# Modal Logics with Counting

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**Abstract.** We present a modal language that includes explicit operators to count the number of elements that a model might include in the extension of a formula, and we discuss how this logic has been previously investigated under different guises. We show that the language is related to graded modalities and to hybrid logics. We illustrate a possible application of the language to the treatment of plural objects and queries in natural language. We investigate the expressive power of this logic via bisimulations, discuss the complexity of its satisfiability problem, define a new reasoning task that retrieves the cardinality bound of the extension of a given input formula, and provide an algorithm to solve it.

## 1 Counting, Modally

Suppose there are at least two apples (say, on the table, but we don't care at the moment where the apples are). First-order logic ($\mathcal{FOL}$) with equality has no problem expressing this fact[1]:

$$\exists x.\exists y.(x \neq y \wedge Apple(x) \wedge Apple(y)).$$

We can actually dispense with equality, if we introduce counting quantifiers [1]

$$\exists^{\geq 2} x.Apple(x).$$

But suppose that we want to dispense with *quantifiers* instead, and count in terms of a propositional (or a modal) language. The following representation seems quite natural (arguably, even more natural than the first-order counterparts with or without counting quantifiers)

$$Apple \geq 2.$$

In this paper we will investigate propositional and modal languages extended with such counting operators. Let us be bold and introduce, already, the formal syntax and semantics of the basic modal logic with counting $\mathcal{MLC}$, the main language we want to explore:

[1] It is well known that $\mathcal{FOL}$ can express any finite counting quantifier.

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**Definition 1 (Syntax).** *Let* $\mathsf{Prop} = \{p_1, p_2, \dots\}$ *(the* propositional symbols*) and* $\mathsf{Rel} = \{r_1, r_2, \dots\}$ *(the* relational symbols*) be disjoint, countable infinite sets. The set* $\mathsf{Forms}$ *of formulas of* $\mathcal{MLC}$ *over signature* $\langle \mathsf{Prop}, \mathsf{Rel}\rangle$ *is defined as:*

$$\mathsf{Forms} ::= \bot \mid p \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \langle r\rangle\varphi \mid (\varphi \geq n) \mid (\varphi \leq n),$$

*for* $p \in \mathsf{Prop}$*,* $r \in \mathsf{Rel}$*,* $\varphi, \varphi_1, \varphi_2 \in \mathsf{Forms}$ *and* $n$ *a natural number. Other Boolean and modal operators are defined as usual, and we define* $(\varphi = n)$ *as* $(\varphi \geq n) \wedge (\varphi \leq n)$*,* $(\varphi > n)$ *as* $(\varphi \geq (n+1))$ *and* $(\varphi < n)$ *as* $(\varphi \leq (n-1))$ *if* $n > 0$ *or* $\bot$ *otherwise.*

We will call $\mathcal{PLC}$ the "propositional fragment," i.e., the fragment obtained by dropping $\langle r\rangle\varphi$. Let us now introduce the semantics.

**Definition 2 (Semantics).** *Given a signature* $\mathcal{S} = \langle \mathsf{Prop}, \mathsf{Rel}\rangle$*, a* model *for* $\mathcal{S}$ *is a tuple* $\langle W, (R_r)_{r\in\mathsf{Rel}}, V\rangle$*, satisfying the following conditions: (i)* $W \neq \emptyset$ *(elements in* $W$ *are called states); (ii) each* $R_r$ *is a binary relation on* $W$ *(usually called accessibility relations); (iii)* $V : \mathsf{Prop} \to 2^W$ *is a labeling function.*

*Given the model* $\mathcal{M} = \langle W, (R_r)_{r\in\mathsf{Rel}}, V\rangle$ *and* $w \in W$*, the semantics for the different operators is defined as follows:*

$$\begin{array}{lll}
\mathcal{M}, w \models p & \iff & w \in V(p), \quad p \in \mathsf{Prop} \\
\mathcal{M}, w \models \neg\varphi & \iff & \mathcal{M}, w \not\models \varphi \\
\mathcal{M}, w \models \varphi \wedge \psi & \iff & \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \langle r\rangle\varphi & \iff & \text{there is } w' \text{ such that } R_r(w, w') \text{ and } \mathcal{M}, w' \models \varphi \\
\mathcal{M}, w \models (\varphi \geq n) & \iff & |\{w \mid \mathcal{M}, w \models \varphi\}| \geq n \\
\mathcal{M}, w \models (\varphi \leq n) & \iff & |\{w \mid \mathcal{M}, w \models \varphi\}| \leq n.
\end{array}$$

*We will say that a formula* $\varphi$ *is* satisfiable*, if there is a model* $\mathcal{M}$ *and a state* $w$ *in its domain such that* $\mathcal{M}, w \models \varphi$*. For a set of formulas* $\Gamma \cup \{\varphi\}$ *we say that* $\Gamma \models \varphi$ *if and only if for any model* $\mathcal{M}$ *and any* $w$ *in its domain* $\mathcal{M}, w \models \Gamma$ *implies* $\mathcal{M}, w \models \varphi$ *(this relation is sometimes called* local entailment*). The* extension $\|\varphi\|^{\mathcal{M}}$ *of a formula* $\varphi$ *in a model* $\mathcal{M}$ *is the set* $\{w \mid \mathcal{M}, w \models \varphi\}$*, and the* theory *of* $w$ *in* $\mathcal{M}$*, notation* $\mathsf{Th}^{\mathcal{M}}(w)$*, is the set* $\{\varphi \mid \mathcal{M}, w \models \varphi\}$*. When the model* $\mathcal{M}$ *is clear from context we will drop the super-indexes. We will write* $\mathcal{M}, w \equiv_{\mathcal{MLC}} \mathcal{M}', w'$ *if* $\mathsf{Th}^{\mathcal{M}}(w) = \mathsf{Th}^{\mathcal{M}'}(w')$*.*

