MODEL-INDEPENDENT BOUNDS FOR OPTION PRICES: A MASS TRANSPORT APPROACH

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Abstract. In this paper we investigate model-independent bounds for exotic options written on a risky asset using infinite-dimensional linear programming methods. Using arguments from the theory of Monge-Kantorovich mass-transport we establish a dual version of the problem that has a natural financial interpretation in terms of semi-static hedging. In particular we prove that there is no duality gap.

Keywords Model-independent pricing, Monge-Kantorovich transport problem, option arbitrage.
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1. Introduction

Since the introduction of the Black-Scholes paradigm, several alternative models which allow to capture the risk of exotic options have emerged: local volatility models, stochastic volatility models, jump-diffusion models, mixed local stochastic volatility models. These models depend on various parameters which can be calibrated more or less accurately to market prices of liquid options (such as vanilla options). This calibration procedure does not uniquely set the dynamics of forward prices which are only required to be (local) martingales according to the no-arbitrage framework. This could lead to a wide range of prices of a given exotic option when evaluated using different models calibrated to the same market data.

In practice, it would be interesting to know lower and upper bounds for exotic options produced by models calibrated to the same market data, and therefore with similar marginals. If bounds are tight enough, they would be used to detect the possibility of arbitrage in market prices, provided these bounds have an interpretation as investment strategies. This problem has already been studied in the case of exotic options written on multi-assets $(S^1, \dots, S^k)$ observed at the same time T [BP02, BHR01b, CDDV08, CO11, HLW05a, HLW05b, LW05, LW04]. Within the class of models with fixed marginals $\left(\mathrm{Law}(S^1_T), \dots, \mathrm{Law}(S^k_T)\right)$ at T, the search for lower/upper bounds involves infinite-dimensional linear programming issues. Analytical expressions have been obtained in the case of basket options [LW05, LW04]. These correspond to the determination of optimal copulas. In practice, these bounds are not tight as the information of marginals is not restrictive enough.

Here we focus on multi-period models and general path-dependent options. This problem, which has not been extensively considered in the literature as far as we know, is more involved as we have to impose that the asset price S_t is a discrete time martingale[1] satisfying marginal restrictions.

In our setting the problem of determining the interval of consistent prices of a given exotic option can be cast as a (primal) infinite-dimensional linear programming problem. We propose a dual problem that has a practically relevant interpretation in terms of trading strategies and prove that there is no duality gap under rather mild regularity assumptions.

Setting. In the following, we fix an exotic option depending only on the value of a single asset S at discrete times $t_1 < \ldots < t_n$ and denote by $\Phi(S_1, \dots, S_n)$ its payoff, where we suppose Φ to be some measurable

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[1] For the sake of simplicity, we assume zero interest rate and no cash/yield dividends. This assumption can be relaxed by considering the process f_t introduced in [HL09] (see equation 14) which has the property to be a local martingale.

function. In the no-arbitrage framework, the standard approach is to postulate a model, that is, a probability measure $\mathbb{Q}$ on $\mathbb{R}^n$ under which the coordinate process $(S_i)_{i=1}^n$

$$S_i : \mathbb{R}^n \to \mathbb{R},\ S_i(s_1, \ldots, s_n) = s_i,\ i = 1, \ldots, n,$$

is required to be a (discrete) martingale in its own filtration. By $S_0 = s_0$ we denote the current spot price. The fair value of Φ is then given as the expectation of the payoff

$$\mathbb{E}_{\mathbb{Q}}[\Phi].$$

Additionally, we impose that our model is calibrated to a continuum of call options with payoffs $\Phi_{i,K}(S_i) = (S_i - K)^+, K \in \mathbb{R}$ at each date t_i and price

$$C(t_i, K) = \mathbb{E}_{\mathbb{Q}}[\Phi_{i,K}] = \int_{\mathbb{R}^+} (s - K)_+ \, d\mathrm{Law}_{S_i}(s). \tag{1}$$

Plainly (1) is tantamount to prescribing probability measures $\mu_1, \ldots, \mu_n$ on the real line[2] such that the *one dimensional marginals* of $\mathbb{Q}$ satisfy

$$\mathbb{Q}^i = \mathrm{Law}_{S_i} = \mu_i \ \text{ for all } i = 1, \ldots, n.$$

Primal formulation. For further reference, we denote by $\mathcal{M}(\mu_1, \ldots, \mu_n)$ the set of all martingale measures $\mathbb{Q}$ on (the pathspace) $\mathbb{R}^n$ having marginals $\mathbb{Q}^1 = \mu_1, \ldots, \mathbb{Q}^n = \mu_n$ and mean s_0. Equivalently, we have $\mathbb{Q} \in \mathcal{M}(\mu_1, \ldots, \mu_n)$ if and only if $\mathbb{E}_{\mathbb{Q}}[S_i | S_1, \ldots, S_{i-1}] = S_{i-1}$ for $i = 2, \ldots, n$ and $\mathbb{E}_{\mathbb{Q}}[\Phi_{i,K}] = C(t_i, K)$ for all $K \in \mathbb{R}$ and $i = 1, \ldots, n$.

Following the tradition customary in the optimal transport literature we concentrate on the lower bound and consider the *primal problem*

$$P = \inf \{\mathbb{E}_{\mathbb{Q}}[\Phi] : \mathbb{Q} \in \mathcal{M}(\mu_1, \ldots, \mu_n)\}. \tag{2}$$

Dual formulation. The dual formulation corresponds to the construction of a *semi-static subhedging strategy* consisting of the sum of a static vanilla portfolio and a delta strategy.[3] More precisely, we are interested in payoffs of the form

$$\Psi_{(u_i),(\Delta_j)}(s_1, \ldots, s_n) = \sum_{i=1}^n u_i(s_i) + \sum_{j=1}^{n-1} \Delta_j(s_1, \ldots, s_j)(s_{j+1} - s_j), \qquad s_1, \ldots, s_n \in \mathbb{R}, \tag{3}$$

where the functions $u_i : \mathbb{R} \to \mathbb{R}$ are μ_i-integrable ($i = 1, \ldots, n$) and the functions $\Delta_j : \mathbb{R}^j \to \mathbb{R}$ are assumed to be bounded measurable ($j = 1, \ldots, (n-1)$).

If these functions lead to a strategy which is subhedging in the sense

$$\Phi \geq \Psi_{(u_i),(\Delta_j)}$$

we have for every pricing measure $\mathbb{Q} \in \mathcal{M}(\mu_1, \ldots, \mu_n)$ the obvious inequality

$$\mathbb{E}_{\mathbb{Q}}[\Phi] \geq \mathbb{E}_{\mathbb{Q}}[\Psi_{(u_i),(\Delta_j)}] = \mathbb{E}_{\mathbb{Q}}\Big[\sum_{i=1}^n u_i(S_i)\Big] = \sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i]. \tag{4}$$

This leads us to consider the *dual problem*

$$D = \sup \Big\{ \sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i] : \exists\, \Delta_1, \ldots, \Delta_{n-1} \text{ s.t. } \Psi_{(u_i),(\Delta_j)} \leq \Phi \Big\}; \tag{5}$$

[2]The cumulative distribution function of μ_i can be read off the call prices through $F_i(K) = \lim_{\varepsilon \downarrow 0} 1/\varepsilon[C(t_i, K - \varepsilon) - C(t_i, K)]$ for $i = 1, \ldots, n$. Concerning the mathematical finance application it would be sufficient to consider strikes $K \geq 0$ and marginals which are concentrated on the positive half-line. We prefer to go with the more general case since the proofs are not more complicated. A technical difference is that call prices satisfy only $\lim_{K \to -\infty} C(t_i, K) - K = s_0$ rather than the simpler $C(t_i, 0) = s_0$ in the case where S is assumed to be non-negative.

