Model-free norm-based fixed structure controller synthesis

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Abstract— **This paper presents a method to perform model-free fixed structure controller synthesis. Based on frequency response data of the plant, the parameters of a predefined controller structure are optimized directly with respect to closed-loop performance specifications. As a result, no parametric plant model is required such that time consuming iterative identification-synthesis procedures can be omitted.**

A framework is presented to both assure stability and optimize closed-loop performance based on frequency response data. Furthermore, based upon these results, a cost-function is formulated that can be exploited to converge from a destabilizing- to the stabilizing controller parameter region. Both the stability guarantee and performance optimization procedures are combined in one optimization algorithm that is illustrated by means of an example.

I. INTRODUCTION

Application of model-based optimal control synthesis methodologies require a low order parametric model of the plant. When only input-output data of a system is available, the most appropriate procedure to obtain such a model is via system identification. However, it has been widely recognized in system identification literature [10], [24] that a model can only approximate the real system behavior. To "tune" the model such that it contains closed-loop relevant aspects, iterative identification-synthesis procedures are proposed [10], [24]. Most recent work also includes closed-loop relevant uncertainty modeling [19], therefore handling misfit via robustness of the controller.

As an alternative for these iterative procedures, this paper considers controller synthesis based upon plant input-output data only without parametrization of the plant. The main advantage of a data-based approach is that the actual parametrization is performed in terms of the controller. As a result, closed-loop relevant aspects are taken into account automatically. This results in a straightforward design approach for norm-based controller synthesis, i.e. no identification-synthesis iterations are required.

Several model-free approaches can be found in literature. In [7], [21], a time domain LQG approach is deduced that is based on a plant description in terms of impulse response coefficients. Similar to this approach, [25] describes an $\mathcal{H}_\infty$ synthesis method for 2-block control problems. Contrary to these time-domain approaches, [1] describes a method to find the set of stabilizing parameters of 3-term controllers. The approach described in [12] contains aspects that can be applied in a non-free setting as well. In the context of fixed structure controller design, the line of work initiated by [13] proposes an iterative experimental scheme to locally optimize the controller parameter set of a predefined controller

structure. Virtual Reference Feedback Tuning (VRFT) is an other method to optimize parameters of a fixed controller structure without the need of a parametric model [3].

This paper proposes a frequency domain method to optimize fixed-structure controllers that are optimal in the sense of closed-loop norm specifications and are tuned based on experimental data only. The main contribution of this paper is a general framework to assure closed-loop stability based on a criterion formulated in terms of frequency response coefficients. A method is given that enables convergence from a destabilizing controller parameter set to a stabilizing parameter set. Furthermore a procedure is described to locally optimize performance for fixed structure controllers in the sense of $\mathcal{H}_\infty$ norm specifications.

The presented work can be seen as an extension and generalization of the work presented in [5]. Specific details about generalizations and extensions compared to results presented previously are given in the text.

The outline of the paper is as follows. Section II starts with the formal problem definition of this paper. Section III described how classical results from complex function theory are exploited and extended to formulate a stability criterion in terms frequency response data. Based on this result, Section IV presents a cost-function that can be used to converge to a stabilizing parameter set. Section V describes a performance optimization procedure that minimizes the largest singular values of the closed-loop frequency response matrices over all frequencies. In Section VI, both stability and performance optimization is combined in one optimization algorithm and applied for controller synthesis. A simulation example is given to illustrate the approach.

