics

- **Correspondence theory**
  - F val $(p \neq 0) \rightarrow (p \oplus p)$ iff $\forall w \in W (wRw)$
  - F val $(p \oplus q) \rightarrow (q \oplus p)$ iff $\forall w, w' \in W (wRw' \rightarrow w'Rw)$
  - F val $(p \neq 0) \rightarrow (p \oplus 1)$ iff $\forall w \in W \exists w' \in W (wRw')$
  - F val $(p \neq 0) \rightarrow (1 \oplus p)$ iff $\forall w \in W \exists w' \in W (w'Rw)$
  - F val $(p \neq 0) \rightarrow (p \oplus 1) \vee (1 \oplus p)$ iff $\forall w \in W \exists w' \in W (wRw' \vee w'Rw)$
  - F val $(1 \oplus 1)$ iff $\exists w, w' \in W (wRw')$
  - F val $(p \neq 0) \wedge (q \neq 0) \rightarrow (p \oplus q)$ iff $\forall w, w' \in W (wRw')$
  - F val $(p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)$ iff R **is connected**

Bisimulation

- Let $M = <W,R,V>$ and $M' = <W',R',V'>$ be models
- A **bisimulation** between M and M' is a binary relation Z between W and W' such that
  - $\forall w \in W \exists w' \in W' (wZw')$
  - $\forall w' \in W' \exists w \in W (wZw')$
  - $\forall w_1, w_2 \in W \exists w_1', w_2' \in W' (w_1 R w_2 \rightarrow w_1' R' w_2')$
  - $\forall w_1', w_2' \in W' \exists w_1, w_2 \in W (w_1' R w_2' \rightarrow w_1 R w_2)$
  - $\forall w \in W \forall w' \in W' (wZw' \rightarrow (w \in V(p) \leftrightarrow w' \in V'(p)))$

Bisimulation

- **Bisimulation theorem**
  - If M and M' are bisimilar then they are modally equivalent
- **Hennessy-Milner theorem**
  - If M and M' are finite and modally equivalent then they are bisimilar
- **Van Benthem characterization theorem**
  - For all 1st-order sentences A like
    - $A ::= R(x_1,x_2) \mid P(x) \mid 0 \mid \neg\phi \mid (\phi_1 \vee \phi_2) \mid \forall x A$
  - A is invariant for bisimulations iff A is equivalent to the standard translation of a modal formula

Axiomatization/completeness

- **Axioms of L_{min}**
  - Identity axioms:
    - $(a=a)$, $(a_1=a_2)\rightarrow(a_2=a_1)$, $(a_1=a_2)\wedge(a_2=a_3)\rightarrow(a_1=a_3)$
  - Congruence axioms:
    - $(a=b)\rightarrow(\neg a=\neg b)$, $(a_1=b_1)\wedge(a_2=b_2)\rightarrow((a_1\vee a_2)=(b_1\vee b_2))$
  - Boolean axioms:
    - $(a=b)$ if a and b are equivalent Boolean terms
    - $(0\neq 1)$

Axiomatization/completeness

- **Axioms of L_{min}**
  - Proximity axioms:
    - $(a \oplus b) \rightarrow (a \neq 0) \wedge (b \neq 0)$
    - $((a \vee b) \oplus c) \leftrightarrow (a \oplus c) \vee (b \oplus c)$
    - $(a \oplus (b \vee c)) \leftrightarrow (a \oplus b) \vee (a \oplus c)$
- **Completeness of L_{min}**: For all modal formulas ϕ, ϕ is provable from the axioms of L_{min} iff ϕ is valid in the class of all frames $F = \langle W, R \rangle$

Axiomatization/completeness

- Let Σ be a set of modal formulas
- **Axioms of L_Σ** are those of L_{min} plus the following
  - Σ-axioms: Every modal formula $\psi(a_1,\ldots,a_n)$ which can be obtained from a modal formula $\psi(p_1,\ldots,p_n)$ of Σ by uniformly substituting the Boolean terms $a_1, \ldots, a_n$ for the Boolean variables $p_1, \ldots, p_n$
- **Completeness of L_Σ**: If Σ is finite then for all modal formulas ϕ, ϕ is provable from the axioms of L_Σ iff ϕ is valid in the nonempty class of all frames $F = \langle W,R \rangle$ such that $\langle W,R \rangle$ val Σ

