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MODE EXPANSION IN TIME DOMAIN FOR CONICAL LINES WITH ANGULAR MEDIUM INHOMOGENEITY

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Abstract—A new modification of the method of Mode Expansion in Time Domain is proposed for studying transient signals propagation in conical lines (including multi-connected ones) with inhomogeneous and time-dependent medium. The method is based on expanding the fields in spherical coordinate system into series of angular dependent modes with mode amplitudes being governed by a system of coupled evolutionary equations. The medium parameters (permittivity and permeability) are taken in a factorized form as a product of angular dependent factor and a factor that depends on time and radial coordinate. The introduced method can be applied to analysis of propagation and radiation in conical-like antennas with dielectric filling.

1. INTRODUCTION

Recently, a great interest has grown in studying transient electromagnetic phenomena with ultrawideband and ultra-short impulse signals. Due to very large frequency band of such signals using Frequency Domain (FD) methods becomes ineffective. That is why a number of Time Domain (TD) methods for solving transient problems emerge. Some of them are based on shifting known from FD methods into TD. Among them is the mode expansion concept. In regular structures the fields can be presented as a sum of independent modes [1]. This presentation can be directly transferred to the time domain as long as the modes are frequency independent, which is the case for homogeneous media only. In the case of transverse inhomogeneous but longitudinally regular structures the modes defined in FD become frequency dependent and thus cannot be directly converted into TD formulation.

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In this case the transition between FD and TD is not straightforward, and the consideration should be made completely in the TD.

The most well-known and universal method in TD that can be applied to the considered inhomogeneous regular structures is the method of Finite-Differences in Time-Domain (FDTD) [2–4]. Since FDTD is a completely numerical method any physical analysis of obtained results is rather complicated. Also, FDTD requires a lot of computational resources especially in case of 3D problems. That is why a more sophisticated method that can make use of structure regularity would be beneficial.

As a more sophisticated alternative the methods based on mode decomposition can be used. For the first time TD method was proposed by Kisunko in [5] for studying transient fields in homogeneous multi-connected waveguides with Perfect Electric Conductor (PEC) walls. The main idea of the method consists in presenting the sought fields in a waveguide as expansion over independent uncoupled modes with mode amplitudes being governed by some evolutionary equations describing evolution of the waveforms with propagation. Later, Tretyakov has formalized and advanced this approach as Mode Basis Method (MBM) or Evolutionary Approach to Electromagnetics [6]. In [7], the MBM was formulated for analysis of transient oscillations in a cavity with homogeneous time-dependent medium. Then in [8], the MBM was formulated for transient fields in a waveguide filled with longitudinally inhomogeneous and time-dependent medium ($\varepsilon = \varepsilon(z, t)$, $\mu = \mu(z, t)$). Such TD methods were used for studying transient oscillations in cavities filled with various media [9–15]. Mode expansion in TD for studying transient problems for cylindrical single- and multi-connected waveguides has also been used in [16–21].

If one moves the side walls of a cylindrical waveguide far away then it is possible to consider such a waveguide as free space. At that the discrete spectrum turns into a continuous one, and the fields are presented as integrals over Bessel modes. In such a way an impulse wavebeams in the free space or in time-dependent layered medium can be considered [22–25].

The mode expansion in time domain method has been applied to guiding wave problems in several possible geometries by Borisov, namely for cylindrical, conical, and sectorial waveguides [26–29]. A similar technique for spherical coordinate system in the free space, radial inhomogeneous medium, and conical lines has been used later by Shlivinski and Heyman [30]. The MBM in spherical coordinate system for time-dependent radial inhomogeneous medium ($\varepsilon = \varepsilon(r, t)$ and $\mu = \mu(r, t)$) was considered in [31, 32].

Further generalization should be aimed at considering both time-radial and angular inhomogeneity along with PEC conical lines. Such a problem in cylindrical waveguides has been studied by authors in [20]. Due to transverse inhomogeneity the modes are coupled in a waveguide, but in spite of that the mode decomposition converges very rapidly in this case [21]. In this paper we are going to apply a similar technique to the most general problem in spherical coordinates described above, i.e., with medium inhomogeneity of kind $\varepsilon(\vec{r},t) = \varepsilon_{\|}(r,t)\varepsilon_{\perp}(\theta,\varphi)$ and $\mu(\vec{r},t) = \mu_{\|}(r,t)\mu_{\perp}(\theta,\varphi)$. The proposed technique has been first reported in [33].

The method of mode expansion in time domain in spherical coordinates was applied to study impulse wave radiation by a bi-conical antenna in [34]. The method proposed in this paper gives the possibility to apply, similar to [34] technique, to analysis of transient radiation of a more general class of ultrawideband antennas like dielectric-filled TEM-horns, which will be the subject of further publications.

2. MODE BASIS IN SPHERICAL COORDINATE SYSTEM

2.1. Problem Statement

The problem geometry is shown in Fig. 1. We consider regular conical lines consisted of several or none PEC cones with a common apex. The cones cut multiple-connected domains on a sphere. The domains are bounded by the contours $L_1, L_2, \ldots$. The space in between the PEC cones is filled with medium described by constitutive relations that can be inhomogeneous, non-stationary and nonlinear. The waveforms

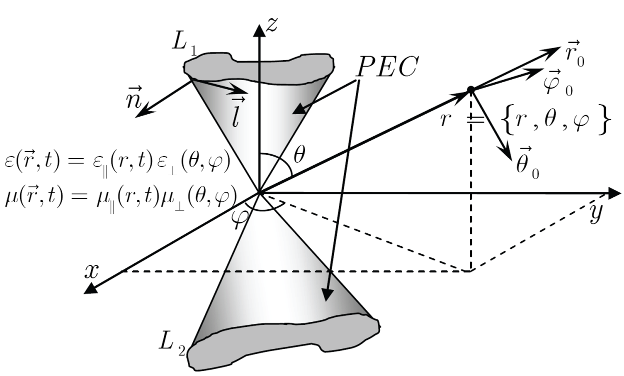

Figure 1. The problem geometry.

of transient electromagnetic fields excited by some specified electric charges and currents are of interest. The sought electromagnetic fields are governed by Maxwell equations:

$$\begin{aligned} &\mathrm{rot}\,\vec{\mathcal{H}} = \partial_t\vec{\mathcal{D}} + \vec{\mathcal{J}}_\sigma + \vec{\mathcal{J}}_0, \quad && -\mathrm{rot}\,\vec{\mathcal{E}} = \partial_t\vec{\mathcal{B}}, \\ &\mathrm{div}\,\vec{\mathcal{D}} = \rho_\sigma + \rho_0, && \mathrm{div}\,\vec{\mathcal{B}} = 0. \end{aligned} \tag{1}$$

Here $\vec{\mathcal{J}}_\sigma$ and ρ_σ are the conductivity current and charge densities; $\vec{\mathcal{J}}_0$ and ρ_0 are the impressed electric currents and charges. The Maxwell equations should be complemented with constitutive equations that can be presented in the following form:

$$\begin{aligned} \vec{\mathcal{D}}\left(\vec{\mathcal{E}}\right) &= \varepsilon_0\vec{\mathcal{E}} + \vec{\mathcal{P}}\left(\vec{\mathcal{E}}\right) = \varepsilon_0\varepsilon\vec{\mathcal{E}} + \vec{P}'\left(\vec{\mathcal{E}}\right), \\ \vec{P}\left(\vec{\mathcal{E}}\right) &= \varepsilon_0\alpha\vec{\mathcal{E}} + \vec{P}'\left(\vec{\mathcal{E}}\right), \quad \varepsilon = 1+\alpha; \\ \vec{\mathcal{B}}\left(\vec{\mathcal{H}}\right) &= \mu_0\left(\vec{\mathcal{H}} + \vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right)\right) = \mu_0\mu\vec{\mathcal{H}} + \mu_0\vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right), \\ \vec{\mathcal{M}}\left(\vec{\mathcal{H}}\right) &= \chi\vec{\mathcal{H}} + \vec{\mathcal{M}}'\left(\vec{\mathcal{H}}\right), \quad \mu = 1+\chi. \end{aligned} \tag{2}$$

ε_0 and μ_0 are the free-space permittivity and permeability respectively.

These equations describe isotropic nondispersive medium that can be inhomogeneous, nonstationary, and nonlinear. The polarization and magnetization are split into a linear part that can be presented by permittivity and permeability of the following factorized form:

$$\begin{aligned} \varepsilon(\vec{r},t) &= \varepsilon_{\|}(r,t)\,\varepsilon_{\perp}(\theta,\varphi); \\ \mu(\vec{r},t) &= \mu_{\|}(r,t)\,\mu_{\perp}(\theta,\varphi); \end{aligned} \quad \vec{r} = \{r,\theta,\varphi\} \tag{3}$$

and the rest of the polarization and magnetization that contains nonlinearity and not accounted inhomogeneity of a more general form; the latter is included into consideration as induced currents that depend on the total field (the primed terms in (2)). These induced sources are joined with the conductivity and impressed currents and charges. This yields the total electric and magnetic (with hats) currents and charges defined as follows:

$$\begin{aligned} &\vec{\mathcal{J}} = \partial_t\vec{\mathcal{P}}' + \vec{\mathcal{J}}_\sigma + \vec{\mathcal{J}}_0, \quad && \rho = -\mathrm{div}\,\vec{\mathcal{P}}' + \rho_\sigma + \rho_0, \quad && \mathrm{div}\,\vec{\mathcal{J}} + \partial_t\rho = 0; \\ &\hat{\vec{\mathcal{J}}} = \mu_0\partial_t\vec{\mathcal{M}}', && \hat{\rho} = -\mu_0\mathrm{div}\,\vec{\mathcal{M}}', && \mathrm{div}\,\hat{\vec{\mathcal{J}}} + \partial_t\hat{\rho} = 0. \end{aligned} \tag{4}$$

Each of the applied, conductive, and induced currents satisfies the continuity equation given above.

