Notre Dame Journal of Formal Logic
Volume 27, Number 4, October 1986

Modal Logics with Functional Alternative Relations

KRISTER SEGERBERG*

Some families of modal logics form lattices; particularly important examples are the set of extensions of a modal logic and the set of normal extensions of a normal logic. One traditional way of studying such lattices, falling back on previous work in algebra, seeks to establish general properties of very big lattices. Thanks to Kit Fine, Wim Blok, Johan van Benthem, and others, this tradition is very much alive.

But there is also an earlier tradition, related to but perhaps possible to distinguish from the one mentioned, where the ambition is to map out in complete detail sufficiently small lattices. The first work in this vein was Scroggs's celebrated [15], followed by Bull's equally celebrated [1]. Other investigations in the same tradition are exemplified by [2], [6], [18], [19], [21], [22]; and works such as [5], [7], [11] also bear on it. In view of how enormously complicated the big lattices are, this tradition can never hope to develop very far. Nevertheless, where it is viable there may still be some interest in seeing it pursued. In this paper we will offer one such example, exploring the lattice of extensions of the normal modal logic $\boldsymbol{KD_c}$, where the schema D_c. $\Diamond \mathbf{A} \supset \Box \mathbf{A}$ is the converse of the well-known "deontic" schema D. $\Box \mathbf{A} \supset \Diamond \mathbf{A}$. At the outset we may note that the only extensions of $\boldsymbol{KD_c}$ other than the Inconsistent Logic (the normal extension of $\boldsymbol{K}$ by $\bot$) which seem to have been described in the literature are $\boldsymbol{KD!} = \boldsymbol{KDD_c}$ (the smallest normal logic to contain both D and D_c), the Trivial Logic (the normal extension of $\boldsymbol{K}$ by the schema $\Box \mathbf{A} \equiv \mathbf{A}$), and the Verum Logic (the normal extension of $\boldsymbol{K}$ by $\Box \bot$). The relationship between these logics is set out by the chart in Fig. 1. This, then, is the map whose white patches we propose to fill.

*The research reported in this paper was prompted by Brian Chellas's interest in functional modal logics, as explained in [4], and the author would like to acknowledge fruitful exchanges with him, Max Cresswell, and Steve Thomason on this topic, in conversation and in correspondence. The results achieved were presented in the author's invited address to the Australasian Logic Conference at the University of Western Australia in May 1983 and were summarized in the abstract [20]. The author wishes to thank Graham Priest and his colleagues for an exciting and well-organized conference.

Received March 10, 1984

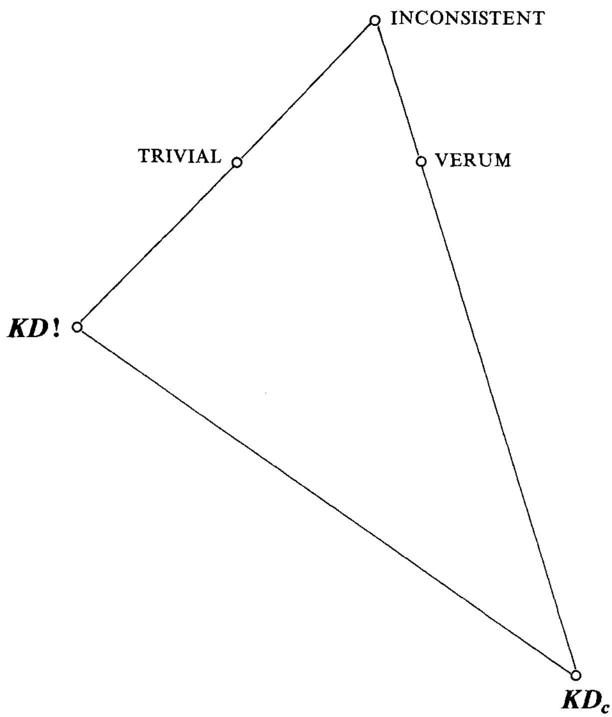

Fig. 1

The paper consists of three sections. In Section 1 we study the class of normal logics between $\boldsymbol{KD_c}$ and the logic $\boldsymbol{KD!}$. In Section 2 we report the main result of the paper that every normal extension of $\boldsymbol{KD_c}$ has the finite model property. This insight is a key to an understanding of the lattice of extensions of $\boldsymbol{KD!}$, and in Section 3 we show how it can be used to gain an understanding of the lattice of extensions of $\boldsymbol{KD_c}$ as well. A noteworthy feature of our investigation has to do with the relationship between normal and nonnormal modal logic: Sections 1 and 2 are concerned with normal logics only, and in Section 3 it is shown how the picture obtained in the first two sections, interesting and informative as it is, is still incomplete, and how the general picture obtained when all extensions are considered, nonnormal as well as normal, is superior.

A background in standard modal logic is assumed. Such a background may be got by referring to the author's [17] or to the more current texts [3], [8], [10]. Some conventions: we use **P** for propositional letters; **A**, **B**, **C** for formulas; i, j, k, l, m, n for natural numbers 0, 1, 2, . . . ; and the set of all natural numbers is denoted by ω.

1 Normal logics between KD_c and $KD!$ A frame $\langle U, R\rangle$ is *serial* if $\forall x \exists y\, xRy$; *functional* if $\forall x \forall y \forall y'$ $(xRy \,\&\, xRy' \Rightarrow .\; y = y')$; *totally functional* if functional and serial; and *partially functional* if functional but not serial. Functional frames and their logics have been studied by Prior [13], [14], von Wright [23], Lemmon and Scott [10], and Segerberg [16]. It is well known that $\boldsymbol{KD}$,

KD$_c$, and ***KD!*** are strongly complete in relational frame semantics, and that every frame for ***KD*** (***KD***$_c$, ***KD!***) is serial (functional, totally functional).

We now describe and introduce notations for all generated functional frames. The finite, generated, partially functional frames are particularly simple as they are of the type $\mathfrak{P}_n = \langle U_n, R_n \rangle$, for $n \geqq 0$, where

$$U_n = \{i : i < n\},$$
$$R_n = \{\langle i, i+1 \rangle : i < n-1\}.$$

Note that we recognize the "empty frame" $\mathfrak{P}_0 = \langle \varnothing, \varnothing \rangle$ as a partially functional frame. This is perhaps unorthodox, but in the present context it is natural, as will be seen below. The finite, generated, totally functional frames are all of the type $\mathfrak{T}_{k,l} = \langle U_{k,l}, R_{k,l} \rangle$, where $k \geqq 0$ but $l > 0$, and

$$U_{k,l} = \{i : i < k + l\},$$
$$R_{k,l} = \{\langle i, i+1 \rangle : i < k + l - 1\} \cup \{\langle k + l - 1, k \rangle\}.$$

This accounts for all types of finite, generated, functional frames. Among infinite, functional frames, the only generated ones are those isomorphic to the "omega frame" $\langle U_\omega, R_\omega \rangle$, where

$$U_\omega = \omega,$$
$$R_\omega = \{\langle i, i+1 \rangle : i < \omega\}.$$

We will write ω for this frame, so in a sense we identify the omega frame with the set of natural numbers; but this convention should not cause any confusion. In what follows we will also identify isomorphic frames. Thus, for example, we will regard ω as the unique infinite, generated, functional frame. The schematic representation in Fig. 2 of all generated, functional frames will perhaps be helpful.

A *logic* is a set of formulas that contains all tautologies and is closed under modus ponens and substitution. A logic is a *normal* (*modal*) *logic* if it contains the scheme $\Box(\mathbf{A} \wedge \mathbf{B}) \equiv \Box\mathbf{A} \wedge \Box\mathbf{B}$ and, in addition, is closed under necessitation (it contains $\Box\mathbf{A}$ whenever it contains $\mathbf{A}$). If $\mathfrak{M}$ is a model, then $Th(\mathfrak{M})$ denotes the *theory determined by* $\mathfrak{M}$; that is, the set of formulas true at every point of $\mathfrak{M}$. We say that $\mathfrak{M}$ is a *model for* a logic L if $L \subseteq Th(\mathfrak{M})$. If $\mathfrak{F}$ is a frame, then $L(\mathfrak{F})$ denotes the normal modal logic determined by $\mathfrak{F}$; that is, the set of formulas valid in $\mathfrak{F}$. Similarly, if C is a class of frames, then $L(C)$ denotes the normal *logic determined by* C; that is, the set of formulas valid in every frame in C. In other words, $L(C) = \bigcap \{L(\mathfrak{F}) : \mathfrak{F} \in C\}$. We say that $\mathfrak{F}$ is a *frame for* a logic L if $L \subseteq L(\mathfrak{F})$. Note that the Inconsistent Logic, the Verum Logic, and the Trivial Logic are identical with $L(\mathfrak{P}_0)$, $L(\mathfrak{P}_1)$, and $L(\mathfrak{T}_{0,1})$, respectively.

The following two results are well-known:

Lemma 1.1 $\quad$ $\boldsymbol{KD!} = L(\omega) = \bigcap \{L(\mathfrak{T}_{k,l}) : k, l < \omega \;\&\; l > 0\}$.

Lemma 1.2 $\quad$ $\boldsymbol{KD}_c = \bigcap \{L(\mathfrak{P}_n) : n < \omega\}$.

