Model Checking for Modal Intuitionistic Dependence Logic

Fan Yang

Department of Mathematics and Statistics
University of Helsinki

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Joint work with Johannes Ebbing, Peter Lohmann,
Institute for Theoretical Computer Science, Leibniz University Hannover

Outline

1. First-order Dependence Logic with Intuitionistic Implication
   - First-order Dependence Logic
   - First-order Intuitionistic Dependence Logic
2. Modal Intuitionistic Dependence Logic
   - Introduction
   - Some Properties of **MID**
3. Complexity of Model Checking Problem for fragments of **MID**
   - **MID**-MC is PSPACE-complete

Characterizing dependence between variables

First Order Quantifiers

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi$$

Henkin Quantifiers (Henkin, 1961)

$$\begin{pmatrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{pmatrix} \phi$$

Independence Friendly Logic (Hintikka, Sandu, 1989)

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 / \{x_1\} \phi$$

Dependence Logic (Väänänen 2007)

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=(x_2, y_2) \wedge \phi)$$

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$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=\!(x_2, y_2) \wedge \phi)$$

Syntax of First-order Dependence Logic (**D**)

$\mathbf{D} = \mathbf{FO} + {=}(t_1, \ldots, t_n)$

Well-formed formulas of **D** (in negation normal form) are given by the following grammar

$$\phi ::= \alpha \mid {=}(t_1, \ldots, t_n) \mid \neg {=}(t_1, \ldots, t_n) \mid \phi \wedge \phi \mid \phi \otimes \phi \mid \forall x \phi \mid \exists x \phi$$

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Semantics of dependence logic: Team semantics

D adopts team semantics, originally introduced by W. Hodges (1997) for IF-logic.

Important points of team semantics:

1. satisfaction is defined w.r.t. sets of assignments (teams) instead of single assignments (Tarskian semantics)
2. the semantics is compositional

Theorem (Enderton, Walkoe, Väänänen)

D *sentences have the same expressive power as sentences of the second order* Σ_1^1 *fragment.*

Intuitionistic Implication [Abramsky, Väänänen, 2009]

- In a general context of W. Hodges' team semantics, Abramsky, Väänänen introduced intuitionistic implication $\to$ and Boolean disjunction $\varovee$.

- $\wedge$, $\to$ satisfy the Galois connection:

$$\phi \wedge \psi \models \chi \iff \phi \models \psi \to \chi$$

- $\to$, $\wedge$, $\varovee$ satisfy axioms of intuitionistic propositional logic.

- first-order intuitionistic dependence logic:

$$\mathbf{ID} = \mathbf{D} + \varovee + \to$$

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First-order Intuitionistic Dependence Logic

Theorem ([Abramsky, Väänänen 2009],[Y. 2010])

Sentences of **ID** *have the same expressive power as sentences of the full second order logic.*

PD and **PID**

- The underlying propositional logic of **D** and **ID** are *propositional dependence logic* (**PD**) and *propositional intuitionistic dependence logic* (**PID**), respectively.

- Syntactically:

$$\mathbf{PD} = \mathbf{CPL} + =\!(p_1, \dots, p_n)$$
$$\mathbf{PID} = \mathbf{IPL} + =\!(p_1, \dots, p_n)$$

- **PID** is essentially equivalent to *inquisitive logic* [Ciardelli and Roelofsen, 2009], studied in the field of linguistics.

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Modal Intuitionistic Dependence Logic

- Modal Dependence Logic (Väänänen 2008)

$$\mathbf{MD} = \text{Modal Logic } (\mathbf{M}) + {=}(p_1, \cdots, p_n)$$

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Well-formed formulas of **MID** (in negation normal form) are defined by the following grammar :

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$\Box = (p_1, \ldots, p_n)$

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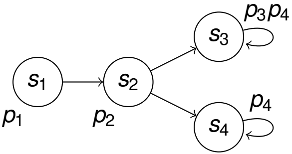
- $\textbf{MD} = \textbf{MID}[\neg, =(\cdot), \wedge, \otimes, \Box, \Diamond]$,

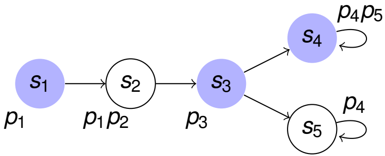
Team Semantics of **MID**

Definition

A *Kripke model* is a triple $K = (S, R, \pi)$ consisting of a nonempty set S, a binary relation $R \subseteq S \times S$, and a labeling function $\pi : S \to \wp(\text{PROP})$.