Substitution of constitutive relations (2)–(4) into Maxwell equation (1) results in:

$$\mathrm{rot}\,\vec{\mathcal{H}} = \partial_t\left(\varepsilon_0\varepsilon_{\|}\varepsilon_{\perp}\vec{\mathcal{E}}\right) + \vec{\mathcal{J}}, \quad -\mathrm{rot}\,\vec{\mathcal{E}} = \partial_t\left(\mu_0\mu_{\|}\mu_{\perp}\vec{\mathcal{H}}\right) + \hat{\vec{\mathcal{J}}}. \tag{5}$$

$$\mathrm{div}\,\left(\varepsilon_0\varepsilon_{\|}\varepsilon_{\perp}\vec{\mathcal{E}}\right) = \rho, \qquad \mathrm{div}\,\left(\mu_0\mu_{\|}\mu_{\perp}\vec{\mathcal{H}}\right) = \hat{\rho}. \tag{6}$$

On the PEC cones the following boundary conditions are satisfied:

$$\vec{l}\cdot\vec{\mathcal{E}}\Big|_L = 0; \quad \vec{n}\cdot\vec{\mathcal{H}}\Big|_L = 0; \quad \vec{r}_0\cdot\vec{\mathcal{E}}\Big|_L = 0 \tag{7}$$

where $\vec{n}$ is the normal to the cone; $\vec{l}$ is the unit tangent vector of the contour L; $\vec{r}_0$ is the ort of the radial coordinate (see Fig. 1).

2.2. Radial-angular Form of Maxwell Equations

Let us split all the vector quantities into an angular 2D vector and a radial component:

$$\vec{\mathcal{E}} = \vec{E} + \vec{r}_0 E_r; \quad \vec{\mathcal{H}} = \vec{H} + \vec{r}_0 H_r; \quad \vec{\mathcal{J}} = \vec{J} + \vec{r}_0 J_r; \quad \hat{\vec{\mathcal{J}}} = \hat{\vec{J}} + \vec{r}_0 \hat{J}_r \tag{8}$$

We also split the divergence and curl operators into angular and radial derivatives, at which the following vector operator will be used:

$$\nabla_\perp = \vec{\theta}_0\partial_\theta + \vec{\varphi}_0\frac{1}{\sin\theta}\partial_\varphi \tag{9}$$

This operator acts by usual rules of vector calculus, though the orts in spherical coordinate system $\vec{r}_0$, $\vec{\theta}_0$ and $\vec{\varphi}_0$ are dependent on angular variables that should be accounted for when applying the operator. In Appendix A, we give the coordinate form of different vector operations that involve operator (9).

Divergent Equation (6) written in angular-radial form yields:

$$\begin{aligned} r^{-2}\partial_r\left(r^2\varepsilon_0\varepsilon_\|\varepsilon_\perp E_r\right) + r^{-1}\nabla_\perp\cdot\left(\varepsilon_0\varepsilon_\|\varepsilon_\perp\vec{E}\right) &= \rho \\ r^{-2}\partial_r\left(r^2\mu_0\mu_\|\mu_\perp H_r\right) + r^{-1}\nabla_\perp\cdot\left(\mu_0\mu_\|\mu_\perp\vec{H}\right) &= \hat{\rho} \end{aligned} \tag{10}$$

Projection of curl Equation (5) onto the radial direction results in:

$$\begin{aligned} r^{-1}\nabla_\perp\cdot\left[\vec{H}\times\vec{r}_0\right] &= \partial_t\left(\varepsilon_0\varepsilon_\|\varepsilon_\perp E_r\right) + J_r \\ -r^{-1}\nabla_\perp\cdot\left[\vec{E}\times\vec{r}_0\right] &= \partial_t\left(\mu_0\mu_\|\mu_\perp H_r\right) + \hat{J}_r \end{aligned} \tag{11}$$

Angular part of curl Equation (5) can be written as follows:

$$\begin{aligned} -r^{-1}\left(\partial_r\left(r\left[\vec{H}\times\vec{r}_0\right]\right) + \left[\vec{r}_0\times\nabla_\perp H_r\right]\right) &= \partial_t\left(\varepsilon_0\varepsilon_\|\varepsilon_\perp\vec{E}\right) + \vec{J} \\ r^{-1}\left(\partial_r\left(r\left[\vec{E}\times\vec{r}_0\right]\right) + \left[\vec{r}_0\times\nabla_\perp E_r\right]\right) &= \partial_t\left(\mu_0\mu_\|\mu_\perp\vec{H}\right) + \hat{\vec{J}} \end{aligned} \tag{12}$$

Equations (10) and (11) can be used for elimination of the radial field components from Equation (12). In this way, we obtain the following second order equations that contain angular components only:

$$\begin{aligned}
&\left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \vec{H} \\
=\; & r^{-1} \mu_\parallel^{-1} \partial_r r^3 \mu_\parallel \left\{ \partial_t \left(\varepsilon_0 \varepsilon_\parallel \vec{E} \right) + r^{-1} \varepsilon_\perp^{-1} \partial_r \left[r \vec{H} \times \vec{r}_0 \right] \right\} \\
& + \varepsilon_\perp^{-1} \mu_\parallel^{-1} \left\{ r^{-1} \partial_r \left(r^3 \mu_\parallel \vec{J} \right) + r \left[\vec{r}_0 \times \nabla_\perp \mu_\perp^{-1} \mu_0^{-1} \hat{\rho} \right] \right\}
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\left[\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0\right] \nabla_\perp \cdot \varepsilon_\perp \vec{E} \\
=\; & r^{-1} \varepsilon_\parallel^{-1} \partial_r r^3 \varepsilon_\parallel \left\{ \partial_t \left(\mu_0 \mu_\parallel \vec{H} \right) + r^{-1} \mu_\perp^{-1} \partial_r \left[\vec{r}_0 \times r \vec{E} \right] \right\} \\
& + \mu_\perp^{-1} \varepsilon_\parallel^{-1} \left\{ r^{-1} \partial_r \left(r^3 \varepsilon_\parallel \hat{\vec{J}} \right) + r \left[\nabla_\perp \varepsilon_\perp^{-1} \varepsilon_0^{-1} \rho \times \vec{r}_0 \right] \right\}
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\nabla_\perp \mu_\perp^{-1} \left[\vec{r}_0 \times \nabla_\perp\right] \cdot \vec{E} \\
=\; & -\mu_0 r^2 \partial_t \mu_\parallel \left\{ r^{-1} \partial_r \left(r \vec{H} \right) + \varepsilon_0 \partial_t \left(\varepsilon_\parallel \varepsilon_\perp \left[\vec{r}_0 \times \vec{E} \right] \right) \right\} \\
& - r \left\{ \mu_0 r \partial_t \mu_\parallel \left[\vec{r}_0 \times \vec{J} \right] + \nabla_\perp \mu_\perp^{-1} \hat{J}_r \right\}
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\nabla_\perp \varepsilon_\perp^{-1} \left[\nabla_\perp \times \vec{r}_0\right] \cdot \vec{H} \\
=\; & -r^2 \partial_t \varepsilon_0 \varepsilon_\parallel \left\{ r^{-1} \partial_r \left(r \vec{E} \right) + \mu_0 \partial_t \left(\mu_\parallel \mu_\perp \left[\vec{H} \times \vec{r}_0 \right] \right) \right\} \\
& - r \left\{ \nabla_\perp \varepsilon_\perp^{-1} J_r + r \varepsilon_0 \partial_t \varepsilon_\parallel \left[\hat{\vec{J}} \times \vec{r}_0 \right] \right\}
\end{aligned} \tag{16}$$

In the expressions as $\nabla_\perp \mu_\perp^{-1} \ldots$, we assume that multiplication has priority over differentiation.

In elimination of the radial components we make essential use of the factorized form of the permittivity and permeability. For example, in order to derive Equation (14) we apply operator $r^{-2} \partial_r \left(r^2 \varepsilon_\parallel \varepsilon_\perp \ldots \right)$ to the second equation in (12), at which the term containing E_r is transformed as follows:

$$\begin{aligned}
& r^{-2} \partial_r \left(r^2 \varepsilon_\parallel \varepsilon_\perp \left[\vec{r}_0 \times \nabla_\perp E_r \right] \right) = \varepsilon_\perp r^{-2} \partial_r \left(r^2 \varepsilon_\parallel \left[\vec{r}_0 \times \nabla_\perp E_r \right] \right) \\
& = \varepsilon_\perp \left[\vec{r}_0 \times \nabla_\perp r^{-2} \partial_r r^2 \varepsilon_\parallel E_r \right] = \varepsilon_\perp \left[\vec{r}_0 \times \nabla_\perp r^{-2} \partial_r r^2 \varepsilon_\perp^{-1} \varepsilon_\perp \varepsilon_\parallel E_r \right] \\
& = \varepsilon_\perp \left[\vec{r}_0 \times \nabla_\perp \varepsilon_\perp^{-1} \left(r^{-2} \partial_r r^2 \varepsilon_\perp \varepsilon_\parallel E_r \right) \right]
\end{aligned}$$

The expression in round brackets is then substituted from the first of Equation (10).

The boundary conditions (7) in the radial-angular form look as follows:

$$\begin{aligned}
\vec{l} \cdot \vec{E} \Big|_L = 0, \qquad & \vec{n} \cdot \vec{H} \Big|_L = 0, \\
\nabla_\perp \cdot \left(\varepsilon_\perp \vec{E} \right) \Big|_L = 0, \quad & \nabla_\perp \cdot \left[\vec{H} \times \vec{r}_0 \right] \Big|_L = 0
\end{aligned} \tag{17}$$

In deriving the second pair of these conditions we exploit Equations (10), (11) and the fact that there are no impressed charges and impressed radial currents on the PEC surfaces [8].

2.3. Angular Derivatives Linear Operators for the Eigenvalue Problems

Let us unite the transverse vectors $\vec{E}$ and $\vec{H}$ into one 4-dimensional vector $X = \mathrm{col}\,(\vec{E}, \vec{H})$. Then let $L_2^4\,(S, \gamma)$ be the Hilbert space of such 4-dimensional real functions that satisfy boundary conditions (17) and are square-integrable with weight $\gamma = \{\varepsilon_\perp, \mu_\perp\}$. The dot product in $L_2^4\,(S, \gamma)$ is defined as integral over the full sphere (dS is a solid angle):

$$(X_1, X_2) = \frac{1}{4\pi} \int_S \left(\varepsilon_\perp \vec{E}_1 \cdot \vec{E}_2 + \mu_\perp \vec{H}_1 \cdot \vec{H}_2\right) dS \tag{18}$$

The normalizing coefficient 4π is the full solid angle and is introduced here for consistency with traditional normalization of spherical harmonics.