We also list some other results of interest for our investigation.

Lemma 1.3 $\quad$ $L(\mathfrak{T}_{k,l})$ *is the smallest normal logic to contain the formula* $\Box^k(\mathbf{P} \; {}^3\!/\!_8 \; \Box^l\mathbf{P})$.

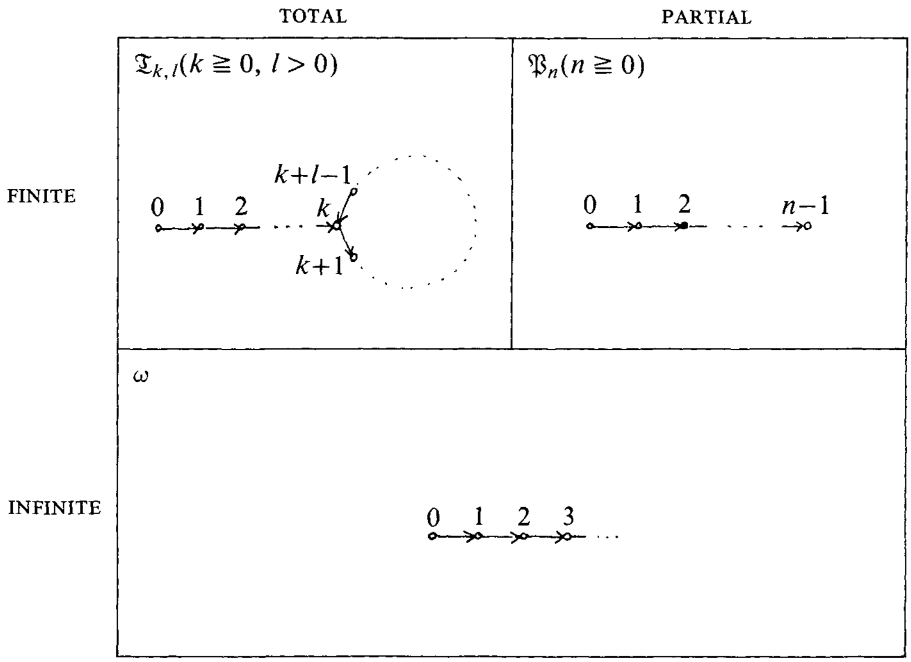

Fig. 2. Schematic inventory of all generated functional frames (up to isomorphism)

Lemma 1.4 *$L(\mathfrak{P}_n)$ is the smallest normal logic to contain the formula* $\bigvee\{\Diamond^i\Box\bot : i < n\}$.

Lemma 1.5 $L(\mathfrak{P}_m) \subseteq L(\mathfrak{P}_n)$ *if and only if* $m \geqq n$.

Lemma 1.6 $L(\mathfrak{T}_{k,l}) \subseteq L(\mathfrak{T}_{k',l'})$ *if and only if* $k \geqq k'$ *and* l' *divides* l.

Proof: First assume that $L(\mathfrak{T}_{k,l}) \subseteq L(\mathfrak{T}_{k',l'})$, for some k, $k' \geqq 0$ and some l, $l' > 0$. If $k < k'$, then let $\mathfrak{M}$ be a model on $\mathfrak{T}_{k',l'}$ in which a certain propositional letter **P** is true only at k. By the assumption and Lemma 1.3, $\Box^k(\mathbf{P} \equiv \Box^l\mathbf{P}) \in L(\mathfrak{T}_{k',l'})$. Hence, in $\mathfrak{M}$, $\Box^l\mathbf{P}$ is true at k. As **P** is true only at k and $\mathfrak{T}_{k',l'}$ is functional, it follows that $l = 0$, a contradiction. Consequently, $k \geqq k'$.

To see that l' divides l, let $\mathfrak{M}$ instead be any model on $\mathfrak{T}_{k',l'}$ such that **P** holds at k' and nowhere else. Note that, since $L(\mathfrak{T}_{k,l})$ is normal, $\Box^i(\mathbf{P} \equiv \Box^l\mathbf{P}) \in L(\mathfrak{T}_{k,l})$, for all $i \geqq k$. From our assumption that $L(\mathfrak{T}_{k,l}) \subseteq L(\mathfrak{T}_{k',l'})$ it then follows that, for all j such that $k' \leqq j < k' + l' - 1$, $\mathbf{P} \equiv \Box^l\mathbf{P}$ holds at j. In particular, $\Box^l\mathbf{P}$ holds at k'. But as **P** is true at k' only, it must be that $k' + l = k' + ml'$, for some $m \geqq 0$. Hence $l = ml'$; and as l, $l' > 0$, also $m > 0$. Consequently, l' divides l. Thus we have proved the first half of the lemma.

For the converse, suppose that $k \geqq k'$ and l' divides l, where l, $l' > 0$. Define a function $f: U_{k',l} \to U_{k',l'}$ as follows:

$$fx = \begin{cases} x, \text{ if } x < k', \\ k' + ((x - k')(\text{mod } l')), \text{ if } k' \leqq x < k' + l. \end{cases}$$

It is clear that f is well-defined. Since l' divides l, $l' \leqq l$, and so f is onto. It is not difficult to verify that f is a p-morphism from $\mathfrak{T}_{k',l}$ to $\mathfrak{T}_{k',l'}$. Actually, the detail that needs checking is that $f(k' + l - 1) = k' + l' - 1$. But this holds iff $(l - 1)(\text{mod } l') = l' - 1$; which holds iff $l - 1 - ml' = l' - 1$, for some $m \geqq 0$; which holds iff $l = (m + 1)l'$, for some $m \geqq 0$; which holds iff l' divides l; and that this holds we have assumed. Therefore, by the P-morphism Theorem, $L(\mathfrak{T}_{k',l}) \subseteq L(\mathfrak{T}_{k',l'})$. Moreover, we have assumed that $k \geqq k'$. Therefore by the Generation Theorem, $L(\mathfrak{T}_{k,l}) \subseteq L(\mathfrak{T}_{k',l})$. The desired result follows. QED

Now consider two other families of formulas (C for Chellas, H for Hughes):

$$C_n. \quad \Box^n \Diamond \mathbf{T};$$
$$H_n. \quad \Diamond^n \Box \mathbf{A} \supset \Box^n \Diamond \mathbf{A}.$$

The question whether there are normal logics between $\boldsymbol{KD_c}$ and $\boldsymbol{KD!}$ was broached at the Waikanae meeting of the Scroggs Society in January 1983. After the meeting Brain F. Chellas observed that $\{\boldsymbol{KD_c}C_n : n < \omega\}$ constitutes a sequence of such logics; and George Hughes, independently, observed that $\{\boldsymbol{KD_c}H_n : n < \omega\}$ does. It is easy to see that the two sequences are the same. We denote $\boldsymbol{KD_c}C_n = \boldsymbol{KD_c}H_n$ by $(\boldsymbol{CH})_n$ and will refer to the members of the sequence $\{(\boldsymbol{CH})_n : n < \omega\}$ as the *Chellas/Hughes logics*. Note that $(\boldsymbol{CH})_{n+1} \subseteq (\boldsymbol{CH})_n$ but $(\boldsymbol{CH})_n \nsubseteq (\boldsymbol{CH})_{n+1}$; that $\boldsymbol{KD_c} \subseteq (\boldsymbol{CH})_n$ but $(\boldsymbol{CH})_n \nsubseteq \boldsymbol{KD_c}$; and that $(\boldsymbol{CH})_0 = \boldsymbol{KD!}$. We will now prove that the Chellas/Hughes logics are in fact all the normal logics between $\boldsymbol{KD_c}$ and $\boldsymbol{KD!}$.

Lemma 1.7 $(\boldsymbol{CH})_n = \boldsymbol{KD!} \cap L(\mathfrak{P}_n)$.

Proof: The $\subseteq$-part is obvious. For the $\supseteq$-part, assume that $\mathbf{A} \notin (\boldsymbol{CH})_n$. We must prove that either $\mathbf{A} \notin \boldsymbol{KD!}$ or $\mathbf{A} \notin L(\mathfrak{P}_n)$. By standard reasoning, $\mathbf{A}$ fails in some generated submodel $\mathfrak{M}$ of the canonical model for $(\boldsymbol{CH})_n$. There are three cases. *Case 1. $\mathfrak{M}$ is infinite.* Then $\mathfrak{M}$ is based on the omega frame ω. But ω is a frame for $\boldsymbol{KD!}$, by Lemma 1.1. Therefore $\mathbf{A} \notin \boldsymbol{KD!}$. *Case 2. $\mathfrak{M}$ is finite and totally functional.* Then there are $k \geqq 0$ and $l > 0$ such that $\mathfrak{T}_{k,l}$ is the frame of $\mathfrak{M}$. Consequently $\mathbf{A} \notin L(\mathfrak{T}_{k,l})$, and so, by Lemma 1.1, $\mathbf{A} \notin \boldsymbol{KD!}$. *Case 3. $\mathfrak{M}$ is finite and partially functional.* Then there is some k such that $\mathfrak{P}_k$ is the frame of $\mathfrak{M}$. Assume that $k > n$. Then $k - n - 1$ is a nonnegative number and hence an element of $\mathfrak{P}_k$. As $\mathfrak{M}$ is a model for $(\boldsymbol{CH})_n$, the formula $\Box^n \Diamond \mathbf{T}$ is true at $k - n - 1$. Hence $\Diamond \mathbf{T}$ is true at $k - 1$, the last element of $\mathfrak{P}_k$. But this is a contradiction, for a last element has no alternatives, and so $\Diamond \mathbf{T}$ cannot be true at it. Consequently, $k \leqq n$. Therefore, by Lemma 1.5, $L(\mathfrak{P}_k) \supseteq L(\mathfrak{P}_n)$. From this and the fact that $\mathbf{A} \notin L(\mathfrak{P}_k)$ it follows that $\mathbf{A} \notin L(\mathfrak{P}_n)$. QED

Corollary 1.8 $\boldsymbol{KD_c} = \bigcap \{(\boldsymbol{CH})_n : n < \omega\}$.