Axiomatization/completeness

- A first extension of L_{min}
  - Let Σ_{sym} be $\{(\boldsymbol{p}\oplus\boldsymbol{q})\rightarrow(\boldsymbol{q}\oplus\boldsymbol{p})\}$
  - The **axioms of $\boldsymbol{L_{\Sigma sym}}$** are those of L_{min} plus every modal formula like $(\boldsymbol{a}\oplus\boldsymbol{b})\rightarrow(\boldsymbol{b}\oplus\boldsymbol{a})$
- **Completeness of $\boldsymbol{L_{\Sigma sym}}$**: For all modal formulas ϕ, ϕ is provable from the axioms of $L_{\Sigma sym}$ iff ϕ is valid in the nonempty class of all frames $F = \langle W,R \rangle$ such that $\forall \boldsymbol{w},\boldsymbol{w'}\in \boldsymbol{W}(\boldsymbol{wRw'}\rightarrow \boldsymbol{w'Rw})$

Axiomatization/completeness

- A second extension of L_{min}
  - Let Σ_{con} be $\{(\boldsymbol{p \neq 0}) \wedge (\boldsymbol{p \neq 1}) \rightarrow (\boldsymbol{p \oplus \neg p})\}$
  - The **axioms of $L_{\Sigma con}$** are those of L_{min} plus every modal formula like $(\boldsymbol{a \neq 0}) \wedge (\boldsymbol{a \neq 1}) \rightarrow (\boldsymbol{a \oplus \neg a})$
- **Completeness of $L_{\Sigma con}$**: For all modal formulas ϕ, ϕ is provable from the axioms of $L_{\Sigma con}$ iff ϕ is valid in the nonempty class of all frames $F = \langle W, R \rangle$ such that ***R* is connected**

Axiomatization/completeness

- **Open problems**
  - Find a set Σ of modal formulas such that $\boldsymbol{L_\Sigma}$ **is not complete** with respect to the nonempty class of all frames *F* = <*W*,*R*> such that <*W*,*R*> val Σ
  - Find a set Σ of modal formulas such that **the class of all frames *F* = <*W*,*R*> such that <*W*,*R*> val Σ is empty**

Canonicity

- Given a set Σ of modal formulas and a maximal L_Σ-consistent set S of modal formulas, the **canonical frame of L_Σ** defined by S is the structure $F_S = <W_S,R_S>$ defined as follows
  - W_S is the set of all maximal consistent sets w of Boolean terms such that for all Boolean terms $a \in w$, $(a \neq 0) \in S$
  - R_S is the binary relation on W_S such that for all $w,w' \in W_S$, wR_Sw' iff for all Boolean terms $a \in w$ and for all Boolean terms $a' \in w'$, $(a \oplus a') \in S$

Canonicity

- Let Σ be a set of modal formulas
  - L_Σ is s**trongly canonical** iff for all maximal L_Σ-consistent sets S of modal formulas, the canonical frame $F_S = <W_S,R_S>$ of L_Σ defined by S validates L_Σ
  - L_Σ is **weakly canonical** iff there exists a maximal L_Σ-consistent set S of modal formulas such that the canonical frame $F_S = <W_S,R_S>$ of L_Σ defined by S validates L_Σ

Canonicity

- **Strong canonicity of L_{min}**: For all maximal L_{min}-consistent sets S of modal formulas, the canonical frame $F_S = \langle W_S, R_S \rangle$ of L_{min} defined by S validates L_{min}
- **Weak canonicity of L_{min}**: There exists a maximal L_{min}-consistent set S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of L_{min} defined by S validates L_{min}

Canonicity

- A first extension of L_{min}
  - Let Σ_{sym} be $\{(\boldsymbol{p}\oplus\boldsymbol{q})\rightarrow(\boldsymbol{q}\oplus\boldsymbol{p})\}$
  - The **axioms of $\boldsymbol{L_{\Sigma sym}}$** are those of L_{min} plus every modal formula like $(\boldsymbol{a}\oplus\boldsymbol{b})\rightarrow(\boldsymbol{b}\oplus\boldsymbol{a})$
- **Strong canonicity of $\boldsymbol{L_{\Sigma sym}}$**: For all maximal $L_{\Sigma sym}$-consistent sets S of modal formulas, the canonical frame $F_S = <W_S,R_S>$ of $L_{\Sigma sym}$ defined by S validates $L_{\Sigma sym}$