The set of Equations (13)–(16) complemented with boundary conditions (17) can be written in operator form. To this aim let's extract from Equations (13) and (15) the angular derivatives and combine them into a linear operator $W_H : \mathscr{D}\,(W_H) \to L_2^4\,(S, \gamma)$ defined as:

$$\begin{gathered} W_H = \begin{pmatrix} 0 & \left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \\ \nabla_\perp \mu_\perp^{-1} \left[\vec{r}_0 \times \nabla_\perp\right] \cdot & 0 \end{pmatrix} \\ \mathscr{D}(W_H) = \left\{ X \in L_2^4\,(S, \gamma) : W_H X \in L_2^4\,(S, \gamma)\,, \vec{l} \cdot \vec{E}\Big|_L = 0, \vec{n} \cdot \vec{H}\Big|_L = 0 \right\} \end{gathered} \tag{19}$$

In a similar way Equations (14) and (16) yield another linear operator $W_E : \mathscr{D}\,(W_E) \to L_2^4\,(S, \gamma)$:

$$\begin{gathered} W_E = \begin{pmatrix} 0 & \nabla_\perp \varepsilon_\perp^{-1} \left[\nabla_\perp \times \vec{r}_0\right] \cdot \\ \left[\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0\right] \nabla_\perp \cdot \varepsilon_\perp & 0 \end{pmatrix} \\ \mathscr{D}(W_E) = \Big\{ X \in L_2^4\,(S, \gamma) : W_E X \in L_2^4\,(S, \gamma)\,, \nabla_\perp \cdot \left(\varepsilon_\perp \vec{E}\right)\Big|_L = 0, \\ \nabla_\perp \cdot \left[\vec{H} \times \vec{r}_0\right]\Big|_L = 0 \Big\} \end{gathered} \tag{20}$$

Thus, the result of action of these operators is defined as:

$$\begin{aligned} W_H X &= \begin{pmatrix} 0 & \left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \\ \nabla_\perp \mu_\perp^{-1} \left[\vec{r}_0 \times \nabla_\perp\right] \cdot & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \\ &= \begin{pmatrix} \left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \vec{H} \\ \nabla_\perp \mu_\perp^{-1} \left[\vec{r}_0 \times \nabla_\perp\right] \cdot \vec{E} \end{pmatrix} \end{aligned} \tag{21}$$

$$W_E X = \begin{pmatrix} 0 & \nabla_\perp \varepsilon_\perp^{-1} \left[\nabla_\perp \times \vec{r}_0\right] \cdot \\ \left[\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0\right] \nabla_\perp \cdot \varepsilon_\perp & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$
$$= \begin{pmatrix} \nabla_\perp \varepsilon_\perp^{-1} \left[\nabla_\perp \times \vec{r}_0\right] \cdot \vec{H} \\ \left[\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0\right] \nabla_\perp \cdot \varepsilon_\perp \vec{E} \end{pmatrix} \quad (22)$$

Now the set of Equations (13)–(17) can be written in a more compact operator form with transverse (angular) derivatives on the left hand side and radial and time derivatives and the sources on the right hand side:

$$W_H X = \begin{pmatrix} r^{-1} \mu_\parallel^{-1} \partial_r r^3 \mu_\parallel \left\{ \partial_t \left(\varepsilon_0 \varepsilon_\parallel \vec{E} \right) + r^{-1} \varepsilon_\perp^{-1} \partial_r \left[r \vec{H} \times \vec{r}_0 \right] \right\} \\ + \varepsilon_\perp^{-1} \mu_\parallel^{-1} \left\{ r^{-1} \partial_r \left(r^3 \mu_\parallel \vec{J} \right) + r \left[\vec{r}_0 \times \nabla_\perp \mu_\perp^{-1} \mu_0^{-1} \hat{\rho} \right] \right\} \\ \\ -\mu_0 r^2 \partial_t \mu_\parallel \left\{ r^{-1} \partial_r \left(r \vec{H} \right) + \varepsilon_0 \partial_t \left(\varepsilon_\parallel \varepsilon_\perp \left[\vec{r}_0 \times \vec{E} \right] \right) \right\} \\ -r \left\{ \mu_0 r \partial_t \mu_\parallel \left[\vec{r}_0 \times \vec{J} \right] + \nabla_\perp \mu_\perp^{-1} \hat{J}_r \right\} \end{pmatrix} \quad (23)$$

$$W_E X = \begin{pmatrix} -r^2 \partial_t \varepsilon_0 \varepsilon_\parallel \left\{ r^{-1} \partial_r \left(r \vec{E} \right) + \mu_0 \partial_t \left(\mu_\parallel \mu_\perp \left[\vec{H} \times \vec{r}_0 \right] \right) \right\} \\ -r \left\{ \nabla_\perp \varepsilon_\perp^{-1} J_r + r \varepsilon_0 \partial_t \varepsilon_\parallel \left[\hat{\vec{J}} \times \vec{r}_0 \right] \right\} \\ \\ r^{-1} \varepsilon_\parallel^{-1} \partial_r r^3 \varepsilon_\parallel \left\{ \partial_t \left(\mu_0 \mu_\parallel \vec{H} \right) + r^{-1} \mu_\perp^{-1} \partial_r \left[\vec{r}_0 \times r \vec{E} \right] \right\} \\ + \mu_\perp^{-1} \varepsilon_\parallel^{-1} \left\{ r^{-1} \partial_r \left(r^3 \varepsilon_\parallel \hat{\vec{J}} \right) + r \left[\nabla_\perp \varepsilon_\perp^{-1} \varepsilon_0^{-1} \rho \times \vec{r}_0 \right] \right\} \end{pmatrix} \quad (24)$$

Note that (23) is equivalent to Equations (13), (15) and (17), while Equation (24) is equivalent to Equations (14), (16) and (17).

The following analog of Green's formula can be derived for integrals over sphere as defined in (18) (See detailed derivation in Appendices B, C):

$$\int_S \left(\vec{A} \cdot \nabla_\perp f \nabla_\perp \cdot \vec{B} - \vec{B} \cdot \nabla_\perp f \nabla_\perp \cdot \vec{A} \right) dS$$
$$= \int_L \left[\left(\vec{n} \cdot \vec{A} \right) \left(\nabla_\perp \cdot \vec{B} \right) - \left(\vec{n} \cdot \vec{B} \right) \left(\nabla_\perp \cdot \vec{A} \right) \right] f \, dl \quad (25)$$

Using this formula one can prove that the introduced operators are symmetric ones in $L_2^4(S, \gamma)$. The symmetry of operators W_H and W_E means that for any $X_1, X_2 \in L_2^4(S, \gamma)$ the following equalities hold (detailed derivation is given in Appendix D):

$$(W_H X_1, X_2) = (X_1, W_H X_2); \quad (W_E X_1, X_2) = (X_1, W_E X_2). \quad (26)$$

Further consideration is aimed at constructing a basis in the whole Hilbert space $L_2^4(S,\gamma)$. It is clear that the angular part of any physically meaningful solution to Maxwell Equations (5), (6) complemented with boundary conditions (7) belongs to the introduced Hilbert space $L_2^4(S,\gamma)$.

Let L^H be the range of operator W_H. Then the kernel of operator W_H is the orthogonal complement to linear subspace L^H in L_2^4. That is, $\left(L^H\right)^\perp = \ker W_H$, and thus $L_2^4(S,\gamma) = L^H \oplus \ker W_H$. Now let L^E be the range of operator W_E, then by analogy $L_2^4(S,\gamma) = L^E \oplus \ker W_E$. It can be proved that $L^H \subset \ker W^E$ and $L^E \subset \ker W^H$, the derivation is similar to that in Appendix D and is left as an exercise for our reader. Now let's define L^T as an intersection of the operators' kernels, i.e., $L^T \equiv \ker W^H \cap \ker W^E$. Finally, based on this definition and taking into consideration the above embeddings the Hilbert space $L_2^4(S,\gamma)$ can be decomposed into direct sum of three orthogonal subspaces $L_2^4(S,\gamma) = L^H \oplus L^E \oplus L^T$. Later, it will be shown that the introduced subspaces L^H, L^E, and L^T correspond to fields of TE-, TM-, and TEM-type respectively. The subspace L^T is empty or has finite dimensionality equal to $N-1$ if the PEC cones contour L is N-connected.

2.4. Basis of Eigenmodes

Let's start our consideration with operator W_H. In the previous subsection we established its symmetry; beside others it means that its spectrum is real-valued, and its eigenfunctions constitute a basis in the range of the operator, which we denoted as subspace $L^H \subset L_2^4(S,\gamma)$. The spectrum is discrete since the operator is of elliptic type and is defined on a finite domain. Additionally, the spectrum is symmetric relative to zero because of the block structure with zero blocks on the main diagonal. It means that if p_m^2 is an eigenvalue that corresponds to eigenfunction $X_m = \mathrm{col}\left(\vec{E}_m, \vec{H}_m\right)$ then $p_{-m}^2 = -p_m^2$ is also an eigenvalue that corresponds to eigenfunction $X_{-m} = \mathrm{col}\left(\vec{E}_m, -\vec{H}_m\right)$. At the end we are interested in expanding individual fields $\vec{E}$ and $\vec{H}$ over the mode functions. Since eigenfunctions X_m and X_{-m} constitute a basis, then $(X_m + X_{-m})/2 = \mathrm{col}\left(\vec{E}_m, 0\right)$ and $(X_m - X_{-m})/2 = \mathrm{col}\left(0, \vec{H}_m\right)$ also constitute a basis. Hence the mode functions $\vec{E}_m$ and $\vec{H}_m$ that correspond to positive eigenvalues $p_m^2 > 0$ can be used for expansion of $\vec{E}$ and $\vec{H}$ fields correspondingly. That is why we write the eigenvalues squared p_m^2. Similar consideration is also relevant to operator W_E.