Proof: The $\subseteq$-part is obvious. For the $\supseteq$-part, suppose that $\mathbf{A} \notin \boldsymbol{KD_c}$. Then, by Lemma 1.2, there is some n such that $\mathbf{A} \notin L(\mathfrak{P}_n)$. Hence, by Lemma 1.7, $\mathbf{A} \notin (\boldsymbol{CH})_n$. QED

Corollary 1.9 *$(\boldsymbol{CH})_n$ has the finite model property.*

Proof: A normal logic is said to have the finite model property (f.m.p.) if and only if it is determined by a class of finite frames. It follows from Lemmas 1.1 and 1.7 that $(\boldsymbol{CH})_n$ is determined by the class $\{\mathfrak{T}_{k,l}: k, l < \omega \ \& \ l > 0\} \cup \{\mathfrak{P}_n\}$. QED

The following result is well known but of special importance to this paper:

Proposition 1.10 (Segerberg [17], cf. Fine [6]) *Let $\mathfrak{M}$ be a distinguishable model based on a finite frame $\mathfrak{F}$. Then, for any logic L, $L \subseteq Th(\mathfrak{M})$ only if $L \subseteq L(\mathfrak{F})$.*

Lemma 1.11 *If L is a normal extension of $\boldsymbol{KD_c}$, then the formula $\Diamond^n\Box\bot$ is consistent in L only if $L \subseteq L(\mathfrak{P}_{n+1})$.*

Proof: Let $L \supseteq \boldsymbol{KD_c}$ be a normal logic. Suppose that $\Diamond^n\Box\bot$ is consistent in L (by which is meant that the set $\{\Diamond^n\Box\bot\}$ is consistent in L). By Lindenbaum's Lemma there is some maximal L-consistent set x of formulas such that $\Diamond^n\Box\bot \in x$. Since x is an element of the canonical model $\mathfrak{M}_L$ for L, it makes sense to speak of the submodel $\mathfrak{M}$ generated from $\mathfrak{M}_L$ by x. By assumption, L extends $\boldsymbol{KD_c}$, so the accessibility relation of $\mathfrak{M}$ is functional. As $\Diamond^n\Box\bot \in x$, the frame of $\mathfrak{M}$ must therefore be (isomorphic to) $\mathfrak{P}_{n+1}$. Thus $\mathfrak{M}$ is a finite distinguishable model for L. Hence, by Proposition 1.10, $\mathfrak{P}_{n+1}$ is a frame for L. QED

Theorem 1.12 *The only normal logics between $\boldsymbol{KD_c}$ and $\boldsymbol{KD!}$ are the Chellas/Hughes logics.*

Proof: Suppose that L is a normal logic such that $\boldsymbol{KD_c} \subseteq L \subseteq \boldsymbol{KD!}$. We wish to prove that either $L = \boldsymbol{KD_c}$ or else $L = (\boldsymbol{CH})_n$. Consider the set

$$J = \{i: \Diamond^i\Box\bot \text{ is consistent in } L\}.$$

There are three cases. *Case 1. J is empty.* Since $0 \notin J$, $\Box\bot$ is inconsistent in L. This means that $\Diamond\top \in L$ and therefore that $\boldsymbol{KD!} \subseteq L$. Thus in this case $L = \boldsymbol{KD!}$. *Case 2. J is nonempty but finite.* Let n be the maximum number in J. Then on the one hand $\Diamond^{n+1}\Box\bot$ is inconsistent in L, whence $\Box^{n+1}\Diamond\top \in L$, and so $(\boldsymbol{CH})_{n+1} \subseteq L$. On the other hand $\Diamond^n\Box\bot$ is consistent in L, whence $L \subseteq L(\mathfrak{P}_{n+1})$ by Lemma 1.11, and so $L \subseteq (\boldsymbol{CH})_{n+1}$ by Lemma 1.7. Thus in this case, $L = (\boldsymbol{CH})_{n+1}$. *Case 3. J is infinite.* Then $\Diamond^n\Box\bot$ is consistent in L, for all n; hence $\Box^n\Diamond\top \notin L$, for all n; hence, by Lemma 1.11, $L \subseteq L(\mathfrak{P}_{n+1})$, for all n; hence, by Lemma 1.2, $L \subseteq \boldsymbol{KD_c}$. Thus in this case $L = \boldsymbol{KD_c}$. QED

Figure 1 can now be updated as shown in Fig. 3. However, the picture is still far from complete.

2 *Normal extensions of $\boldsymbol{KD_c}$*

In the preceding section we determined the normal logics between $\boldsymbol{KD_c}$ and $\boldsymbol{KD!}$. In this section we will go one step further and map out all the normal extensions of $\boldsymbol{KD_c}$. The pivotal result on which everything else turns is this analogue of Proposition 1.10:

Proposition 2.1 *Let $\mathfrak{M}$ be a distinguishable model based on the omega frame ω. Then, for any logic L, $L \subseteq Th(\mathfrak{M})$ only if $L \subseteq L(\omega)$.*

Proof: Suppose that $\mathfrak{M}$ is a distinguishable model on ω. Let L be a logic such that

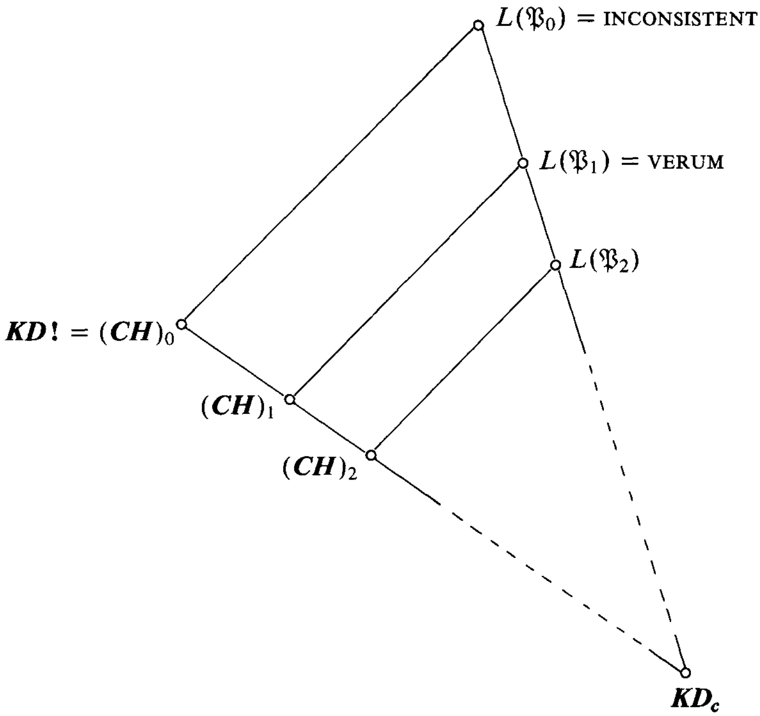

Fig. 3

(1) $L \subseteq Th(\mathfrak{M})$.

Suppose that $\mathbf{A}$ is any formula such that

(2) $\mathbf{A} \notin L(\omega)$.

It will be enough to prove that $\mathbf{A} \notin L$. Now, by (2), $\mathbf{A}$ will fail at 0 in some model $\mathfrak{M}^*$ on ω:

(3) $\mathfrak{M}^* \nvDash_0 \mathbf{A}$.

That $\mathfrak{M}$ is distinguishable means by definition that, whenever $m \neq n$, then there is some formula $C_{m,n}$ such that $\mathfrak{M} \vDash_m C_{m,n}$ but $\mathfrak{M} \nvDash_n C_{m,n}$. For any m, define the formula

$$C_m =_{df} \bigwedge\{C_{m,n} : m \neq n \;\&\; n \leqq \deg \mathbf{A}\},$$

where $\bigwedge$ stands for finite conjunction and deg $\mathbf{A}$ is the modal degree of $\mathbf{A}$ (that is, the maximum number of nested modal operators in $\mathbf{A}$). Then, for all m, $n \leqq \deg \mathbf{A}$, we have

$$\mathfrak{M} \vDash_n C_m \text{ iff } m = n.$$

The following defines a substitution function s: for every propositional letter $\mathbf{P}$,

$$\mathbf{P}^s =_{df} \bigvee\{C_m : m \leqq \deg \mathbf{A} \;\&\; \mathfrak{M}^* \vDash_m \mathbf{P}\},$$

where $\bigvee$ stands for finite disjunction. We now make the following claim· for all formulas **B** and all natural numbers m,

(4) if $m + \deg \mathbf{B} \leqq \deg \mathbf{A}$, then $\mathfrak{M}^* \models_m \mathbf{B}$ iff $\mathfrak{M} \models_m \mathbf{B}^s$.