[3]Similar strategies are considered in [DH07, Cou07] where they are used to subreplicate a European option based on finitely many given call options.

which, by (4), satisfies

$$P \geq D. \tag{6}$$

Semi-static subhedging. The dual formulation corresponds to the construction of a semi-static subhedging portfolio consisting of static vanilla options $u_i(S_i)$ and investments in the risky asset according to the self-financing trading strategy $(\Delta_i(S_1, \ldots, S_i))_{i=1}^{n-1}$.

We note the financial interpretation of inequality (6): suppose somebody offers the option Φ at a price $p < D$. Then there exists $(u_i), (\Delta_j)$ with $\Psi_{(u_i),(\Delta_j)} \leq \Phi$ with price $\sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i]$ strictly larger than p. Buying Φ and going short in $\Psi_{(u_i),(\Delta_j)}$, the arbitrage can be locked in.

The crucial question is of course if (6) is sharp, i.e. if every option priced below P allows for an arbitrage by means of semi-static subhedging. In Theorem 1 below we show that this is the case under relatively mild assumptions.

Of course it is a classical theme of Mathematical Finance that the extremal martingale prices of a financial derivative correspond to the minimal resp. maximal initial capital necessary for sub-/super-replication. This is precisely the replication theorem of mathematical finance, which is a corollary of the fundamental theorem of asset pricing. The novelty of our contribution is that we establish a robust, *model-free* version of this result.

Main result.

Theorem 1. *Assume that $\mu_1, \ldots, \mu_n$ are Borel probability measures on $\mathbb{R}$ such that $\mathcal{M}(\mu_1, \ldots, \mu_n)$ is non-empty. Let $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ be a lower semi-continuous function such that*

$$\Phi(s_1, \ldots, s_n) \geq -K \cdot (1 + |s_1| + \ldots + |s_n|) \tag{7}$$

on $\mathbb{R}^n$ for some constant K. Then there is no duality gap, i.e. $P = D$. Moreover, the primal value P is attained, i.e. there exists a martingale measure $\mathbb{Q} \in \mathcal{M}(\mu_1, \ldots, \mu_n)$ such that $P = \mathbb{E}_{\mathbb{Q}}[\Phi]$.

The dual supremum is in general not attained (cf. Proposition 4.1 below).

Our approach to this result is based on the duality theory of optimal transport which is briefly introduced in Section 2; the actual proof will be given in Section 3 with the help of the Min-Max Theorem of decision theory. We conclude this introductory section by a short discussion of the content of Theorem 1.

The assumption $\mathcal{M}(\mu_1, \ldots, \mu_n) \neq \emptyset$ excludes the degenerate case in which no calibrated market model exists. For the existence of a martingale measure having marginals $\mu_1, \ldots, \mu_n$ it is necessary and sufficient that these measures possess the same finite first moments and increase in the *convex order*, i.e. $\mathbb{E}_{\mu_1}\phi \leq \ldots \leq \mathbb{E}_{\mu_n}\phi$ for each convex function $\phi : \mathbb{R} \to \mathbb{R}$ (cf. [Str65]).[4]

Having the financial interpretation in mind, it is important that the value D of the dual problem remains unchanged if a smaller set of subhedging strategies $\Psi_{(u_i),(\Delta_j)}$ is used. In the proof of Theorem 1 we show that it is sufficient to consider functions $u_1, \ldots, u_n$ which are linear combinations of finitely many call options (plus one position in the bond resp. the stock); at the same time $\Delta_1, \ldots, \Delta_{n-1}$ can be taken to be continuous and bounded. This means that for every $\varepsilon > 0$ there exist $b, c_{i,l}, K_{i,l} \in \mathbb{R}, i = 1, \ldots, n, l = 1, \ldots, m_i$, $\Delta_j \in C_b(\mathbb{R}^j)$, $j = 0, \ldots, n-1$ such that

$$b + \sum_{i=1}^{n} \sum_{l=1}^{m_i} c_{i,l}(s_i - K_{i,l})_+ + \sum_{j=0}^{n-1} \Delta_j(s_1, \ldots, s_j)(s_{j+1} - s_j) \leq \Phi(s_1, \ldots, s_n), \tag{8}$$

and the corresponding price

$$p = b + \sum_{i=1}^{n} \sum_{l=1}^{m_i} c_{i,l} C(t_i, K_{i,l}) \tag{9}$$

[4]In more financial terms this means that $C(t, K)$ is increasing in t for each fixed $K \in \mathbb{R}$.

is ε-close to the primal value P.

Condition (7) could be somewhat relaxed. For instance it is sufficient to demand that the function Φ is bounded from below by the sum of integrable functions. However, in this case it is necessary to allow for dual strategies that use European options beyond call options and we will not pursue this further.

We conclude this introductory section by noting that an *upper* bound for the price of the option Φ can be given by means of *semi-static (super)hedging*. Applying Theorem 1 to the function $-\Phi$ we obtain that this bound is sharp:

Corollary 1.1. *Assume that $\mu_1, \dots, \mu_n$ are Borel probability measures on $\mathbb{R}$ such that $\mathcal{M}(\mu_1, \dots, \mu_n)$ is non-empty. Let $\Phi : \mathbb{R}^n \to [-\infty, \infty)$ be an upper semi-continuous function such that*

$$\Phi(s_1, \dots, s_n) \leq K \cdot (1 + |s_1| + \dots + |s_n|) \tag{10}$$

on $\mathbb{R}^n$ for some constant K. Then there is no duality gap

$$P = \sup\left\{\mathbb{E}_{\mathbb{Q}}\Phi : \mathbb{Q} \in \mathcal{M}(\mu_1, \dots, \mu_n)\right\} = \inf\left\{\sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] : \exists\, \Delta_1, \dots, \Delta_{n-1} \text{ s.t. } \Psi_{(u_i),(\Delta_j)} \geq \Phi\right\} = D.$$

The supremum is attained, i.e. there exists a maximizing martingale measure.

Comparison with previous results. The main novelty of our approach is that we apply the theory of optimal transport in mathematical finance, more specifically, to obtain robust model-independent bounds on option prices. A time-continuous analysis of the present connection between optimal transport and mathematical finance is contained in the parallel work to the present one by Galichon, Henry-Labordère, Touzi [GHLT11] (see also [HLT12]) where a stochastic control approach is used.

We point out that the problem of model independent pricing is classically approached in the literature by means of the Skorohkod embedding problem, see the informative survey paper by Hobson [Hob11]. Also the notion of semi-static hedges is well-established (see for instance [Hob11, Section 2.6]).

The problem of robust pricing in a multi-period setting has previously been studied in the case of specific exotic options. Hodges and Neuberger [NH00] are mainly interested in the case of Barrier options. Albrecher, Mayer and Schoutens produce an explicit bound (based on conditioning arguments) in the case of an Asian option in discrete time and give a feasible subreplicating strategy associated to it [AMS08]. The problem to explicitly give the optimal lower/upper bounds seems harder and remains open to the best of our knowledge. A numerical implementation of our dual approach in the Asian option setting is given in [HL11].

In the continuous-time setting, the problem has been treated for instance in the case of lockback options [Hob98], variance/volatility options [CL10, CW11, HK12] and double-(no) touch options [CO09, CO11]. These solutions are mainly based on Skorokhod-stopping techniques and differ from our approach also in that only the marginal at the maturity is incorporated. Extensions to the multi-marginal case are addressed in [BHR01a, MY02, HP02, HLT12].

