In M. Bilkova and L. Behounek, editors.
The Logica Yearbook 2004, pp.213-222.
Prague: Filosofia, 2005.

Modality and Paraconsistency

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Paraconsistent logic was born in the vicinity of modal logic. Moreover, as every other non-classical logicians, paraconsistentists have very often flirted with modalities. The first known system of paraconsistent logic was in fact defined as a fragment of $S5$, in the late 40s. But a fragment of a modal system is not necessarily a modal system. I will show here, indeed, that Jaśkowski's **D2** is not a modal logic, in the contemporary usual meaning of the term. By contrast, I will also show, subsequently, that any non-degenerate normal modal system is inherently paraconsistent.

1 What is a paraconsistent logic?

Classical logic is maculated by many irrelevancies. The enterprise of paraconsistency was designed so as to help cleansing a particular stain, by eschewing the so-called Principle of Explosion:

(PE) $\forall\alpha\forall\beta(\alpha, \neg\alpha \Vdash \beta)$.

According to (PE), contradictions are malicious creatures: Whenever they are present in a theory, anything goes, any statement is equally derivable.

In contemporary times, one of the most notorious insurgents against the Aristotelean doctrine that contradictions should be avoided for ontological, logical or psychological reasons was the Polish logician Jan Łukasiewicz (1910). But it was only a few so many years later that one of his disciples, Stanisław Jaśkowski (1948), would really inaugurate the study of non-trivial inconsistent formal systems.

Jaśkowski's proposal was that of a *discussive system*, 'a system which cannot be said to include theses that express opinions in agreement with one another'. To obtain such a system every statement was to be preceded by the reservation 'in accordance with the opinion of one of the participants in the discussion', or

*This study was partially supported by FCT (Portugal) and FEDER (European Union), namely, via the Project FibLog POCTI / MAT / 37239 / 2001 of the Center for Logic and Computation (IST, Portugal) and the FCT grant SFRH / BD / 8825 / 2002. I am grateful to the audience of the **Logica 2004** for their welcoming reaction to the present note, and to Walter Carnielli for his comments on an early version of it. In the meanwhile, the second part of this material has been more thoroughly investigated in Marcos (2004).

‘for a certain admissible meaning of the terms used’. These ideas were initially implemented with the help of the modal logic $S5$ into a sort of ‘pre-discussive’ system J, which was such that

$$\Gamma \Vdash_J \alpha \ \text{ iff } \ \Diamond\Gamma \vDash_{S5} \Diamond\alpha.$$

Obviously, J defines a paraconsistent logic. A very weak one, however. As it is easy to see, the consequence relation of J is essentially single-premised, as $\Gamma \Vdash_J \alpha$ iff $\gamma \Vdash_J \alpha$, for some $\gamma \in \Gamma$. There are in J no typically multiple-premised rules, thus, such as *modus ponens*. To fix that weakness, Jaśkowski was to propose a sort of preprocessing of the usual classical connectives, by recursively translating:

1. $p^* = p$, for every atomic variable p;
2. $(\neg\alpha)^* = \neg\alpha^*$;
3. $(\alpha \vee \beta)^* = \alpha^* \vee \beta^*$;
4. $(\alpha \wedge \beta)^* = \alpha^* \wedge \Diamond\beta^*$;
5. $(\alpha \supset \beta)^* = \Diamond\alpha^* \supset \beta^*$.

While clause 5, defining a ‘discussive implication’, belongs to Jaśkowski (1948), clause 4, defining a ‘discussive conjunction’, belongs to Jaśkowski (1949). The main ‘discussive’ logic **D2** was then put forward by setting

$$\Gamma \Vdash_{\mathbf{D2}} \alpha \ \text{ iff } \ \Gamma^* \Vdash_J \alpha^*.$$

It is straightforward to check that **D2** is a paraconsistent extension of the positive fragment of classical logic (that is, the logical constants $\vee$, $\wedge$ and $\supset$ in **D2** behave just like their classical homonyms). Notice that without clause 4 this observation about the positive fragment of classical logic would not be fully true, for the resulting logic would fail negation introduction, that is, it would fail $\alpha, \beta \Vdash (\alpha \wedge \beta)$, as it happens with J. There are indeed a few systems of paraconsistent logic that have this ‘non-adjunctive’ character. Any defense about this having been a feature desired and cherished by Jaśkowski seems to depend however on not having read his 1949 two-pages paper (and that disgracefully applies to most discussivists from the literature).