So, the eigenvalue problem for operator W_H has the following form:

$$W_H X_m^H = p_m^2 X_m^H,\ X_m^H = \begin{pmatrix} \vec{E}_m^H \\ \vec{H}_m^H \end{pmatrix} \Rightarrow \begin{cases} [\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}] \nabla_\perp \cdot \mu_\perp \vec{H}_m^H = p_m^2 \vec{E}_m^H \\ \nabla_\perp \mu_\perp^{-1} [\vec{r}_0 \times \nabla_\perp] \cdot \vec{E}_m^H = p_m^2 \vec{H}_m^H \\ \vec{l} \cdot \vec{E}_m^H \Big|_L = 0,\ \vec{n} \cdot \vec{H}_m^H \Big|_L = 0 \end{cases} \tag{27}$$

This vector eigenvalue boundary problem can be reduced to a simpler scalar problem. To this aim let's introduce scalar functions Φ_m^H and Ψ_m^H that are related with the corresponding vector eigenfunctions $\vec{E}_m^H$ and $\vec{H}_m^H$ as follows:

$$\vec{E}_m^H = p_m^{-1} \varepsilon_\perp^{-1} \left[\nabla_\perp \Phi_m^H \times \vec{r}_0 \right]; \quad \vec{H}_m^H = p_m^{-1} \nabla_\perp \Psi_m^H. \tag{28}$$

Substituting (28) into (27) yields the following scalar boundary eigenvalue problem:

$$\begin{cases} \nabla_\perp \cdot \mu_\perp \nabla_\perp \Psi_m^H + p_m^2 \mu_\perp \Phi_m^H = 0 \\ \nabla_\perp \cdot \varepsilon_\perp^{-1} \nabla_\perp \Phi_m^H + p_m^2 \mu_\perp \Psi_m^H = 0 \\ \frac{\partial \Phi_m^H}{\partial \vec{n}} \Big|_L = 0 \qquad \frac{\partial \Psi_m^H}{\partial \vec{n}} \Big|_L = 0 \end{cases} \tag{29}$$

Substituting vector eigenfunctions $\vec{E}_m^H$ and $\vec{H}_m^H$ in the form (28) into formulae (10) and (11) one can easily find that in case of no sources the radial component of electric field vanishes $E_r^H \equiv 0$. Hence, we have established that the angular components of electromagnetic field from subspace L^H correspond to TE-waves.

Similar consideration for operator W_E yields the following vector boundary eigenvalue problem:

$$W_E X_n^E = q_n^2 X_n^E,$$
$$X_n^E = \begin{pmatrix} \vec{E}_n^E \\ \vec{H}_n^E \end{pmatrix} \Rightarrow \begin{cases} \nabla_\perp \varepsilon_\perp^{-1} [\nabla_\perp \times \vec{r}_0] \cdot \vec{H}_n^E = q_n^2 \vec{E}_n^E \\ [\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0] \nabla_\perp \cdot \varepsilon_\perp \vec{E}_n^E = q_n^2 \vec{H}_n^E \\ \nabla_\perp \cdot \left(\varepsilon_\perp \vec{E}_n^E \right) \Big|_L = 0,\ \nabla_\perp \cdot \left[\vec{H}_n^E \times \vec{r}_0 \right] \Big|_L = 0 \end{cases} \tag{30}$$

that can be reduced to a scalar problem by introducing scalar eigenfunctions Ψ_n^E and Φ_n^E:

$$\vec{E}_n^E = q_n^{-1} \nabla_\perp \Psi_n^E, \quad \vec{H}_n^E = q_n^{-1} \mu_\perp^{-1} \left[\vec{r}_0 \times \nabla_\perp \Phi_n^E \right] \tag{31}$$

Substituting this form into (30) results in the following scalar boundary eigenvalue problem:

$$\begin{cases} \nabla_\perp \cdot \mu_\perp^{-1} \nabla_\perp \Phi_n^E + q_n^2 \varepsilon_\perp \Psi_n^E = 0 \\ \nabla_\perp \cdot \varepsilon_\perp \nabla_\perp \Psi_n^E + q_n^2 \varepsilon_\perp \Phi_n^E = 0 \\ \Psi_n^E \big|_L = 0 \quad \Phi_n^E \big|_L = 0 \end{cases} \tag{32}$$

It can also be shown that the eigenfunctions in the form (31) being substituted into (10) and (11) results in vanishing radial component of magnetic field $H_r^H \equiv 0$ provided that there are no sources. Thus, the angular components of electromagnetic field from subspace L^E correspond to TM-waves.

In order to construct a basis in subspace L^T we should consider problems (27) and (30) jointly assuming that the eigenvalue is equal to zero (that corresponds to kernels of the operators):

$$\begin{cases} W_H X_k^T = 0 \\ W_E X_k^T = 0 \end{cases},$$

$$X_k^T = \begin{pmatrix} \vec{E}_k^T \\ 0 \end{pmatrix} \cup \begin{pmatrix} 0 \\ \vec{H}_k^T \end{pmatrix} \Rightarrow \begin{matrix} \begin{cases} \left[\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0\right] \nabla_\perp \cdot \varepsilon_\perp \vec{E}_k^T = 0 \\ \nabla_\perp \mu_\perp^{-1} \left[\vec{r}_0 \times \nabla_\perp\right] \cdot \vec{E}_k^T = 0 \\ \vec{l} \cdot \vec{E}_k^T \Big|_L = 0 \quad \nabla_\perp \cdot \left(\varepsilon_\perp \vec{E}_k^T\right) \Big|_L = 0 \end{cases} \\ \begin{cases} \left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \vec{H}_k^T = 0 \\ \nabla_\perp \varepsilon_\perp^{-1} \left[\nabla_\perp \times \vec{r}_0\right] \cdot \vec{H}_k^T = 0 \\ \vec{n} \cdot \vec{H}_k^T \Big|_L = 0 \quad \nabla_\perp \cdot \left[\vec{H}_k^T \times \vec{r}_0\right] \Big|_L = 0 \end{cases} \end{matrix} \tag{33}$$

Acting similarly to consideration in paper [8] one can prove that each subsystem in (33) has exactly $N-1$ linearly independent solutions, where N is the number of contours bounding the PEC cones (connectivity of contour L). Similar to the previous consideration, we introduce scalar eigenfunctions Φ_k^T and Ψ_k^T as:

$$\vec{E}_k^T = \nabla_\perp \Phi_k^T, \quad \vec{H}_k^T = \left[\vec{r}_0 \times \mu_\perp^{-1} \nabla_\perp \Psi_k^T\right]. \tag{34}$$

This results in two scalar eigenvalue boundary problems with inhomogeneous boundary conditions:

$$\begin{cases} \nabla_\perp \cdot \varepsilon_\perp \nabla_\perp \Phi_k^T = 0 \\ \Phi_k^T \big|_{L_j} = c_k^j \end{cases} \qquad \begin{cases} \nabla_\perp \cdot \mu_\perp^{-1} \nabla_\perp \Psi_k^T = 0 \\ \Psi_k^T \big|_{L_j} = d_k^j \end{cases} \tag{35}$$

where c_k^j and d_k^j are some constants at the jth contour L_j for the kth eigenfunction. In order to find all $N-1$ linearly independent solutions it is convenient to chose $c_k^N = 0$, $c_k^j = \delta_{kj}$ for $k, j = \overline{1, N-1}$ and similarly for d_k^j.

These eigenfunctions in form (34) being substituted into Equations (10) and (11) taking into consideration (35) result in vanishing radial components of both electric and magnetic fields $E_r^T \equiv H_r^T \equiv 0$, thus subspace L^T corresponds to TEM-waves as well as to electrostatic and magnetostatic fields that can exist in such multi-connected transmission lines. The static fields can be presented as a superposition of contradirectional TEM-modes.

2.5. Orthogonality Conditions

At the beginning of the previous subsection we have shown that the block structure of the operators allows us to treat electric $\vec{E}$ and magnetic $\vec{H}$ components of the eigenfunctions separately. The Hilbert space of 4-dimensional vectors $L_2^4(S,\gamma)$ can be presented as exterior product of Hilbert spaces of "electric" and "magnetic" vectors $L_2^4(S,\gamma) = L_2^2(S,\varepsilon) \otimes L_2^2(S,\mu)$. The dot product in electric vector functional space $L_2^2(S,\varepsilon)$ is defined with weight $\varepsilon_\perp$:

$$\left(\vec{E}_1, \vec{E}_2\right)_e = \frac{1}{4\pi} \int_S \varepsilon_\perp \vec{E}_1 \cdot \vec{E}_2 dS \tag{36}$$

while in the magnetic vectors functional space $L_2^2(S,\mu)$ we use weight $\mu_\perp$:

$$\left(\vec{H}_1, \vec{H}_2\right)_h = \frac{1}{4\pi} \int_S \mu_\perp \vec{H}_1 \cdot \vec{H}_2 dS \tag{37}$$

Thus, the original dot product (18) in $L_2^4(S,\gamma) = L_2^2(S,\varepsilon) \otimes L_2^2(S,\mu)$ can be presented as:

$$(X_1, X_2) = \left(\vec{E}_1, \vec{E}_2\right)_e + \left(\vec{H}_1, \vec{H}_2\right)_h \tag{38}$$

From symmetry of operators W_H, W_E and the fact that $L^H \subset \operatorname{Ker} W_E$, $L^E \subset \operatorname{Ker} W_H$, $L^T = \operatorname{Ker} W_E \cap \operatorname{Ker} W_H$ the following orthogonality conditions follow for the eigenfunctions of the eigenvalue boundary problems (27), (30), and (33):

$$\left(\vec{E}_m^A, \vec{E}_n^B\right)_e = \delta_{mn}\delta_{AB}, \ \left(\vec{H}_m^A, \vec{H}_n^B\right)_h = \delta_{mn}\delta_{AB}, \ A, B \in \{H, E, T\} \tag{39}$$

Substituting the expressions of vector basis functions via scalar basis functions (28) into (39) one can establish bi-orthogonality relations for the scalar basis functions Φ_m^H and Ψ_m^H:

$$\left(\Psi_m^H, \Phi_n^H\right)_h = \frac{1}{4\pi} \int_S \mu_\perp \Psi_m^H \Phi_n^H dS = \delta_{mn} \tag{40}$$

It can be proved that functions $\left\{\Psi_m^H, \Phi_n^H\right\}$ constitute a bi-orthogonal basis in the Hilbert space $L_2^1(S,\mu)$ of scalar real-valued functions satisfying Neumann conditions at contour L with inner product defined by (40).