II. PROBLEM FORMULATION

In order to property formulate the problem statement, the following definitions are introduced:

Definition 1:

Define $s \in \mathbb{C}$, whereas $\mathbb{C}^+ := \{s \mid \mathrm{Re}(s) > 0\}$ and $\mathbb{C}^- := \{s \mid \mathrm{Re}(s) < 0\}$. Let $P(s)$ be a real rational function $\mathbb{C} \mapsto \mathbb{C}$ analytic in $\mathbb{C}^+$. $C(s, \theta)$ is a real rational complex function $\mathbb{C} \mapsto \mathbb{C}$ with coefficients $\theta \in \mathbb{R}$ to be optimized. Ω is a set of real-values equidistant points ω_i. The points P_i and $C_i(\theta)$ are defined as $P_i := \{P(j\omega_i) \mid \omega_i \in \Omega\}$ and $C_i := \{C(j\omega_i, \theta) \mid \omega_i \in \Omega\}$ respectively

The problem definition considered in this paper can be stated as:

Problem statement 1:

<u>*Given:*</u> *evaluations P_i of the unknown transfer function $P(s)$, where P_i represents the frequency response behavior of the*

data-generating system obtained on the frequency grid $\omega_i \in \Omega$.

Find: the parameters θ of a predefined controller structure $C(s, \theta)$ that result in a feedback interconnection $\mathrm{LFT}\big(P(s), C(s, \theta)\big)$ that is optimal in the sense of (1):

$$\min_{\theta \in \Theta_{stabilizing}} \| \mathrm{LFT}\big(P(s), C(s, \theta)\big)\|_\infty \qquad (1)$$

where $\Theta_{stabilizing}$ represents the set of all controller parameters that correspond to stabilizing controllers.

The problem statement represents the fixed structure controller optimization problem that is performed based on frequency response data of the plant. This frequency response data can be obtained via means of experiments [14].

A. Approach

Problem 1 describes an optimization problem in terms of the $\mathcal{H}_\infty$ norm of the closed-loop system. On the other hand, the plant behavior is only known at finite grid points in the set Ω. Under the assumption of smoothness of the closed-loop transfer functions, i.e. the poles of the closed-loop transfer function are contained in $\{\mathbb{C}_\varsigma^- \mid \varsigma < 0\}$, Eq.(1) can be approximated with:

$$\min_{\theta \in \Theta_{stabilizing}} \max_{\omega_i} \quad \bar{\sigma}\big(\mathrm{LFT}(P(j\omega_i), C(j\omega_i))\big) \qquad (2)$$

where $\bar{\sigma}$ represents the maximum singular value at ω_i. The approach to solve this problem is twofold:

1) Since the set $\Theta_{stabilizing}$ is not known on beforehand, the approach is to describe the set of stabilizing controller parameters in terms of a test on the frequency response points of the closed-loop. Hence, the optimization over the set $\Theta_{stabilizing}$ can be replaced with an optimization under an additional stability constraint formulated in terms of the frequency response points.

2) The controller parameters appear in a non-convex manner in the linear fractional transformation described in (2). Via a local linearization approach, that will be described in Sec.V, a local performance optimization can be performed. By increasing the number of starting points, the chance to convergence to global optimum increases.

The first aspect is discussed in more detail. To check stability of the closed-loop system, the Youla parameter Q is exploited [8], [26]. This Youla parameter describes a bijective mapping between set of stabilizing controllers and set of stable systems.

This mapping can be evaluated point wise such that:

$$\mathcal{Q} = \{ Q_i = C_i(I + P_iC_i)^{-1} \mid C(s) \in \mathcal{C}_{stab}\} \quad (3)$$

$$\mathcal{C}_{stab} = \{ C_i = Q_i(I - P_iQ_i)^{-1} \mid Q(s) \in \mathcal{Q}\} \qquad (4)$$

With $\mathcal{C}_{stab}$ the set of all stabilizing controllers. It is well known that $\mathcal{Q} = \mathcal{RH}_\infty$ [8]. Section III will derive a criterion to verify stability of $Q(s)$ based on it's frequency response coefficients Q_i.

Remark that the Youla parametrization can be applied in the generalized plant framework as well [26]. Without loss of generality, P is considered to be SISO to reduce notational aspects.