It should be clear from Definitions 1 and 2 that $\mathcal{MLC}$ is indeed the basic modal logic $\mathcal{ML}$ [2] extended with the counting operators. We will be mainly discussing extensions of $\mathcal{ML}$ for simplicity. We could have naturally added the counting operators to any modal logic, e.g., temporal logic with counting.

The $\mathcal{MLC}$ language and, in particular, its sublanguage $\mathcal{PLC}$ have been investigated under different guises. $\mathcal{PLC}$ is introduced as the logic $S5_n$ by Fine in [3] where the, by now well studied, notion of *graded modalities* was introduced. The semantic definition of the graded modality $\langle r\rangle_n\varphi$ is given by the condition

$$\mathcal{M}, w \models \langle r\rangle_n\varphi \iff |\{w' \mid R_r(w, w') \text{ and } \mathcal{M}, w' \models \varphi\}| \geq n.$$

$S5_n$ is the logic obtained when the $\langle r\rangle_n$ operator is restricted to models where $R_r$ is interpreted as an equivalence relation. Now, if $R_r$ is the universal relation, then $\langle r \rangle_n \varphi$ is trivially equivalent to $(\varphi \geq n)$. But a well known result (see, e.g. [2]) establishes that the modal logic of the universal relation coincides with the modal logic obtained when we only require the accessibility relation to be an equivalence relation. The main contribution of [3] is to provide sound and complete axiomatizations for these languages. The original results of Fine were extended by van der Hoek and de Rijke in [4,5]. In addition to providing further axiomatizations, investigating normal forms, and establishing the complexity of the satisfiability problem for different logics with graded modalities, the authors propose these languages as a modal framework where some ideas from the Theory of Generalized Quantifiers [6] could be investigated by means of modal tools.

The relation between $\mathcal{MLC}$ and graded modalities was also discovered in the field of description logics. In this area, graded modalities are called *cardinality restrictions* and Baader *et al.* investigate in [7] *concept cardinality restrictions* which coincide exactly with the counting operators we defined. Interestingly, they decide to add concept cardinality restrictions not as operators of the concept language, but as a more expressive kind of terminological axioms, and they remark that they can express classical terminological axioms of the form $\varphi \sqsubseteq \psi$. $\varphi \sqsubseteq \psi$ is satisfied in the model if the interpretation of $\varphi$ is a subset of the interpretation of $\psi$, and indeed this is the case exactly when $((\varphi \wedge \neg\psi) \leq 0)$. The main contribution of [7] is the definition of sound, complete and terminating tableaux calculus for these languages. A detailed complexity analysis of their satisfiability problem and optimal tableau calculi are given in [8].

Another way of explaining why counting operators can express terminological axioms is realizing that they can express the universal modality $\mathsf{A}\varphi$ [9]:

$$\mathcal{M}, w \models \mathsf{A}\varphi \iff \text{for all } w', \mathcal{M}, w' \models \varphi.$$

$\mathsf{A}\varphi$ is equivalent to $((\neg\varphi) \leq 0)$, and $\varphi \sqsubseteq \psi$ is equivalent to $\mathsf{A}(\varphi \rightarrow \psi)$. Actually, counting modalities can also express *nominals* (i.e., special propositional symbol whose interpretations are restricted to singleton subsets of the domain) by just stating $(p = 1)$ for $p$ a propositional symbol, and hence they can be considered also as hybrid logics [10].

In this article, we provide new results about the $\mathcal{MLC}$ language. Our first contribution is conceptual, rather than technical, and it can be simple put as follows. The counting operators $(\varphi \geq n)$ and $(\varphi \leq n)$ are interesting on their own, independently of their relation with graded modalities. They are *global* operators (with a behavior similar to the universal modality or satisfiability operators), and they can be naturally combined with local operators (as is commonly done in hybrid languages). They are also modular, and they can naturally be added to any modal language. In a slogan: counting operators are the modal counterpart of first-order counting quantifiers.

In Section 2 we show how $\mathcal{MLC}$ can be used as representation language in a natural language application modeling queries including plurals. In Section 3 we will investigate the expressive power of $\mathcal{MLC}$ using a suitable notion of bisimulation. In Section 4 we first discuss the complexity of the satisfiability problem, drawing from previously known results, we then introduce a new reasoning task and devise an algorithm to solve it.

## 2 Representing Plurals in Natural Language

We discuss here a possible representation of plurals and references in $\mathcal{MLC}$, intended to be used in natural language processing tasks such as reference resolution or generation as is done in, e.g., [11]. The idea is to represent the information introduced in a discourse as a set of $\mathcal{MLC}$ formulas $\Gamma$, and to be able to express and answer queries of the form "how many of a certain kind of objects are there?" in this context. This representation does not aim to solve all the issues concerning the use of plurals in natural language (e.g., the distributive versus collective readings of certain adjectives when applied to sets of objects), which are known to be difficult to model [12]. For further details see, for example, [13].