A result similar to our findings was recently proved for forward-start options by Hobson and Neuberger [HN12]. In the terminology of Corollary 1.1 they show that $P = D$ in the case where $n = 2$ and the payoff function is given by

$$\Phi(s_1, s_2) = |s_2 - s_1|.$$

In contrast to our paper, their approach is more constructive and they obtain maximizers for the dual problem in particular cases. Here some care is needed in certain (pathological) situations, see Proposition 4.1 below.

2. Optimal Transport

In the usual theory of Monge-Kantorovich optimal transport[5] one considers two probability spaces (X_1, μ_1), (X_2, μ_2) and the problem is to find a "cheap" way of transporting μ_1 to μ_2. Following Kantorovich, a transport plan is formalized as probability measure π on $X_1 \times X_2$ which has X_1-marginal μ_1 and X_2-marginal μ_2.

We will come back to the two dimensional case in Section 4 below; for now we turn to the *multidimensional version* of the transport problem which will be the main tool in our proof of Theorem 1. Subsequently we consider probability measures $\mu_1, \dots, \mu_n$ on the real line[6] which have finite first moments. The set $\Pi(\mu_1, \dots, \mu_n)$ of *transport plans* consists of all Borel probability measures on $\mathbb{R}^n$ with marginals $\mu_1, \dots, \mu_n$. A *cost function* is a measurable function $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ which is bounded from below in the sense that there exist μ_i-integrable functions u_i, $i = 1, \dots, n$ such that

$$\Phi \geq u_1 \oplus \dots \oplus u_n, \tag{11}$$

where $u_1 \oplus \dots \oplus u_n(x_1, \dots, x_n) := u_1(x_1) + \dots + u_n(x_n)$. Given a cost function Φ and a transport plan π the *cost functional* is defined as

$$I_\pi(\Phi) = \int_{\mathbb{R}^n} \Phi \, d\pi \,. \tag{12}$$

Note that this integral is well defined (assuming possibly the value $+\infty$) by (11). The *primal Monge-Kantorovich problem* is then to minimize $I_\pi(\Phi)$ over the set of all transport plans $\pi \in \Pi(\mu_1, \dots, \mu_n)$.

Given μ_i-integrable functions u_i, $i = 1, \dots, n$, such that

$$\Phi \geq u_1 \oplus \dots \oplus u_n, \tag{13}$$

we have for every transport plan π

$$\int \Phi \, d\pi \geq \int u_1 \oplus \dots \oplus u_n \, d\pi = \int u_1 \, d\mu_1 + \dots + \int u_n \, d\mu_n. \tag{14}$$

The *dual* part of the Monge-Kantorovich problem is to maximize the right side of (14) over a suitable class of functions satisfying (13).

Starting already with Kantorovich, there has been a long line of research on the question in which setting the optimal values of primal and dual problem agree, we refer the reader to [Vil09, page 88f.] for an account of the history of the problem. For our intended application, we need to restrict the dual maximizers to functions in

$$\mathcal{S} = \Big\{ u : \mathbb{R} \to \mathbb{R} : u(x) = a + bx + \sum_{i=1}^{m} c_i (x - k_i)_+, \; a, b, c_i, k_i \in \mathbb{R} \Big\},$$

i.e., we will employ the following Monge-Kantorovich duality theorem.

Proposition 2.1. *Let $\Phi : \mathbb{R}^n \to (-\infty, \infty]$ be a lower semi-continuous function satisfying*

$$\Phi(s_1, \dots, s_n) \geq -K \cdot (1 + |s_1| + \dots + |s_n|) \tag{15}$$

on $\mathbb{R}^n$ for some constant K and let $\mu_1, \dots, \mu_n$ be probability measures on $\mathbb{R}$ having finite first moments. Then

$$P_{MK}(\Phi) = \inf\{I_\pi(\Phi) : \pi \in \Pi(\mu_1, \dots, \mu_n)\} = \sup\Big\{ \sum_{i=1}^{n} \int u_i \, d\mu_i : u_1 \oplus \dots \oplus u_n \leq \Phi, \; u_i \in \mathcal{S} \Big\} = D_{MK}(\Phi) \,.$$

The dual bound D_{MK} could be realized by holding a static position in European options with respective maturity date t_i and payoff u_i. This static portfolio with intrinsic value $\sum_{i=1}^n u_i$ and market value $\sum_{i=1}^n \mathbb{E}_{\mu_i}[u_i]$ subreplicates the payoff Φ at maturity.

We postpone the proof of Proposition 2.1 to the Appendix and continue with our discussion.

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[5] See [Vil03, Vil09] for an extensive account on the theory of optimal transportation.

[6] Most of the basic results are equally true for polish probability spaces $(X_1, \mu_1), \dots, (X_n, \mu_n)$, but we do not need this generality here.

The set of transport plans $\Pi(\mu_1,\ldots,\mu_n)$ carries a natural topological structure: it is a compact convex subset of the space of finite (signed) Borel measures equipped with the weak topology induced by the bounded continuous functions $C_b(\mathbb{R}^n)$. (Compactness of $\Pi(\mu_1,\ldots,\mu_n)$ is essentially a consequence of Prohorov's theorem, for a proof we refer the reader to [Vil09, Lemma 4.4].)

Subsequently we want to study the set of transport plans which are also martingales. Therefore we will assume from now on that the measures $\mu_1,\ldots,\mu_n$ are in convex order such that $\mathcal{M}(\mu_1,\ldots,\mu_n)$ is a nonempty subset of $\Pi(\mu_1,\ldots,\mu_n)$. It will be crucial for our purposes that also $\mathcal{M}(\mu_1,\ldots,\mu_n)$ is compact in the weak topology. To establish this we need two auxiliary lemmas.

Lemma 2.2. *Let $c:\mathbb{R}^n\to\mathbb{R}$ be continuous and assume that there exists a constant K such that*

$$|c(x_1,\ldots,x_n)|\le K(1+|x_1|+\ldots+|x_n|)$$

for all $x_1\in X_1,\ldots,x_n\in X_n$. Then the mapping

$$\pi\mapsto\int_{\mathbb{R}^n}c\,d\pi$$

is continuous on $\Pi(\mu_1,\ldots,\mu_n)$.

Proof. Since we assume that $\mu_1,\ldots,\mu_n$ have finite first moments, $\int_{\mathbb{R}^n\setminus[-a,a]^n}c\,d\pi$ converges to 0 uniformly in $\pi\in\Pi(\mu_1,\ldots,\mu_n)$ as $a\to\infty$. □

Lemma 2.3. *Let $\pi\in\Pi(\mu_1,\ldots,\mu_n)$. Then the following are equivalent.*

(1) $\pi\in\mathcal{M}(\mu_1,\ldots,\mu_n)$.
(2) For $1\le j\le n-1$ and for every continuous bounded function $\Delta:\mathbb{R}^j\to\mathbb{R}$ we have

$$\int_{\mathbb{R}^n}\Delta(x_1,\ldots,x_j)(x_{j+1}-x_j)\,d\pi(x_1,\ldots,x_n)=0.$$

Proof. Plainly, (1) asserts that whenever $A\subseteq R^j$, $j=1,\ldots,(n-1)$ is Borel measurable, then

$$\int_{\mathbb{R}^n}I_A(x_1,\ldots,x_j)(x_{j+1}-x_j)\,d\pi(x_1,\ldots,x_n)=0.$$

Using standard approximation techniques one obtains that this is equivalent to (2). □

Proposition 2.4. *The set $\mathcal{M}(\mu_1,\ldots,\mu_n)$ is compact in the weak topology.*

Proof. Since $\mathcal{M}(\mu_1,\ldots,\mu_n)$ is contained in the compact set $\Pi(\mu_1,\ldots,\mu_n)$ it is sufficient to prove that it is closed. By Lemma 2.3, $\mathcal{M}(\mu_1,\ldots,\mu_n)$ is the intersection of the sets

$$\Big\{\pi\in\Pi(\mu_1,\ldots,\mu_n):\int_{\mathbb{R}^n}f(x_1,\ldots,x_j)(x_{j+1}-x_j)\,d\pi(x_1,\ldots,x_n)=0\Big\},\tag{16}$$

where $j=1,\ldots,n-1$ and $f:\mathbb{R}^j\to\mathbb{R}$ runs through all continuous bounded support functions. By Lemma 2.2 the sets in (16) are closed. □

3. Proof of Theorem 1

Our argument combines a Monge-Kantorovich duality theorem (in the form of Proposition 2.1) with the following Min-Max theorem of decision theory which we cite here from [Str85, Thm. 45.8] (another reference is [AH96, Thm. 2.4.1]).