Teams: sets of possible worlds

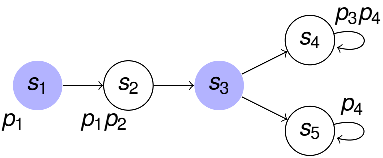


team $T = \{ s_1, s_3, s_4 \}$

Operations on teams

For any set T of states in a Kripke model K, we define

$$R(T) = \{s \in K \mid \exists s' \in T, \text{ s.t. } s'Rs\},$$



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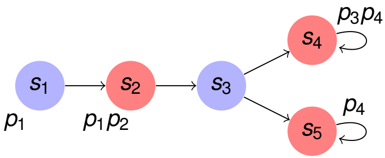
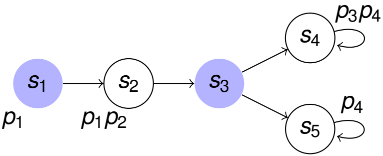


Operations on teams

Let T be a team. Define

$$\langle T \rangle = \{T' \mid T' \subseteq R(T) \text{ and } \forall s \in T,\ R(s) \cap T' \neq \emptyset\}.$$



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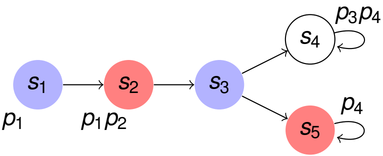

$$\{s_2, s_5\} \in \langle \{s_1, s_3\} \rangle$$

Clearly, $R(T) \in \langle T \rangle$.

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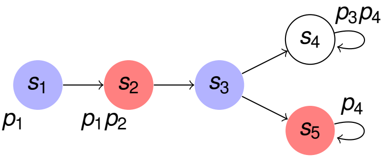

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Let $K = (S, R, \pi)$ be a Kripke model, $T \subseteq S$ a team. Define

- $K, T \models \Box\varphi$ iff $K, R(T) \models \varphi$
- $K, T \models \Diamond\varphi$ iff there exists nonempty $T' \in \langle T \rangle$ such that $K, T' \models \varphi$

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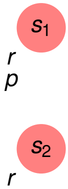

$T = \{s_1, s_2\}$

$K, T \models r$

$K, T \not\models \neg p$

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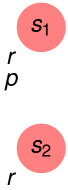
- $K, T \models \neg {=}(p_1, \cdots, p_n, q)$ iff $T = \emptyset$



$T = \{s_1, s_2\}$

$K, T \not\models {=}(r, p) \qquad K, T \models {=}(r)$

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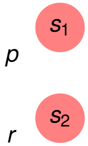
- $K, T \models \varphi \wedge \psi$ iff $K, T \models \varphi$ and $K, T \models \psi$
- $K, T \models \varphi \otimes \psi$ iff there are teams T_1, T_2 with $T = T_1 \cup T_2$ such that $K, T_1 \models \varphi$ and $K, T_2 \models \psi$



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- $K, T \models \varphi \varovee \psi$ iff $K, T \models \varphi$ or $K, T \models \psi$
- $K, T \models \varphi \rightarrow \psi$ iff for any subteam $T' \subseteq T$ if $K, T' \models \varphi$ then $K, T' \models \psi$.

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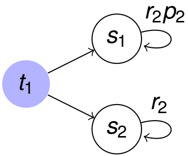
Lemma (Empty team property)

Empty team satisfies all formulas ϕ of **MID** *in any Kripke model K, namely $K, \emptyset \models \phi$.*

Example



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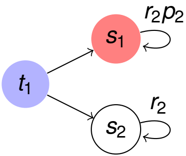

$T = \{t_1\}$

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$T_0 = \{s_1\}$

$K, T_0 \models r_2 \rightarrow =(p_2)$

Simple properties

$$=\!(p_1, \cdots, p_n, q) \equiv \big(\, =\!(p_1) \wedge \cdots \wedge =\!(p_n) \,\big) \rightarrow =\!(q)$$

$$=\!(p) \equiv p \ovee \neg p$$

$$\neg p \equiv p \rightarrow \bot;$$

$$\neg =\!(p_1, \cdots, p_n) \equiv \; =\!(p_1, \cdots, p_n) \rightarrow \bot;$$

Important Properties of **MID**

Theorem (Downwards Closure)

For any formula ϕ of **MID***, if $K, T \models \phi$ and $T' \subseteq T$, then $K, T' \models \phi$.*

Definition (Flatness)

We say that ϕ is *flat* if for all Kripke models K and teams T

$$K, T \models \phi \iff (K, \{s\} \models \phi \text{ for all } s \in T).$$

Theorem

Formulas without any occurrences of $=\!(\cdot)$ and $\varovee$ are flat, namely **MID**$[\neg, \wedge, \otimes, \Box, \Diamond]$ *formulas are flat.*

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Theorem

Satisfaction of **MID** *is invariant under generated submodels, p-morphic images, disjoint unions.*

Important Properties of **MD**

[Sevenster 2009]:

Theorem

On singleton teams, $\mathbf{MD} = \mathbf{M}$*.*

There is a translation from **MD** into the usual modal logic. But the translation causes an exponential blow-up in the size of the formulas.

Theorem

On arbitrary teams, $\mathbf{MD} > \mathbf{M}$*.*

Important Properties of **MD**

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There is a translation from **MD** into the usual modal logic. But the translation causes an exponential blow-up in the size of the formulas.

Theorem

On arbitrary teams, **MD** $>$ **M**.