The ‘asymmetric’ looks of clauses 4 and 5 have been criticized here and there. Based on the facts that the formulas $\Diamond(\Diamond\alpha \supset \beta)$ and $\Diamond(\Diamond\alpha \supset \Diamond\beta)$ are equivalent inside the modal logic $S4$ (a fragment of $S5$) and that the formulas $\Diamond(\alpha \wedge \Diamond\beta)$, $\Diamond(\Diamond\alpha \wedge \beta)$ and $\Diamond(\Diamond\alpha \wedge \Diamond\beta)$ are all equivalent inside $S5$, the following alternatives to the above clauses have been proposed:

4.1 $(\alpha \wedge \beta)^* = \Diamond\alpha^* \wedge \beta^*$;

4.2 $(\alpha \wedge \beta)^* = \Diamond\alpha^* \wedge \Diamond\beta^*$;

5.1 $(\alpha \supset \beta)^* = \Diamond\alpha^* \supset \Diamond\beta^*$.

Now, while it is true that any choice of preprocessing translation would have the same effect for the positive fragment of **D2** (it would still coincide with the positive fragment of classical logic), the same is not true for the full logic, when the interaction of negation with the other connectives is considered. It is not true thus that different translation clauses 'would have just the same consequences', as claimed in Priest (2002, section 5.2). Different choices of discussive conjunction and discussive implication would in fact define logics distinct from **D2**. This phenomenon will be carefully illustrated in Section 3 of the present note.

Other usual classical connectives can be easily defined in **D2**, such as bi-implication: $(\alpha \equiv \beta) \stackrel{\text{def}}{=} (\alpha \supset \beta) \wedge (\beta \supset \alpha)$. Moreover, a classical negation $\sim$ can be defined in **D2** by setting $\sim\alpha \stackrel{\text{def}}{=} \alpha \supset \neg(\alpha \vee \neg\alpha)$ (hint: check that $\Diamond p$ and $\Diamond{\sim}p$ cannot be both true and cannot be both false in a given world of a model of $S5$). The logic **D2** can also define a consistency connective $\circ\alpha \stackrel{\text{def}}{=} (\sim\alpha) \vee (\sim\neg\alpha)$, in the sense of Carnielli and Marcos (2002), that is, a logical constant that says when explosion can be recovered, through the following Gentle Principle of Explosion:

(GPE) $\exists\alpha\exists\beta(\circ\alpha, \alpha \nVdash \beta$ and $\circ\alpha, \neg\alpha \nVdash \beta)$, while $\forall\alpha\forall\beta(\circ\alpha, \alpha, \neg\alpha \Vdash \beta)$.

The fact that **D2** enjoys (GPE) makes it qualify as an **LFI**, a Logic of Formal Inconsistency (more specifically, in this case, a **dC**-system based on classical logic). Consistent reasoning can often be recaptured from inside inconsistent logics, and the ability of doing just that is in fact a fundamental feature of **LFI**s. More precisely, if $\vDash_{\mathbf{CPL}}$ is the consequence relation of Classical Propositional Logic, the following *Derivability Adjustment Theorem* can then be proven:

(DAT) $(\Gamma \vDash_{\mathbf{CPL}} \beta)$ iff $\exists\Sigma(\circ\Sigma, \Gamma \Vdash_{\mathbf{D2}} \beta)$.