The claim is proved by induction on **B**. The basic step goes through thanks to the definition of s. The inductive step is trivial in the Boolean cases. In the modal case we rely on the fact that $x + \deg(\Box\mathbf{C}) = x + 1 + \deg \mathbf{C}$, for all **C**. Thus in the latter case—assuming as the induction hypothesis that (4) holds for **B**—we have $\mathfrak{M}^* \models_m \Box\mathbf{B}$ iff $\mathfrak{M}^* \models_{m+1} \mathbf{B}$ iff $\mathfrak{M} \models_{m+1} \mathbf{B}^s$ iff $\mathfrak{M} \models_m \Box\mathbf{B}^s$ iff $\mathfrak{M} \models_m (\Box\mathbf{B})^s$.

From (3) and (4) it follows that $\mathfrak{M} \not\models_0 \mathbf{A}^s$. By (1), therefore, $\mathbf{A}^s \notin L$. But L is closed under substitution. Consequently, $\mathbf{A} \notin L$, as we wanted to show. QED

We are now ready for the main result of this section, which holds the key to an understanding of the structure of the entire class of normal extensions of $\boldsymbol{KD_c}$:

Theorem 2.2 *Every normal extension of* $\boldsymbol{KD_c}$ *has the finite model property.*

Proof: Suppose, by way of contradiction, that there is a normal logic $L \supseteq \boldsymbol{KD_c}$ which lacks the f.m.p. This means that there must be some nontheorem of L that fails in no finite model for L. But L is normal, so every nontheorem of L fails somewhere in the canonical model $\mathfrak{M}_L$ for L. Consequently, $\mathfrak{M}_L$ has an infinite, generated submodel $\mathfrak{M}$ which is a model for L: $L \subseteq Th(\mathfrak{M})$. As L extends $\boldsymbol{KD_c}$, $\mathfrak{M}$ is functional. In other words, $\mathfrak{M}$ is based on the omega frame ω. Moreover, as submodel of a canonical model, $\mathfrak{M}$ is certainly distinguishable. By Proposition 2.1, therefore, $L \subseteq L(\omega)$. Hence, by Lemma 1.1, $L \subseteq \boldsymbol{KD!}$. By Theorem 1.13, either $L = (\boldsymbol{CH})_n$, for some n, or else $L = \boldsymbol{KD_c}$. In either case L has the f.m.p., by Corollary 1.9 in the former case, by Lemma 1.2 in the latter; and so in either case we have a contradiction. QED

Corollary 2.3 *If L is a normal extension of* $\boldsymbol{KD_c}$ *not contained in* $\boldsymbol{KD!}$, *then L is determined by a finite class of finite frames.*

Proof: Let $L \supseteq \boldsymbol{KD_c}$ be a normal logic. By Theorem 2.2, L is determined by a class C of finite, functional frames. Assume that C contains infinitely many frames. It will be enough to show that this assumption entails that $L \subseteq \boldsymbol{KD!}$.

Let **A** be any formula such that $\mathbf{A} \notin \boldsymbol{KD!}$. Then there is a model $\mathfrak{M}$ based on the omega frame ω such that

(1) $\mathfrak{M} \not\models_0 \mathbf{A}$.

Since C is assumed to be infinite, C must contain arbitrarily large frames of type $\mathfrak{T}_{k,l}$ or $\mathfrak{P}_n$. Let $\mathfrak{F}$ be any frame in C with deg **A** elements. Let $\mathfrak{M}'$ be a model on $\mathfrak{F}$ which copies the behavior in $\mathfrak{M}$ of propositional letters on $0, 1, \ldots, \deg \mathbf{A}$. In other words, for all propositional letters **P** and all natural numbers $i \leqq \deg \mathbf{A}$,

$$\mathfrak{M}' \models_i \mathbf{P} \text{ iff } \mathfrak{M} \models_i \mathbf{P}.$$

A straightforward inductive argument shows that, for all formulas **B** and all natural numbers i,

(2) if $i + \deg \mathbf{B} \leqq \deg \mathbf{A}$, then $\mathfrak{M}' \models_i \mathbf{B}$ iff $\mathfrak{M} \models_i \mathbf{B}$.

From (1) and (2) it follows that $\mathfrak{M}' \nvDash_0 \mathbf{A}$. But $\mathfrak{F}$ is a frame for L, so evidently $\mathbf{A} \notin L$. Thus we have established that $L \subseteq \boldsymbol{KD!}$. QED

Corollary 2.4 *Every normal extension of* $\boldsymbol{KD_c}$ *is determined by some class of finite, functional frames.*

Corollary 2.5 *Every proper normal extension of* $\boldsymbol{KD!}$ *is a finitely many-valued logic.*

Proof: By Corollary 2.3, if $L \supsetneqq \boldsymbol{KD!}$ is a normal logic, then L is determined by a finite class of finite frames. In algebraic terms—see, for example, Lemmon [9]—this means that L has a finite characteristic matrix. QED

Thus every proper normal extension of $\boldsymbol{KD!}$ can be viewed as the intersection of finitely many logics of type $L(\mathfrak{T}_{k,l})$. This means that $\boldsymbol{KD!}$ has a relatively unusual property:

Corollary 2.6 $\boldsymbol{KD!}$ *is pretabular.*

Proof: A normal logic is defined as pretabular if it lacks a finite characteristic matrix but every proper normal extension has one. The result follows from Lemma 1.1 and Corollary 2.5. QED

The last two corollaries in conjunction with Lemma 1.6 afford a good understanding of the structure of the lattice of normal extensions of $\boldsymbol{KD!}$. Interestingly enough, this understanding will help us to understand also the more complicated lattice of normal extensions of $\boldsymbol{KD_c}$. In order to substantiate this claim, it is useful to introduce the notion of a normal slice. For each n, let f_n be the function on the class of normal extensions of $\boldsymbol{KD!}$ defined by the condition

$$f_n(L) =_{df} L \cap L(\mathfrak{P}_n).$$

By the n-th *normal slice*, denoted by S_n, we mean the range of f_n. Thus S_0, the zeroth slice, is simply the set of normal extensions of $\boldsymbol{KD!}$ (as well as the domain of f_n, for every n).

Lemma 2.7 S_n *consists of the normal logics between (and including)* $(\boldsymbol{CH})_n$ *and* $L(\mathfrak{P}_n)$.

Proof: We wish to prove that

$$S_n = \{L : L \text{ is normal } \& \ (\boldsymbol{CH})_n \subseteq L \subseteq L(\mathfrak{P}_n)\}.$$

There are two halves to the proof. For the first one, take any $L \in S_0$. We must now prove that $f_n(L)$, obviously a normal logic, contains $(\boldsymbol{CH})_n$ and is contained in $L(\mathfrak{P}_n)$. Now $\Box^n\Diamond\mathbf{T}$ is a thesis of $\boldsymbol{KD!}$ and hence of L; it is also a thesis of $L(\mathfrak{P}_n)$. Therefore $\Box^n\Diamond\mathbf{T} \in f_nL$, and so $(\boldsymbol{CH})_n \subseteq f_n(L)$. That $f_n(L) \subseteq L(\mathfrak{P}_n)$ is trivial.

For the other half of the proof, let L be any normal logic such that $(\boldsymbol{CH})_n \subseteq L \subseteq L(\mathfrak{P}_n)$. By Corollary 2.4 there is a class C of finite functional frames such that $L = L(C)$. Let C^+ and C^- be the classes of frames of C which are totally functional and partially functional, respectively. Evidently, $C^+ \cap C^- = \varnothing$ and $C^+ \cup C^- = C$, so we note that $L = L(C^+) \cap L(C^-)$. Since $L \subseteq L(\mathfrak{P}_n)$, there is no loss of generality to assume that $\mathfrak{P}_n \in C^-$. Hence

(1) $L(C^-) \subseteq L(\mathfrak{P}_n)$.

On the other hand, since $(\boldsymbol{CH})_n \subseteq L$ and thus $\Box^n\Diamond\top \in L$, it is clear that $\mathfrak{P}_k \in C^-$ only if $k \leqq n$. In other words, $C^- \subseteq \{\mathfrak{P}_k : k \leqq n\}$, so $L(C^-) \supseteq \bigcap\{L(\mathfrak{P}_n): k \leqq n\}$. Hence, by Lemma 1.5,

(2) $L(C^-) \supseteq L(\mathfrak{P}_n)$.

Putting (1) and (2) together, we conclude that $L(C^-) = L(\mathfrak{P}_n)$ and hence that $L = L(C^+) \cap L(\mathfrak{P}_n)$. Therefore, as $L(C^+) \in S_0$, we have shown that $f_n(L(C^+)) = L$. That is to say, $L \in S_n$. QED

Lemma 2.8 *Every proper normal extension of* $\boldsymbol{KD_c}$ *belongs to a unique normal slice.*

Proof: Uniqueness: Suppose that some normal logic belongs to both S_m and S_n. Then there are $L, L' \in S_0$ such that $L \cap L(\mathfrak{P}_m) = L' \cap L(\mathfrak{P}_n)$. Now, L is a normal extension of $\boldsymbol{KD!}$, and so $\Box^i\Diamond\top \in L$, for all i. Therefore, for all i, $\Box^i\Diamond\top \in L \cap L(\mathfrak{P}_m)$ if and only if $i \geqq m$, and $\Box^i\Diamond\top \in L' \cap L(\mathfrak{P}_n)$ if and only if $i \geqq n$. Consequently, $m = n$.