Complexity results of **MD**

Theorem (Sevenster 2009)

Satisfiability problem for **MD** *is NEXPTIME-complete.*

Theorem (Ebbing, Lohmann 2011)

Model checking problem for **MD** *is NP-complete.*

MID Model Checking Problem

Definition

Let $\mathcal{L}$ be a sublogic (fragment) of **MID.** The *model checking problem* for $\mathcal{L}$ is defined as the decision problem of the set

$$\mathcal{L}\text{−MC} := \left\{ \langle K, T, \varphi \rangle \;\middle|\; \begin{array}{l} K = (S, R, \pi) \text{ is a Kripke model, } T \subseteq S, \\ \varphi \in \mathcal{L} \text{ and } K, T \models \varphi \end{array} \right\}$$

Complexity results for fragments of **MD**-MC (Ebbing, Lohmann 2010)

Operators						**Complexity**	**Ref.**
$\Box$	$\Diamond$	$\wedge$	$\otimes$	$\neg$	$=(\cdot)$		
$*$	$*$	$+$	$+$	$*$	$+$	NP	[E& L 2011]
$+$	$*$	$*$	$+$	$*$	$+$	NP	[E& L 2011]
$*$	$+$	$*$	$*$	$*$	$+$	NP	[E& L 2011]
$*$	$-$	$*$	$-$	$*$	$*$	in P	[E& L 2011]
$*$	$*$	$*$	$*$	$*$	$-$	in P	[CES 1986]

$+$: operator present $-$: operator absent
$*$: complexity independent of operator

Computational Complexity of **MID** Model Checking

Theorem

Model checking problem for **MID** *is PSPACE-complete.*

Complexity results for fragments of **MID**-MC

Operators								Complexity	Method/Ref.
$\Box$	$\Diamond$	$\wedge$	$\otimes$	$\neg$	$\ovee$	$\rightarrow$	$=(\cdot)$		
$*$	$*$	$+$	$+$	$*$	$*$	$+$	$+$	PSPACE	reduct. from **TQBF**
$*$	$*$	$+$	$+$	$*$	$+$	$+$	$*$	PSPACE	see above
$*$	$+$	$+$	$*$	$*$	$+$	$+$	$*$	PSPACE	see above
$*$	$-$	$+$	$-$	$*$	$+$	$+$	$*$	coNP	reduct. from **TAUT**
$*$	$*$	$*$	$*$	$*$	$-$	$*$	$-$	in P	[CES 1986]
$*$	$*$	$+$	$+$	$*$	$*$	$-$	$+$	NP	[E& L 2011]
$+$	$*$	$*$	$+$	$*$	$*$	$-$	$+$	NP	[E& L 2011]
$*$	$+$	$*$	$*$	$*$	$*$	$-$	$+$	NP	[E& L 2011]
$*$	$-$	$*$	$-$	$*$	$*$	$-$	$*$	in P	[E& L 2011]
$*$	$*$	$*$	$*$	$*$	$*$	$-$	$-$	in P	[E& L 2011]

$+$: operator present $-$: operator absent
$*$: complexity independent of operator

In the rest of the talk, we will prove:

Theorem

MID-*MC is PSPACE complete.*

- **MID**-MC is in PSPACE.
- **MID**-MC is PSPACE-hard.

Theorem

MID-*MC is in PSPACE.*

PSPACE algorithm of **MID**-MC

```
check(K = (S, R, π), φ, T):
case φ

when φ = p
  foreach s ∈ T
    if not p ∈ π(s) then
      return false
  return true

when φ = ¬p
  foreach s ∈ T
    if p ∈ π(s) then
      return false
  return true

when φ = =(p1, ..., pn, q)
  foreach (s, s') ∈ T × T
    if π(s) ∩ {p1, ..., pn} = π(s') ∩ {p1, ..., pn} then
      if (q ∈ π(s) and not q ∈ π(s')) or (not q ∈ π(s) and q ∈ π(s')) then
        return false
  return true

when φ = ¬ =(p1, ..., pn)
  if S = ∅
    return true
  return false

when φ = ψ ⊗ χ
  existentially guess two sets of states T1, T2 ⊆ S
  if not T1 ∪ T2 = T then
    return false
  return (check(K, T1, ψ) and check(K, T2, χ))
```

PSPACE algorithm of **MID**-MC (cont.)

when $\varphi = \psi \varovee \chi$
  return (check(K, T, ψ) **or** check(K, T, ψ))

when $\varphi = \psi \wedge \chi$
  return (check(K, T, ψ) **and** check(K, T, χ))

when $\varphi = \Box\psi$
  $T' := \emptyset$
  foreach $s' \in S$
    foreach $s \in T$
      if $(s, s') \in R$ then
        $T' := T' \cup \{s'\}$
  return check(K, T', ψ)

when $\varphi = \Diamond\psi$
  existentially guess a **set** of states $T' \subseteq S$
    foreach $s \in T$
      if there is no $s' \in T'$ with $(s, s') \in R$ then
        return false
  return check(K, T', ψ)

when $\varphi = \psi \rightarrow \chi$
  universally guess a **set** of states $T' \subseteq T$
  if (**not** check(K, ψ, T') **or** check(K, χ, T'))
    return true
  return false

Next, we show

Theorem

MID *-MC is PSPACE-hard.*

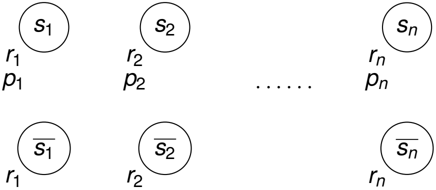
Proof. Reduction from TQBF.