The above result says that, even though **D2** fails the 'consistency presupposition' that is typical of classical logic, any classical inference can be recovered if a sufficient number of 'consistency assumptions' are added to the set of premises. We will see several examples of derivability adjustments in the next sections. Clearly, yet another way of recovering **CPL** from inside **D2** is by taking the new classical negation of **D2** into account. If $(\alpha)^{\neg,\sim}$ denotes a translation that changes any occurrence of the paraconsistent $\neg$ by its classical counterpart $\sim$, leaving the rest of the formula as it is, it is easy to check that $(\Gamma \vDash_{\mathbf{CPL}} \beta)$ iff $(\Gamma^{\neg,\sim} \Vdash_{\mathbf{D2}} \beta^{\neg,\sim})$. This direct translation is an alternative to the addition of further premises promoted by the derivability adjustments of the set $\circ\Sigma$, in the above (DAT).

It should be remarked that semantical features of a given logic are usually not inherited by its proper fragments. Thus, while classical logic is two-valued, other many-valued logics can only be given a two-valued semantics at the cost of their truth-functionality, and intuitionistic logic is not even a finitely-valued logic. Typically, in fact, non-classical fragments of **CPL** will have connectives that are not classically expressible —such as an intuitionistic or a paraconsistent negation. The realization that many-valued logics can all be embedded into certain modal logics, as in Demri (2000), do not make them any more 'modal' than they were before, and, as we will see in Section 3, the fact that **D2** is introduced through an embedment into the modal logic $S5$ does NOT make this system a 'modal logic', in the contemporary usual meaning of the term.

2 What is a modal logic?

Unfortunately, there is no generally agreed definition of the term 'modal logic'. Fortunately, however, this situation has not hindered the enormous advance of the studies in that area. Among the most solid achievements of those studies one should certainly count the modern development of Kripke-like semantics. There is nowadays a plethora of modal systems available. What do they have in common, if anything? I will assume here that the most fundamental feature of modal logics, common to both the usual models of normal modal logics and the minimal models of non-normal modal logics (see Chellas, 1980, chap. 7) consists in the validity of the so-called 'replacement property'. The validity of such a logical property coincides in fact with the abstract property that Wójcicki (1988, chap. 5) calls 'self-extensionality' and shows to be the characterizing feature of the logics that have 'an adequate frame semantics'. I will briefly recall in this section what this property means.

Let $\alpha \dashv\Vdash \beta$ abbreviate the combination of $\alpha \Vdash \beta$ with $\beta \Vdash \alpha$ —this is to say that α and β are *equivalent* formulas. In any logic with a classic-like bi-implication $\equiv$, as all the logics we will be mentioning in the present study, asserting $\alpha \dashv\Vdash \beta$ is the same as asserting $\Vdash \alpha \equiv \beta$. Let $\varphi(p)$ denote a formula in which the variable p occurs, and $\varphi(p/\delta)$ denote the formula obtained from φ by uniformly substituting all occurrences of p by the formula δ. Given a pair of formulas α and β, say that they are *indiscernible* if, for every formula $\varphi(p)$, one has that $\varphi(p/\alpha) \dashv\Vdash \varphi(p/\beta)$. In particular, indiscernible formulas are equivalent (to see that, take $\varphi(p)$ as p itself). An explicit definition, such as those we have been writing since the last section with the help of the extra-logical symbol '$\stackrel{\text{def}}{=}$' simply postulates that the formula at the left-hand side of that symbol should be treated as indiscernible from the formula at the right-hand side of that same symbol. Now, a logic enjoying the *replacement property* is a logic for which every pair of equivalent formulas is indiscernible, that is, a logic in which $\alpha \dashv\Vdash \beta$ implies $\varphi(p/\alpha) \dashv\Vdash \varphi(p/\beta)$, for any formula φ. It should be clear that this property allows us to replace any occurrence of a subformula by an equivalent expression, while derivability is preserved.