III. Closed-loop stability condition

Closed-loop stability is a primary requirement to be met during synthesis of feedback control systems. Since the dynamical behavior of the plant is only described by experimental input- output-data, the location of closed-loop poles can not be obtained by directly computation. Hence, closed-loop stability has to be obtained based upon frequency response points only. In this section, we formulate a criterion to guarantee that all poles are contained in $\mathbb{C}^-$ based on frequency response data. It has to be emphasized that we do not try to identify the location of the poles itself.

Although the transfer function of the data-generating system is unknown, frequency response points are obviously coupled to the locations of the poles by the frequency response function of the system. Mathematical properties of real rational functions, i.e. transfer functions under consideration, are exploited to uncover poles located in $\mathbb{C}^+$ based on frequency response points. From a mathematical point of view, a real rational complex function belongs to the class of holomorphic functions. For this class of functions, the imaginary part and real parts of a complex function do not behave as independent variables. Based on this elementary property, the Cauchy Residue Theorem is well known. Based upon this theorem, the derivation of the stability condition is started.

As a first step, it is assumed that frequency response data are given for a continuum of frequency points. Afterwards, it is checked under which conditions this relation can be discretised. We start by stating the main result of this section.

Theorem 1. *A strictly proper system $Q(s)$ is stable if and only if:*

$$\int_{-\infty}^{\infty} \frac{Q(j\omega)}{s - p^*} \, j \, d\omega = 0, \quad \forall p^* \in \mathbb{C}^- \qquad (5)$$

where p^* is a manually added pole which is located in $\mathbb{C}^-$.

Proof. A detailed proof of Thm. 1 can be found in [5].

$\square$

The proof Thm.1 is based onto Cauchy Residue Theorem which is exploited extensively in literature and has been the basis for the derivation of many analytical properties [9], [18], [23]. It is however important to remark that during these derivations, stability is a pre-assumed property rather than a guaranteed property of the transfer function. Contrary to this assumption, Thm.1 presents a condition that guarantees stability. This is possible by adding a series of poles in $\mathbb{C}^-$. The resulting stability condition is very interesting from a data-based control synthesis perspective.

The presented approach can be considered as an extension of the work presented in [5] by proposing a method to compute of a gradient towards the stabilizing controller parameter region as will be described in Section IV.

Remark 1:
- As a practical rule of thumb, it is sufficient to chose the number of test points p^* much larger than the expected order of $Q(s)$.
- Due to practical constraints, frequency response data is only available on the finite frequency points contained in Ω. In order to be able to exploit Theorem 1, (5) has to be approximated by a discrete series approximation.

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We refer to the convergence proof presented in [5] which proves that (5) can be well approximated by a finite series approximation under the assumption that $\frac{|\operatorname{Re}(p_i)|}{\Delta} >> 1$. Here, $\operatorname{Re}(p_i)$ represents the real part of the poles of $Q(s)$ and Δ is the spacing between the discrete equidistant grid points contained in Ω.

IV. GRADIENT TO STABILIZING REGION

The use of many starting points $\theta_0 \in \Theta_{stabilizing}$ are helpful in trying to find the global optimum of the fixed structure controller synthesis problem via gradient based optimization. However, the set of stabilizing controllers $\Theta_{stabilizing}$ is commonly unknown on beforehand. Even when one stabilizing controller is known, the entire set of stabilizing controller controllers can not be derived since this set is not convex and even not necessary connected.

The theory given in Section III is exploited to check if a chosen starting point θ_0 satisfies $\theta_0 \in \Theta_{stab}$. However, local minima with high performance could possibly be located in a small stabilizing parameter sets. This require a high density of starting points in order find these region. For these cases, it is valuable to have an algorithm that converges from a destabilizing starting point θ_0 to a neighboring stabilizing parameter-set. As a result, the density of starting points can be reduced which subsequently lowers computational costs.

Theorem 1 presents a qualitative test for stability. However, to compute a gradient, a qualitative measure is required. Construction of such a cost-function is not trivial since all system norms explode if poles move to the imaginary axis. Hence, choosing a cost-function that is based on a norm, results in a barrier for poles moving from $\mathbb{C}^+$ to $\mathbb{C}^-$.