Theorem 2. *Let K,T be convex subsets of vector spaces V_1 resp. V_2, where V_1 is locally convex and let $f:K\times T\to\mathbb{R}$. If*

(1) K is compact,
(2) $f(.,y)$ is continuous and convex on K for every $y\in T$,
(3) $f(x,.)$ is concave on T for every $x\in K$

then

$$\sup_{y\in T}\inf_{x\in K} f(x,y) = \inf_{x\in K}\sup_{y\in T} f(x,y).$$

Proof of Theorem 1. As we want to show that the subhedging portfolios can be formed using just call options, we will restrict ourselves to dual candidates $\Psi_{(u_i),(\Delta_j)}$ satisfying $u_i \in \mathcal{S}, i = 1,\dots,n$ (and $\Delta_j \in C_b(\mathbb{R}^j), j = 1,\dots,n-1$).

If the assertion of Theorem 1 holds true for a function Φ and if $u_1,\dots,u_n \in \mathcal{S}$ then the assertion carries over to $\Phi' = \Phi + u_1 \oplus \dots \oplus u_n$. Therefore we may assume without loss of generality that $\Phi \geq 0$.

Moreover for now we make the additional assumption that $\Phi \in C_b(\mathbb{R}^n)$; we will get rid of this extra condition later.

We will apply Theorem 2 to the compact convex set $K = \Pi(\mu_1,\dots,\mu_n)$, the convex set $T = C_b(\mathbb{R}) \times \dots \times C_b(\mathbb{R}^{n-1})$ of $(n-1)$-tuples of continuous bounded functions on $\mathbb{R}^j, j = 1,\dots,(n-1)$ and the function

$$f(\pi,(\Delta_j)) = \int \Phi(x_1,\dots,x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1,\dots,x_j)(x_{j+1}-x_j)\, d\pi(x_1,\dots,x_n). \tag{17}$$

Clearly the assumptions of Theorem 2 are satisfied, the continuity of $f(.,(\Delta_j))$ on $\Pi(\mu_1,\dots,\mu_n)$ being a consequence of Lemma 2.2.

We then find

$$D \geq \sup_{u_i\in\mathcal{S},\, \Delta_j\in C_b(\mathbb{R}^j),\, \Psi_{(u_i),(\Delta_j)}\leq\Phi} \sum_{i=1}^n \int u_i\, d\mu_i \tag{18}$$

$$= \sup_{\Delta_j\in C_b(\mathbb{R}^j)} \sup_{u_i\in\mathcal{S},\, \sum_{i=1}^n u_i(x_i)\leq\Phi(x_1,\dots,x_n)-\sum_{j=1}^{n-1}\Delta_j(x_1,\dots,x_j)(x_{j+1}-x_j)} \sum_{i=1}^n \int u_i\, d\mu_i \tag{19}$$

$$= \sup_{\Delta_j\in C_b(\mathbb{R}^j)} \inf_{\pi\in\Pi(\mu_1,\dots,\mu_n)} \int \Phi(x_1,\dots,x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1,\dots,x_j)(x_{j+1}-x_j)\, d\pi(x_1,\dots,x_n) \tag{20}$$

$$= \inf_{\pi\in\Pi(\mu_1,\dots,\mu_n)} \sup_{\Delta_j\in C_b(\mathbb{R}^j)} \int \Phi(x_1,\dots,x_n) - \sum_{j=1}^{n-1} \Delta_j(x_1,\dots,x_j)(x_{j+1}-x_j)\, d\pi(x_1,\dots,x_n) \tag{21}$$

$$= \inf_{\mathbb{Q}\in\mathcal{M}(\mu_1,\dots,\mu_n)} \int \Phi(x_1,\dots,x_n)\, d\mathbb{Q} = P. \tag{22}$$

Here Proposition 2.1 (applied to the cost function $\Phi(x_1,\dots,x_n) - \sum_{j=1}^{n-1}\Delta_j(x_1,\dots,x_i)(x_{j+1}-x_j)$) is used to show the equality between (19) and (20) and the equality of (20) and (21) is guaranteed by Theorem 2. Finally let us justify the equality between (21) and (22): indeed if π is not a martingale measure, then by Lemma 2.3 for some j there is a function Δ_j such that

$$B = \int \Delta_j(x_1,\dots,x_j)(x_{j+1}-x_j)\, d\pi(x_1,\dots,x_n)$$

does not vanish. By appropriately scaling Δ the value of B can be made arbitrarily large.

Next assume that $\Phi : \mathbb{R}^n \to [0,\infty]$ is merely lower semi-continuous and pick a sequence of bounded continuous functions $\Phi_1 \leq \Phi_2 \leq \dots$ such that $\Phi = \sup_{k\geq 0}\Phi_k$. In the following paragraph we will write $P(\Phi), D(\Phi), P(\Phi_k)$, resp. $D(\Phi_k)$ to emphasize the dependence on the cost function. For each k pick $\mathbb{Q}_k \in \Pi(\mu_1,\dots,\mu_n)$ such that

$$P(\Phi_k) \geq \int \Phi\, d\mathbb{Q}_k - 1/k.$$

Passing to a subsequence if necessary, we may assume that $(\mathbb{Q}_k)$ converges weakly to some $\mathbb{Q} \in \Pi(\mu_1, \dots, \mu_n)$. Then

$$P(\Phi) \leq \int \Phi \, d\mathbb{Q} = \lim_{m\to\infty} \int \Phi_m \, d\mathbb{Q} = \lim_{m\to\infty} \left(\lim_{k\to\infty} \int \Phi_m \, d\mathbb{Q}_k \right) \tag{23}$$

$$\leq \lim_{m\to\infty} \left(\lim_{k\to\infty} \int \Phi_k \, d\mathbb{Q}_k \right) = \lim_{k\to\infty} P(\Phi_k).$$

Since $P(\Phi_k) \leq P(\Phi)$ it follows that $D(\Phi) \geq D(\Phi_k) = P(\Phi_k) \uparrow P(\Phi)$.

It remains to prove that the optimal value of the primal problem is attained. To establish this, we use the lower semi-continuity of $\int \Phi \, d\pi$ on $\Pi(\mu_1, \dots, \mu_n)$: if a sequence of measures (π_k) in $\Pi(\mu_1, \dots, \mu_n)$ converges weakly to a measure π, then

$$\liminf_{k\to\infty} \int \Phi \, d\pi_k \geq \int \Phi \, d\pi. \tag{24}$$

We refer the reader to [Vil09, Lemma 4.3] for a proof of this assertion.