Canonicity

- A second extension of L_{min}
  - Let Σ_{con} be $\{(p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)\}$
  - The **axioms of $L_{\Sigma con}$** are those of L_{min} plus every modal formula like $(a \neq 0) \wedge (a \neq 1) \rightarrow (a \oplus \neg a)$
- **Non strong canonicity of $L_{\Sigma con}$**: There exists a maximal $L_{\Sigma con}$-consistent sets S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of $L_{\Sigma con}$ defined by S does not validate $L_{\Sigma con}$
- **Weak canonicity of $L_{\Sigma con}$**: There exists a maximal $L_{\Sigma con}$-consistent sets S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of $L_{\Sigma con}$ defined by S validates $L_{\Sigma con}$

Canonicity

- **Open problems**
  - Find a set Σ of modal formulas such that $\boldsymbol{L_\Sigma}$ **is not weakly canonical**
  - Find a syntactic condition on the sets Σ of modal formulas implying that $\boldsymbol{L_\Sigma}$ **is strongly canonical**
  - Find a syntactic condition on the sets Σ of modal formulas implying that $\boldsymbol{L_\Sigma}$ **is weakly canonical**

Decidability/complexity

- Let **sat-L_{min}** be the following decision problem
  - Input: A modal formula ϕ
  - Output: Determine if there exists a model $M = <W,R,V>$ such that M sat ϕ
- **Complexity of sat-L_{min}**: sat-L_{min} is NP-complete

Decidability/complexity

- Let Σ be a set of modal formulas
- Let **sat-L_Σ** be the following decision problem
  - Input: A modal formula ϕ
  - Output: Determine if there exists a model $M = \langle W,R,V \rangle$ such that M sat ϕ and $\langle W,R \rangle$ val Σ
- **Complexity of sat-L_Σ (upper bound)**: If Σ is finite then sat-L_Σ is in NEXPTIME
- **Complexity of sat-L_Σ (lower bound)**: If the class of all frames $F = \langle W,R \rangle$ such that $\langle W,R \rangle$ val Σ is nonempty then sat-L_Σ is NP-hard

Decidability/complexity

- A first extension of L_{min}
  - Let Σ_{sym} be $\{(\boldsymbol{p}\oplus\boldsymbol{q})\rightarrow(\boldsymbol{q}\oplus\boldsymbol{p})\}$
  - The **axioms of** $\boldsymbol{L_{\Sigma sym}}$ are those of L_{min} plus every modal formula like $(\boldsymbol{a}\oplus\boldsymbol{b})\rightarrow(\boldsymbol{b}\oplus\boldsymbol{a})$
- **Complexity of sat-**$\boldsymbol{L_{\Sigma sym}}$: sat-$L_{\Sigma sym}$ is NP-complete

Decidability/complexity

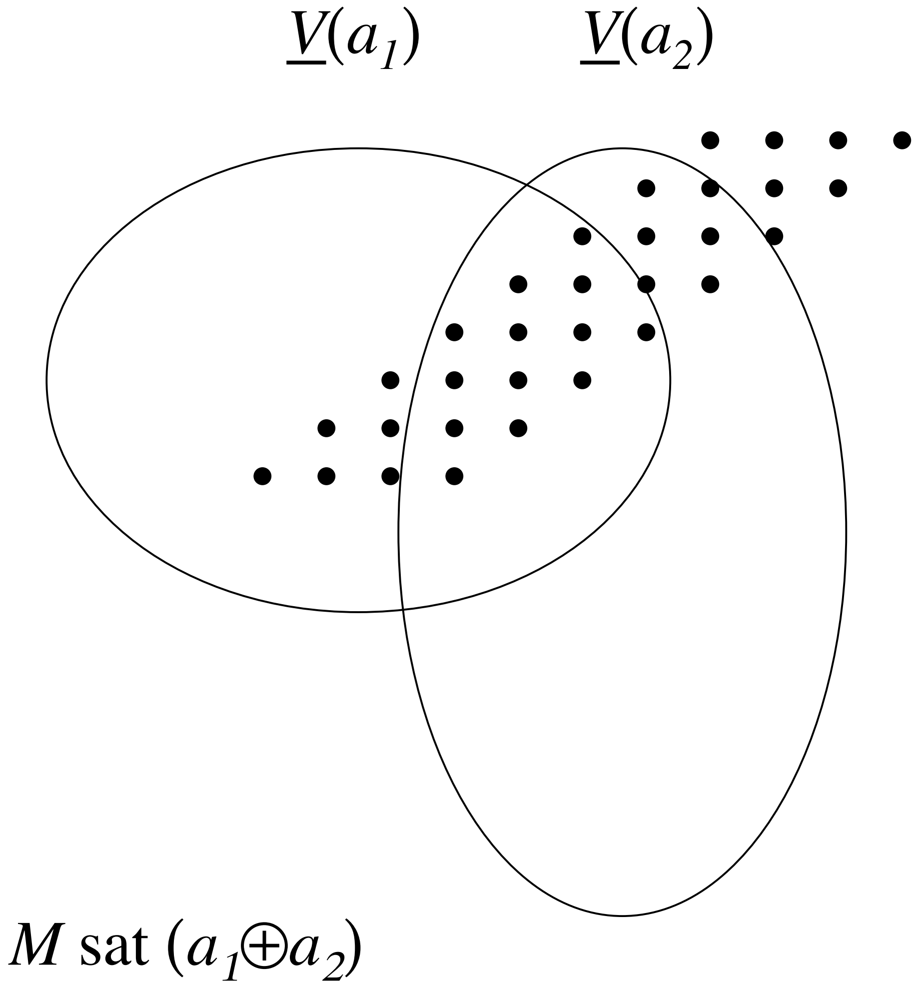
- A second extension of L_{min}
  - Let Σ_{con} be $\{(\boldsymbol{p \neq 0}) \wedge (\boldsymbol{p \neq 1}) \rightarrow (\boldsymbol{p \oplus \neg p})\}$
  - The **axioms of $L_{\Sigma con}$** are those of L_{min} plus every modal formula like $(\boldsymbol{a \neq 0}) \wedge (\boldsymbol{a \neq 1}) \rightarrow (\boldsymbol{a \oplus \neg a})$
- **Complexity of sat-$L_{\Sigma con}$**: sat-$L_{\Sigma con}$ is PSPACE-complete