Kripke model K_φ

Definition

Let $\varphi(p_1, \cdots, p_n)$ be a formula of **CPL**. We define a Kripke model $K_\varphi = (S_\varphi, R_\varphi, \pi_\varphi)$ by letting

$$\begin{aligned} S_\varphi &:= \{s_1, \ldots, s_n, \overline{s_1}, \ldots, \overline{s_n}\}, \\ R_\varphi &:= \emptyset, \\ \pi_\varphi(s_i) &:= \{r_i, p_i\}, \\ \pi_\varphi(\overline{s_i}) &:= \{r_i\}. \end{aligned}$$



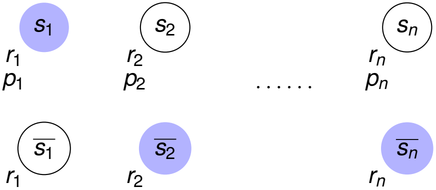
σ vs $T_{\varphi,\sigma}$

Let $\sigma : Prop \to \{\top, \bot\}$ be a valuation of **CPL.** The team $T_{\varphi,\sigma}$ of K_φ *induced by* σ is defined as

$$T_{\varphi,\sigma} = \{s_i \in S_\varphi \mid \sigma(p_i) = \top\} \cup \{\overline{s_i} \in S_\varphi \mid \sigma(p_i) = \bot\}.$$

Example:

- σ: $p_1 \mapsto \top \quad p_2 \mapsto \bot \quad \ldots \quad p_n \mapsto \bot$
- $T_{\varphi,\sigma} = \{s_1, \overline{s_2}, \ldots, \overline{s_n}\}$



Definition

Let r be a fixed propositional variable.

For every formula φ of **CPL** in negation normal form without any occurrence of r, we inductively define a formula $\varphi^{\rightarrow}$ of **MID** as follows:

$$
\begin{aligned}
p^{\rightarrow} &:= r \rightarrow p, \\
(\neg p)^{\rightarrow} &:= r \rightarrow \neg p, \\
(\varphi \wedge \psi)^{\rightarrow} &:= \varphi^{\rightarrow} \wedge \psi^{\rightarrow}, \\
(\varphi \vee \psi)^{\rightarrow} &:= \varphi^{\rightarrow} \ovee \psi^{\rightarrow}.
\end{aligned}
$$

In team $T_{\varphi,\sigma}$ of the Kripke model K_φ,

MID formula $\varphi^{\to}$ behaves like the **CPL** formula φ under valuation σ.

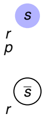
Lemma

For any formula φ and any valuation σ of **CPL***,*

$$\sigma(\varphi) = \top \iff K_\varphi, T_{\varphi,\sigma} \models \varphi^{\to}.$$

Proof. By induction on φ.
Case $\varphi = p$: We have that $S_p = \{s, \bar{s}\}$ and that

$$\begin{aligned}\sigma(p) = \top &\iff T_{p,\sigma} = \{s\} \\ &\iff K_p, T_{p,\sigma} \models r \to p.\end{aligned}$$



Complexity of **MID** Model Checking

Theorem

***MID**-MC is PSPACE-hard.*

Proof.

We give a polynomial-time reduction from TQBF to **MID**-MC.

Let $\psi = \forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n \, \varphi$ with φ quantifier-free (thus in **CPL**) be a QBF instance. The corresponding **MID**-MC instance is defined as $(K, T_0, f(\psi))$ where

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- $K = (S, R, \pi)$, where $S = \bigcup_{1 \le i \le n} S_i$, $R = \bigcup_{1 \le i \le n} R_i$, for $1 \le i \le n/2$

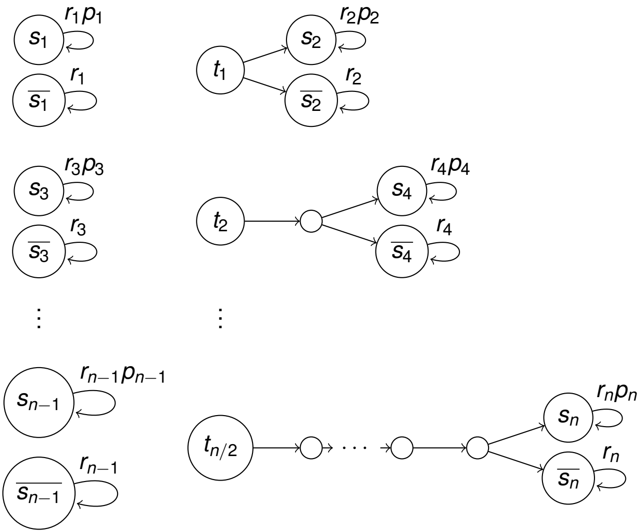
$$
\begin{aligned}
S_{2i-1} &= \{s_{2i-1}, \overline{s_{2i-1}}\} \\
S_{2i} &= \{s_{2i}, \overline{s_{2i}}\} \cup \{t_i\} \cup \{t_{i1}, \cdots, t_{i(i-1)}\} \\
R_{2i-1} &= \{(s_{2i-1}, s_{2i-1}), (\overline{s_{2i-1}}, \overline{s_{2i-1}})\} \\
R_{2i} &= \{(t_i, t_{i1}), (t_{i1}, t_{i2}), \cdots, (t_{i(i-2)}, t_{i(i-1)})\} \\
&\quad \cup \{(t_{i(i-1)}, s_{2i}), (t_{i(i-1)}, \overline{s_{2i}})\} \\
&\quad \cup \{(s_{2i}, s_{2i}), (\overline{s_{2i}}, \overline{s_{2i}})\} \\
\pi(s_j) &= \{r_j, p_j\}, \text{ for } 1 \le j \le n \\
\pi(t) &= \emptyset, \text{ for } t \notin \{s_j, \overline{s_j} \mid 1 \le j \le n\};
\end{aligned}
$$