Modal logics, just as classical logic, enjoy the replacement property, and so they are such that $\alpha_1 \dashv\Vdash \alpha_2$ and $\beta_1 \dashv\Vdash \beta_2$ provide sufficient conditions for $(\alpha_1 \wedge \beta_1) \dashv\Vdash (\alpha_2 \wedge \beta_2)$ or $\Diamond\alpha_1 \dashv\Vdash \Diamond\alpha_2$. As it can be seen in Theorems 44, 78 and 124 of Carnielli, Coniglio, and Marcos (2005), there are many paraconsistent logics that FAIL the replacement property.

3 D2 is not a modal logic

The discussive logic **D2** has a very long and dramatic story (see Ciuciura, 1999). And it is not over yet. Besides non-adjunctiveness, another common obsession of discussivists concerns the alleged 'modal character' of **D2**. This section will exhibit a few properties of the logic **D2** and of some of its close relatives, and then show that none of these logics, nor their deductive fragments, nor their paraconsistent extensions (if they exist), can enjoy the replacement property.

Inside a proper fragment of classical logic, the classical connectives can certainly not all be interdefined as usual. So, in **D2** the paraconsistent negation cannot interact with the other connectives such as classical negation does. Consider the following inferences:

(ID1)	$(\neg\alpha \supset \beta) \Vdash (\alpha \vee \beta)$
(ID2)	$(\alpha \vee \beta) \Vdash (\neg\alpha \supset \beta)$
(ID3)	$\neg(\neg\alpha \supset \beta) \Vdash \neg(\alpha \vee \beta)$
(ID4)	$\neg(\alpha \vee \beta) \Vdash \neg(\neg\alpha \supset \beta)$
(ID5)	$(\alpha \supset \beta) \Vdash \neg(\alpha \wedge \neg\beta)$
(ID6)	$\neg(\alpha \wedge \neg\beta) \Vdash (\alpha \supset \beta)$
(ID7)	$\neg(\alpha \supset \beta) \Vdash (\alpha \wedge \neg\beta)$
(ID8)	$(\alpha \wedge \neg\beta) \Vdash \neg(\alpha \supset \beta)$
(ID9)	$\neg(\neg\alpha \wedge \neg\beta) \Vdash (\alpha \vee \beta)$
(ID10)	$(\alpha \vee \beta) \Vdash \neg(\neg\alpha \wedge \neg\beta)$
(ID11)	$\neg(\neg\alpha \vee \neg\beta) \Vdash (\alpha \wedge \beta)$
(ID12)	$(\alpha \wedge \beta) \Vdash \neg(\neg\alpha \vee \neg\beta)$

It is easy to use the semantics of $S5$, based on reflexive, symmetric and transitive frames, to check that (ID1), (ID4), (ID9) and (ID11) are a consequence of reflexivity, while (ID7) and (ID8) are a consequence of symmetry and transitivity. Now, to prove the remaining inferences in **D2**, some derivability adjustments are in order (recall Section 1): To recover (ID2) and (ID6) one needs to add $\circ\alpha$ to the set of premises; to recover (ID5) and (ID10) one needs to add $\circ\beta$; in the case of (ID3) and (ID12), adding either $\circ\alpha$ or $\circ\beta$ will do.

One can now also readily show the difference between the various possible clauses for a preprocessing translation, as proposed in Section 1. Combining the three versions of the translation clause 4 and the two versions of the translation clause 5 there will be at most 5 distinct alternatives to the logic **D2**. And, in fact, there are. While all these logics agree in validating (ID1), (ID7), (ID9) and (ID11), none of them validates (ID2), (ID10) and (ID12). The logics based on the original clause 5 validate (ID4) but not (ID3), the other logics do exactly the contrary; the logics based on clause 5 also validate (ID8), while the others do not. The logics based on the original clause 4 fail (ID5) and (ID6), all the remaining logics validate (ID6). Finally, (ID5) is validated exactly by those logics that substitute clause 4.1 for clause 4. These 6 possible 'discussive logics' are thus all different, and each of them allows for its own derivability adjustments, in each case (exercise).