Decidability/complexity

- **Open problems**
  - Find a set Σ of modal formulas such that **sat-L_Σ is EXPTIME-complete**
  - Find a set Σ of modal formulas such that **sat-L_Σ is NEXPTIME-complete**
  - Find a set Σ of modal formulas such that **sat-L_Σ is not decidable**

Topological interpretation

- A **topological model** is a structure of the form $M = <X,T,V>$ where
  - $<X,T>$ is a topological space
  - V associates a regular closed subset $V(p)$ of $<X,T>$ to each Boolean variable p
- $\underline{V}$ associates a regular closed subset $\underline{V}(a)$ of $<X,T>$ to each Boolean term a
  - $\underline{V}(p) = V(p)$
  - $\underline{V}(0) = \emptyset$, $\underline{V}(\neg a) = Cl(X/\underline{V}(a))$, $(a_1 \vee a_2) = \underline{V}(a_1) \cup \underline{V}(a_2)$
- Remark that
  - $\underline{V}(1) = X$, $\underline{V}(a_1 \wedge a_2) = Cl(In(\underline{V}(a_1) \cap \underline{V}(a_2)))$


$\underline{V}(a_1)$
$\underline{V}(a_2)$
M sat $(a_1 \oplus a_2)$

Topological interpretation

- **Satisfiability** of ϕ in M = <X,T,V> is defined by:
  - M sat $(\boldsymbol{a_1 \oplus a_2})$ iff $\underline{V}(a_1) \cap \underline{V}(a_2) \neq \emptyset$
  - Not M sat 0, M sat $\neg\phi$ iff not M sat ϕ, M sat $(\phi_1 \vee \phi_2)$ iff M sat ϕ_1 or M sat ϕ_2
  - M sat $(\boldsymbol{a_1 = a_2})$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
- Remark that
  - M sat 1, M sat $(\phi_1 \wedge \phi_2)$ iff M sat ϕ_1 and M sat ϕ_2

Topological interpretation

- A first topological extension of L_{min}
  - Let $\Sigma_{ref,sym}$ be $\{(\boldsymbol{p \neq 0}) \rightarrow (\boldsymbol{p \oplus p}), (\boldsymbol{p \oplus q}) \rightarrow (\boldsymbol{q \oplus p})\}$
  - The **axioms of** $\boldsymbol{L_{\Sigma ref,sym}}$ are those of L_{min} plus every modal formula like $(\boldsymbol{a \neq 0}) \rightarrow (\boldsymbol{a \oplus a}), (\boldsymbol{a \oplus b}) \rightarrow (\boldsymbol{b \oplus a})$
- **Completeness of** $\boldsymbol{L_{\Sigma ref,sym}}$: For all modal formulas ϕ, ϕ is provable from the axioms of $L_{\Sigma ref,sym}$ iff ϕ is valid in the class of all topological models iff ϕ is valid in the class of all frames $F = \langle W,R \rangle$ such that $\forall \boldsymbol{w} \in \boldsymbol{W}(\boldsymbol{wRw})$ and $\forall \boldsymbol{w,w'} \in \boldsymbol{W}(\boldsymbol{wRw'} \rightarrow \boldsymbol{w'Rw})$
- **Complexity of sat-**$\boldsymbol{L_{\Sigma ref,sym}}$: sat-$L_{\Sigma ref,sym}$ is NP-complete