If $P = \infty$, the infimum is trivially attained, so assume $P < \infty$ and pick a sequence $(\mathbb{Q}_k)$ in $\mathcal{M}(\mu_1, \dots, \mu_n)$ such that $P = \lim_k \int \Phi \, d\mathbb{Q}_k$. As $\mathcal{M}(\mu_1, \dots, \mu_n)$ is compact, $(\mathbb{Q}_k)$ converges to some measure $\mathbb{Q}$ along a subsequence and $\mathbb{Q}$ is a primal minimizer by (24). □

As we have just seen, the existence of a primal optimizer $\mathbb{Q}$ is basically a consequence of the compactness of the set of all martingale transport plans. The dual set of sub-hedges does not exhibit nice compactness properties and as we already mentioned the dual supremum is not necessarily attained (Proposition 4.1 below.) Although we are not able to give a positive criterium in this direction, it seems worthwhile to comment on the consequences of attainment of the dual problem.

Assume that there exists a dual maximizer, i.e. that the there exist μ_i integrable functions u_i and continuous bounded functions Δ such that the corresponding subhedge (cf. (3)) satisfies

$$\Psi_{(u_i),(\Delta_j)} \leq \Phi \tag{25}$$

and

$$\sum_{i=1}^{n} \mathbb{E}_{\mu_i}[u_i] = P.$$

Let $\mathbb{Q}$ be a primal optimizer, i.e. a martingale measure satisfying the given marginal constraints as well as $\mathbb{E}_{\mathbb{Q}}[\Phi] = P$. Then we have

$$0 \leq \mathbb{E}_{\mathbb{Q}}[\Phi - \Psi_{(u_i),(\Delta_j)}] = P - D = 0.$$

As a consequence, equality holds $\mathbb{Q}$-a.s. in (25). The financial interpretation is that under the market model $\mathbb{Q}$, the payoff Φ is perfectly replicated through the semi-static hedge corresponding to $(u_i), (\Delta_j)$.

4. Further analysis in the two dimensional case.

Throughout this section we focus on the two-period case, i.e. $n = 2$. We start with two examples which illustrate (the general) Theorem 1. Then we show that the dual supremum is not necessarily attained. Finally we explain a conjugacy relation which is relevant for the dual problem and resembles a well-known concept from the classical theory of optimal transport.

4.1. **A numerical example: forward-start options.** We consider the problem to find optimal upper and lower bounds for forward-start options with payoffs

$$\Phi_K(s_1, s_2) = (s_2 - Ks_1)^+, \quad K = 0.5, \dots, 1.5.$$

Recently, Hobson and Neuberger [HN12] have obtained interesting results on model-independent bounds for the forward-start straddle $|s_2 - s_1|$. Since $|s_2 - s_1| = 2(s_2 - s_1)^+ - (s_2 - s_1)$, this is equivalent to the case $K = 1$, $\Phi_1(s_1, s_2) = (s_2 - s_1)^+$. An unfortunate feature is that no fully explicit solution is known for generic measures μ_1 and μ_2. In [HN12, Section 9] numerical upper bounds are obtained in the cases where μ_1, μ_2 are given as uniform resp. log-normal distributions.

We will consider the cases of different strikes and laws μ_1, μ_2 inferred from market data. By using a linear programming algorithm, we have computed numerically the optimal lower and upper bounds for different values of K.

The measures μ_1 and μ_2 are deduced from the prices of call options written on the DAX (pricing date = 2nd Feb. 2012) with $t_1 = 1$ year and $t_2 = 1.5$ years with $m = 18$ strikes ranging from 30% to 200% of the current spot price s_0. The dual for the upper bound reads as (setting $K_{1,0} = K_{2,0} = 0$)

$$D = \inf_{b, c_{i,l}, \Delta} b + \sum_{i=1}^{2} \sum_{l=0}^{m} c_{i,l} C(t_i, K_{i,l}) \tag{26}$$

$$\text{s.t. } F(s_1, s_2) := b + \sum_{i=1}^{2} \sum_{l=0}^{m} c_{i,l}(s_i - K_{i,l})_+ + \Delta(s_1)(s_2 - s_1) \geq (s_2 - Ks_1)^+, \ (s_1, s_2) \in \mathbb{R}_+^2. \tag{27}$$

The additional term $\Delta_0(s_0)(s_1 - s_0)$ has been incorporated by considering a vanilla option at t_1 with a zero strike. Note that the function $s_2 \mapsto F(s_1, s_2) - (s_2 - Ks_1)^+$ is piecewise linear with respect to s_2 and therefore attains its extremal values at the points $s_2 = \{K_{2,j}\}_{j=1,\dots,m}$, $s_2 = 0$, $s_2 = \infty$, $s_2 = Ks_1$. The above constraints therefore reduce to $m+3$ constraints parametrized by s_1. As a consequence this low-dimensional linear program can be efficiently implemented by using a classical simplex algorithm [PTVF07] and by discretizing the spot value s_1 on a space grid. We have compared the upper and lower bounds against the prices produced by models commonly used by practitioners (see Fig. 1): the local volatility model (in short LV) [Dup94], Bergomi's model [Ber05] which is a two-factor variance curve model and finally the local Bergomi model [HL09] which has the property to be perfectly calibrated to vanilla smiles at t_1 and t_2. The LV and local Bergomi models have been calibrated to the DAX implied volatility market. The Bergomi model has been calibrated to the variance-swap term structure. As expected, the prices as produced by the LV and local Bergomi models – consistent with the marginals μ_1 and μ_2 – are within our bounds.[7]
We have also plotted $F(s_1, s_2)$ as a function of s_1 and s_2 for the at-the-money forward-start option (i.e. $K = 1$, see Fig. 2) to check the super-replication strategy.

Our result shows that forward-start options are poorly constrained by vanilla smiles. As a conclusion, the practice in the old-quant community to calibrate stochastic volatility models on vanilla smiles to price exotic options (depending strongly on forward volatility) is inappropriate.

Additional numerical examples are investigated in a companion paper [HL11]. In the case of Asian options the bounds are tighter, indicating that this option can be fairly well hedged with vanilla options.

We would like to highlight that for general exotic options, our dual bound can be framed into a large-scale semi-infinite linear program whose numerical implementation requires advanced simplex algorithm such as a primal-dual algorithm within a cutting-plane algorithm [HL11].

4.2. **Analysis of a theoretical example.** We consider a forward-start straddle with payoff function $\Phi(S_1, S_2) = |S_2 - S_1|$; as above we assume that the marginal laws μ_1, μ_2 are fixed. As mentioned before, Hobson and

[7] We would like to emphasize that the lower/upper bounds corresponding to different strikes K are attained by different martingale measures. This is not the case if we do not include the martingality constraint as in this case the upper/lower bounds are attained by the co-monotone resp. anti-monotone coupling for each strike K (see for instance [Vil03, Section 2.2.2]).

FIGURE 1. Lower/Upper bounds versus (local) Bergomi and LV models for forward-start options (quoted in Black-Scholes volatility ×100). Parameters for the Bergomi model: $\sigma = 2$, $k_1 = 4$, $k_2 = 0.125$, $\rho = 34.55\%$, $\rho_{SX} = -76.84\%$, $\rho_{SX} = -86.40\%$

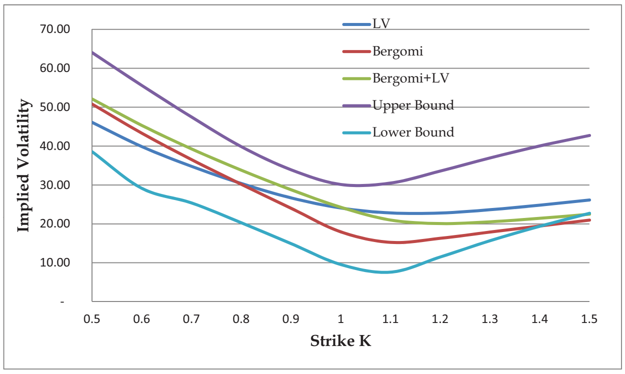

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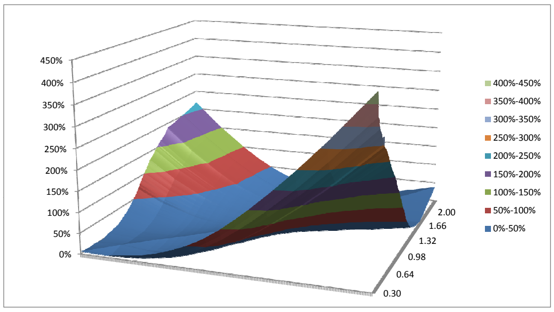
FIGURE 2. Super-replication strategy for $K = 1$: $F(s_1, s_2)$ as a function of $\frac{s_1}{s_0}$ and $\frac{s_2}{s_0}$.