Existence: Suppose that L is a proper normal extension of $\boldsymbol{KD_c}$. Define

$$J = \{i : \Box^i\Diamond\bot \text{ is consistent in } L\}.$$

As in the proof of Theorem 1.12 there are three cases, and analogous arguments apply. If J is empty, then $\boldsymbol{KD!} \subseteq L$, and so $L \in S_0$ (Case 1 in the proof of Theorem 1.12). If J is infinite, then $L \subseteq \boldsymbol{KD_c}$, contradicting the assumption that L properly extends $\boldsymbol{KD_c}$ (Case 3 in the proof of Theorem 1.12). If J is neither empty nor infinite we have a more complicated situation (Case 2 in the proof of Theorem 1.12). In this case there must be some number n such that

(1) $\Box^i\Diamond\top \in L$, for all $i > n$,
(2) $L \subseteq L(\mathfrak{P}_{n+1})$.

Now consider the class C of normal extensions of L. It is clear that $\bigcap C$ exists and is a normal logic; thus $\bigcap C$ is the smallest normal extension of L. In order to prove that $L \in S_{n+1}$ it will be enough to prove that $L = \bigcap C \cap L(\mathfrak{P}_{n+1})$. That $L \subseteq \bigcap C \cap L(\mathfrak{P}_{n+1})$ follows from (2) and the definition of C. To see that $L \supseteq \bigcap C \cap L(\mathfrak{P}_{n+1})$, assume that $\mathbf{A} \notin L$. By Theorem 2.2, $\mathbf{A}$ fails on some finite frame $\mathfrak{F}$ for L, which then has to be functional. If $\mathfrak{F}$ is totally functional, then $\mathfrak{F} = \mathfrak{T}_{k,l}$, for some $l > 0$. Evidently $\mathfrak{T}_{k,l} \in C$, and so $\bigcap C \subseteq L(\mathfrak{T}_{k,l})$. Consequently, $\mathbf{A} \notin \bigcap C$. On the other hand, if $\mathfrak{F}$ is partially functional, then $\mathfrak{F} = \mathfrak{P}_k$, for some k, and so $\mathbf{A} \notin L(\mathfrak{P}_k)$. But $\Box^{k-1}\Diamond\top \notin L(\mathfrak{P}_k)$, and so (1) implies that $k \leqq n + 1$. Hence, by Lemma 1.5, $\mathbf{A} \notin L(\mathfrak{P}_{n+1})$. QED

In other words, the set of normal slices partitions the set of proper normal extensions of $\boldsymbol{KD_c}$. We will now investigate the structure of each normal slice.

Lemma 2.9 *Let L and L′ be normal extensions of* $\boldsymbol{KD!}$. *If* $L - L' \neq \varnothing$, *then* $f_n(L) - f_n(L') \neq \varnothing$.

Proof: Suppose that $L, L' \in S_0$. If $L - L' \neq \varnothing$, take any $\mathbf{A} \in L - L'$. Consider the formula $\mathbf{A}^* = \mathbf{A} \vee \bigvee\{\Diamond^i\Box\bot : i < n\}$. Since $\mathbf{A} \in L$ and $\bigvee\{\Diamond^i\Box\bot : i < n\} \in L(\mathfrak{P}_n)$, it is clear that $\mathbf{A}^* \in L \cap L(\mathfrak{P}_n)$; that is, $\mathbf{A}^* \in f_n(L)$. By previous results, there is a class C of totally functional frames such that $L' = L(C)$. Since $\mathbf{A} \notin L'$, there is some frame $\mathfrak{F} \in C$ such that $\mathbf{A} \notin L(\mathfrak{F})$. As $\mathfrak{F}$ is totally functional, $\bigvee\!\!\bigvee\{\Diamond^i\Box\bot : i < n\} \notin L(\mathfrak{F})$. Therefore $\mathbf{A}^* \notin L(\mathfrak{F})$, and so $\mathbf{A}^* \notin L' \cap L(\mathfrak{P}_n)$; that is, $\mathbf{A}^* \notin f_n(L')$. Consequently, $\mathbf{A}^* \in f_n(L') - f_n(L')$. QED

Lemma 2.10 *If L is a normal extension of **KD!**, then $L = L(C)$ if and only if $f_n(L) = L(C \cup \{\mathfrak{P}_n\})$.*

Proof: The only-if-part is immediate. For the if-part, assume that $f_n(L) = L(C \cup \{\mathfrak{P}_n\})$. Then $f_n L = L(C) \cap L(\mathfrak{P}_n)$. By definition, $f_n(L(C)) = L(C) \cap L(\mathfrak{P}_n)$, so $f_n(L) = f_n(L(C))$. Hence, by Lemma 2.9, $L = L(C)$. QED

Theorem 2.11 *f_n is a lattice isomorphism from S_0 to S_n.*

Proof: By Lemma 2.9, f_n is one-to-one. By definition, it is onto. Suppose that $L, L' \in S_0$. Then $L \subseteq L'$ implies $L \cap L(\mathfrak{P}_n) \subseteq L' \cap L(\mathfrak{P}_n)$, whence $f_n(L) \subseteq f_n(L')$. On the other hand, if $L \not\subseteq L'$, then $f_n(L) \not\subseteq f_n(L')$, by Lemma 2.9. QED

The upshot of this examination is that the set of normal extensions of $\boldsymbol{KD_c}$ consists of $\boldsymbol{KD_c}$ itself and denumerably many isomorphic normal slices. Thus the chart of normal extensions of $\boldsymbol{KD_c}$ given in Fig. 3 can now be improved on as shown in Fig. 4. The latter is complete in the sense that it intimates all normal extensions of $\boldsymbol{KD_c}$.

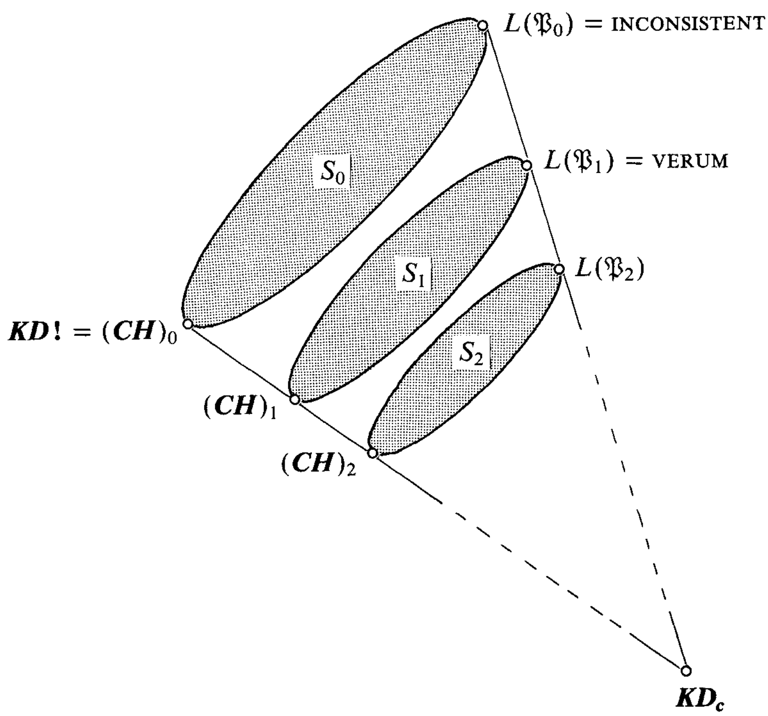

Fig. 4

3 Quasi-normal extensions of $\boldsymbol{KD_c}$ The preceding account of the normal extensions of $\boldsymbol{KD_c}$ leaves us with some gaps and some puzzles. Among the gaps are some unanswered questions concerning logics that are not normal: are there nonnormal extensions of $\boldsymbol{KD_c}$? If so, where are they? In particular, are there any between $(\boldsymbol{CH})_n$ and $(\boldsymbol{CH})_{n+1}$? Among the puzzles is the fact—presented in more detail below—that whereas all extensions of $\boldsymbol{KD!} = (\boldsymbol{CH})_0$ and of $(\boldsymbol{CH})_1$ are normal, $(\boldsymbol{CH})_2$ has a nonnormal extension ([17], pp. 192 ff): why does the dividing line go between 1 and 2 and not between 0 and 1? Another puzzle is the unique position of $\boldsymbol{KD_c}$: why is that the only logic to fall outside the normal slice system?