Topological interpretation

- A second topological extension of L_{min}
  - Let $\Sigma_{ref,sym,con}$ be $\{(\boldsymbol{p \neq 0}) \rightarrow (\boldsymbol{p \oplus p}), (\boldsymbol{p \oplus q}) \rightarrow (\boldsymbol{q \oplus p}), (\boldsymbol{p \neq 0}) \wedge (\boldsymbol{p \neq 1}) \rightarrow (\boldsymbol{p \oplus \neg p})\}$
  - The **axioms of $L_{\Sigma ref,sym,con}$** are those of L_{min} plus every modal formula like $(\boldsymbol{a \neq 0}) \rightarrow (\boldsymbol{a \oplus a})$, $(\boldsymbol{a \oplus b}) \rightarrow (\boldsymbol{b \oplus a})$, $(\boldsymbol{a \neq 0}) \wedge (\boldsymbol{a \neq 1}) \rightarrow (\boldsymbol{a \oplus \neg a})$
- **Completeness of $L_{\Sigma ref,sym,con}$**: For all modal formulas ϕ, ϕ is provable from the axioms of $L_{\Sigma ref,sym,con}$ iff ϕ is valid in the class of all connected topological models iff ϕ is valid in the class of all frames $F = <W,R>$ such that $\boldsymbol{\forall w \in W (wRw)}$, $\boldsymbol{\forall w,w' \in W (wRw' \rightarrow w'Rw)}$ and ***R* is connected**
- **Complexity of sat-$L_{\Sigma ref,sym,con}$**: sat-$L_{\Sigma ref,sym,con}$ is PSPACE-complete

Conclusion

- We have considered logics based on a language that contains the operators $\oplus$ and $=$
  - $a ::= p \mid 0 \mid \neg a \mid (a_1 \vee a_2)$
  - $\phi ::= (\boldsymbol{a_1 \oplus a_2}) \mid 0 \mid \neg\phi \mid (\phi_1 \vee \phi_2) \mid (\boldsymbol{a_1 = a_2})$
  - M sat $(\boldsymbol{a_1 \oplus a_2})$ iff there exists $w \in \underline{V}(a_1)$ such that for some $w' \in R(w)$, $w' \in \underline{V}(a_2)$
  - M sat $(\boldsymbol{a_1 = a_2})$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
- $(\boldsymbol{a_1 \oplus a_2})$ is equivalent to $\langle U \rangle (a_1 \wedge \langle R \rangle a_2)$ and $(\boldsymbol{a_1 = a_2})$ is equivalent to $[U](a_1 \leftrightarrow a_2)$

Conclusion

- We might also consider logics based on the more general language that contains the $\otimes$ as well
  - $a ::= p \mid 0 \mid \neg a \mid (a_1 \vee a_2)$
  - $\phi ::= (a_1 \oplus a_2) \mid \boldsymbol{(a_1 \otimes a_2)} \mid 0 \mid \neg\phi \mid (\phi_1 \vee \phi_2) \mid (a_1 = a_2)$
  - M sat $(a_1 \oplus a_2)$ iff there exists $w \in \underline{V}(a_1)$ such that for some $w' \in R(w)$, $w' \in \underline{V}(a_2)$
  - M sat $(a_1 = a_2)$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
  - M sat $\boldsymbol{(a_1 \otimes a_2)}$ iff there exists $w \in \underline{V}(a_1)$ such that for each $w' \in R(w)$, $w' \in \underline{V}(a_2)$
- $(a_1 \oplus a_2)$ is equivalent to $\langle U\rangle(a_1 \wedge \langle R\rangle a_2)$, $(a_1 = a_2)$ is equivalent to $[U](a_1 \leftrightarrow a_2)$ and $\boldsymbol{(a_1 \otimes a_2)}$ is equivalent to $\langle U\rangle(a_1 \wedge [R]a_2)$