- $T_0 = \{s_i, \overline{s_i} \mid 1 \le i \le n,\ i \text{ odd}\} \cup \{t_i \mid 1 \le i \le n/2\}$;

- f : QBF $\to$**MID** is the reduction function defined by

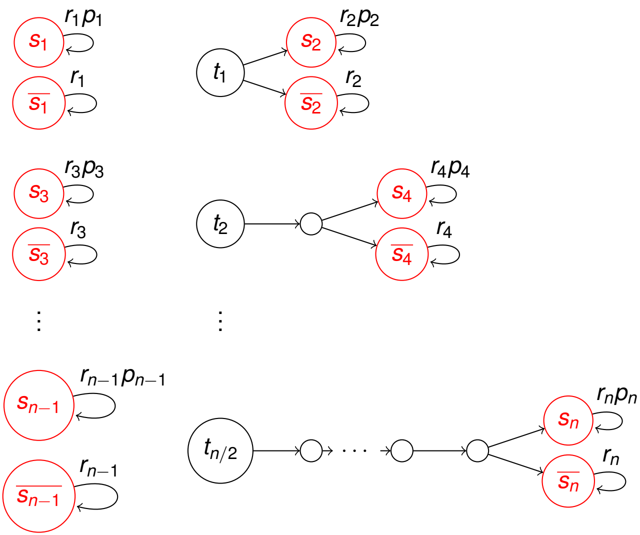
$$
\begin{aligned}
f(\psi) = \Big((r_1 \to\!=\!(p_1)) \to \Diamond \\
\Big((r_3 \to\!=\!(p_3)) \to \Diamond \\
\cdots \qquad \cdots \to \Diamond \\
\Big((r_{n-1} \to\!=\!(p_{n-1})) \to \Diamond \varphi^{\to} \Big) \Big) \ldots \Big) \Big).
\end{aligned}
$$

Model K



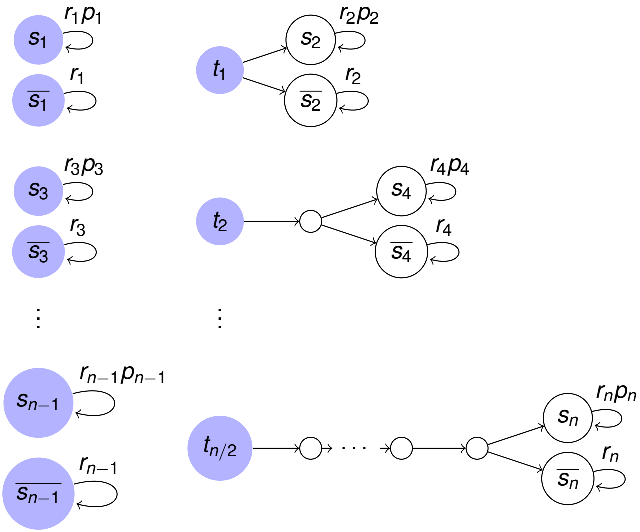
$r_1 p_1$
s_1
r_1
$\overline{s_1}$
t_1
$r_2 p_2$
s_2
r_2
$\overline{s_2}$
$r_3 p_3$
s_3
r_3
$\overline{s_3}$
t_2
$r_4 p_4$
s_4
r_4
$\overline{s_4}$
$r_{n-1} p_{n-1}$
s_{n-1}
r_{n-1}
$\overline{s_{n-1}}$
$t_{n/2}$
$\cdots$
$r_n p_n$
s_n
r_n
$\overline{s_n}$

Model K



r1p1
s1
r1
s̄1
t1
r2p2
s2
r2
s̄2
r3p3
s3
r3
s̄3
t2
r4p4
s4
r4
s̄4
rn−1pn−1
sn−1
rn−1
s̄n−1
tn/2
rnpn
sn
rn
s̄n

Model K



s_1
$r_1 p_1$
$\overline{s_1}$
r_1
t_1
s_2
$r_2 p_2$
$\overline{s_2}$
r_2
s_3
$r_3 p_3$
$\overline{s_3}$
r_3
t_2
s_4
$r_4 p_4$
$\overline{s_4}$
r_4
$\vdots$
$\vdots$
s_{n-1}
$r_{n-1} p_{n-1}$
$\overline{s_{n-1}}$
r_{n-1}
$t_{n/2}$
$\cdots$
s_n
$r_n p_n$
$\overline{s_n}$
r_n

It suffices to show:

for any QBF formula $\psi = \forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n \, \varphi$,

$$\psi \in \mathsf{TQBF} \iff K, T_0 \models f(\psi).$$

An example

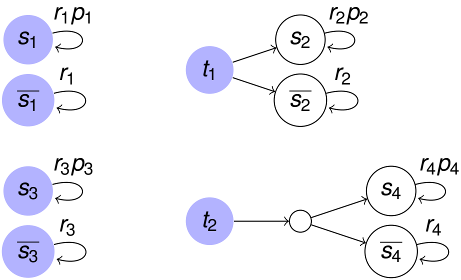
Let $\psi = \forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi$ be a QBF formula with φ quantifier-free.