To gain a deeper understanding of the situation we now proceed to analyze the full set of extensions of $\boldsymbol{KD_c}$. First we review some terminology, some of it established, some of it new. A modal logic is *quasi-normal* if it contains the minimal normal modal logic $\boldsymbol{K}$ and is closed under modus ponens and substitution. Thus a quasi-normal logic is normal if and only if it is closed under necessitation. If $\mathfrak{F}$ is a frame generated by some element t, then we will write $L_t(\mathfrak{F})$ for the set of formulas which hold at t in every model on $\mathfrak{F}$. Thus $L(\mathfrak{F}) = \bigcap \{L_t(\mathfrak{F}) : t \in \text{dom } \mathfrak{F}\}$, where of course dom $\mathfrak{F}$ is the domain of $\mathfrak{F}$. It will prove convenient sometimes to use the symbol "$*$" as a generic name for the generator of a generated frame. Thus, if t is the generator of $\mathfrak{F}$, then $L_t(\mathfrak{F}) = L_*(\mathfrak{F})$. By convention, $L_*(\emptyset)$ will denote the Inconsistent Logic. It is easy to check that $L_*(\mathfrak{F})$ is always a quasi-normal logic. Similarly, if C is a class of generated frames, then $L_*(C) = \bigcap \{L_*(\mathfrak{F}) : \mathfrak{F} \in C\}$ is also always a quasi-normal logic. By analogy, if $\mathfrak{M}$ is a model, then we might write $Th_*(\mathfrak{M})$ for the set $\{\mathbf{A} : \mathfrak{M} \models_* \mathbf{A}\}$ of formulas true in $\mathfrak{M}$ at $*$. Thus if L is a logic such that $L \subseteq L_*(\mathfrak{M})$, then $Th_*(\mathfrak{M})$ will be a maximal L-consistent set.

Let us first establish the existence of nonnormal logics in the area under study.

Lemma 3.1 *For* $m > 0$, $\Diamond^{m-1}\Box\bot \in L_*(\mathfrak{P}_n)$ *if and only if* $m = n$.

Corollary 3.2 *If K is any class of generated, partially functional frames, then, for* $m > 0$, $\Box^{m-1}\Diamond\top \in L_*(K)$ *if and only if* $\mathfrak{P}_m \notin K$.

Lemma 3.3 $L_*(\mathfrak{P}_n)$ *is normal if and only if* $n \leqq 1$.

Proof: $L_*(\mathfrak{P}_0) = L(\mathfrak{P}_0)$ and $L_*(\mathfrak{P}_1) = L(\mathfrak{P}_1)$, so $L_*(\mathfrak{P}_0)$ and $L_*(\mathfrak{P}_1)$ are normal. Assume that $n > 1$. By Lemma 3.1,

(1) $\Diamond^{n-1}\Box\bot \in L_*(\mathfrak{P}_n)$.

By Lemma 3.1, also $\Diamond^{n-1}\Box\bot \notin L_*(\mathfrak{P}_{n-1})$. It is clear that $\mathfrak{P}_{n-1}$ is isomorphic to $\mathfrak{P}_n^1$, the subframe of $\mathfrak{P}_n$ generated by the element 1. Therefore, $\Diamond^{n-1}\Box\bot \notin L_1(\mathfrak{P}_n)$, and so

(2) $\Box\Diamond^{n-1}\Box\bot \notin L_*(\mathfrak{P}_n)$.

It follows from (1) and (2) that $L_*(\mathfrak{P}_n)$ is nonnormal. QED

As we tend to be particularly interested in logics that are normal, it is important to develop criteria for normality.

Theorem 3.4 *If $\mathfrak{F}$ is any generated frame, then for $L_*(\mathfrak{F})$ to be normal it is sufficient that, for all alternatives u of $*$, there be a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}^u$ taking $*$ to u.*

Proof: Assume the condition which is claimed to be sufficient. Suppose that $\mathbf{A} \in L_*(\mathfrak{F})$. Let u be any alternative of the generator $*$. According to the hypothesis, there is a p-morphism $f: \mathfrak{F} \longrightarrow \mathfrak{F}^u$ such that $u = f^*$. Hence, by the P-morphism Theorem, $\mathbf{A} \in L_u(\mathfrak{F}^u)$, and so, by the Generation Theorem, $\mathbf{A} \in L_u(\mathfrak{F})$. Consequently, $\Box\mathbf{A} \in L_*(\mathfrak{F})$. QED

Theorem 3.5 *If $\mathfrak{F}$ is a generated, functional frame, then for $L_*(\mathfrak{F})$ to be normal it is necessary and sufficient that, if $*$ has a distinct immediate successor s, then there is a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}^s$.*

Proof: If $\mathfrak{F}$ is the empty frame, the theorem is trivial. Therefore assume that $\mathfrak{F}$ is not empty.

Sufficiency: First suppose that $*$ lacks an immediate successor other than itself. Then $\mathfrak{F}$ must be either $\mathfrak{T}_{0,1}$ or $\mathfrak{P}_0$. It is well known that $L_*(\mathfrak{T}_{0,1}) = L(\mathfrak{T}_{0,1})$, the Trivial Logic, and $L_*(\mathfrak{P}_0) = L(\mathfrak{P}_0)$, the Verum Logic, are normal. Next suppose that $*$ has an immediate successor, s. Moreover, suppose that there is a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}^s$. The result will follow from Theorem 3.4. If $f^* = s$, then the theorem can be applied directly. Therefore suppose that $f^* \neq s$. As f is onto there must be some x in $\mathfrak{F}$ such that $fx = s$. Both $\mathfrak{F}$ and $\mathfrak{F}^s$ are generated, so—if R stands for the accessibility relation in both cases—there are m and n such that $*R^m x$ and $sR^n f*$. Since f is a p-morphism, we also have $f*R^m fx$; that is, $f*R^m s$. Therefore, $sR^{m+n}s$. If $m = 0$ or $n = 0$, then $f* = s$, contrary to assumption. Therefore $m > 0$ and $n > 0$, and *a fortiori* $m + n > 1$. This means that $sR^l s$, for some $l > 0$. Since R is functional, this in turn means that the set $\{y : \exists i < l(sR^i y)\}$ comprises all elements of $\mathfrak{F}^s$; and this set is finite. Hence $\mathfrak{F}^s$ is of type $\mathfrak{T}_{0,l}$, and so $\mathfrak{F}$ is of type $\mathfrak{T}_{1,l}$ or $\mathfrak{T}_{0,l}$. In either case it is easy to define a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}^s$ which takes $*$ to s. Now Theorem 3.4 can be applied.

Necessity: Suppose that $\mathfrak{F} = \langle U, R\rangle$ is generated and functional, and that $L_*(\mathfrak{F})$ is normal. Since we identify isomorphic frames, $\mathfrak{F}$ is one of the frames listed in Fig. 2. In particular, U is a nonempty set of natural numbers, $*$ is 0, and the immediate successor of $*$, if it exists, is 1. If $\mathfrak{F}$ is only partially functional, then $\mathfrak{F} = \mathfrak{P}_n$, for some $n > 0$. But, by Lemma 3.3, the only case in which $L_*(\mathfrak{P}_n)$ is consistent and normal is when $n = 1$. In this case the condition of the theorem is satisfied, for 0 has no immediate successor in $\mathfrak{P}_1$. Therefore assume that $\mathfrak{F}$ is totally functional. Either $\mathfrak{F} = \mathfrak{T}_{k,l}$, for some $k \geqq 0$ and $l > 0$, or else $\mathfrak{F} = \omega$. In the former case define a function f on U_{k+1} by the condition

$$fi = \begin{cases} i + 1, & \text{if } i < k + l - 1, \\ k, & \text{if } i = k + l - 1. \end{cases}$$

In the latter case define a function f on ω by the condition

$$fi = i + 1.$$

In either case it is easy to check that f is a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}^1$. QED

The following corollary does for the finite totally functional frames what Lemma 3.3 did for the partially functional ones:

Corollary 3.6 *For all* $k \geqq 0$ *and* $l > 0$, $L_*(\mathfrak{T}_{k,l}) = L(\mathfrak{T}_{k,l})$ *is normal.*

It is worth noting that $L(\mathfrak{P}_n) = \bigcap \{L_*(\mathfrak{P}_i) : i \leqq n\}$ and $\boldsymbol{KD_c} = \bigcap \{L_*(\mathfrak{P}_i) : i < \omega\}$. This result can be generalized as follows:

Theorem 3.7 *Let C be a class of finite, generated, functional frames. Then the following three conditions are equivalent:*

(i) $L_*(C)$ *is normal.*
(ii) *For all n, if* $\mathfrak{P}_n \in C$, *then* $L_*(C) \subseteq L(\mathfrak{P}_n)$.
(iii) *For all n, if* $\mathfrak{P}_n \in C$, *then for all* $i \leqq n$, $L_*(C) \subseteq L_*(\mathfrak{P}_i)$.

Proof: Let C be any class of finite, generated, functional frames.

(i) ⇒ (ii): Suppose that $L_*(C)$ is normal. Take any $\mathfrak{P}_n \in C$. Assume that

(1) $\mathbf{A} \in L_*(C)$;
(2) $\mathbf{A} \notin L(\mathfrak{P}_n)$.

It will be enough to derive a contradiction. By (2), there is some i such that $0 < i \leqq n$ and

(3) $\mathbf{A} \notin L(\mathfrak{P}_n)$.

Since $n - i \geqq 0$, the notation $\Box^{n-i}\mathbf{A}$ is meaningful; and it is clear from (3) that

(4) $\Box^{n-i}\mathbf{A} \notin L_*(\mathfrak{P}_n)$.

