MODAL LOGICS OF SPACE

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***Pictures**: dynamically in real-time on the blackboard!*

1 The interest of Space

When thinking about the physical world, modal logicians have taken Time as their main theme, because it fits so well with an interest in the flow of information and computation. Spatial logics have been footnotes to the tradition – even though the axiomatic method was largely geometrical. An exception is Tarski's early work on deviant geometrical primitives, and his decidable first-order axiomatization of elementary geometry. Today Space remains intriguing – both for mathematical reasons, and given the amount of work in CS and AI on visual reasoning and image processing. These two concerns are by no means the same, but both involve logic of spatial structures.

2 Levels of structure

Studying space can be done at many different levels, depending on one's special interest. One must choose some grain level of mathematical structure (topological, affine, metrical, or yet other) providing the right invariances (homeomorphism, affine transformations, etc.). But given all this, why not 'just do geometry'? At each of these levels, a logician will try to design some appropriate language that brings out interesting laws, preferably in a calculus of some reasonable complexity. Major concern in logic: *The Balance* between Expressive Power and Computational Complexity. We will look at a number of spatial languages, going from simple to more complex.

3 Modal logic and topology

Started by Tarski and McKinsey. Let **M** be a topological space plus a valuation:

$\mathbf{M}, s \models []\phi$ iff s is in the topological interior of $[[\phi]]^{\mathbf{M}}$
or, $\exists O \in \mathbf{O}$: $s \in O$ & $\forall t \in O$: $\mathbf{M}, t \models \phi$

Relevant valid S4-principles:

$[]\phi \to \phi$ $\qquad$ $[][]\phi \leftrightarrow []\phi$
$[](\phi \& \psi) \leftrightarrow []\phi \& []\psi$ $\qquad$ $[]\phi \vee []\psi \leftrightarrow []([]\phi \vee []\psi)$

<u>General Completeness Theorem</u>

A modal formula is valid iff it is provable in S4.

This is proved by the usual modal completeness arguments. In a strange historical inversion, the following semantic result is earlier, and mathematically much deeper.

Special Completeness Theorem

The modal logic of any metric space without isolated points equals S4.

In particular, S4 is the complete logic of any space IR^n. Recent nice completeness proof by Mints, for Cantor Space: i.e., the binary tree with its natural topology, extended to IR by Kremer, Mints & Ting, Aiello, van Benthem & Bezhaneshvili.

In addition to classical completeness results, there is modal model theory of space. E.g., this language suggests *bisimulation games* between Spoiler and Duplicator, comparing points across topological models. A basic episode goes like this:

> Spoiler chooses a current point s in one model plus an open neighbourhood U ,
> Duplicator responds with an open neighbourhood V of the other current point t ,
> Spoiler chooses a point v in V, Duplicator chooses a point u in U, making u–v
> the new match. Duplicator loses if the two points differ qua atomic properties.

Topo-bisimulations: the corresponding relations between topological spaces with a valuation, that encode winning strategies for Duplicator. Here is one direction:

whenever s E t , then, if $s \in U \in \mathbf{O}$, there exists a V
with (a) $t \in V \in \mathbf{O}'$ and (b) $\forall v \in V\ \exists u \in U$: u E v .

These zigzag clauses are related to continuous functions. Topo-bisimulation is a coarse variant of homeomorphism. It *preserves truth of all modal formulas*, on the analogy of ordinary modal logic over graph models. But it does not preserve connectedness of topological spaces, for which we need stronger modal languages.

Issue: modal analysis of just what is preserved by *continuous maps*. Cf. Aiello & van Benthem 2002, and van Benthem 2000 on broader background in Chu Spaces.

Further issues: *landscape of stronger topological logics above S4*. Example: determining logics of reasonable sets that occur in 'images'. E.g., 'serial sets': finite unions of convex sets on the real line. A typical valid principle now:

$$\neg\phi \;\&\; \langle\rangle\phi \;\rightarrow\; \langle\rangle[\,]\phi$$

Complete logic = that of the '2-fork' (NP-complete). Open questions for various 'image patterns' abound. E.g., axiomatize the complete modal logic of unions of rectangles in 2D, 3D,...: just done by van Benthem, Bezhanishvili, and Gehrke.

4 Logical strengthenings

This brings tools from modern "extended modal languages (cf. 'hybrid modal logic') to the analysis of spatial patterns. Example: add *universal/existential modalities* to define the *global* predicates used in the Calculus of Regions: 'A⊆B' as U(A→B) . This also defines, e.g., connectedness of topological spaces:

U([]pv[]q) & Ep & Eq → E(p&q) (no non-empty open sets partition the space)

Extended *topo-bisimulation*: domain and range now have to be *total*. Continuous functions are 'simulations' preserving 'universal statements' in this extended modal language. *Completeness theorems*. Shehtman recently axiomatized the modal logic of connected spaces. Open question: the complete logic(s) of the spaces IR^n . Another modal theme which generalizes: *correspondence* analysis. In particular, the above axiom says, on standard modal models, that every two points are connected by some finite chain of moves {forward, backward} in the ordering.

Other logical strengthenings: analogues of *temporal operators*. E.g., UNTILφψ:

there is an open neighbourhood of the current point which is totally φ,
and its boundary points all satisfy ψ.

This allows us much more expressiveness over regions.

5 Geometrical strengthenings

Now, we change the 'similarity type' of the models.

Affine extensions Primitive ternary *betweenness*, and a convexity modality:

M, s |= Cφ iff s lies in between two points satisfying φ
or, ∃t, u: Bs, tu & **M**, t|=φ & **M**, u|=φ

Now differences of dimensionality between spaces IR^n do show up in the logic:

CCφ ↔ Cφ , etc.