Neuberger [HN12] treat the problem to find a market model which *maximizes* the price $\mathbb{E}_{\mathbb{Q}}[\Phi(S_1, S_2)]$; specific examples are worked out in detail.

Here we focus on the problem to minimize $\mathbb{E}_{\mathbb{Q}}[\Phi(S_1, S_2)]$ in a concrete example. The marginals μ_1, μ_2 are defined by the respective densities (where we write λ for the Lebesgue measure)

$$\frac{d\mu_1}{d\lambda}(s_1) = \frac{1}{2}\mathbb{1}_{[-1,1]}, \quad \frac{d\mu_2}{d\lambda}(s_1) = \frac{2+s_1}{3}\mathbb{1}_{[-2,-1]} + \frac{1}{3}\mathbb{1}_{[-1,1]} + \frac{2-s_1}{3}\mathbb{1}_{[1,2]},$$

cf. Figure 3 below. Recall that the primal, resp. dual problem is then given by

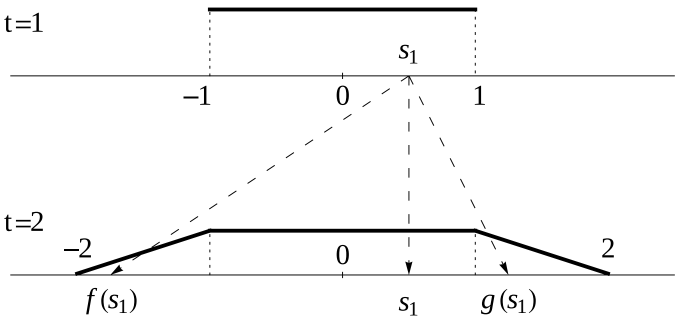
$$P = \inf_{\mathbb{Q}\in\mathcal{M}(\mu_1,\mu_2)} \mathbb{E}_{\mathbb{Q}}[|S_2 - S_1|], \qquad D = \sup_{u_1,u_2:\exists\Delta, u_1(s_1)+u_2(s_2)+\Delta(s_1)(s_2-s_1)\le|s_2-s_1|} \mathbb{E}_{\mu_1}[u_1] + \mathbb{E}_{\mu_2}[u_2].$$

By Theorem 1 we know that there is no duality gap, i.e. $P = D$. Our aim is to determine the primal minimizer $\mathbb{Q}$ as well as dual maximizers u_1, u_2, Δ. We follow the common procedure of guessing and verification: i.e. making various (unjustified) assumptions we will first produce explicit candidates. Then it is possible to verify rigorously that these candidates indeed solve the given problem.

Due to Hobson [Hob12] (see also ([BJ12, Section 6]) one expects that the primal minimizer $\mathbb{Q}$ has a very particular structure: Writing $(\mathbb{Q}_{s_1})_{s_1\in[-1,1]}$ for the disintegration[8] of $\mathbb{Q}$ w.r.t. μ_1, each measure $\mathbb{Q}_{s_1}$ will be concentrated on three points. More specifically we *guess*[9] that there exist monotone decreasing functions $f\colon [-1,1] \to [-2,-1], g\colon [-1,1] \to [1,2]$ such that $\operatorname{supp}(\mathbb{Q}_{s_1}) = \{f(s_1), s_1, g(s_1)\}$.

Figure 3 depicts the measures μ_1, μ_2 and for each particle starting in $s_1 \in [-1,1]$ the possible positions $f(s_1), s_1, g(s_1)$ at time $t = 2$.

Figure 3. Marginals and Support of Primal Optimizer



I.e., as much mass as possible remains at its place, the rest is either moved to the interval on the left of $[-1,1]$ (via f), or to the right (via g). For $s_1 \in [-1,1]$, we write the measure $\mathbb{Q}_{s_1}$ as

$$\mathbb{Q}_{s_1} = a(s_1)\delta_{f(s_1)} + b(s_1)\delta_{s_1} + c(s_1)\delta_{g(s_1)}, \quad \text{where } a(s_1) + b(s_1) + c(s_1) = 1.$$

Taking for granted that f, g are sufficiently smooth, the marginal conditions on $\mathbb{Q}$ translate to

$$\tfrac{d\mu_1}{d\lambda}(s_1)a(s_1) = (-f'(s_1))\tfrac{d\mu_2}{d\lambda}(f(s_1)), \quad \tfrac{d\mu_1}{d\lambda}(s_1)b(s_1) = \tfrac{d\mu_2}{d\lambda}(s_1), \quad \tfrac{d\mu_1}{d\lambda}(s_1)c(s_1) = (-g'(s_1))\tfrac{d\mu_2}{d\lambda}(g(s_1)).$$

Thus $b(s_1) = 2/3$ and $a(s_1), c(s_1)$ can be expressed in terms of the functions f, g, i.e.

$$a(s_1) = -f'(s_1)\tfrac{d\mu_2}{d\lambda}(f(s_1))/\tfrac{d\mu_1}{d\lambda}(s_1), \qquad c(s_1) = -g'(s_1)\tfrac{d\mu_2}{d\lambda}(g(s_1))/\tfrac{d\mu_1}{d\lambda}(s_1).$$

From $a(s_1) + b(s_1) + c(s_1) = 1$, we obtain for $s_1 \in [-1,1]$ the equation

$$\tfrac{d\mu_2}{d\lambda}(f(s_1))(-f'(s_1)) + \tfrac{d\mu_2}{d\lambda}(g(s_1))(-g'(s_1)) = \frac{1}{3}\tfrac{d\mu_1}{d\lambda}(s_1) = \frac{1}{6}. \tag{28}$$

The martingale property is expressed by $f(s_1)a(s_1) + s_1 b(s_1) + g(s_1)c(s_1) = s_1$. In terms of f, g this amounts to

$$\tfrac{d\mu_2}{d\lambda}(f(s_1))f(s_1)(-f'(s_1)) + \tfrac{d\mu_2}{d\lambda}(g(s_1))g(s_1)(-g'(s_1)) = \frac{s_1}{3}\tfrac{d\mu_1}{d\lambda}(s_1) = \frac{s_1}{6}. \tag{29}$$

Adding the initial conditions $f(1) = -2$ and $g(1) = 1$, the differential equations (28), (29) have the unique solution

$$f(s_1) = -(3 + s_1)/2, \qquad g(s_1) = (3 - s_1)/2. \tag{30}$$

These functions f, g determine a martingale measure $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ which is our candidate optimizer for the primal problem.

[8]In probabilistic terms, the measure $\mathbb{Q}_{s_1}$ is the conditional distribution of S_2 under $\mathbb{Q}$ given that $S_1 = s_1$.

[9]We emphasize that while this simple guess works in the present setting, the situation is more subtle for general distributions.