As we saw in the previous section, $\mathcal{MLC}$ enables us to assert the cardinality of a proposition in the model. For example, $\Gamma = \{(Apple \wedge Red) = 2\}$ represents the sentence "there are two red apples", and the query "how many $(Apple \wedge Red)$?" should return "2". But suppose that we want to *refer* to "two red apples" (i.e., we don't know how many red apples are there in total, but we want to refer to two of them). For the representation of this kind of reference we need to be able to *name* the referred group of object by, for example, introducing a new propositional symbol $a_1$ and adding to $\Gamma$ the formula[2]:

$$\text{“two red apples”: } (a_1 = 2) \wedge (a_1 \sqsubseteq (Apple \wedge Red))$$

In this case, a query "how many $(Apple \wedge Red)$?" cannot be answered (i.e., is undefined) since the total number of apples in the model is not known. But the query "how many $a_1$?" should return "2".

If now we add that there are also two green apples and want to refer to that group, we need to introduce another propositional symbol $a_2$ and add to $\Gamma$:

$$\text{“two green apples”: } (a_2 = 2) \wedge (a_2 \sqsubseteq (Apple \wedge Green))$$

Now, the number of apples that are in the group formed by $a_1$ and $a_2$ (i.e., $a_1 \vee a_2$) is also undefined because nothing prevents those two sets from overlapping. If we explicitly say that the group are disjoint ($a_1 \sqsubseteq \neg a_2$) or that the colors are mutually exclusive ($Green \sqsubseteq \neg Red$) for that we should be able to answer "4".

Suppose that now we learn that "three of the apples are rotten." This reference creates a new group containing all the apples mentioned up to now:

$$(a_3 \sqsubseteq (a_1 \vee a_2)) \;\wedge\; ((a_1 \vee a_2) \sqsubseteq a_3)$$

And then assert that three of them are rotten by adding to $\Gamma$ $(a_3 \wedge Rotten) = 3)$. If we further discover that all the red apples are rotten ($a_1 \sqsubseteq Rotten$), querying for "how many green apples are rotten," i.e., "how many $(a_2 \wedge Rotten)$" will returns "1".

In Section 4 we introduce the inference task of counting that corresponds to the finite cardinality queries we just discussed. But first, in the next section, we investigate in detail the expressive power of $\mathcal{MLC}$.

[2] Remember that $\varphi \sqsubseteq \psi$ is a short hand for $\mathsf{A}(\varphi \to \psi)$ or, equivalently, $(\varphi \wedge \neg\psi) \leq 0$.

## 3 The Expressive Power of $\mathcal{MLC}$

To get more familiar with the language, let us start with some examples of what can be expressed in $\mathcal{MLC}$. We can, for example, fix the size of the model to any finite cardinality by setting

$$(\top = n)$$

for $n$ a natural number. The formula also shows that, if numbers are coded in binary, then neither $\mathcal{MLC}$ nor $\mathcal{PLC}$ has the polysize model property.

**Proposition 1.** *If numbers are coded in binary, then there are formulas in $\mathcal{PLC}$ (and hence also in $\mathcal{MLC}$) whose only models are exponentially larger.*

Notice that counting operators can be nested. For example $((p \geq 1) \geq 1)$ is a well formed formula, which it is actually equivalent to $(p \geq 1)$. But, as it is discussed in [4], every formula in $\mathcal{MLC}$ is equivalent to a formula where each counting operator appears under the scope of neither modal nor counting operators. The proof uses the fact that for any counting subformula $\sigma$ appearing in a formula $\varphi$ we have that the following is valid

$$\varphi[\sigma] \leftrightarrow (\sigma \rightarrow \varphi[\sigma/\top]) \wedge (\neg\sigma \rightarrow \varphi[\sigma/\bot])$$

Other operators with a global semantics, like the universal modality $\mathsf{A}$ or satisfiability operators $i$:, have the same property. Notice though, that the formula we obtain after extracting all counting operators can be exponentially larger. If we only require equi-satisfiability (and not equivalence), we can use the method of [14] to obtain a formula which is only polynomially larger. We will return to this issue in Section 4.

As we mentioned in the introduction, the hybrid logic $\mathcal{H}(\mathsf{A})$ (the basic modal logic extended with nominals and the universal modality [10]) is a sublogic of $\mathcal{MLC}$, as the language can express nominals and the universal modality. It can even express the difference modality $\mathsf{D}\varphi$ [15] with semantics

$$\mathcal{M}, w \models \mathsf{D}\varphi \iff \text{there is } w' \neq w \text{ and } \mathcal{M}, w' \models \varphi$$

as $\mathsf{D}\varphi$ is equivalent to $(\varphi \rightarrow (\varphi \geq 2)) \wedge (\neg\varphi \rightarrow (\varphi \geq 1))$. On the other hand, the expressive power of counting and graded modalities is incomparable. We will establish this in Theorem 3 using a suitable notion of bisimulation for $\mathcal{MLC}$ that we now introduce

**Definition 3 (Bisimulation).** *A* bisimulation *between two models $\mathcal{M} = \langle W, (R_r)_{r\in\mathsf{Rel}}, V\rangle$ and $\mathcal{M}' = \langle W', (R'_r)_{r\in\mathsf{Rel}}, V'\rangle$ is a non-empty binary relation $E$ between their domains (that is, $E \subseteq W \times W'$) such that whenever $wEw'$ we have:*