A cost-function that defines the required gradient can be constructed by exploiting the theory presented in the previous section. The novel idea here is to define the cost-function as a fraction of 2-norms. This eliminates the barrier and creates a gradient towards Θ_{stab}. This will be discussed in a formal manner.

The approach is as follows. As a first step, the gradient is computed in terms of the poles of $Q(s)$. Afterwards, the gradient in terms of θ is computed by taking partial derivative of Q along controller parameters θ.

The main result of this section is directly posed by the following theorem:

Lemma 2. *Given a transfer-function with one conjugate pole pair. Then the gradient of the cost-function ϕ, defined as:*

$$\phi = \frac{\|Q_\delta^*\|_2^2}{\|Q\|_2^2} \qquad (6)$$

represents a global region of attraction towards the stabilizing region. Here,

$$Q_\delta^*(j\omega) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{Q(j\omega)}{j\omega - p^*} \, j \, d\omega, \quad p^* \in j\omega + \delta, \delta \in \mathbb{R}^- \qquad (7)$$

Proof. Write $Q(s)$ in its partial fraction expansion as:

$$Q(s) = \sum_i \frac{\alpha_i}{s - p_i} \qquad (8)$$

with α_i the residue corresponding to p_i. The 2-norm of Q

can be obtained via [4], [6]:

$$\|Q\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} Q(j\omega)Q^\dagger(-j\omega)d\omega \qquad (9)$$

where $Q^\dagger$ represents the complex conjugate of Q which can be obtained by taking the complex conjugate of all coefficients.

Under the assumption that $Q(j\omega)$ has at least relative degree 2, the integral described in (9) can be computed by considering a contour integral over the Nyquist D-contour and applying Cauchy's Residue Theorem [16]. Hence, (9) can be computed by taking the sum of the residues of the integrant corresponding the poles in $\mathbb{C}^+$. Via this method, it can be derived that [4]:

$$\|Q\|_2^2 = \sum_n \sum_m \frac{\alpha_n \alpha_m^\dagger}{p_n + p_m^\dagger} \qquad (10)$$

The same approach can be used to compute $\|Q_\delta^*\|_2$. However by Theorem 1 it can be shown that the value of $\|Q_\delta^*\|_2^2$ depends on the location of the poles of Q.

$$Q_\delta^*(j\omega) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{Q(j\omega)}{s - p^*} j d\omega, \quad p^* = j\omega + \delta, \delta \in \mathbb{R}^- \qquad (11)$$

$$= \sum_{p_i \in \mathbb{C}^+} \frac{\alpha_i}{j\omega - (p_i - \delta)} \qquad (12)$$

Hence:

$$\|Q_\delta^*\|_2^2 = \sum_{p_n \in \mathbb{C}^+} \sum_{p_m \in \mathbb{C}^+} \frac{\alpha_n \alpha_m^\dagger}{(p_n - \delta) + (p_m^\dagger - \delta)} \qquad (13)$$

By substitution of (9) and (13), the following expression can be found for ϕ:

$$\phi = \frac{\sum_{p_n \in \mathbb{C}^+} \sum_{p_m \in \mathbb{C}^+} \frac{\alpha_n \alpha_m^\dagger}{(p_n - \delta) + (p_m^\dagger - \delta)}}{\sum_n \sum_m \frac{\alpha_n \alpha_m^\dagger}{(p_n + p_m^\dagger)}} \qquad (14)$$

which can be simplified if Q contains one conjugated pole pair such that $p_2 = p_1^\dagger$ and $\alpha_2 = \alpha_1^\dagger$ which gives:

$$\phi = \frac{\operatorname{Re}(p)(\operatorname{Re}(p)^2 + \operatorname{Im}(p)^2)}{(\operatorname{Re}(p) - \delta)\left((\operatorname{Re}(p) - \delta)^2 + \operatorname{Im}(p)^2\right)} \qquad (15)$$