A dual optimizer (u_1, u_2, Δ) consists of functions $u_1, u_2, \Delta : \mathbb{R} \to \mathbb{R}$ satisfying

$$u_1(s_1) + \Delta(s_1)(s_2 - s_1) \leq |s_2 - s_1| - u_2(s_2) \tag{31}$$

for all $(s_1, s_2) \in \mathbb{R}^2$. From the considerations at the end of Section 3 we know that equality should hold in (31) for all (s_1, s_2) in the support of the primal minimizer. Since we anticipate that $\mathbb{Q}$ is this minimizer, we expect equality in (31) for $s_2 \in \{f(s_1), s_1, g(s_1)\}$, $s_1 \in [-1, 1]$. Writing $u_{2,l} := u_2|_{(-\infty,-1]}$, $u_{2,m} := -u_1|_{[-1,1]}$ and $u_{2,r} := u_2|_{[1,\infty)}$, this amounts to

$$\begin{aligned} u_1(s_1) + \Delta(s_1)(f(s_1) - s_1) &= |f(s_1) - s_1| - u_{2,l}(f(s_1)), \\ u_1(s_1) + \Delta(s_1)(s_1 - s_1) &= |s_1 - s_1| - u_{2,m}(f(s_1)) \qquad \Longleftrightarrow\ u_1(s_1) = -u_{2,m}(s_1) \\ u_1(s_1) + \Delta(s_1)(g(s_1) - s_1) &= |g(s_1) - s_1| - u_{2,r}(g(s_1)), \end{aligned} \tag{32}$$

for $s_1 \in [-1, 1]$. Furthermore it is reasonable to assume that for fixed $s_1 \in [-1, 1]$, the affine function $y \mapsto u_1(s_1) + \Delta(s_1)(s_2 - s_1)$ is tangential to the function $y \mapsto |s_2 - s_1| - u_2(s_2)$ if s_2 equals $f(s_1)$ resp. $g(s_1)$. This leads us to identify the slope $\Delta(s_1)$ of this affine function with the derivatives of the right hand side for $s_2 \in \{f(s_1), g(s_1)\}$

$$\begin{aligned} \Delta(s_1) &= \partial_{s_2}\Big((s_1 - s_2) - u_{2,l}(s_2)\Big)\Big|_{s_2=f(s_1)} = -1 - u'_{2,r}(f(s_1)) \\ \Delta(s_1) &= \partial_{s_2}\Big((s_2 - s_1) - u_{2,r}(s_2)\Big)\Big|_{s_2=g(s_1)} = 1 - u'_{2,l}(g(s_1)). \end{aligned} \tag{33}$$

The equations (32) resp. (33) are solved by

$$\begin{aligned} &u_1(s_1) = (9 - 5{s_1}^2)/6 = -u_{2,m}(s_1), \quad \Delta(s_1) = -2s_1/3. \\ &u_{2,l}(s_2) = -3 - 3s_2 - 2s_2^2/3, \quad u_{2,r}(s_2) = -3 + 3s_2 - 2s_2^2/3. \end{aligned}$$

Setting $u_2 = u_{2,l}\mathbb{1}_{(-\infty,1]} + u_{2,m}\mathbb{1}_{[-1,1]} + u_{2,r}\mathbb{1}_{[1,\infty)}$, we have thus found a "reasonable" candidate solution for the dual problem. It is then straightforward to verify that (u_1, u_2, Δ) is admissible, i.e., satisfies (31).

To verify that $\mathbb{Q}$ resp. (u_1, u_2, Δ) are in fact solutions of the primal resp. dual problem we evaluate the corresponding functionals

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[|S_2 - S_1|] &= \int\!\!\int |s_2 - s_1|\, d\mathbb{Q}_{s_1}(s_2)\, d\mu_1(s_1) = \frac{1}{3}, \\ \mathbb{E}_{\mu_1}[u_1] + \mathbb{E}_{\mu_2}[u_2] &= \int_{-1}^{1} u_1(s_1)\tfrac{d\mu_1}{d\lambda}(s_1)\, ds_1 + \int_{-2}^{2} u_2(s_2)\tfrac{d\mu_2}{d\lambda}(s_2)\, ds_2 = \frac{1}{3}. \end{aligned}$$

Hence $P = \frac{1}{3} = D$ and we conclude that $\mathbb{Q}$ resp. (u_1, u_2, Δ) are indeed the desired solutions.

4.3. **Non-Existence of dual maximizers.** In the classical optimal transport problem, the optimal value of the dual problem is attained provided that the cost function is bounded ([Kel84, Theorem 2.14]) or satisfies appropriate moment conditions ([AP03, Therorem 2.3]).

This is not the case in our present setting as Proposition 4.1 shows that the dual supremum (5) is not necessarily attained even if μ_1, μ_2 are compactly supported. Our counterexample fits into the framework[10] of [HN12], i.e. we consider two periods and an exotic option with payoff $-|S_2 - S_1|$.

Proposition 4.1. *Let μ_2 be the uniform distribution on the interval $[0, 2]$ and $\Phi(s_1, s_2) = -|s_2 - s_1|$. There exists a measure μ_1, concentrated on countably many atoms, such that the (finite) dual value is not attained.*

Moreover, there do not exist functions $u_1, u_2, \Delta : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} u_1(s_1) + u_2(s_2) + \Delta(s_1)(s_2 - s_1) &\leq -|s_2 - s_1|, \quad \textit{for all } (s_1, s_2) \in \mathbb{R}^2, \\ u_1(s_1) + u_2(s_2) + \Delta(s_1)(s_2 - s_1) &= -|s_2 - s_1|, \quad \textit{for } \mathbb{Q}\textit{-a.a. } (s_1, s_2) \in \mathbb{R}^2, \end{aligned} \tag{34}$$

where $\mathbb{Q}$ is a minimizer of the primal problem.

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[10]Formally Hobson and Neuberger are interested to *maximize* the payoff of $|S_2 - S_1|$ while we are interested to *minimize* the payoff $-|S_2 - S_1|$. Mathematically, the two problems are of course the same. We haven chosen the latter formulation to be consistent with the notation in our main result Theorem 1.

In the proof of Proposition 4.1 we will use the following auxiliary result.

Lemma 4.2. *Assume that μ_1, μ_2 are probability measures on $\mathbb{R}$ having finite first moments, let $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ and fix $s \in \mathbb{R}$. The following are equivalent.*

(i) The call prices $\mathbb{E}_\mathbb{Q}[(S_1 - s)_+] = \int (s_1 - s)^+ \, d\mu_1(s_1)$ and $\mathbb{E}_\mathbb{Q}[(S_2 - s)_+] = \int (s_2 - s)^+ \, d\mu_2(s_2)$ are equal.
(ii) If $S_1 \le s$, then $S_2 \le s$ and if $S_1 > s$ then $S_2 \ge s$ $\mathbb{Q}$-a.s.

In particular, if (ii) holds for one measure in $\mathcal{M}(\mu_1, \mu_2)$, then it applies to all elements of $\mathcal{M}(\mu_1, \mu_2)$.