**Atomic harmony:** *$w$ and $w'$ satisfy the same propositional symbols.*
**Zig:** *if $R_r wv$ then there exists a point $v' \in W'$ such that $vEv'$ and $R'_r w'v'$.*
**Zag:** *if $R'_r w'v'$ then there exists a point $v \in W$ such that $vEv'$ and $R_r wv$.*
**Bijectivity:** *$E$ contains a bijection between $W$ and $W'$.*

*For two models $\mathcal{M}$ and $\mathcal{M}'$ and two elements $w$ and $w'$ in their respective domains, we write $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ if there exists a bisimulation between $\mathcal{M}, w$ and $\mathcal{M}', w'$ linking $w$ and $w'$.*

**Theorem 1.** *If $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ then $\mathcal{M}, w$ and $\mathcal{M}', w'$ satisfy the same formulas of $\mathcal{MLC}$.*

*Proof.* Assume there is a bisimulation $E$ between $\mathcal{M}$ and $\mathcal{M}'$. Because of Atomic harmony, Zig and Zag, we know that $E$ preserves all formulas of the basic modal language [2]. We only need to consider the counting operators.

Suppose then that $\varphi = (\psi \geq n)$ and let $f$ be one bijection that by definition is contained in the bisimulation linking $\mathcal{M}$ and $\mathcal{M}'$. Assume that $\mathcal{M}, w \models (\psi \geq n)$. By inductive hypothesis $f(\|\psi\|^{\mathcal{M}}) \subseteq \|\psi^{\mathcal{M}'}\|$ and because $f$ is a injective $|f(\|\psi\|^{\mathcal{M}})| \geq n$, hence $\mathcal{M}', w' \models (\psi \geq n)$. For the other direction, assume $\mathcal{M}', w' \models (\psi \geq n)$. Because $f$ is a bijection we can consider $f^{-1}(\|\psi\|^{\mathcal{M}'})$ which has size greater than $n$ , and by inductive hypothesis we know that it is a subset of $\|\psi\|^{\mathcal{M}}$. Hence $\mathcal{M}, w \models (\psi \geq n)$. The case for $\varphi = (\psi \leq n)$ is similar.

As usual, the converse is not necessarily true but it holds on finite models.

**Theorem 2.** *Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ be two finite models and $(w, w') \in W \times W'$, $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ if and only if $\mathcal{M}, w \equiv_{\mathcal{MLC}} \mathcal{M}', w'$.*

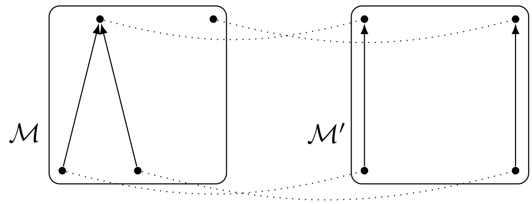
*Proof.* The implication from left to right is given by Theorem 1. For the other implication, we have to prove that $\equiv_{\mathcal{MLC}}$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ that links $w$ and $w'$. Atomic harmony, Zig and Zag are proved in the standard way (see [2]). To prove that $\equiv_{\mathcal{MLC}}$ contains a bijection reason as follows.

Consider every pair of subsets $(C, C')$, $C \subseteq W$, $C' \subseteq W'$ such that for all $(a, b) \in C \times C'$, $\mathcal{M}, a \equiv_{\mathcal{MLC}} \mathcal{M}', b$. There is at least one such pair by hypothesis. Enumerate these pairs as $(C_1, C'_1), \ldots, (C_n, C'_n)$ (as the model is finite there is only a finite number of them), and let $\Sigma_1, \ldots, \Sigma_n$ be such that $\Sigma_i = \mathsf{Th}(a)$ for some $a \in C_i \cup C'_i$ (by construction all elements in $C_i \cup C'_i$ satisfy the same formulas of $\mathcal{MLC}$). Now choose for each $i$, $\varphi_i \in \Sigma_i$ such that for all $j \neq i$, $\varphi_i \notin \Sigma_j$. Notice that $|C_i| = |\|\varphi_i\|^{\mathcal{M}}|$ and that $|C'_i| = |\|\varphi_i\|^{\mathcal{M}'}|$, we want to prove that $|C_i| = |C'_i|$. But by hypothesis $\mathcal{M}, w \equiv_{\mathcal{MLC}} \mathcal{M}', w'$, and then $\mathcal{M}, w \models \varphi_i = n$ if and only if $\mathcal{M}', w' \models \varphi_i = n$.

As $C_i$ and $C'_i$ have the same cardinality we can define an injective function $f : \bigcup C_i \to \bigcup C'_i$, such that for $a \in C_i, f(a) \in C'_i$. It only rests to prove that $f$ is total and surjective.

Suppose there is $a \in W$ such that $a \notin \bigcup C_i$, then there is no element $a'$ in $W'$ such that $\mathcal{M}, a \equiv_{\mathcal{MLC}} \mathcal{M}', a'$. For each $a'_i \in W'$, let $\varphi_i$ be a formula such that $\varphi_i \in \mathsf{Th}(a)$ but $\varphi_i \notin \mathsf{Th}(a')$. But then $\mathcal{M}, w \models (\bigwedge \varphi_i \geq 1)$ while $\mathcal{M}, w' \not\models (\bigwedge \varphi_i \geq 1)$ contradicting hypothesis. In a similar way we can prove that $f$ is surjective.