The function ϕ is plotted as function of $\operatorname{Re}(p)$ in Fig.1. It can be proved that:

$$\frac{d\phi}{d\operatorname{Re}(p)} = \frac{-\delta(3a^4 - 6a^3\delta + 3a^2\delta^2 + b^2\delta^2 + b^4)}{(a - \delta)^2(a^2 - 2a\delta + \delta^2 + b^2)^2} \qquad (16)$$

with $a = \operatorname{Re}(p)$ and $b = \operatorname{Im}(p)$. Since $\delta < 0$ and $\operatorname{Re}(p) > 0$, the denominator is never negative and the nominator is always positive. Hence, function is monotonically increasing in $\mathbb{C}^+$. This results in a gradient towards stabilizing set which proves Lemma 2. $\square$

The result presented in Lemma 2 can be generalized towards the case that Q contains several pole pairs, albeit with local convergence properties instead of global convergence.

Lemma 3. *Given an arbitrary real rational transfer function $Q = \sum_i \frac{\alpha_i}{s - p_i}$. Then the gradient of the cost-function ϕ presented in (6) represent a local region of attraction towards the stabilizing region.*
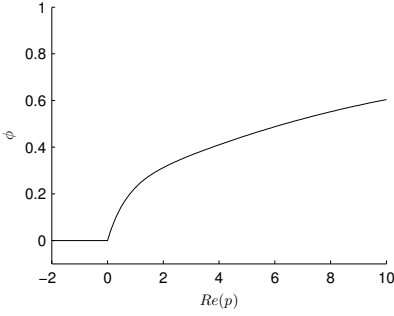
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Fig. 1. Behavior of the function ϕ for $\delta = -2$

By observing (14), it can be seen that ϕ is not only reduced by moving unstable poles to imaginary axis but also by moving stable poles to the imaginary axis which increases the denominator term. However, if instable poles are located close to the imaginary axis, ϕ is dominated by the corresponding term $\frac{\alpha\alpha^\dagger}{p+p^\dagger}$. Hence, the steepest descend direction is expected to shift the corresponding poles over the imaginary axis. This result in a local region of attraction for instable poles located closely near the imaginary axis.

Remark 2: To enable practical evaluation of ϕ based on experimental data, the integrals required to compute the $\|Q_\delta^*\|_2^2$ and $\|Q\|_2^2$ have to be approximated by a series approximation. Via the approach described in [5], the integrals can be approximated by discrete sums.

V. NORM MINIMIZATION

Complementary to the results presented in Section III and IV that focus on stability, this section will focus on performance optimization. Both parts are combined into one optimization algorithm presented in Section VI.

According to the problem statement and approach presented in Section II, the $\mathcal{H}_\infty$ norm minimization problem can be approximated by a point wise minimization of the maximum singular values over all $\omega_i \in \Omega$. This section presents a method to compute a gradient in the parameter space that reduces the maximum singular value.

We start by making (2) more explicit by rewriting the plant using the generalized plant framework given by:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad (17)$$

where $[w, u]^T$ are the exogenious disturbances and controller output respectively and $[z, y]^T$ represent the performance variables and controller input.

To reduce the complexity of derivations, the Youla parameter is introduced which appears affine in the performance optimization problem:

$$\min \max_{\omega_i} \bar{\sigma}\Big(P_{11}(j\omega_i) + P_{12}(j\omega_i)Q(j\omega_i)P_{21}(j\omega_i)\Big) \quad (18)$$

Minimization of the maximum singular values over all frequencies is achieved by the following optimization problem:

$$\min \quad \gamma \qquad (19)$$

$$s.t.: \quad T_i \succ 0, \quad \forall \omega_i \in \Omega \qquad (20)$$

with

$$T_i = \begin{bmatrix} \gamma I & P_{11_i} + P_{12_i}Q_iP_{21_i} \\ (P_{11_i} + P_{12_i}Q_iP_{21_i})^\dagger & \gamma I \end{bmatrix} \qquad (21)$$

Although the problem is affine in Q, the controller parameters appear in a non-linear manner in this set of matrix inequalities. This makes that the optimization problem in this form is not directly useful for optimization.