But by (1) and—this is where the assumption of normality comes in—some number of applications of the rule of necessitation, $\Box^{n-i}\mathbf{A} \in L_*(C)$. That $\mathfrak{P}_n \in C$ implies that $L_*(C) \subseteq L_*(\mathfrak{P}_n)$, and so $\Box^{n-i}\mathbf{A} \in L_*(\mathfrak{P}_n)$, contradicting (4).

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (i): Assume condition (iii). Suppose that

(1) $\mathbf{A} \in L_*(C)$.

Take any $\mathfrak{F} \in C$. If $\mathfrak{F}$ is totally functional, then from (1) it follows by Corollary 3.6 that $\Box\mathbf{A} \in L_*(\mathfrak{F})$. If, on the other hand, $\mathfrak{F}$ is partially functional, then there is some n such that $\mathfrak{F} = \mathfrak{P}_n$. If $n = 0$, then it is trivial that $\Box\mathbf{A} \in L_*(\mathfrak{F})$, for $L(\mathfrak{P}_0)$ is the Inconsistent Logic. If $n > 0$, then $n - 1$ exists, and by condition (iii) $L_*(C) \subseteq L_*(\mathfrak{P}_{n-1})$. Therefore $\mathbf{A} \in L_*(\mathfrak{P}_{n-1})$, which implies that $\Box\mathbf{A} \in L_*(\mathfrak{P}_n)$; for $\mathfrak{P}_{n-1}$ is isomorphic to $\mathfrak{P}_n^1$. Thus in any case $\Box\mathbf{A} \in L_*(C)$, so $L_*(C)$ is normal. QED

Many results in normal modal logic have counterparts in quasi-normal modal logic. As the difference between normal and quasi-normal has to do with validity and not with truth at a point in a model, proofs can often be taken over with little or no change. We will now discuss some that are of special interest to us.

Proposition 3.8 *Let* $\mathfrak{M}$ *be a distinguishable model based on a frame* $\mathfrak{F}$ *that either is finite or else is the omega frame* ω. *Then, for every logic* L, $L \subseteq Th_*(\mathfrak{M})$ *only if* $L \subseteq L_*(\mathfrak{F})$.

Proof: This result is a variation of Proposition 1.9 and Proposition 2.1. QED

Theorem 3.9 *For every quasi-normal extension Q of $\boldsymbol{KD_c}$ there is a class C of nonempty, finite, generated, functional frames such that $Q = L_*(C)$.*

Proof: If Q is the Inconsistant Logic, then $Q = L_*(\varnothing)$. Therefore suppose that Q is a consistent quasi-normal extension of $\boldsymbol{KD_c}$. Suppose that $\mathbf{A} \notin Q$. It will be enough to exhibit a finite frame $\mathfrak{F}$ such that $\mathbf{A} \notin L_*(\mathfrak{F})$ and $Q \subseteq L_*(\mathfrak{F})$. Assume that $Q \cup \{\neg \mathbf{A}\}$ is inconsistent in $\boldsymbol{KD_c}$. Then there would be some $\mathbf{B}_0, \dots, \mathbf{B}_{m-1} \in Q$ such that $\neg(\mathbf{B}_0 \wedge \dots \wedge \mathbf{B}_{m-1} \wedge \neg \mathbf{A}) \in \boldsymbol{KD_c}$. As Q extends $\boldsymbol{KD_c}$, and quasi-normal logics admit truth-functional reasoning, we conclude that $\mathbf{A} \in Q$; which contradicts our assumption. Consequently, $Q \cup \{\neg \mathbf{A}\}$ is consistent in $\boldsymbol{KD_c}$. By Lindenbaum's Lemma there is a maximal $\boldsymbol{KD_c}$-consistent set x such that $\mathbf{A} \notin x$. Now x is a member of $\mathfrak{M}_{\boldsymbol{KD_c}}$, the canonical model for $\boldsymbol{KD_c}$. Let $\mathfrak{M}$ be the submodel of $\mathfrak{M}_{\boldsymbol{KD_c}}$ generated by x. Then it is true to say that $Q \subseteq L_*(\mathfrak{M})$ but $\mathbf{A} \notin L_*(\mathfrak{M})$. But $\mathfrak{M}$ is certainly distinguishable. Moreover, because of the syntactic strength of $\boldsymbol{KD_c}$, $\mathfrak{M}$ is functional and hence, being generated, is based either on a nonempty, finite frame or on ω. In either case, Proposition 3.8 is applicable, and we conclude that $Q \subseteq L_*(\mathfrak{F})$, where $\mathfrak{F}$ is the frame of $\mathfrak{M}$.

If $\mathfrak{F}$ is finite, our task is complete. Therefore assume $\mathfrak{F}$ is not finite. Then $\mathfrak{F} = \omega$. But from Lemma 1.1 and Corollary 3.6 we infer that $Q \subseteq \bigcap \{L_*(\mathfrak{T}_{k,l}) : k \geqq 0 \ \&\ l > 0\}$ and $\mathbf{A} \notin \bigcap \{L_*(\mathfrak{T}_{k,l}) : k \geqq 0 \ \&\ l > 0\}$. Consequently, there must be some $k \geqq 0$ and $l > 0$ such that $Q \subseteq L_*(\mathfrak{T}_{k,l})$ and $\mathbf{A} \notin L_*(\mathfrak{T}_{k,l})$. QED

Corollary 3.10 *Every extension of $(\boldsymbol{CH})_1$ is normal. In particular, every extension of $\boldsymbol{KD!}$ is normal.*

Proof: By Theorem 3.9, if $(\boldsymbol{CH})_1 \subseteq Q$, then there is a class C of finite, generated, functional frames such that $Q = L_*(C)$. Since by assumption $\Box^i \Diamond \mathbf{T} \in Q$, for all $i > 0$, it is clear that every nonempty member of C either is totally functional or else is $\mathfrak{P}_1$. Therefore, by Corollary 3.6 and Lemma 3.3, Q is normal. QED

The following result is perhaps surprising:

Theorem 3.11 *The n-th normal slice, S_n, contains all quasi-normal logics between $(\boldsymbol{CH})_n$ and $L(\mathfrak{P}_n)$.*

Proof: Let Q be any quasi-normal logic such that $(\boldsymbol{CH})_n \subseteq Q \subseteq L(\mathfrak{P}_n)$. By Theorem 3.9, there is a class of finite, generated, functional frames such that $Q = L_*(C)$. Let k be any number, and suppose that $\mathfrak{P}_k \in C$. As $(\boldsymbol{CH})_n \subseteq Q$, we have $\Box^i \Diamond \mathbf{T} \in Q$, for all $i \geqq n$. It follows from Lemma 3.1 that $k < n$. By Lemma 1.5, then, $L(\mathfrak{P}_k) \supseteq L(\mathfrak{P}_n)$. As $Q \subseteq L(\mathfrak{P}_n)$, we conclude that $Q \subseteq L(\mathfrak{P}_k)$. By Theorem 3.7, this shows that Q is normal. QED

We will now generalize the notion, introduced in Section 2, of a normal slice. First we define, for each class K of nonempty, generated, partially functional (and hence finite) frames, a function f_K on the set S_0 of normal extensions of $\boldsymbol{KD!}$:

$$f_K(L) =_{df} L \cap L_*(K).$$

The range f_K is called the *slice determined by* K, and we denote it by S_K. This terminology well agrees with that laid down in Section 2, for the n-th normal slice coincides with the slice determined by the class $\{\mathfrak{P}_i : 0 < i \leqq n\}$; and, as we saw in Theorem 3.11, this slice contains only normal logics and therefore must be said to have earned its name of normal. Note in particular that S_0 in the old notation is identical to $S_\varnothing$ in the new. In what follows, let us agree to use K, possibly superscripted, as a parameter over nonempty classes of generated, partially functional (hence finite) frames.

Lemma 3.12 *$K \subseteq K'$ if and only if $L_*(K) \supseteq L_*(K')$.*

Proof: The only-if-part is trivial. For the if-part, suppose that $\mathfrak{P}_n \in K - K'$, for some $n > 0$. Since $n - 1$ exists, $\Box^{n-1}\Diamond\mathbf{T}$ is a formula and, by Corollary 3.2, is a thesis of $L_*(K')$ but not of $L_*(K)$. QED

Corollary 3.13 *If $K \neq K'$, then $L_*(K) \neq L_*(K')$.*

Corollary 3.14 *If $K \neq K'$, then $S_K \neq S_{K'}$.*

Proof: Recall that $L(\varnothing)$ is the Inconsistent Logic. Clearly, $f_K(L(\varnothing)) = L(\varnothing) \cap L_*(K) = L_*(K)$ and $f_{K'}(L(\varnothing)) = L(\varnothing) \cap L_*(K') = L_*(K')$ are the strongest logics in S_K and $S_{K'}$, respectively. The result then follows from Corollary 3.13. QED

Corollary 3.15 *There are nondenumerably many nonnormal extensions of* $\boldsymbol{KD_c}$.

Proof: By Corollary 3.13 there are nondenumerably many quasi-normal extensions of $\boldsymbol{KD_c}$. In Section 2 we established that $\boldsymbol{KD_c}$ has only denumerably many normal extensions. QED

Next we will embark on an analysis of the system of slices which will generalize that of the normal slice system in Section 2.