Notice that $\mathcal{MLC}$-bisimulations are not isomorphisms. The following two models, for example, are $\mathcal{MLC}$-bisimilar but not isomorphic.

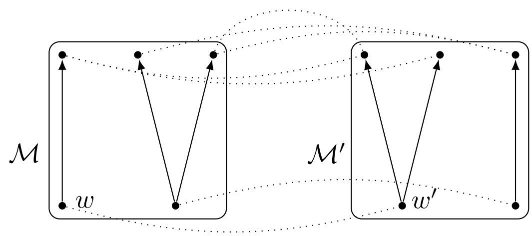



$\mathcal{M}$ and $\mathcal{M}'$ can be differentiated by the first order sentences $\exists x.\forall y.(\neg R(x,y) \wedge \neg R(y,x))$. But there is no $\mathcal{MLC}$ formula which is globally true in one model but false in the other. On the other hand, [6] proves that every sentence of first-order logic with equality and only monadic propositional symbols is equivalent to the translation of a formula in $\mathcal{PLC}$.

We now return to the comparison of $\mathcal{MLC}$ and graded modalities.

**Theorem 3.** *The expressive power of counting modalities and graded modalities is incomparable (when interpreted on the set of all possible models).*

*Proof.* Consider the following two models $\mathcal{M}$ and $\mathcal{M}'$. It is not difficult to verify that the dotted arrows defines a $\mathcal{MLC}$-bisimulation.




$\mathcal{M}, w \not\models \langle r \rangle_2 \top$ while $\mathcal{M}', w' \models \langle r \rangle_2 \top$ while no formula of $\mathcal{MLC}$ can differentiate $w$ and $w'$[3]. For the other direction, just consider a model with one state and another model with two states. Clearly, the models cannot be distinguished using graded modalities (as they can only count the number of successors) but the counting forma formula $(\top \leq 1)$ differentiates them.

## 4 Inference in $\mathcal{MLC}$

The complexity of the satisfiability problem for $\mathcal{MLC}$ and $\mathcal{PLC}$ have been studied in the literature. As we mention in Section 3, when dealing with complexity we should take care of whether numbers are coded in unary or binary. Let us call $\mathcal{L}^u$ and $\mathcal{L}^b$ the unary and binary coding, respectively, for either $\mathcal{MLC}$ or $\mathcal{PLC}$. Then, the previously established results are as follows.

[3] The proof goes through using the same models even if we add past operators to the language, as the bisimulation shown also satisfies the standard conditions $Zig^{-1}$ and $Zag^{-1}$ which preserve past operators [2].

**Theorem 4.** *1. $\mathcal{PLC}^u$-SAT is NP-complete [5].*
*2. $\mathcal{MLC}^u$-SAT is ExpTime-complete [16,8].*
*3. $\mathcal{PLC}^b$-SAT is NP-hard and in PSpace [4].*
*4. $\mathcal{MLC}^b$-SAT is ExpTime-hard and in 2-NExpTime [8].*

*Proof.* Hardness in all cases is clear, we only comment on the upper bounds. The proof of 1) is via the polysize model property. The proof of 2) is by a linear satisfiability preserving translation into $\mathcal{H}(\mathsf{A})$ as we will show below. The proof of 3) is by a direct algorithm that solves satisfiability. The proof of 4) is by a linear satisfiability preserving translation into $C^2$, first order logic with only two variables and counting quantifiers.

The satisfiability problem is not our main focus here, although it is going to be an essential part of the following inference task "exactly how many $\varphi$ states are implied by the theory $\Gamma$?" Formally

**Definition 4.** *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas in $\mathcal{MLC}$, we define the function $|\varphi|$ in $\Gamma$ as follows*[4]

$$|\varphi| \text{ in } \Gamma = \begin{cases} n & \text{if } \Gamma \models (\varphi = n) \text{ and } \Gamma \text{ consistent} \\ \text{undefined} & \text{otherwise} \end{cases}$$

For instance, given $\Gamma = \{(p = 2), (q = 3), (\neg(p \leftrightarrow \neg q) \leq 0)\}$, we have that $|p \vee q|$ in $\Gamma$ will be defined as 5.

We will show an algorithm that solves this task using any model building procedure. In particular we will show how model building algorithms for $\mathcal{H}(\mathsf{A})$ like those proposed in [17,18] can be used. We introduce first the notion of *negation normal form* for $\mathcal{MLC}$.