It can however be proved that (20) can be locally approximated by a set of Linear Matrix Inequalities (LMI's). Based upon these LMI's, methods from interior points LMI solvers can be used to compute the gradient of the direction that reduces the maximum singular value. The proposed optimization algorithm is as follows:

1) Locally approximated (21) around the current parameter $\theta(k)$ with an LMI.
2) Compute the gradient via LMI optimization tools and make one iteration to obtain $\theta(k+1)$.

The proposed approach has similarities with the path following algorithm proposed by [11]. However, contrary to this approach, the assumption of small deviations is automatically fulfilled since the closed-loop system is linearized around $\theta(k)$ obtained from the previous iteration. This omits the assumption of low-authority controllers.

The following Lemma is used to prove that (20) can be locally approximated by a set of LMI's.

Lemma 4. *A non-linear matrix inequality, as described in (20), can be locally approximated around $\bar{\theta}$ by:*

$$T(\theta) = T(\bar{\theta}) + \sum_n \frac{\partial T(\bar{\theta})}{\partial \theta_n}(\theta_n - \bar{\theta}_n) + \epsilon \qquad (22)$$

where ϵ converges to zero if $\bar{\theta} - \theta$ is sufficiently small, if $det(I + PC)$ is sufficiently large and if a controller structure is chosen such that the frequency response function $C(j\omega)$ behaves smooth with respect to θ.

In order to prove Lemma 4, T_i is written as a Taylor expansion. To compute this expansion, matrix differential calculus is exploited. We use the notation introduced by [20] and further described in [15].

According to [15], [20], a matrix derivative is notated by:

$$\frac{dF(X)}{dX} := \frac{\partial \text{vec } F(X)}{\partial \text{vec } X^T} \qquad (23)$$

Using this definition, the chain rule for $Y = F(X)$ respectively equals:

$$\frac{dG(F(X))}{dX} = \frac{dG(Y)}{dY}\frac{dY}{dX} \qquad (24)$$

Furthermore, it can be derived that the derivative of the inverse of a matrix equals [15]:

$$\frac{dF^{-1}}{dX} = -F^{-1}\frac{dF}{dX}F^{-1} \qquad (25)$$

Based upon these derivations, Lemma 4 can be proven.

Proof. Using (24), the first order derivative of $T(j\omega_i)$ can be computed as:

$$\frac{dT_i(\theta)}{d\theta} = \frac{dT_i(Q_i)}{dQ_i}\frac{dQ_i(C_i)}{dC_i}\frac{dC_i(\theta)}{d\theta} \qquad (26)$$

The right hand terms are computed separately:

$$\frac{dT_i}{dQ_i} = \begin{bmatrix} 0 & P_{12_i}IP_{21_i} \\ P_{12_i}IP_{21_i} & 0 \end{bmatrix} \qquad (27)$$

Without loss of generality, it is assumed that C is a SISO controller to reduce notation. As a consequence, Q is SISO

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such that:

$$Q_i(C_i) = C_i(I + P_{22_i}C_i)^{-1} \qquad (28)$$

$$\frac{dQ_i}{dC_i} = -(1 + P_{22_i}C_i)^{-1}P_{22_i}(1 + P_{22_i}C_i)^{-1} \qquad (29)$$

The partial derivative of $\frac{dC_i(\theta)}{d\theta}$ depends on the chosen controller structure and is therefore not given explicitly. However, in order to obtain a smooth relation between C_i and θ, a minimum distance between the poles of C and the imaginary axis is required.