Then

$$f(\psi) = (r_1 \rightarrow =(p_1)) \rightarrow \Diamond\big((r_3 \rightarrow =(p_3)) \rightarrow \Diamond\varphi^{\rightarrow}\big)$$

Claim:

$$\psi \in \text{TQBF} \iff K, T_0 \models f(\psi).$$



Proof of Claim "$\Longrightarrow$"

"$\Longrightarrow$": Suppose $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi \in$ TQBF, i.e. $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi \equiv \top$.

We will show that

$$K, T_0 \models (r_1 \rightarrow {=}(p_1)) \rightarrow \Diamond((r_3 \rightarrow {=}(p_3)) \rightarrow \Diamond \varphi^{\rightarrow})$$

$T_0 = \{$ blue points $\}$

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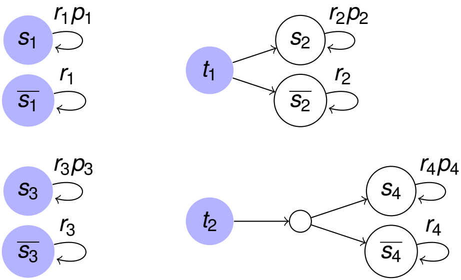

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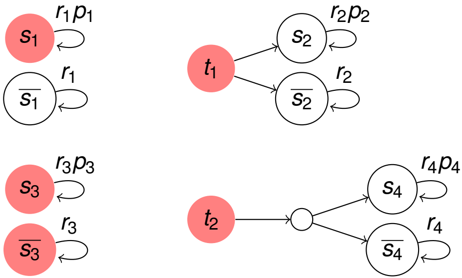


$T_1 = \{$ red points $\}$

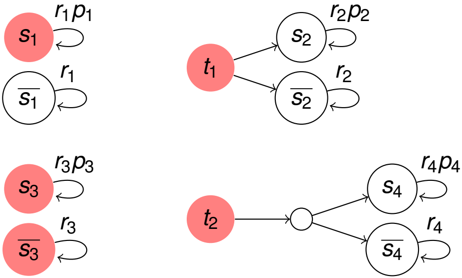
$K, T_1 \models r_1 \rightarrow\!=\!(p_1)$

Proof of Claim "$\Longrightarrow$"

"$\Longrightarrow$": Suppose $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi \equiv \top$.

Claim:

$$K, T_0 \models (r_1 \rightarrow =(p_1)) \rightarrow \Diamond((r_3 \rightarrow =(p_3)) \rightarrow \Diamond \varphi^{\rightarrow})$$



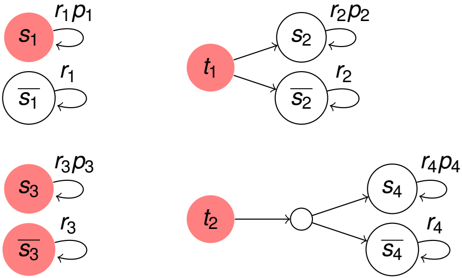
$T_1 = \{$ red points $\}$
$K, T_1 \models r_1 \rightarrow =(p_1)$
σ: $x_1 \mapsto \top$

Proof of Claim "$\Longrightarrow$"

"$\Longrightarrow$": Suppose $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi \equiv \top$.

Claim:

$$K, T_0 \models (r_1 \to\!=(p_1)) \to \Diamond((r_3 \to\!=(p_3)) \to \Diamond\varphi^{\to})$$



$T_1 = \{$ red points $\}$
$K, T_1 \models r_1 \to\!=(p_1)$
σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$

Proof of Claim "$\Longrightarrow$"

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Claim:

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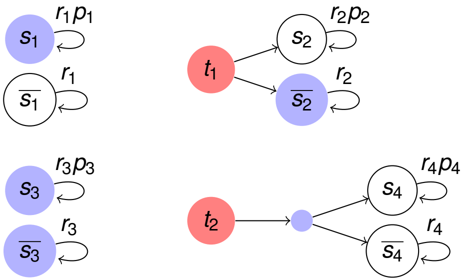


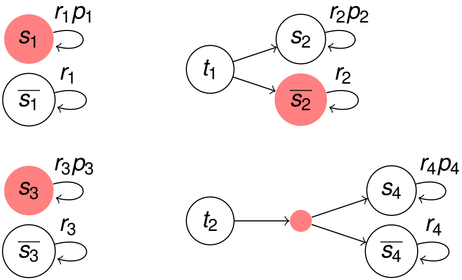
$T_2 = \{$ blue points $\}$

σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$

"$\Longrightarrow$": Suppose $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi \equiv \top$.