We will now check that all the 6 logics above fail the replacement property. Notice first that the definitions of the bi-implication, the classical negation and the consistency connective used in Section 1 work the same for any of the above logics. In particular, for the classical negation $\sim$, defined by setting $\sim\alpha \stackrel{\text{def}}{=} \alpha \supset \neg(\alpha \vee \neg\alpha)$, the theory $\{\alpha, \sim\alpha\}$ is explosive, and the formulas α and $\sim\sim\alpha$ are logically indiscernible. Now, by (ID7), an inference validated by all the above logics, we have that $\neg\sim\gamma \Vdash \gamma \wedge \neg\neg(\gamma \vee \neg\gamma)$. By conjunction elimination, a rule valid in the positive fragment of classical logic, $\neg\sim\gamma \Vdash \gamma$. But a classical negation is explosive, thus $\forall\gamma\forall\beta(\gamma, \sim\gamma \Vdash \beta)$. In that case we also have, by transitivity of deduction (the cut rule), that $\neg\sim\gamma, \sim\gamma \Vdash \beta$, for arbitrary formulas γ and β. In particular, we have $\neg\sim\sim\alpha, \sim\sim\alpha \Vdash \beta$, taking γ as $\sim\alpha$. Again, considering the properties of classical negation we have that $\alpha \dashV\Vdash \sim\sim\alpha$. To proceed by absurdity, if the replacement property did hold good for any of the above logics one could then conclude that $\neg\alpha \dashV\Vdash \neg\sim\sim\alpha$. From this and the cut rule one would finally derive $\neg\alpha, \alpha \Vdash \beta$, and the logic would not be paraconsistent, as we know it is.

4 Modal logics are paraconsistent

Can paraconsistent logics enjoy the replacement property at all? And can they have appropriate 'natural' modal semantics? How natural? The answer to those disquietudes is doubly positive, as we will see in this section. First: Yes, there are paraconsistent logics enjoying full replacement. Second: Yes, one does not need to adventure into strange new territories to find them. We had a modal paraconsistent negation around all the time, when we were dealing with usual normal modal logics —and there is an infinite number of the latter.

Béziau (2002, 2005) has been calling attention to that, recently: Just as much as intuitionistic negation has its standard modal interpretation in terms of a certain translation into $S4$ that interprets this negation by $\sim\Diamond$, a dual paraconsistent negation is obtained if one interprets it by using $\Diamond\sim$. The idea in reality is anything but new, and it has been deeply explored in between the mid-70s and the 80s (check specially Došen 1986 and Vakarelov 1989). This section will sketch the big picture —for many more details, an emphasis on duality, and proofs of all claims, check Marcos (2004).

Normal modal logics are extensions of the logic K: In their usual language, they admit the necessitation rule and propagate necessity through conjunctions. They also enjoy the replacement property, by their very design. The most obvious degenerate examples of normal modal logics are characterized by frames that are such that every world can access only itself or no other world. Now, it is not difficult to verify that, for any non-degenerate normal modal logic, a connective defined by setting $\neg\alpha \stackrel{\text{def}}{=} \Diamond\sim\alpha$ is a (sub-classical) paraconsistent negation, that is: (a) It only has positive properties that are also enjoyed by classical negation; (b) it has enough negative properties so that it qualifies as a 'minimally decent negation', in the sense of Marcos (2005b); (c) it is not explosive. Moreover, any such logic is in fact a Logic of Formal Inconsistency, and a

dC-system (recall Section 1) where a consistency connective can be defined, for instance, by setting $\circ\alpha \stackrel{\text{def}}{=} \alpha \supset {\sim}\neg\alpha$.