Lemma 3.16 *Suppose that K is finite. Then, for all L, $L' \in S_0$, if $L \nsubseteq L'$ then $f_K(L) \nsubseteq f_K(L')$.*

Proof: Let K be finite and suppose that L and L' are normal extensions of $\boldsymbol{KD!}$. Assume that $\mathbf{A} \in L - L'$. Since K is finite, either K is empty or else there is a greatest number $m > 0$ such that $\mathfrak{P}_m \in K$. In either case, $\Box^m\bot \in L_*(K)$. From this and the fact that L' is normal it follows that the formula $\mathbf{A} \vee \Box^m\bot$ is a thesis of $f_K(L) = L \cap L_*(K)$ but not of $f_K(L') = L' \cap L_*(K)$. QED

Lemma 3.17 *If $K \nsubseteq K'$, then, for all L, $L' \in S_0$, $f_K(L) \nsupseteq f_{K'}(L')$.*

Proof: Suppose that $K \nsubseteq K'$. Then there is some $m > 0$ such that $\mathfrak{P}_m \in K - K'$. Consequently, by Corollary 3.2, $\Box^{m-1}\Diamond\mathbf{T}$ is a thesis of $L_*(K')$ but not of $L_*(K)$. Let L and L' be any normal extension of $\boldsymbol{KD!}$. Then $\Box^{m-1}\Diamond\mathbf{T}$ is a thesis of $f_{K'}(L') = L' \cap L_*(K')$ but not of $f_K(L) = L \cap L_*(K)$. QED

We are now able to prove that, just as the normal slice system partitions the class of all normal extensions of $\boldsymbol{KD_c}$, so the general slice system partitions the class of all extensions of $\boldsymbol{KD_c}$.

Theorem 3.18 *Every extension of* $\boldsymbol{KD_c}$ *belongs to a unique slice.*

Proof: Uniqueness: Suppose that some logic belongs to slices S_K and $S_{K'}$. Then there must be some logics $L, L' \in S_0$ such that $f_K(L) = f_{K'}(L')$. By Lemma 3.17, twice applied, $K = K'$.

Existence: Let Q be any extension of $\boldsymbol{KD_c}$. By Theorem 3.9 there is some class C of nonempty, finite, generated, functional frames such that $Q = L_*(C)$. Let C^+ and C^- be the classes of totally functional and partially functional frames in C, respectively. Then $Q = L_*(C^+) \cap L_*(C^-)$. By Corollary 3.6, $L_*(C^+)$ is a normal extension of $\boldsymbol{KD!}$. Consequently, $Q = f_{C^-}(L_*(C^+))$, and so Q belongs to the slice determined by C^-. QED

The following two theorems settle the internal structure of each slice:

Theorem 3.19 *Every slice determined by a finite class is isomorphic to* S_0.

Proof: Suppose that K is finite. Then f_K is one-to-one, by Lemma 3.16; onto S_K, by definition of f_K; and order-preserving both ways by the same kind of argument as in the proof of Theorem 2.11. QED

Theorem 3.20 *Every slice determined by an infinite class is a singleton, the only element of which is a logic between* $\boldsymbol{KD_c}$ *and* $\boldsymbol{KD!}$.

Proof: Suppose that K is infinite. First we establish the second part of the theorem. That $\boldsymbol{KD_c} \subseteq L_*(K)$ needs no proof. Take any $\mathbf{A} \notin \boldsymbol{KD!}$. Then $\mathbf{A}$ fails in some model on ω. Since K is infinite, it contains arbitrarily large, finite, generated, partially functional frames. Pick any $\mathfrak{P}_m \in K$ where $m \geqq \deg \mathbf{A}$. It is easy to see that $\mathbf{A} \notin L_*(\mathfrak{P}_m)$, and so $\mathbf{A} \notin L_*(K)$. Thus we have shown that $L_*(K) \subseteq \boldsymbol{KD!}$. The first part of the theorem is a consequence of this result, for now we may infer that $f_K(L) = L \cap L_*(K) = L_*(K)$, for every $L \in S_0$. QED

The relationship between logics in different slices is settled by the following two theorems:

Theorem 3.21 *Suppose that K is finite. Then, for all L, $L' \in S_0$, $f_K(L) \subseteq f_{K'}(L')$ if and only if $L \subseteq L'$ and $K \supseteq K'$.*

Proof: The if-part follows from Lemma 3.12, the only-if-part from Lemmas 3.16 and 3.17. QED

Theorem 3.22 *Suppose that K is infinite. Then, for all L, $L' \in S_0$, $f_K(L) \subseteq f_{K'}(L')$ if and only if $K \supseteq K'$.*

Proof: If K is infinite, then, by Theorem 3.20, $L_*(K) \subseteq \boldsymbol{KD!}$. This means that $f_K(L) \subseteq f_{K'}(L')$ iff $L \cap L_*(K) \subseteq L' \cap L_*(K')$ iff $L_*(K) \subseteq L_*(K')$ iff, by Lemma 3.12, $K \supseteq K'$. QED

We have reached the end of our investigation. The puzzles mentioned at the beginning of this section have been given an explanation by being put into perspective. Complexity unfortunately makes it too difficult to represent the insights gained in a final chart that could replace Fig. 4. Nevertheless, the last several theorems, together with Lemmas 1.5 and 1.6, can be used to answer any question about the structure of the set of extensions of $\boldsymbol{KD_c}$.

Perhaps the following final remarks will also be of some help. For any

$L \in S_0$, let us define the *finite-class stratum determined by* L as the set $\{f_K(L) : K \text{ is finite}\}$ and the *infinite-class stratum* as the set of logics occurring in any slice determined by an infinite class. Then every extension of $\boldsymbol{KD_c}$ belongs to a unique stratum. Anyone who would like to try to draw an improved chart should contemplate the following facts, the proofs of which are left to the reader: Every finite-class stratum is isomorphic with the lattice of co-finite sets of numbers; the infinite-class stratum is isomorphic with the lattice of co-infinite sets of natural numbers; and the set of logics between $\boldsymbol{KD_c}$ and ***KD!*** is isomorphic with the lattice of all sets of natural numbers.

REFERENCES

[1] Bull, R. A., "That all normal extensions of S4.3 have the finite model property," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 12 (1966), pp. 341–344.

[2] Byrd, Michael, "The extensions of $BAlt_3$ revisited," *Journal of Philosophical Logic*, vol. 7 (1978), pp. 407–413.

[3] Chellas, Brian F., *Modal Logic*, Cambridge University Press, Cambridge, 1980.

[4] Chellas, Brian F., "$KG^{k,l,m,n}$ and the *efmp*," *Logique et Analyse*, vol 26 (1983), pp. 255–262.

[5] Esakia, L. and V. Meskhi, "Five critical systems," *Theoria*, vol. 48 (1977), pp. 52–60.

[6] Fine, Kit, "The logics containing S4.3," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 17 (1971), pp. 371–376.

[7] Fine, Kit, "An ascending chain of S4 logics," *Theoria*, vol. 40 (1974), pp. 110–116.

[8] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 1968.

[9] Lemmon, E. J., "Algebraic semantics for modal logic," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 46–65, 191–218.

[10] Lemmon, E. J., *The 'Lemmon' Notes: An Introduction to Modal Logic*, in collaboration with Dana Scott, Blackwell, Oxford, 1977.

[11] Maksimova, L. L., "Pretabular extensions of Lewis's S4" [in Russian], *Algebra i logika*, vol. 14 (1975), pp. 28–55.

[12] Nagle, M. C. and S. K. Thomason, "Extensions of K5," *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 102–109.

[13] Prior, A. N., *Time and Modality*, Clarendon Press, Oxford, 1957.

[14] Prior, A. N., *Past, Present and Future*, Clarendon Press, Oxford, 1967.

[15] Scroggs, Schiller Joe, "Extensions of the Lewis system S5," *The Journal of Symbolic Logic*, vol. 16 (1951), pp. 112–120.

[16] Segerberg, Krister, "On the logic of 'tomorrow'," *Theoria*, vol. 33 (1967), pp. 45–52.

[17] Segerberg, Krister, *An Essay in Classical Modal Logic*, Uppsala 1971.

[18] Segerberg, Krister, "On the extensions of S4.4," *The Journal of Symbolic Logic,"* vol. 36 (1971), pp. 590–591.

[19] Segerberg, Krister, "Two Scroggs theorems," *The Journal of Symbolic Logic*, vol. 36 (1971), pp. 697–698.

[20] Segerberg, Krister, "A small corner of the big lattice of modal logics," *The Journal of Symbolic Logic*, vol. 49 (1984) p. 1428.

[21] Sobociński, Bolesław, "Certain extensions of modal system S4," *Notre Dame Journal of Formal Logic*, vol. 11 (1970), pp. 347–368.

[22] Ulrich, Dolph and Michael Byrd, "The extensions of $BAlt_3$," *Journal of Philosophical Logic*, vol. 6 (1977), pp. 109–117.

[23] von Wright, Georg Henrik. " 'And next'," *Acta Philosophica Fennica*, fasc. 18 (1965), pp. 293–304.

University of Auckland
Private Bag, Auckland
New Zealand