**Definition 5.** *Given $\varphi \in$ Forms the* negation normal form *of $\varphi$ is obtained applying the following rules*

$$\begin{aligned} \neg\neg\varphi &\rightsquigarrow \varphi \\ \neg(\varphi_1 \wedge \varphi_2) &\rightsquigarrow (\neg\varphi_1) \vee (\neg\varphi_2) \\ \neg(\varphi_1 \vee \varphi_2) &\rightsquigarrow (\neg\varphi_1) \wedge (\neg\varphi_2) \\ \neg\langle r\rangle\varphi &\rightsquigarrow [r]\neg\varphi \\ \neg[r]\varphi &\rightsquigarrow \langle r\rangle\neg\varphi \\ \neg(\varphi \geq 0) &\rightsquigarrow \bot \\ \neg(\varphi \geq n) &\rightsquigarrow \varphi \leq (n-1) \text{ for } n > 0 \\ \neg(\varphi \leq n) &\rightsquigarrow \varphi \geq (n+1) \end{aligned}$$

As we mentioned in Section 3, every formula in $\mathcal{MLC}$ is equivalent to a formula where each counting operators has been *extracted* and it appears under the scope of neither modal nor counting operators. Each $\mathcal{MLC}$ formula is equivalent to its extracted, negation normal form. Let $\mathcal{MLC}^{en}$ be set of extracted formulas of $\mathcal{MLC}$ in negation normal form. We now present a translation from $\mathcal{MLC}^{en}$ to

[4] We recall that the implication $\models$ is to be taken as local, see Definition 2.

$\mathcal{H}(\mathsf{A})$ formulas, which follows a very similar procedure to the one presented by Tobies for Description Logics in [8]. $\mathsf{Tr}_\pi$ works by traversing formulas and adding new nominals so that counting claims are preserved ($\pi$ is used to ensure that we always introduce new nominals, initially $\pi$ is set to the empty string; $i{:}\varphi$ is a satisfiability statement defined in $\mathcal{H}(\mathsf{A})$ as $\mathsf{A}(\neg i \vee \varphi)$).

$$
\begin{aligned}
\mathsf{Tr}_\pi(p) &= p \\
\mathsf{Tr}_\pi(\neg\varphi) &= \neg\mathsf{Tr}_\pi(\varphi) \\
\mathsf{Tr}_\pi(\varphi \wedge \psi) &= \mathsf{Tr}_{\pi 0}(\varphi) \wedge \mathsf{Tr}_{\pi 1}(\psi) \\
\mathsf{Tr}_\pi(\varphi \vee \psi) &= \mathsf{Tr}_{\pi 0}(\varphi) \vee \mathsf{Tr}_{\pi 1}(\psi) \\
\mathsf{Tr}_\pi(\langle r\rangle\varphi) &= \langle r\rangle\mathsf{Tr}_\pi(\varphi) \\
\mathsf{Tr}_\pi([r]\varphi) &= [r]\mathsf{Tr}_\pi(\varphi) \\
\mathsf{Tr}_\pi(\varphi \geq n) &= (\textstyle\bigwedge_{1\leq i<j\leq n} x_i^\pi{:}\neg x_j^\pi) \wedge (\textstyle\bigwedge_{1\leq i\leq n} x_i^\pi{:}\varphi) \\
\mathsf{Tr}_\pi(\varphi \leq n) &= \mathsf{A}(\neg\varphi \vee \textstyle\bigvee_{1\leq i\leq n} x_i^\pi)
\end{aligned}
$$

in particular $Tr_\pi(\varphi \geq 0) = \top$ and $Tr_\pi(\varphi \leq 0) = \mathsf{A}(\neg\varphi)$.

Let us call $\varphi^{\mathcal{H}_\pi}$ the formula obtained from the $\mathcal{MLC}$ formula $\varphi$ by first extracting counting operators, transforming into negation normal form, and applying $\mathsf{Tr}_\pi$; we write $\varphi^{\mathcal{H}}$ when $\pi$ is the empty prefix.

Suppose now that $\mathcal{M}$ is a model satisfying $\varphi^{\mathcal{H}}$ returned by the model builder. We will show that counting has not been affected by the translation.

**Definition 6.** *We call a model $\mathcal{M}'$ a* naming extension *of $\mathcal{M}$ if it is a conservative extension of $\mathcal{M}$ for an extended language that only adds nominals.*

**Theorem 5.** *Let $\varphi \in \mathcal{MLC}$, and $\pi$ an arbitrary prefix. Then $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w \models \varphi^{\mathcal{H}_\pi}$ for $\mathcal{M}'$ a naming extension of $\mathcal{M}$.*

*Proof.* We can disregard the extraction and negation normal form steps of the transformation since they are equivalence preserving.

[$\Rightarrow$] The atomic, negation and modal connectors cases are immediate. For any model $\mathcal{M}$ let us represent as $\mathcal{M}{+}N$ any naming extension of $\mathcal{M}$ where $N$ is the function that assigns nominals to elements of the domain of $\mathcal{M}$. Assume $\mathcal{M}, w \models \varphi_1 \wedge \varphi_2$, i.e., $\mathcal{M}, w \models \varphi_1$ and $\mathcal{M}, w \models \varphi_2$. By induction hypothesis $\mathcal{M}{+}N_1, w \models \varphi_1^{\mathcal{H}_{\pi 0}}$ and $\mathcal{M}{+}N_2, w \models \varphi_2^{\mathcal{H}_{\pi 1}}$. As $N_1$ and $N_2$ are defined on different nominals we can obtain $N = N_1 \cup N_2$ and we have $\mathcal{M}{+}N, w \models \varphi_1^{\mathcal{H}_{\pi 0}} \wedge \varphi_2^{\mathcal{H}_{\pi 1}}$, and hence $\mathcal{M}{+}N, w \models (\varphi_1 \wedge \varphi_2)^{\mathcal{H}_\pi}$. The case for $\varphi_1 \vee \varphi_2$ is handled similarly.