Similarly, it can be shown that the scalar basis functions Φ_n^E and Ψ_n^E form a bi-orthogonal basis in space $L_2^1(S,\varepsilon)$ of scalar real-valued functions satisfying Dirichlet conditions at contour L. Substitution of (31) into (39) results in the following bi-orthogonality conditions and a definition for inner product in $L_2^1(S,\varepsilon)$:

$$\left(\Psi_m^E, \Phi_n^E\right)_e = \frac{1}{4\pi} \int_S \varepsilon_\perp \Psi_m^E \Phi_n^E dS = \delta_{mn} \tag{41}$$

2.6. Expansion of the Sought Fields and Sources over the Mode Basis

We have constructed a basis in each subspace of L_2^4 and thereby have a full set of modes for expanding the sought fields, initial conditions, and source functions. Thus, the angular part of the sought fields can be presented as an expansion over the introduced vector mode functions:

$$\begin{aligned}\varepsilon_0^{1/2}\vec{E}(r,\theta,\varphi,t) &= r^{-1}\sum_{A\in\{H,E,T\}}\sum_m e_m^A(r,t)\,\vec{E}_m^A(\theta,\varphi)\\ &= r^{-1}\left(\sum_m e_m^H(r,t)\,\vec{E}_m^H(\theta,\varphi)+\sum_n e_n^E(r,t)\,\vec{E}_n^E(\theta,\varphi)\right.\\ &\left.+\sum_k e_k^T(r,t)\,\vec{E}_k^T(\theta,\varphi)\right)\end{aligned}\tag{42}$$

$$\begin{aligned}\mu_0^{1/2}\vec{H}(r,\theta,\varphi,t) &= r^{-1}\sum_{A\in\{H,E,T\}}\sum_m h_m^A(r,t)\,\vec{H}_m^A(\theta,\varphi)\\ &= r^{-1}\left(\sum_m h_m^H(r,t)\,\vec{H}_m^H(\theta,\varphi)+\sum_n h_n^E(r,t)\,\vec{H}_n^E(\theta,\varphi)\right.\\ &\left.+\sum_k h_k^T(r,t)\,\vec{H}_k^T(\theta,\varphi)\right)\end{aligned}\tag{43}$$

The radial component of electric field E_r is a scalar real-valued function satisfying Dirichlet conditions at L. Therefore, it belongs to the space $L_2^1(S,\varepsilon)$. That is why we can expand E_r in terms of basis functions $\{\Phi_n^E\}$ with $\{\Psi_n^E\}$ being projectors. Thus, we have:

$$\varepsilon_0^{1/2}E_r(r,\theta,\varphi,t) = r^{-2}\sum_n e_n^r(r,t)\,q_n\Phi_n^E(\theta,\varphi)\tag{44}$$

The eigenvalues q_n are introduced into this formula for convenience in further derivation. Similarly, the magnetic field radial component H_r belongs to the space $L_2^1(S,\mu)$ and can be expanded over eigenfunctions $\{\Phi_m^H\}$:

$$\mu_0^{1/2}H_r(r,\theta,\varphi,t) = r^{-2}\sum_m h_m^r(r,t)\,p_m\Phi_m^H(\theta,\varphi)\tag{45}$$

The expansion coefficients $e_m^H, e_n^E, e_k^T, h_m^H, h_n^E, h_k^T, e_n^r, h_m^r$ in (42)–(45) depend on radial coordinate and time. Let's name them as mode amplitudes (angular and radial correspondingly).

2.7. Projection of Initial Conditions onto the Mode Basis

The problem under study (1) should also be supplemented with some (probably zero) initial conditions $\vec{\mathcal{H}}\Big|_{t=0} = \vec{\mathcal{H}}_0(\vec{r})$, $\vec{\mathcal{E}}\Big|_{t=0} = \vec{\mathcal{E}}_0(\vec{r})$. By projecting these fields onto the mode basis we obtain initial conditions for the mode amplitudes. Using orthogonality conditions (39)–(41) and definitions of mode expansions (42)–(45) one can find the following expressions for the mode amplitudes at $t = 0$:

$$\begin{aligned} e_m^A(r,0) &= \varepsilon_0^{1/2}\tfrac{r}{4\pi}\textstyle\int_S \vec{E}(r,\theta,\varphi,0)\cdot\vec{E}_m^A(\theta,\varphi)\varepsilon_\perp dS \\ h_m^A(r,0) &= \mu_0^{1/2}\tfrac{r}{4\pi}\textstyle\int_S \vec{H}(r,\theta,\varphi,0)\cdot\vec{H}_m^A(\theta,\varphi)\mu_\perp dS \quad A\in\{H,E,T\} \\ e_n^r(r,0) &= \varepsilon_0^{1/2}\tfrac{r^2}{4\pi}\textstyle\int_S E_r(r,\theta,\varphi,0)\,\Psi_n^E(\theta,\varphi)q_n^{-1}\varepsilon_\perp dS \\ h_m^r(r,0) &= \mu_0^{1/2}\tfrac{r^2}{4\pi}\textstyle\int_S H_r(r,\theta,\varphi,0)\,\Psi_m^H(\theta,\varphi)p_m^{-1}\mu_\perp dS \end{aligned} \tag{46}$$

3. SYSTEM OF EVOLUTIONARY WAVEGUIDE EQUATIONS

Next, we need to obtain governing equations for the mode amplitudes. To this aim we substitute the expansions of angular field components (42), (43) into the Maxwell equations in radial-angular form (13)–(16). The obtained functional equations are then projected onto the basis of angular vector basis functions using the defined dot product (36) or (37) in order to obtain a system of PDE for the angular mode amplitudes. Due to orthogonality conditions (39) some of the matrices resulted from the projection are identity matrices. Since the expansion functions are the eigenfunctions of the angular derivative operators some coefficient matrices are diagonal matrices of eigenvalues. Finally, due to presence of $\varepsilon_\perp, \mu_\perp$ in the right-hand side of (13)–(16) some coefficient matrices (L, K) are full, and their definitions are given later (see (60)). As a result, the following System of Evolutionary Waveguide Equations (SEWE) has been derived:

$$\partial_r r^2 \mu_\| \left\{ \partial_\tau \left(\varepsilon_\| e_m^H\right) + \sum_{A\in\{H,E,T\}} \sum_{m'} L_{mm'}^{HA} \partial_r h_{m'}^A \right\} - \mu_\| p_m^2 h_m^H$$
$$= \tfrac{1}{4\pi} \textstyle\int_S \vec{f}_1(\vec{r},t)\cdot\vec{E}_m^H dS \tag{47}$$

$$\partial_r r^2 \varepsilon_\| \left\{ \partial_\tau \left(\mu_\| h_n^E\right) + \sum_{A\in\{H,E,T\}} \sum_{n'} L_{n'n}^{AE} \partial_r e_{n'}^A \right\} - \varepsilon_\| q_n^2 e_n^E$$
$$= \tfrac{1}{4\pi} \textstyle\int_S \vec{f}_2(\vec{r},t)\cdot\vec{H}_n^E dS \tag{48}$$

$$r^2 \partial_\tau \mu_\parallel \left\{ \partial_r h_m^H + \sum_{A \in \{H,E,T\}} \sum_{m'} K_{m'm}^{AH} \partial_\tau \left(\varepsilon_\parallel e_{m'}^A \right) \right\} + p_m^2 e_m^H$$
$$= \tfrac{1}{4\pi} \int_S \vec{f}_3 \left(\vec{r}, t \right) \cdot \vec{H}_m^H dS \tag{49}$$

$$r^2 \partial_\tau \varepsilon_\parallel \left\{ \partial_r e_n^E + \sum_{A \in \{H,E,T\}} \sum_{n'} K_{nn'}^{EA} \partial_\tau \left(\mu_\parallel h_{n'}^A \right) \right\} + q_n^2 h_n^E$$
$$= \tfrac{1}{4\pi} \int_S \vec{f}_4 \left(\vec{r}, t \right) \cdot \vec{E}_n^E dS \tag{50}$$

$$\partial_r r^2 \mu_\parallel \left\{ \partial_\tau \left(\varepsilon_\parallel e_k^T \right) + \sum_{A \in \{H,E,T\}} \sum_{k'} L_{kk'}^{TA} \partial_r h_{k'}^A \right\} = \tfrac{1}{4\pi} \int_S \vec{f}_1 (\vec{r}, t) \cdot \vec{E}_k^T dS \tag{51}$$

$$\partial_r r^2 \varepsilon_\parallel \left\{ \partial_\tau \left(\mu_\parallel h_k^T \right) + \sum_{A \in \{H,E,T\}} \sum_{k'} L_{k'k}^{AT} \partial_r e_{k'}^A \right\} = \tfrac{1}{4\pi} \int_S \vec{f}_2 (\vec{r}, t) \cdot \vec{H}_k^T dS \tag{52}$$

$$\partial_\tau \mu_\parallel \left\{ \partial_r h_k^T + \sum_{A \in \{H,E,T\}} \sum_{k'} K_{k'k}^{AT} \partial_\tau \left(\varepsilon_\parallel e_{k'}^A \right) \right\} = r^{-2} \tfrac{1}{4\pi} \int_S \vec{f}_3 (\vec{r}, t) \cdot \vec{H}_k^T dS \tag{53}$$

$$\partial_\tau \varepsilon_\parallel \left\{ \partial_r e_k^T + \sum_{A \in \{H,E,T\}} \sum_{k'} K_{kk'}^{TA} \partial_\tau \left(\mu_\parallel h_{k'}^A \right) \right\} = r^{-2} \tfrac{1}{4\pi} \int_S \vec{f}_4 \left(\vec{r}, t \right) \cdot \vec{E}_k^T dS \tag{54}$$

Then we exclude radial components from Equations (10), (11) and substitute there the field expansions (42), (43). We express the vector basis functions via scalar basis functions (28), (31), (34). Finally, these scalar functional equations are projected onto the corresponding bi-orthogonal scalar basis functions using a proper dot product definition (40), (41) based on the type of the boundary conditions that correspond to the equation terms. As a result, we obtained another 4 evolutionary equations:

$$\sum_{m'} L_{m'm}^{HH} \partial_r e_{m'}^H + \partial_\tau \left(\mu_\parallel h_m^H \right) = r \tfrac{1}{4\pi} \int_S \left(\nabla_\perp \cdot \varepsilon_0^{1/2} \hat{\vec{J}} \right) p_m^{-1} \Psi_m^H dS \tag{55}$$

$$\partial_r e_m^H + \sum_{m'} K_{mm'}^{HH} \partial_\tau \left(\mu_\parallel h_{m'}^H \right) = r \tfrac{1}{4\pi} \int_S \left(\nabla_\perp \cdot \varepsilon_0^{1/2} \hat{\vec{J}} \right) p_m^{-1} \Phi_m^H dS \tag{56}$$

$$\sum_{n'} L_{nn'}^{EE} \partial_r h_{n'}^E + \partial_\tau \left(\varepsilon_\parallel e_n^E \right) = r \tfrac{1}{4\pi} \int_S \left(\nabla_\perp \cdot \mu_0^{1/2} \vec{J} \right) q_n^{-1} \Psi_n^E dS \tag{57}$$

$$\partial_r h_n^E + \sum_{n'} K_{n'n}^{EE} \partial_\tau \left(\varepsilon_\parallel e_{n'}^E \right) = r \tfrac{1}{4\pi} \int_S \left(\nabla_\perp \cdot \mu_0^{1/2} \vec{J} \right) q_n^{-1} \Phi_n^E dS \tag{58}$$