Other valid principles involve interaction with the topology: e.g., C[]p → []Cp , not vice versa. *Complete logics*: unknown!

Better version (fully distributive):

M, s |= Cφψ iff ∃t, u: Bs, tu & **M**, t|=φ & **M**, u|=ψ

Examples of modal thinking: *bisimulation* for affine geometry. New distinctions: e.g., when is the bisimulation contraction *planar*?

Nice example: real geometric content of special modal principles. Associativity of the binary betweenness modality corresponds to Pasch' Axiom!

Metric extensions Introduce one more ternary predicate of *relative nearness*:

Nx, yz y is closer to x than z is to x

This satisfies some nice principles, such as various 'Triangle Inequalities':

$$(Nx, yz \ \& \ Nz, xy) \ \rightarrow Ny, xz$$

In a full first-order language, N gives all of elementary geometry. Open questions include even: axiomatize complete universal first-order theory of relative nearness.

Connections with 'similarity models' for conditional logics with relative preference semantics. *Completeness* unknown! In philosophical logic, no complete conditional logics are known of specific mathematical similarity structures.

6 Modal topology and geometry

The general program explores 'modal fragments' of complex logics like Tarski's elementary geometry (or its universal monadic second-order extension). Languages try to combine reasonable expressive power ('useful patterns') with low complexity of relevant tasks: model checking, model comparison, theorem proving.

7 Mathematical morphology

This is a relatively new theory of shapes in image processing. Regions are sets of vectors, with two key 'Minkowski operations' (in addition to ordinary Booleans):

Addition	A+B	$\{x \mid \exists y \in A, z \in B: x = y+z\}$
Subtraction	A–B	$\{x \mid \forall y \in B, x+y \in B\}$

Motivation: manipulate, and 'smoothen' figures. Examples of valid principles:

$$(X \cup Y) + S = (X+S) \cup (Y+S)$$

$$(A \cup B) \rightarrow C \ = \ (A \rightarrow C) \cap (B \rightarrow C)$$

'Morphological opening': $X \text{ o } S = \text{def } (S \rightarrow X) + S$

satisfiesa typical law like $(X \text{ o } S) \text{ o } S \ = \ X \text{ o } S$

Uses in describing images: normal forms. Typically *non-valid*: $A + A = A$.
This is unlike earlier projective space semantics for relevant or quantum logics.

This suggests a perhaps surprising logic connection:

<u>Theorem</u> Morphological addition and subtraction are a model for *multiplicative linear logic*, with + for *product*, and A–B for the *implication* B→A.

Illustration: (X o S) o S = X o S amounts to the literally LL-derivable sequents

$$(S \rightarrow X) + S \ \Rightarrow \ (S \ \rightarrow \ ((S \rightarrow X) + S)) \ + S$$

$$(S \ \rightarrow \ ((S \rightarrow X) + S)) \ + S \ \Rightarrow \ (S \rightarrow X) + S$$

Completeness: open problem. Ditto for language extensions ('linear closure').

Modal alternative: 'Arrow Logic', with binary modalities for product, and converse for the linear minus. This generalises better than linear logic, but less familiar.

Further striking resemblances with extended modal logic. Hybrid logic: use of special letters for single vectors, and laws which depend on them. Example:

> $S \rightarrow (X + \{t\}) = (S \rightarrow X) + \{t\}$: from right to left, this is derivable in LL as the general law $(S \rightarrow X) \bullet A \Rightarrow S \rightarrow (X \bullet A)$. The converse of this is not LL-derivable, but it only works because, for singletons $\{t\}$, we have the special principle $S \Leftarrow\Rightarrow (S+\{t\}) - \{t\}$, which we have to 'inject' into an otherwise fine LL-derivation to get the desired result.

The upshot. Mathematical morphology was developed independently in CS in the early 1980s. But it is an interesting sort of linear and modal logic over vector spaces – and hence a nice illustration of the ubiquity of modal structures in space.

8 Alternatives: straight first- and second-order logics of space

Ian Pratt's lecture! See for instance the work by Pratt and Schoop on polygons and other structures, Randall on space in robotics, or Goldblatt on orthogonality geometry of space-time. Much of this continues the line of Tarski's 'Elementary Geometry'. Also relevant: Andréka & Németi's recent work on logics of relativistic space-time, using suitable fragments of first-order languages with an algebraic (or after all) modal flavour. Workshop 'Logic and Physics', *Utrecht 12 January 2008*.

9 Coda: space, diagrams, and logic

Related, but different tradition: information in visual patterns. Reasoning with diagrams, intermediate of language and space. Cognitive counterparts: parietal cortex between visual cortex and language areas ('The Logical Brain in Action', Knauff 2007, *Topoi* 'Logic and Psychology', van Benthem & Hodges, eds.).

10 References

(a) M. Aiello, 2002, *Spatial Reasoning: theory and practice*, dissertation, Institute for Logic, Language and Computation & ISIS, Amsterdam.

(b) M. Aiello, I. Pratt & J. van Benthem, eds., 2007, *Handbook of Spatial Logics*, Springer, Dordrecht – and many chapters therein.

(c) J. van Benthem, P. Blackburn & F. Wolter, eds., 2006, *Handbook of Modal Logic*, Elsevier, Amsterdam – though with chapters on time, but not on space!

(d) G. Kerdiles, 2001, *Saying It with Pictures*: *a Logical Landscape of Conceptual Graphs*, Dissertation, Institute for Logic, Language and Computation, University of Amsterdam.