Furthermore, not only is it possible to start from a (non-degenerate) normal modal logic and define operators that represent a paraconsistent negation and a consistency connective, it is also possible to do it the other way around. Indeed, consider as before the language of positive classical logic to be written over the connectives $\wedge$, $\vee$ and $\supset$, whose interpretation is the standard one over a Kripke-like modal structure, and add to that a negation $\neg$ to be interpreted by assuming, for worlds x and y of a model $\mathcal{M}$ with an accessibility relation R:

$$\models_x^{\mathcal{M}} \neg\alpha \;\text{ iff }\; (\exists y)(x\mathsf{R}y \;\text{ and }\; \not\models_y^{\mathcal{M}} \alpha).$$

In that case, a classical negation could be recovered simply by defining ${\sim}\alpha \stackrel{\text{def}}{=} \alpha \supset \neg(\alpha \supset \alpha)$. The other usual modal connectives would then be obtained by setting $\Diamond\alpha \stackrel{\text{def}}{=} \neg{\sim}\alpha$ and $\Box\alpha \stackrel{\text{def}}{=} {\sim}\neg\alpha$. Alternatively, the consistency connective could also be taken as primitive, by assuming:

$$\models_x^{\mathcal{M}} \circ\alpha \;\text{ iff }\; \models_x^{\mathcal{M}} \alpha \text{ implies } (\forall y)(\text{if } x\mathsf{R}y \text{ then } \models_y^{\mathcal{M}} \alpha).$$

In that case, a classical negation could alternatively be defined by setting ${\sim}\alpha \stackrel{\text{def}}{=} \alpha \to (\alpha\wedge(\neg\alpha\wedge\circ\alpha))$. The significance of this 'consistency connective' in a modal language deprived of a paraconsistent negation was put into proof in Marcos (2005a), where an interpretation was proposed for it as a connective expressing the notion of an 'essential truth' —as opposed to a merely 'accidental' one.

On what concerns some of the usual inferences (recall (ID1)–(ID12), from Section 3) that interrelate the distinct connectives of the positive classical logic by way of the new modal paraconsistent negation presented heretofore, it should be noted that none of them is validated if one considers the semantics of the minimal normal modal logic K. However, both (ID1) and (ID4) are validated if one considers the reflexivity condition that characterizes the logic KT, while both (ID9) and (ID11) are validated given the symmetry condition that characterizes KB. None of the other inferences is valid inside a non-degenerate modal logic.[1] To validate any of the latter, some derivability adjustments (recall Section 1) are needed, with the help of the above defined consistency connective. Indeed, it might be noticed that the logic K can recover (ID2) and (ID8) by the addition of $\circ\alpha$ as a new 'consistency assumption', and it can recover (ID7) by the addition of $\circ{\sim}\alpha$ as such an assumption. Moreover, the logic KT can recover (ID5) by the addition of $\circ\beta$, and it can recover both (ID10) and (ID12) by the addition of both $\circ\alpha$ and $\circ\beta$. Finally, (ID6) can be recovered in KB by the addition of KB, and (ID3) can be recovered in $S4$ by the addition of both $\circ\alpha$ and $\circ\beta$.

There are of course other studies in which paraconsistent negations are endowed with modal interpretations, such as those involving the so-called 'Routley star', in the context of relevance logics, where a ternary accessibility relation is

[1] A mistake has thus remained in Béziau (2005), where (ID10) is said to be validated in $S5$.

used in giving truth conditions to some connectives (see Dunn, 1993). In comparison with those, the above straightforward interpretation of paraconsistent negation inside normal modal logics gains in simplicity what it loses in generality. In the present approach, at any rate, it has been shown that one could either start from the usual language of normal modal logics and define the paraconsistent-related connectives, or else start from the latter and then reintroduce the usual modal connectives. From that perspective, it should be clear to the reader that modal logics could alternatively be regarded as the study of certain modal-like inconsistency-tolerant systems. Instead of qualifying the truth of judgements in terms of belief or tense or duty or whatever other received adverbial expression, modal logic would have its role thus in the study of a more general 'theory of opposition', for the sake of those who believe that Aristotle is possibly not dead.

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