Here $\tau = ct$, c is the velocity of light in the free space. The source functions in the right-hand side of Equations (47)–(54) are defined as follows:

$$\begin{aligned}
\vec{f}_1(\vec{r},t) &= r^2\left[\nabla_\perp \mu_\perp^{-1}\mu_0^{-1/2}\hat{\rho}\times\vec{r}_0\right] - \partial_r\left(r^3\mu_\parallel\mu_0^{1/2}\vec{J}\right)\\
\vec{f}_2(\vec{r},t) &= r^2\left[\vec{r}_0\times\nabla_\perp\varepsilon_\perp^{-1}\varepsilon_0^{-1/2}\rho\right] - \partial_r\left(r^3\varepsilon_\parallel\varepsilon_0^{1/2}\hat{\vec{J}}\right)\\
\vec{f}_3(\vec{r},t) &= r^3\partial_\tau\left(\mu_\parallel\mu_\perp\left[\mu_0^{1/2}\vec{J}\times\vec{r}_0\right]\right) - r^2\mu_\perp\nabla_\perp\mu_\perp^{-1}\varepsilon_0^{1/2}\hat{J}_r\\
\vec{f}_4(\vec{r},t) &= r^3\partial_\tau\left(\varepsilon_\parallel\varepsilon_\perp\left[\vec{r}_0\times\varepsilon_0^{1/2}\hat{\vec{J}}\right]\right) - \varepsilon_\perp\nabla_\perp\varepsilon_\perp^{-1}\mu_0^{1/2}J_r
\end{aligned} \tag{59}$$

The infinite constant matrices of SEWE coefficients L and K describe mode coupling that occurs due to the presence of $\varepsilon_\perp, \mu_\perp$ in the right-hand side of (13)–(16). They are defined as follows:

$$\begin{aligned}
L_{mn}^{AB} &= \tfrac{1}{4\pi}\textstyle\int_S \vec{z}_0\cdot\left[\vec{E}_m^A\times\vec{H}_n^B\right]dS\\
K_{mn}^{AB} &= \tfrac{1}{4\pi}\textstyle\int_S \vec{z}_0\cdot\left[\vec{E}_m^A\times\vec{H}_n^B\right]\varepsilon_\perp\mu_\perp dS
\end{aligned}\qquad A,B\in\{H,E,T\} \tag{60}$$

In case of angular homogeneous structure these matrices become identity matrices, and the SEWE degenerates a set of uncoupled equations similar to those presented in [30–32].

The sums in SEWE (47)–(54) are written in a shorten notation that can be unrolled as follows:

$$\begin{aligned}
&\sum_{A\in\{H,E,T\}}\sum_{m'} L_{mm'}^{HA}\partial_z h_{m'}^A\\
&= \sum_{m'} L_{mm'}^{HH}\partial_z h_{m'}^H + \sum_{m'} L_{mm'}^{HE}\partial_z h_{m'}^E + \sum_{m'} L_{mm'}^{HT}\partial_z h_{m'}^T
\end{aligned} \tag{61}$$

We further shorten the sums using matrix notation like this (prime denotes transposition):

$$\mathbf{L}^{HE}\partial_z\mathbf{h}^E = \sum_{m'} L_{mm'}^{HE}\partial_z h_{m'}^E, \quad \mathbf{L}^{HT'}\partial_z\mathbf{e}^H = \sum_{k'} L_{k'k}^{HT}\partial_z e_{k'}^H \tag{62}$$

It should be noted that 6 out of 18 possible matrices (60) are zeros:

$$\mathbf{L}^{ET} = \mathbf{L}^{TH} = \mathbf{L}^{EH} = \mathbf{K}^{TE} = \mathbf{K}^{HT} = \mathbf{K}^{HE} = \mathbf{0} \tag{63}$$

For matrices (60) the following relations can be proved:

$$\begin{aligned}
\sum_A \mathbf{K}^{BA}\mathbf{L}^{CA'} &= \sum_A \mathbf{L}^{AB'}\mathbf{K}^{AC} = \sum_A \mathbf{K}^{AB'}\mathbf{L}^{AC} = \sum_A \mathbf{L}^{BA}\mathbf{K}^{CA'}\\
&= \begin{cases}\mathbf{0}, & B\neq C\\ \mathbf{I}, & B=C\end{cases}\qquad A,B,C\in\{H,E,T\}
\end{aligned} \tag{64}$$

Here $\mathbf{0}$ is null matrix, and $\mathbf{I}$ is identity matrix. Substituting (63) into (64) further results:

$$\mathbf{L}^{AA}\mathbf{K}^{AA'}=\mathbf{K}^{AA}\mathbf{L}^{AA'}=\mathbf{K}^{AA'}\mathbf{L}^{AA}=\mathbf{L}^{AA'}\mathbf{K}^{AA}=\mathbf{I},\ A\in\{H,E,T\} \quad (65)$$

The shortened formulas (64) in a full form look like:

$$\sum_{A}\mathbf{L}^{BA}\mathbf{K}^{CA'}=\delta_{BC}\Rightarrow\sum_{n}L_{mn}^{HE}K_{kn}^{EE}+\sum_{n}L_{mn}^{HH}K_{kn}^{EH}+\sum_{n}L_{mn}^{HT}K_{kn}^{ET}=0 \quad (66)$$

System of Equations (47)–(58) is overdetermined, i.e., some of the equations are linearly dependent. For example, using (65) one can easily prove equivalence of Equations (55) and (56), and the same is true for Equations (57), (58). In a particular problem this SEWE should be shortened to a suitable reduced form using the coefficient matrix properties (63)–(65).

The obtained SEWE allows us to determine only the angular mode amplitudes. In order to find the radial mode amplitudes we need some more equations. To this aim we make use of Equations (10), (11). First, we substitute mode expansions of both angular (42), (43) and radial (44), (45) field components into Equations (10), (11). Then, we express all the vector basis functions via scalar basis functions using definitions (28), (31), and (34). Finally these scalar functional equations are projected onto the corresponding bi-orthogonal scalar basis functions using a proper dot product definition (40), (41) based on the type of the boundary conditions that correspond to the equation terms. This derivation results in the following differential equations that can be easily integrated provided that the angular mode amplitudes are found:

$$\begin{aligned}\partial_r\left(\varepsilon_{\|}e_n^r\right)&=\varepsilon_{\|}e_n^E+r^2\tfrac{1}{4\pi}\textstyle\int_S\varepsilon_0^{-1/2}\rho\,q_n^{-1}\Psi_n^E dS\\ \partial_r\left(\mu_{\|}h_m^r\right)&=\mu_{\|}h_m^H+r^2\tfrac{1}{4\pi}\textstyle\int_S\mu_0^{-1/2}\hat{\rho}\,p_m^{-1}\Psi_m^H dS\end{aligned}\quad (67)$$

$$\begin{aligned}\partial_\tau\left(\varepsilon_{\|}e_n^r\right)&=-\sum_{n'}L_{nn'}^{EE}h_{n'}^E-r^2\tfrac{1}{4\pi}\textstyle\int_S\mu_0^{1/2}J_r\,q_n^{-1}\Psi_n^E dS\\ \partial_\tau\left(\mu\parallel h_m^r\right)&=-\sum_{m'}L_{m'm}^{HH}e_{m'}^H-r^2\tfrac{1}{4\pi}\textstyle\int_S\varepsilon_0^{1/2}\hat{J}_r\,p_m^{-1}\Psi_m^H dS\end{aligned}\quad (68)$$

Thus, we derived a complete set of governing equations for the mode amplitudes, which allows us to solve particular problems with given sources, initial conditions or initial boundary-value conditions. A summary of all the stages of the proposed method is given in the next section.

4. ALGORITHM OF SOLVING A PROBLEM BY THE METHOD OF MODE EXPANSION IN TIME DOMAIN

The sought fields in the problem under study can be obtained using the following algorithm:

- Solve scalar eigenvalue problems (29), (32) and (35). As a result, we find eigenvalues p_m^2, q_n^2 and scalar eigenfunctions $\Psi_m^A(\theta,\varphi), \Phi_n^A(\theta,\varphi), \quad A \in \{H, E, T\}$
- Construct the vector eigenfunctions $\vec{E}_m^A(\theta,\varphi), \vec{H}_n^A(\theta,\varphi), \quad A \in \{H, E, T\}$ via obtained scalar eigenfunctions using formulae (28), (31), and (34).
- Calculate the nonzero SEWE coefficient matrices $\mathbf{L}^{AB}$ and $\mathbf{K}^{AB}$ ($A, B \in \{H, E, T\}$) using formulae (60).
- Calculate the integrals in the right-hand side of SEWE (47)–(58) and (67), (68) using known source functions $\vec{\mathcal{J}}_0(\vec{r}, t)$ and $\rho_0(\vec{r}, t)$ and auxiliary definitions (59).
- Find initial conditions for the mode amplitudes by projecting the initial conditions for the fields using formula (46).
- Solve the initial problem for SEWE (47)–(58) for angular mode amplitudes. As a result we find mode amplitudes $e_m^A(r,t), h_m^A(r,t) \quad A \in \{H, E, T\}$ for $t > 0$.
- Integrate equations (67), (68) with found angular mode amplitudes in the right-hand side. As a result we determine the radial mode amplitudes $e_n^r(r,t), h_m^r(r,t)$.
- Substitute the obtained angular and radial mode amplitudes and basis functions into the field expansions (42)–(45) and use these formulae for calculating sought electromagnetic fields at arbitrary space position and time instance.