Assume $\mathcal{M}, w \models \varphi \geq n$, i.e., there exist $n$ different states $v_1$ to $v_n$ such that for all $1 \leq i \leq n$, $\mathcal{M}, v_i \models \varphi$. For any $\pi$, choose $N = \bigcup_{1\leq i\leq n}(x_i^\pi, v_i)$ to obtain $\mathcal{M}{+}N, w \models (\bigwedge_{1\leq i<j\leq n} x_i^\pi{:}\neg x_j^\pi) \wedge (\bigwedge_{1\leq i\leq n} x_i^\pi{:}\varphi)$ as needed.

Now, assume $\mathcal{M}, w \models \varphi \leq n$. Let $v_1$ to $v_m$ ($m \leq n$) be all the states of $\mathcal{M}$ satisfying $\varphi$. For any $\pi$, introduce $n$ nominals $x_1^\pi$ to $x_n^\pi$ and a mapping $N$ such that for $1 \leq i \leq n$ there exists $j$, $1 \leq j \leq m$ such that $(x_i^\pi, v_j) \in N$ (two nominals can be true in the same state). Then $\mathcal{M}{+}N, u \models \neg\varphi \vee \bigvee_{1\leq i\leq n} x_i$ for $u$ an arbitrary state, and $\mathcal{M}{+}N, w \models \varphi^{\mathcal{H}}$.

[$\Leftarrow$] Let $\varphi \in \mathcal{MLC}$ and $\pi$ an arbitrary prefix, and $\mathcal{M}'$ a naming extension of $\mathcal{M}$ such that $\mathcal{M}', w \models \varphi^{\mathcal{H}_\pi}$. If $\varphi$ is a modal formula the implication is trivial.

Assume $\mathcal{M}', w \models (\varphi \geq n)^{\mathcal{H}_\pi}$. By definition $\mathcal{M}', w \models (\bigwedge_{1\leq i<j\leq n} x_i^\pi{:}\neg x_j^\pi) \wedge (\bigwedge_{1\leq i\leq n} x_i^\pi{:}\varphi)$. Since $x_1^\pi$ to $x_n^\pi$ are all true at different states $\mathcal{M}, w \models \varphi \geq n$.

Assume $\mathcal{M}', w \models (\varphi \leq n)^{\mathcal{H}(\pi)}$, i.e., $\mathcal{M}', w \models \mathsf{A}(\neg\varphi \vee \bigvee_{1\leq i\leq n} x_i^\pi)$. Then an arbitrary $u$ of $\mathcal{M}'$, $\mathcal{M}', u \models \neg\varphi \vee \bigvee_{1\leq i<n} x_i^\pi$. Hence, either $\mathcal{M}', u \models \neg\varphi$ or $\mathcal{M}', u \models x_i^\pi$ for a given $i \in [[1..m]]$, ie $\{u\} = V(x_i^\pi)$ for $i \in [[1..m]]$. So there can not be more than $n$ distinct states satisfying $\varphi$ in $\mathcal{M}'$ and $\mathcal{M}, w \models \varphi \leq n$. □

Thus we can say that for a given $\mathcal{MLC}$ formula $\varphi$, a model of $\varphi^{\mathcal{H}}$ is a model of $\varphi$. We can now present the algorithm that carries out the reasoning task of counting. Given $P$ a decision procedure for $\mathcal{H}(\mathsf{A})$, $\Gamma$ a finite set of $\mathcal{MLC}$ formulas and $\varphi$ a $\mathcal{MLC}$ formula:

**if** $P(\Gamma^{\mathcal{H}})$ returns UNSAT **then**
    **return** 'undefined'
**else**
    let n = $|\|\varphi\|^{\mathcal{M}}|$ for $\mathcal{M}$ a model returned by $P$
    **if** $P((\Gamma \wedge \neg(\varphi = n))^{\mathcal{H}})$ returns UNSAT **then**
        **return** n
    **else**
        **return** 'undefined'
    **end if**
**end if**

Intuitively, our counting algorithm uses a model of the theory $\Gamma$ to have a candidate answer $n$ to the question "how many $\varphi$ are implied by $\Gamma$?". We then test satisfiability of $(\Gamma \wedge \neg(\varphi = n))^{\mathcal{H}}$ to get the answer.

**Theorem 6.** *The algorithm above computes $|\varphi|$ in $\Gamma$.*

Our solving of the counting task relies essentially on the satisfiability problem and on the model building task carried out by the previously mentioned decision procedures. Another way of carrying this out would be to go the proof-theoretic way and directly try to derive the cardinality of $\varphi$ given a theory $\Gamma$. However, this involves using an axiomatization of $\mathcal{MLC}$ which we currently lack, so given the tools we have, the satisfiability-based approach seems more adequate.

A more feasible alternative would be to solve the satisfiability problem in $\mathcal{MLC}$ directly. As some tableaux systems for Decription Logics with global counting already exist [7], a dedicated calculus for $\mathcal{MLC}$ seems easy to obtain. For a practical implementation, combining tableaux with arithmetic reasoning, as it has been done in [19,20,21], seems a good direction to take. The idea is to separate the counting constraints of the tableau and solve them with a constraint programming or a linear integer programming system. Thus unsatisfiable tableaux can be found efficiently even for large cardinality constraints.

## 5 Conclusions

In this paper we investigated various aspects of modal logics containing the counting quantifiers $(\varphi \geq n)$ and $(\varphi \leq n)$, motivated by the natural language application of representing and querying plural objects in a discourse.