Claim:

$$K, T_0 \models (r_1 \to\!=\!(p_1)) \to \Diamond((r_3 \to\!=\!(p_3)) \to \Diamond \varphi^{\to})$$



$T_3 = \{$ red points $\}$

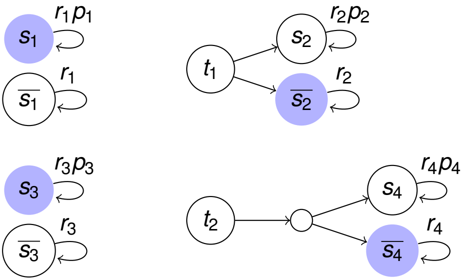
σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$, $x_3 \mapsto \top$

Proof of Claim "$\Longrightarrow$"

"$\Longrightarrow$": Suppose $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \varphi \equiv \top$.

Claim:

$$K, T_0 \models (r_1 \rightarrow =(p_1)) \rightarrow \Diamond((r_3 \rightarrow =(p_3)) \rightarrow \Diamond \varphi^{\rightarrow})$$



$T_4 = \{$ blue points $\}$

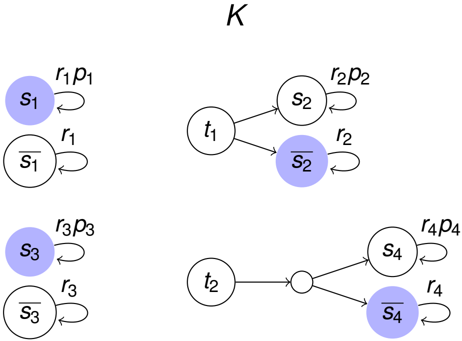
σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$, $x_3 \mapsto \top$, $x_4 \mapsto \bot$

$\sigma(\varphi) = \top$

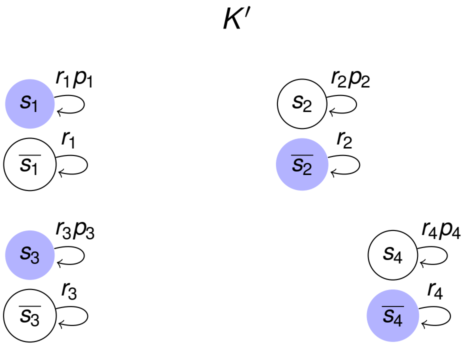
It suffices to show $K, T_4 \models \varphi^{\rightarrow}$.

Proof of Claim "$\Longrightarrow$"

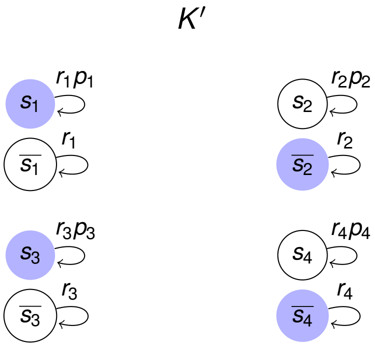
But now,



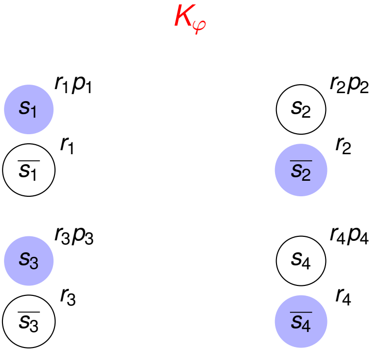
Proof of Claim "$\Longrightarrow$"



Proof of Claim "$\Longrightarrow$"



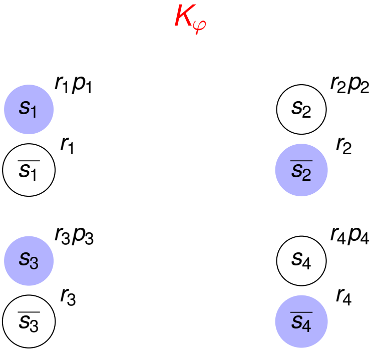
Proof of Claim “$\Longrightarrow$”



$T_4 = T_{\varphi,\sigma} = \{$ blue points $\}$ with

- σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$, $x_3 \mapsto \top$, $x_4 \mapsto \bot$
- $\sigma(\varphi) = \top$

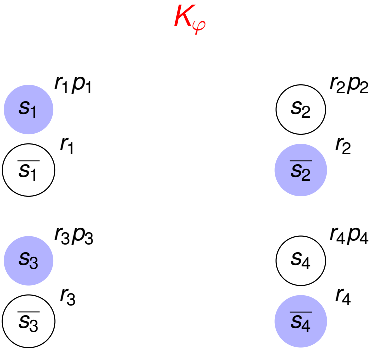
Proof of Claim "$\Longrightarrow$"



$T_4 = T_{\varphi,\sigma} = \{$ blue points $\}$ with

- $\sigma\colon x_1 \mapsto \top,\ x_2 \mapsto \bot,\ x_3 \mapsto \top,\ x_4 \mapsto \bot$
- $\sigma(\varphi) = \top$