5. CONCLUSIONS

We presented a new method for calculating transient electromagnetic fields in conical transmission lines and free space filled with inhomogeneous non-stationary medium that can be described by permittivity and permeability of factorized form (3).

The proposed method is based on expansion of the fields over basis of frequency independent eigenmodes of the structure. The introduced modes satisfy all the boundary conditions at medium discontinuities in angular coordinates. This results in a fast convergence of the series that are obtained from field expansion over such modes. This fact is not directly demonstrated in the paper, but it was formerly established by us for a similar consideration for cylindrical inhomogeneous transmission lines [20, 21, 36].

The considered approach is a generalization of earlier developed time domain mode methods. In case of angular homogeneous structure the developed system of evolutionary waveguide Equations (47)–(58), (67), (68) degenerates to a set of uncoupled equations similar to those presented by Shlivinski and Heyman in [30] and by Dumin et al. in [31, 32].

This paper contains only derivation of the general theory of the proposed method. An example of its application will be given in a subsequent paper that will analyze transient radiation of a dielectric-field TEM-horn. The consideration will be based on the described method along with method of mode matching in time domain [34].

APPENDIX A. OPERATOR $\nabla_\perp$ IN SPHERICAL COORDINATE SYSTEM

In spherical coordinate system the orts $\left\{\vec{r}_0, \vec{\theta}_0, \vec{\varphi}_0\right\}$ depend on angular coordinate, that is why using vector derivative operators like (9) requires special attention. The ort derivatives are given as follows:

	$\partial/\partial r$	$\partial/\partial\theta$	$\partial/\partial\varphi$
$\vec{r}_0$	0	$\vec{\theta}_0$	$\sin\theta\,\vec{\varphi}_0$
$\vec{\theta}_0$	0	$-\vec{r}_0$	$\cos\theta\,\vec{\varphi}_0$
$\vec{\varphi}_0$	0	0	$-\sin\theta\,\vec{r}_0-\cos\theta\,\vec{\theta}_0$

(A1)

In the main body of the paper we introduced symbolical operator (9); besides trivial application to a scalar field like $\nabla_\perp\rho$ and $[\vec{r}_0\times\nabla_\perp]\,A_r$ it is also used in more complicated vector operations: $\nabla_\perp\cdot\vec{A}$, $\nabla_\perp\cdot\left[\vec{A}\times\vec{r}_0\right]$, and $[\vec{r}_0\times\nabla_\perp]\cdot\vec{A}$. Specific coordinate form of these operations in spherical coordinate system looks as follows:

$$\begin{aligned}
\nabla_\perp\cdot\vec{A} &= \left(\vec{\theta}_0\partial_\theta+\frac{\vec{\varphi}_0}{\sin\theta}\partial_\varphi\right)\cdot\left(\vec{\theta}_0A_\theta+\vec{\varphi}_0A_\varphi\right)\\
&= \left(\vec{\theta}_0\cdot\vec{\theta}_0\right)\partial_\theta A_\theta+\left(\vec{\theta}_0\cdot\partial_\theta\vec{\theta}_0\right)A_\theta+\left(\vec{\theta}_0\cdot\vec{\varphi}_0\right)\partial_\theta A_\varphi+\left(\vec{\theta}_0\cdot\partial_\theta\vec{\varphi}_0\right)A_\varphi\\
&\quad+\frac{1}{\sin\theta}\Big[\left(\vec{\varphi}_0\cdot\vec{\theta}_0\right)\partial_\varphi A_\theta+\left(\vec{\varphi}_0\cdot\partial_\varphi\vec{\theta}_0\right)A_\theta+(\vec{\varphi}_0\cdot\vec{\varphi}_0)\partial_\varphi A_\varphi+(\vec{\varphi}_0\cdot\partial_\varphi\vec{\varphi}_0)A_\varphi\Big]\\
&= \partial_\theta A_\theta+\frac{1}{\sin\theta}\left[\cos\theta\,A_\theta+\partial_\varphi A_\varphi\right]=\frac{1}{\sin\theta}\left[\partial_\theta\left(\sin\theta\,A_\theta\right)+\partial_\varphi A_\varphi\right]
\end{aligned}\tag{A2}$$

Similar derivation yields:

$$\begin{aligned}\nabla_\perp\cdot\left[\vec{A}\times\vec{r}_0\right] &= \left(\vec{\theta}_0\partial_\theta+\frac{\vec{\varphi}_0}{\sin\theta}\partial_\varphi\right)\cdot\left[\vec{\theta}_0A_\theta+\vec{\varphi}_0A_\varphi\times\vec{r}_0\right]\\ &= \frac{1}{\sin\theta}\left[\partial_\theta\left(\sin\theta\,A_\varphi\right)-\partial_\varphi A_\theta\right] \qquad\text{(A3)}\\ \left[\vec{r}_0\times\nabla_\perp\right]\cdot\vec{A} &= \left[\vec{r}_0\times\left(\vec{\theta}_0\partial_\theta+\frac{\vec{\varphi}_0}{\sin\theta}\partial_\varphi\right)\right]\cdot\left(\vec{\theta}_0A_\theta+\vec{\varphi}_0A_\varphi\right)\\ &= \frac{1}{\sin\theta}\left[\partial_\theta\left(\sin\theta\,A_\varphi\right)-\partial_\varphi A_\theta\right] \qquad\text{(A4)}\end{aligned}$$

Thus, the following forms are equivalent:

$$\begin{aligned}\nabla_\perp\cdot\left[\vec{A}\times\vec{r}_0\right] &= \left[\vec{r}_0\times\nabla_\perp\right]\cdot\vec{A}=-\nabla_\perp\cdot\left[\vec{r}_0\times\vec{A}\right]=-\left[\nabla_\perp\times\vec{r}_0\right]\cdot\vec{A}\\ &= \frac{1}{\sin\theta}\left[\partial_\theta\left(\sin\theta\,A_\varphi\right)-\partial_\varphi A_\theta\right] \qquad\text{(A5)}\end{aligned}$$

In order to present curl operator in radial-tangential form using the introduced symbolic operators let's start from coordinate presentation:

$$\begin{aligned}\operatorname{rot}\vec{\mathcal{A}} &= \operatorname{rot}\left(\vec{r}_0A_r+\vec{A}\right)=\frac{\vec{r}_0}{r\sin\theta}\left[\partial_\theta\left(\sin\theta A_\varphi\right)-\partial_\varphi A_\theta\right]\\ &+\frac{\vec{\theta}_0}{r}\left[\frac{1}{\sin\theta}\partial_\varphi A_r-\partial_r\left(rA_\varphi\right)\right]+\frac{\vec{\varphi}_0}{r}\left[\partial_r\left(rA_\theta\right)-\partial_\theta A_r\right] \qquad\text{(A6)}\end{aligned}$$

Now taking into consideration that $(\sin\theta)^{-1}\left[\partial_\theta\left(\sin\theta A_\varphi\right)-\partial_\varphi A_\theta\right]=\nabla_\perp\cdot\left[\vec{A}\times\vec{r}_0\right]$, $\left(\vec{\theta}_0\left(\sin\theta\right)^{-1}\partial_\varphi-\vec{\varphi}_0\partial_\theta\right)A_r=-\left[\vec{r}_0\times\nabla_\perp\right]A_r$ and $\left[\vec{\varphi}_0\partial_r\left(rA_\theta\right)-\vec{\theta}_0\partial_r\left(rA_\varphi\right)\right]=\partial_r\left(r\left[\vec{r}_0\times\vec{A}\right]\right)$, we can present rot as follows:

$$\operatorname{rot}\vec{\mathcal{A}}=\frac{1}{r}\left\{\vec{r}_0\nabla_\perp\cdot\left[\vec{A}\times\vec{r}_0\right]-\left[\vec{r}_0\times\nabla_\perp\right]A_r+\partial_r\left(r\left[\vec{r}_0\times\vec{A}\right]\right)\right\}$$

APPENDIX B. GAUSS-OSTROGRADSKY FORMULA FOR ANGULAR DOMAINS

Here, we are going to derive formula (25). First, we prove correctness of the following two-dimension Gauss-Ostrogradsky theorem in spherical coordinates:

$$\int\limits_S\nabla_\perp\cdot\vec{F}\,dS=\int\limits_L\vec{F}\cdot\vec{n}\,dl \qquad\text{(B1)}$$

Here the operator $\nabla_\perp$ is defined by expression (9), $\vec{F}(\theta,\varphi)=\vec{\theta}_0F_\theta(\theta,\varphi)+\vec{\varphi}_0F_\varphi(\theta,\varphi)$, $S=\{(\theta,\varphi):\theta,\varphi\in\Lambda\}$ is some solid angle of the sphere, L is the contour (possibly multi-connected) bounding the solid angular domain S, $\vec{n} = n_\theta \vec{\theta}_0 + n_\varphi \vec{\varphi}_0$ is the outward normal to the contour L. In order to prove formula (B1) we make use of the general formulation of Stokes theorem [35] that can be formulated as: Let ω be $n-1$-differential form with compact support on an oriented smooth manifold Ω of dimension n and $\partial\Omega$ be the boundary of Ω with its induced orientation, then:

$$\int\limits_{\Omega} d\omega = \int\limits_{\partial\Omega} \omega \tag{B2}$$

Let $\omega = \vec{F} \cdot \vec{n}\, dl$ is 1-form then we can see that:

$$\omega = \vec{F} \cdot \vec{n} dl = \left[\vec{\theta}_0 F_\theta + \vec{\varphi}_0 F_\varphi\right] \cdot \vec{n} \sqrt{d\theta^2 + \sin^2\theta d\varphi^2} = F_\theta \sin\theta d\varphi - F_\varphi d\theta \tag{B3}$$

Taking the exterior differential from (B3) one can obtain 2-form $d\omega$ as follows:

$$d\omega = d\left(F_\theta \sin\theta\, d\varphi - F_\varphi d\theta\right) = \frac{\partial\left(F_\theta \sin\theta\right)}{\partial\theta} d\theta \wedge d\varphi - \frac{\partial F_\varphi}{\partial\varphi} d\varphi \wedge d\theta \tag{B4}$$

Substituting formulae (B3) and (B4) into (B2) taking into consideration that $\Omega = S$ and $\partial\Omega = L$ we obtain:

$$\int\limits_{S} \frac{1}{\sin\theta} \left[\frac{\partial\left(F_\theta \sin\theta\right)}{\partial\theta} + \frac{\partial F_\varphi}{\partial\varphi}\right] \sin\theta\, d\theta\, d\varphi = \int\limits_{L} \vec{F} \cdot \vec{n}\, dl \tag{B5}$$

Noticing that $dS = \sin(\theta)\, d\theta d\varphi$ and accounting for (A2) we arrive at (B1) that completes the proof.