These quantifiers have been introduced before in different areas (generalized quantifiers, modal logics, and description logics), and some of their previously known properties have been outlined (existence of extracted normal forms, complexity of the satisfiability problem, etc.). In this article we investigate expressive power and inference.

With respect to the former, we introduce the notion of $\mathcal{MLC}$ bisimulations, prove that it preserves $\mathcal{MLC}$ formulas and that it characterizes $\mathcal{MLC}$-equivalent finite models. A natural next step would be to investigate "van Benthem characterization" results [22]. I.e., to verify whether any formula of the first-order language with equality (in the appropriate signature) invariant under $\mathcal{MLC}$ bisimulations is equivalent to the translation of an $\mathcal{MLC}$ formula. We strongly conjecture that this is the case.

With respect to inference, we defined a new task that given a theory $\Gamma$ and a formula $\varphi$ returns the cardinality of the extension of $\varphi$ in any model of $\Gamma$ if such cardinality is fixed to be a finite natural number. We show that this task can be solved in terms of a calculus for the hybrid logic $\mathcal{H}(\mathsf{A})$ that can return a model for any satisfiable formula (e.g., tableaux based calculi as those defined by [17,18]). The proposed algorithm involves a translation into $\mathcal{H}(\mathsf{A})$ that might return an exponentially larger formula even when numbers are coded in unary. We conjecture that the polynomial satisfiability preserving translation of [14] could be used instead (but assuming, again, that numbers are coded in unary). The complexity of the problem when numbers are coded in binary is open. As we mentioned in Section 4, the complexity of satisfiability for $\mathcal{MLC}$ and $\mathcal{PLC}$ when numbers are coded in unary has been established [5,16,8]. On the other hand, to our knowledge the problem is still open when numbers are given in binary.

## References

1. Mostowski, A.: On a generalization of quantifiers. Fundamenta Mathematicae 44, 12–36 (1957)
2. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press, Cambridge (2001)
3. Fine, K.: In so many possible worlds. Notre Dame Journal of Formal Logics 13(4), 516–520 (1972)
4. van der Hoek, W., de Rijke, M.: Generalized quantifiers and modal logic. Journal of Logic, Language and Information 2(1), 19–58 (1993)
5. van der Hoek, W., de Rijke, M.: Counting objects. Journal of Logic and Computation 5(3), 325–345 (1995)
6. Westerståhl, D.: Quantifiers in formal and natural languages. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. IV, pp. 1–1331. Reidel, Dordrecht (1989)
7. Baader, F., Buchheit, M., Hollunder, B.: Cardinality restrictions on concepts. Artificial Intelligence 88(1-2), 195–213 (1996)
8. Tobies, S.: Complexity results and practical algorithms for logics in Knowledge Representation. PhD thesis, LuFG Theoretical Computer Science, RWTH-Aachen (2001)
9. Goranko, V., Passy, S.: Using the universal modality: Gains and questions. Journal of Logic and Computation 2(1), 5–30 (1992)

10. Areces, C., ten Cate, B.: Hybrid logics. In: Blackburn, P., Wolter, F., van Benthem, J. (eds.) Handbook of Modal Logics, pp. 821–868. Elsevier, Amsterdam (2006)
11. Varges, S., Deemter, K.V.: Generating referring expressions containing quantifiers. In: Proc. of IWCS 2006 (2005)
12. Asher, N., Wang, L.: Ambiguity and anaphora with plurals in discourse. In: Proc. of Semantics and Linguistic Theory 13 (SALT 13), University of Washington, Seattle, Washington (2003)
13. Franconi, E.: A treatment of plurals and plural quantifications based on a theory of collections. In: Minds and Machines, pp. 453–474 (1993)
14. Areces, C., Gorín, D.: Coinductive models and normal forms for modal logics. Logic Journal of the IGPL (to appear, 2010)
15. de Rijke, M.: The modal logic of inequality. The Journal of Symbolic Logic 57(2), 566–584 (1992)
16. Areces, C., Blackburn, P., Marx, M.: The computational complexity of hybrid temporal logics. Logic Journal of the IGPL 8(5), 653–679 (2000)
17. Bolander, T., Blackburn, P.: Termination for hybrid tableaus. Journal of Logic and Computation 17(3), 517–554 (2007)
18. Kaminski, M., Schneider, S., Smolka, G.: Terminating tableaux for graded hybrid logic with global modalities and role hierarchies. In: Giese, M., Waaler, A. (eds.) TABLEAUX 2009. LNCS (LNAI), vol. 5607, pp. 235–249. Springer, Heidelberg (2009)
19. Ohlbach, H.J., Koehler, J.: Modal logics, description logics and arithmetic reasoning. Artif. Intell. 109(1-2), 1–31 (1999)
20. Haarslev, V., Timmann, M., Möller, R.: Combining tableaux and algebraic methods for reasoning with qualified number restrictions. In: Proc. of Description Logics 2001, pp. 152–161 (2001)
21. Faddoul, J., Farsinia, N., Haarslev, V., Möller, R.: A hybrid tableau algorithm for ALCQ. In: Proc. of ECAI 2008, pp. 725–726. IOS Press, Amsterdam (2008)
22. van Benthem, J.: Modal correspondence theory. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. 2, pp. 167–247. Springer, Heidelberg (1984)