Proof of Claim "$\Longrightarrow$"



$T_4 = T_{\varphi,\sigma} = \{$ blue points $\}$ with

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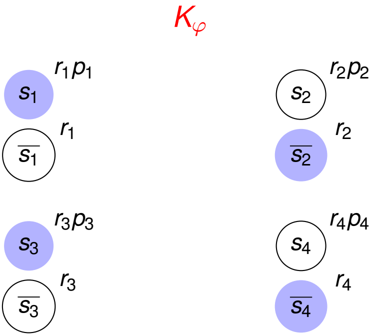
Proof of Claim "$\Longrightarrow$"

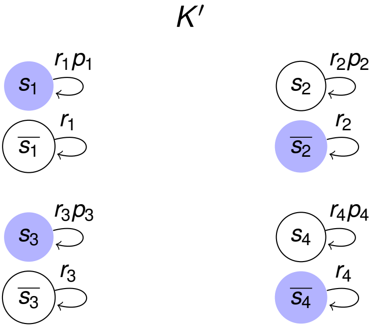


$T_4 = T_{\varphi,\sigma} = \{$ blue points $\}$ with σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$, $x_3 \mapsto \top$, $x_4 \mapsto \bot$

Hence

$$\sigma(\varphi) = \top$$
$$\Longrightarrow K_\varphi, T_{\varphi,\sigma} \models \varphi^{\to}$$

Proof of Claim "$\Longrightarrow$"

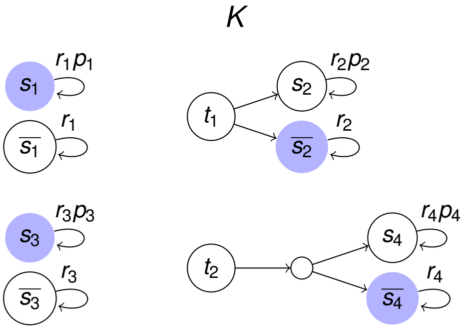


$T_4 = T_{\varphi,\sigma} = \{$ blue points $\}$ with σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$, $x_3 \mapsto \top$, $x_4 \mapsto \bot$

Hence

$$\begin{aligned} & \sigma(\varphi) = \top \\ \Longrightarrow & K_\varphi, T_{\varphi,\sigma} \models \varphi^{\rightarrow} \\ \Longrightarrow & K', T_4 \models \varphi^{\rightarrow}, \text{ since } \varphi^{\rightarrow} \text{ is modality-free} \end{aligned}$$

Proof of Claim "$\Longrightarrow$"



$T_4 = T_{\varphi,\sigma} = \{$ blue points $\}$ with σ: $x_1 \mapsto \top$, $x_2 \mapsto \bot$, $x_3 \mapsto \top$, $x_4 \mapsto \bot$
Hence

$$
\begin{aligned}
& \sigma(\varphi) = \top \\
\Longrightarrow & K_\varphi, T_{\varphi,\sigma} \models \varphi^{\rightarrow} \\
\Longrightarrow & K', T_4 \models \varphi^{\rightarrow}, \text{ since } \varphi^{\rightarrow} \text{ is modality-free} \\
\Longrightarrow & K, T_4 \models \varphi^{\rightarrow}, \text{ since } K' \text{ is a generated submodel of } K \quad \square
\end{aligned}
$$

The other direction "$\Longleftarrow$" is proved symmetrically.

Hence

Theorem

MID *-MC is PSPACE-hard.*

Theorem

MID *-MC is PSPACE-complete.*

The other direction "$\Longleftarrow$" is proved symmetrically.

Hence

Theorem

MID *-MC is PSPACE-hard.*

Theorem

MID *-MC is PSPACE-complete.*

Complexity results for fragments of **MID**-MC

Operators								Complexity	Method/Ref.
$\Box$	$\Diamond$	$\wedge$	$\otimes$	$\neg$	$\ovee$	$\rightarrow$	$=(\cdot)$		
$*$	$*$	$+$	$+$	$*$	$*$	$+$	$+$	PSPACE	reduct. from **TQBF**
$*$	$*$	$+$	$+$	$*$	$+$	$+$	$*$	PSPACE	see above
$*$	$+$	$+$	$*$	$*$	$+$	$+$	$*$	PSPACE	see above
$*$	$-$	$+$	$-$	$*$	$+$	$+$	$*$	coNP	reduct. from **TAUT**
$*$	$*$	$*$	$*$	$*$	$-$	$*$	$-$	in P	[CES 1986]
$*$	$*$	$+$	$+$	$*$	$*$	$-$	$+$	NP	[E& L 2011]
$+$	$*$	$*$	$+$	$*$	$*$	$-$	$+$	NP	[E& L 2011]
$*$	$+$	$*$	$*$	$*$	$*$	$-$	$+$	NP	[E& L 2011]
$*$	$-$	$*$	$-$	$*$	$*$	$-$	$*$	in P	[E& L 2011]
$*$	$*$	$*$	$*$	$*$	$*$	$-$	$-$	in P	[E& L 2011]

$+$: operator present $-$: operator absent
$*$: complexity independent of operator

That’s all.

Thank you!