APPENDIX C. GREEN'S FORMULA FOR ANGULAR DOMAINS

Based on formula (B1) the Green's formula for angular domains (25) can be proved easily:

$$\begin{aligned}
&\int\limits_{\Gamma} \left[\left(\vec{n} \cdot \vec{A}\right)\left(\nabla_\perp \cdot \vec{B}\right) - \left(\vec{n} \cdot \vec{B}\right)\left(\nabla_\perp \cdot \vec{A}\right)\right] f\, dl \\
&= \int\limits_{\Gamma} \left[\vec{A}\left(\nabla_\perp \cdot \vec{B}\right) - \vec{B}\left(\nabla_\perp \cdot \vec{A}\right)\right] f \cdot \vec{n} dl = \int\limits_{\Omega} \nabla_\perp \cdot \left\{\left[\vec{A}\left(\nabla_\perp \cdot \vec{B}\right) - \vec{B}\left(\nabla_\perp \cdot \vec{A}\right)\right] f\right\} d\Omega \\
&= \int\limits_{\Omega} \left\{\left(\nabla_\perp \cdot \vec{A}\right)\left(\nabla_\perp \cdot \vec{B}\right) f + \vec{A} \cdot \nabla_\perp f \nabla_\perp \cdot \vec{B} - \left(\nabla_\perp \cdot \vec{B}\right)\left(\nabla_\perp \cdot \vec{A}\right) f - \vec{B} \cdot \nabla_\perp f \left(\nabla_\perp \cdot \vec{A}\right)\right\} d\Omega \\
&= \int\limits_{\Omega} \left\{\vec{A} \cdot \nabla_\perp f \nabla_\perp \cdot \vec{B} - \vec{B} \cdot \nabla_\perp f \left(\nabla_\perp \cdot \vec{A}\right)\right\} d\Omega
\end{aligned}$$

APPENDIX D. PROOF OF SYMMETRY OF OPERATORS W_H, W_E

Finally, we apply formula (25) for proving symmetry of the introduced differential operators (26). Using definitions of operator W_H (19) and dot product (18) one can derive:

$$
\begin{aligned}
&(W_H X_1, X_2) - (X_1, W_H X_2)\\
&= \frac{1}{4\pi}\int_S \left\{\varepsilon_\perp \vec{E}_2 \cdot \left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \vec{H}_1 + \mu_\perp \vec{H}_2 \cdot \nabla_\perp \mu_\perp^{-1} [\vec{r}_0 \times \nabla_\perp] \cdot \vec{E}_1 \right\} dS\\
&- \frac{1}{4\pi}\int_S \left\{\varepsilon_\perp \vec{E}_1 \cdot \left[\vec{r}_0 \times \varepsilon_\perp^{-1} \nabla_\perp \mu_\perp^{-1}\right] \nabla_\perp \cdot \mu_\perp \vec{H}_2 + \mu_\perp \vec{H}_1 \cdot \nabla_\perp \mu_\perp^{-1} [\vec{r}_0 \times \nabla_\perp] \cdot \vec{E}_2 \right\} dS\\
&= \frac{1}{4\pi}\int_S \left\{\left[\vec{E}_2 \times \vec{r}_0\right] \cdot \nabla_\perp \mu_\perp^{-1} \nabla_\perp \cdot \mu_\perp \vec{H}_1 - \mu_\perp \vec{H}_1 \cdot \nabla_\perp \mu_\perp^{-1} \nabla_\perp \cdot \left[\vec{E}_2 \times \vec{r}_0\right]\right\} dS\\
&+ \frac{1}{4\pi}\int_S \left\{\mu_\perp \vec{H}_2 \cdot \nabla_\perp \mu_\perp^{-1} \nabla_\perp \cdot \left[\vec{E}_1 \times \vec{r}_0\right] - \left[\vec{E}_1 \times \vec{r}_0\right] \cdot \nabla_\perp \mu_\perp^{-1} \nabla_\perp \cdot \mu_\perp \vec{H}_2\right\} dS
\end{aligned}
$$

Formula (25) with $f = \mu_\perp^{-1}$ allows us to turn these integrals over solid angle into integrals over bounding contours where the boundary conditions can be applied to vanish the terms:

$$
\begin{aligned}
&(W_H X_1, X_2) - (X_1, W_H X_2)\\
&= \frac{1}{4\pi}\int_L \left[\left(\vec{n} \cdot \left[\vec{E}_2 \times \vec{r}_0\right]\right)\left(\nabla_\perp \cdot \mu_\perp \vec{H}_1\right) - \left(\vec{n} \cdot \mu_\perp \vec{H}_1\right)\left(\nabla_\perp \cdot \left[\vec{E}_2 \times \vec{r}_0\right]\right)\right] \mu_\perp^{-1} dl\\
&+ \frac{1}{4\pi}\int_L \left[\left(\vec{n} \cdot \mu_\perp \vec{H}_2\right)\left(\nabla_\perp \cdot \left[\vec{E}_1 \times \vec{r}_0\right]\right) - \left(\vec{n} \cdot \left[\vec{E}_1 \times \vec{r}_0\right]\right)\left(\nabla_\perp \cdot \mu_\perp \vec{H}_2\right)\right] \mu_\perp^{-1} dl\\
&= \left|\vec{r}_0 \times \vec{n} = \vec{l}\right| = \frac{1}{4\pi}\int_L \left[\left(\vec{l} \cdot \vec{E}_2\right)\left(\nabla_\perp \cdot \mu_\perp \vec{H}_1\right) - \mu_\perp \left(\vec{n} \cdot \vec{H}_1\right)\left(\nabla_\perp \cdot \left[\vec{E}_2 \times \vec{r}_0\right]\right)\right] \mu_\perp^{-1} dl\\
&+ \frac{1}{4\pi}\int_L \left[\mu_\perp \left(\vec{n} \cdot \vec{H}_2\right)\left(\nabla_\perp \cdot \left[\vec{E}_1 \times \vec{r}_0\right]\right) - \left(\vec{l} \cdot \vec{E}_1\right)\left(\nabla_\perp \cdot \mu_\perp \vec{H}_2\right)\right] \mu_\perp^{-1} dl\\
&= \left| \left.\vec{l} \cdot \vec{E}\right|_L = \left.\vec{n} \cdot \vec{H}\right|_L = 0 \right| = 0
\end{aligned}
$$

Similar derivation for operator W_E (20) yields:

$$
\begin{aligned}
&(W_E X_1, X_2) - (X_1, W_E X_2) \\
&= \frac{1}{4\pi}\iint_S \left\{ \varepsilon_\perp \vec{E}_2 \cdot \nabla_\perp \varepsilon_\perp^{-1} [\nabla_\perp \times \vec{r}_0] \cdot \vec{H}_1 + \mu_\perp \vec{H}_2 \cdot [\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0] \nabla_\perp \cdot \varepsilon_\perp \vec{E}_1 \right\} dS \\
&- \frac{1}{4\pi}\iint_S \left\{ \varepsilon_\perp \vec{E}_1 \cdot \nabla_\perp \varepsilon_\perp^{-1} [\nabla_\perp \times \vec{r}_0] \cdot \vec{H}_2 + \mu_\perp \vec{H}_1 \cdot [\mu_\perp^{-1} \nabla_\perp \varepsilon_\perp^{-1} \times \vec{r}_0] \nabla_\perp \cdot \varepsilon_\perp \vec{E}_2 \right\} dS \\
&= \frac{1}{4\pi}\iint_S \left\{ \varepsilon_\perp \vec{E}_2 \cdot \nabla_\perp \varepsilon_\perp^{-1} \nabla_\perp \cdot \left[\vec{r}_0 \times \vec{H}_1\right] - \left[\vec{r}_0 \times \vec{H}_1\right] \cdot \nabla_\perp \varepsilon_\perp^{-1} \nabla_\perp \cdot \varepsilon_\perp \vec{E}_2 \right\} dS \\
&+ \frac{1}{4\pi}\iint_S \left\{ \left[\vec{r}_0 \times \vec{H}_2\right] \cdot \nabla_\perp \varepsilon_\perp^{-1} \nabla_\perp \cdot \varepsilon_\perp \vec{E}_1 - \varepsilon_\perp \vec{E}_1 \cdot \nabla_\perp \varepsilon_\perp^{-1} \nabla_\perp \cdot \left[\vec{r}_0 \times \vec{H}_2\right] \right\} dS \\
&= \frac{1}{4\pi}\int_L \left[\left(\vec{n} \cdot \varepsilon_\perp \vec{E}_2\right)\left(\nabla_\perp \cdot \left[\vec{r}_0 \times \vec{H}_1\right]\right) - \left(\vec{n} \cdot \left[\vec{r}_0 \times \vec{H}_1\right]\right)\left(\nabla_\perp \cdot \varepsilon_\perp \vec{E}_2\right) \right] \varepsilon_\perp^{-1} dl \\
&+ \frac{1}{4\pi}\int_L \left[\left(\vec{n} \cdot \left[\vec{r}_0 \times \vec{H}_2\right]\right)\left(\nabla_\perp \cdot \varepsilon_\perp \vec{E}_1\right) - \left(\vec{n} \cdot \varepsilon_\perp \vec{E}_1\right)\left(\nabla_\perp \cdot \left[\vec{r}_0 \times \vec{H}_2\right]\right) \right] \varepsilon_\perp^{-1} dl \\
&= \left| \; \left(\nabla_\perp \cdot \varepsilon_\perp \vec{E}\right)\Big|_L = \left(\nabla_\perp \cdot \left[\vec{r}_0 \times \vec{H}\right]\right)\Big|_L = 0 \right| = 0
\end{aligned}
$$

In the latter case the second pair of boundary conditions (17) was used in order to vanish the result.

In case when there are no PEC domains that determine the contour L one can chose, as such a domain, any zero vicinity of some nonsingular point. In such a case the integrals vanish due to the fact that the integrands remain bounded while the contour length tends to zero as the domain deforms continuously to a point.

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