In a similar manner, high order derivatives can be computed. It can be shown that the contribution of high order derivatives in the Taylor expansion converge to zero if $\lim_{n\to\infty} \frac{1}{n!}(det(I + PC))^n \neq \infty$ and $C(j\omega)$ is behaving smooth with respect to the parameters θ.

This shows that for small perturbations $(\bar{\theta} - \theta)$, the Taylor expansion is dominated by the first order derivative which finishes the proof. $\qquad \square$

Given the LMI approximation formulated in (22), the gradient can be computed using gradient computation of interior points LMI barrier solvers. The gradient can be computed by [2], [17]:

$$g_{n,i} = tr(S_i^{-1}\frac{\partial T_i(\bar{\theta})}{\partial \theta_n}) \qquad (30)$$

where $g_{n,i}$ denotes the gradient in the direction of θ_n for frequency ω_i and $S = T_i(\bar{\theta}) - \sum_n \frac{\partial T_i(\bar{\theta})}{\partial \theta_n}$. The gradient of the entire problem formulated in (20) can be obtained by taking the sum of the gradient obtained for all frequencies. Remark 3: The number of frequency points contained in Ω is typically large. This makes that the optimization algorithm has to take into account a large number of LMI constraints. Due to the barrier-function used to compute the gradient of the LMI, the gradient will be dominated by frequencies where $\bar{\sigma}$ is relatively large. Hence, from a practical point of view, it is sufficient to take into account LMI's corresponding to frequencies with large singular values only. This can significantly reduce computational costs.

VI. OPTIMIZATION PROCEDURE

This section combines the stability condition described in Section III with the performance optimization step described in Section V. It is shown that it is allowed to split the optimization process into two sequential steps to makes optimization procedure easier to implement.

The following Lemma is used for this purpose:

Lemma 5. *During minimization of a closed-loop norm that implies robust stability, e.g. the four-block control problem [22], stability is automatically maintained.*

Proof. If $\det(I + PC) = 0$, one of the closed-loop poles of the system is located on the imaginary axis. Since no pole-zero cancelations occur simultaneously in all terms of the performance criterion, the norm of the closed-loop system goes to infinity if poles cross the imaginary axis. This can not occur since $\bar{\sigma}$ is minimized. $\qquad \square$

Application of Lemma 5 allows the following sequential procedure:

1) Test if the starting point θ_0 corresponds to a stabilizing controller using Theorem 1.
   yes: proceed with step 2).
   no: converge to a neighboring stabilizing region with the approach presented in Section IV.
2) Once a stabilizing parameter set is reached, the performance optimization procedure discussed in Section V is applied. By sequentially linearizing and making a small steps in the steepest descent direction, the maximum singular value over all frequencies is minimized.

Some practical remarks have to be made about this procedure. Since LMI interior points solvers exploit barrier functions [2], [17], the optimization can only be started using a feasible starting point. These points can be easily achieved by choosing:

$$\gamma = \max_{\omega_i} \bar{\sigma}(P_{11} + P_{12}Q(\theta)P_{21}) + \epsilon \qquad (31)$$

where ϵ represents a small offset required to avoid numerical problems.

The usual approach to apply concordance of barriers is not applicable in this setting due to the sequential linearization procedure. However by applying (31) after each iteration, it is guaranteed that convergence speed is maintained.

The linearized matrix inequality only holds in the vicinity of $\bar{\theta}$. If the norm of $(\theta(k + 1) - \theta(k))$ is large, performance can be deteriorated or even a parameter-value outside $\Theta_{stabilizing}$ can be found. By checking high order terms of Taylor approximation, applying proper step-size control (e.g. via backtracking [2]), and regularly checking stability, it is expected that these problems do not occur in practice.

VII. SIMULATION STUDY

To illustrate the proposed approach, a simulation example is given that contains both the stability and performance optimization step. The following system is considered for evaluation:

$$P = \frac{0.0002s^2 + 0.1585s + 0.25}{1.2s^4 + 0.7629s^3 + 550s^2 + 0.1585s + 0.5} \qquad (32)$$

To mimic experimental data, points P_i are generated by substitution of $s = j\omega_i$ into $P(s)$ with $\Omega = \{n\Delta \mid \Delta = 0.1, n \in \{-1000, \ldots, -1, 0, 1, \ldots 1000\}\}$. It has to be emphasized that this data is only generated from the transfer function to allow evaluation with the analytical results. In practice, such data can be obtained from the experimental setup by frequency response experiments [14].

The chosen controller structure with the parameters to be optimized is given by:

$$C(s, \theta) = k_p + k_d s \qquad (33)$$

with $\theta = [k_p, k_d]^T$ is the parameter vector to be optimized. Fig.2 depicts the cost-function ϕ is computed via (6) for $\delta = -5$. It can be observed that, although Q contains several pole pairs, the gradient of ϕ is in the direction of the stabilizing region as depicted in Fig.3. Fig.3 show the top-view of ϕ over the parameter k_p and k_d. The dark region corresponds to $\phi < 0.01$ and therefore corresponds to the stabilizing parameter region. The analytical boundary of the stabilizing region is depicted by the black dots. It can be observed that the data-based stability boundary matches very well with that analytical result.
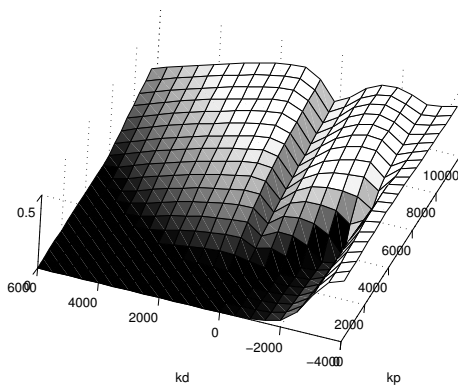
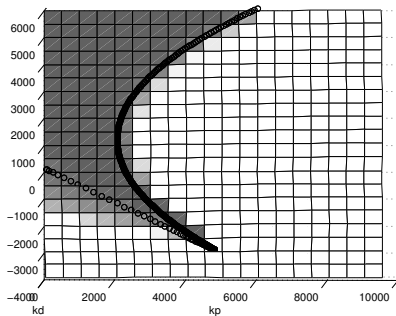**4034**

Fig. 2. ϕ as function k_p and k_d

Fig. 3. Top view of Fig.2 with analytical computed boundary of $\Theta_{stabilizing}$

One iteration of the performance optimization step is depicted in Fig.4. Each ellipse depicts the non-feasible region of T_i. The arrow represents one iteration computed using (22). It can be observed that the computed iteration converges to the region of improved performance, i.e. a region of lower γ.
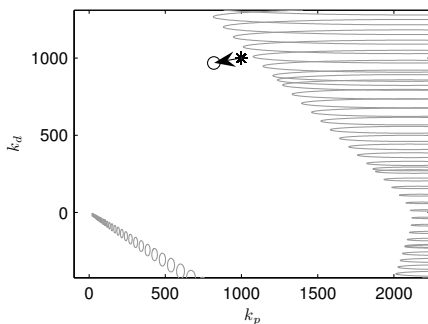

Fig. 4. Feasible set of controller optimization problem with iteration towards region of improved performance.

VIII. CONCLUSIONS

This paper presents a method to optimize the parameters of a fixed order controller based on frequency response data. The resulting controllers are optimal with respect to the $\mathcal{H}_\infty$ norm of the closed-loop system. This approach has the advantage that no low-order parametric plant model is required such that time-consuming plant identification-controller synthesis iterations can be omitted.

Both generic tools for stability and performance optimization are presented that can be applied by using frequency response data only. A stability criterion in terms of frequency response points is given and cost-function for convergence to a stabilizing parameter set is presented. Stability and performance optimization are combined in one optimization algorithm. Simulation results are given to illustrate and validate the presented approach.

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