Proof. Given a random variable X and a measurable set A we write $\mathbb{E}_\mathbb{Q}[X, A] = \mathbb{E}_\mathbb{Q}[X \mathbb{1}_A]$. Then we have

$$\mathbb{E}_\mathbb{Q}[(S_2 - s)^+, S_1 > s] \ge \mathbb{E}_\mathbb{Q}[S_2 - s, S_1 > s], \tag{35}$$

$$\mathbb{E}_\mathbb{Q}[(S_2 - s)^+, S_1 \le s] \ge 0, \tag{36}$$

where equality holds equality holds in (35) if and only if $S_1 > s \Rightarrow S_2 > s$ $\mathbb{Q}$-a.s. and in (36) if and only if $S_1 \le s \Rightarrow S_2 \le s$ $\mathbb{Q}$-a.s. Using (in deriving the last line) that S is a $\mathbb{Q}$-martingale we thus obtain

$$\begin{aligned}
\mathbb{E}_\mathbb{Q}[(S_2 - s)^+] &= \mathbb{E}_\mathbb{Q}[(S_2 - s)^+, S_1 > s] + \mathbb{E}_\mathbb{Q}[(S_2 - s)^+, S_1 \le s] \\
&\ge \mathbb{E}_\mathbb{Q}[S_2 - s, S_1 > s] + 0 \\
&= \mathbb{E}_\mathbb{Q}[S_1 - s, S_1 > s] + \mathbb{E}_\mathbb{Q}[(S_1 - s)^+, S_1 \le s] = \mathbb{E}_\mathbb{Q}[(S_1 - s)^+],
\end{aligned}$$

with equality holding true if and only if (ii) is satisfied. □

We also make the following trivial observation:

Lemma 4.3. *Let $c, d, x \in \mathbb{R}, c < x \le d$, let m be a measure on $(c, d]$ and set $\alpha = m((c, d])$. Then the product-measure $\delta_x \otimes m$ is the unique measure on $(c, d]^2$ which has $\alpha\delta_x$ as first marginal and m as second marginal.*

Proof of Proposition 4.1. Denote by λ the Lebesgue measure on the real line and set $\mu_2 = \frac{1}{2}\lambda_{[0,2]}$. Define

$$a_n = \tfrac{1}{2}\Big(\textstyle\sum_{i=1}^{n-1} \frac{1}{i^2} + \sum_{i=1}^{n} \frac{1}{i^2} \Big), n \ge 1 \tag{37}$$

$$\bar{a} = \tfrac{1}{2}\Big(\textstyle\sum_{i=1}^{\infty} \frac{1}{i^2} + 2 \Big) = \frac{\pi^2}{12} + 1, \tag{38}$$

$$\mu_1 = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{i^2} \delta_{a_i} + \frac{1}{2}\Big(2 - \frac{\pi^2}{6}\Big)\delta_{\bar{a}}. \tag{39}$$

We claim that $\mathcal{M}(\mu_1, \mu_2)$ consists of the single element

$$\mathbb{Q} = \frac{1}{2} \sum_{n=1}^{\infty} \delta_{a_n} \otimes \lambda\big|_{\left(\sum_{i=1}^{n-1} \frac{1}{i^2}, \sum_{k=1}^{n} \frac{1}{i^2}\right]} + \frac{1}{2}\delta_{\bar{a}} \otimes \lambda\big|_{\left(\frac{\pi^2}{6}, 2\right]}. \tag{40}$$

Note that a_n is defined to be the midpoint of the the interval $\left(\sum_{i=1}^{n-1} \frac{1}{i^2}, \sum_{k=1}^{n} \frac{1}{i^2}\right]$, $n \in \mathbb{N}$; likewise $\bar{a}$ is the midpoint of $\left(\frac{\pi^2}{6}, 2\right]$. Therefore $\mathbb{Q}$ is indeed an element of $\mathcal{M}(\mu_1, \mu_2)$.

To prove that $\mathbb{Q}$ is the only element of $\mathcal{M}(\mu_1, \mu_2)$ we first observe that for $s \in S = \{\sum_{i=1}^{n} \frac{1}{i^2} : n \ge 0\} \cup \{\frac{\pi^2}{6}, 2\}$

$$\mathbb{Q}([0, s]^2 \cup (s, 2]^2) = 1. \tag{41}$$

Lemma 4.2 yields that (41) applies to an arbitrary measure $\widetilde{\mathbb{Q}} \in \mathcal{M}(\mu_1, \mu_2)$. As S is countable, it follows that

$$1 = \widetilde{\mathbb{Q}}\left(\bigcap_{s \in S} \big([0, s]^2 \cup (s, 2]^2 \big) \right).$$

We also note that

$$\bigcap_{s \in S} \big([0, s]^2 \cup (s, 2]^2 \big) = \bigcup_{n=1}^{\infty} \Big(\sum_{i=1}^{n-1} \frac{1}{i^2}, \sum_{i=1}^{n} \frac{1}{i^2} \Big]^2 \cup (\tfrac{\pi^2}{6}, 2]^2 =: \Gamma.$$

Applying Lemma 4.3 with $(c,d] = \left(\sum_{i=1}^{n-1}\frac{1}{i^2}, \sum_{i=1}^{n}\frac{1}{i^2}\right]$, $n \in \mathbb{N}$ resp. $(c,d] = (\frac{\pi^2}{6}, 2]$ it follows that $\mathbb{Q}$ is the only measure satisfying $\mathbb{Q}(\Gamma) = 1$ and having marginals μ_1, μ_2. Since $\widetilde{\mathbb{Q}}(\Gamma) = 1$, we conclude that $\widetilde{\mathbb{Q}} = \mathbb{Q}$. Thus we have indeed $\mathcal{M}(\mu_1, \mu_2) = \{\mathbb{Q}\}$.

According to the short discussion preceding Proposition 4.1 it is sufficient to show that (34) can not be verified. Striving for a contradiction, we assume that there exist $u_1, u_2, \Delta\colon \mathbb{R} \to \mathbb{R}$ such that (34) (with the respect to the measure $\mathbb{Q}$) holds true.

Setting $d_n = u_1(a_n), k_n := \Delta(a_n)$, $n \in \mathbb{N}$ we obtain

$$d_n + k_n(s_2 - a_n) + |s_2 - a_n| \le -u_2(s_2) \tag{42}$$

for $s_2 \in \mathbb{R}$ with equality holding for λ-almost all $s_2 \in [\sum_{i=1}^{n-1}\frac{1}{i^2}, \sum_{i=1}^{n}\frac{1}{i^2}]$. Applying this with n resp. $n+1$ yields

(43)

$$d_n + k_n(s_2 - a_n) + |s_2 - a_n| \le -u_2(s_2) = d_{n+1} + k_{n+1}(s_2 - a_{n+1}) + |s_2 - a_{n+1}| \quad \text{for } s_2 \in [\textstyle\sum_{i=1}^{n}\frac{1}{i^2}, \sum_{i=1}^{n+1}\frac{1}{i^2}],$$
$$d_n + k_n(s_2 - a_n) + |s_2 - a_n| = -u_2(s_2) \ge d_{n+1} + k_{n+1}(s_2 - a_{n+1}) + |s_2 - a_{n+1}| \quad \text{for } s_2 \in [\textstyle\sum_{i=1}^{n-1}\frac{1}{i^2}, \sum_{i=1}^{n}\frac{1}{i^2}].$$

Note that as these inequalities appeal to piecewise linear functions it is not necessary to exclude exceptional null-sets, in particular

$$d_n + k_n(y_0 - a_n) + |y_0 - a_n| = d_{n+1} + k_{n+1}(y_0 - a_{n+1}) + |y_0 - a_{n+1}|$$

for $y_0 = \sum_{i=1}^{n}\frac{1}{i^2}$. It follows that the slope of $s_2 \mapsto d_n + k_n(s_2 - a_n) + |s_2 - a_n|$ is smaller or equal than the one of $s_2 \mapsto d_{n+1} + k_{n+1}(s_2 - a_{n+1}) + |s_2 - a_{n+1}|$ at the point $s_2 = y_0$, i.e.

$$k_n + 1 \le k_{n+1} - 1.$$

Hence $k_n \ge (k_1 - 2) + 2n$.

Applying (43) for $s_2 = a_{n+1}$ we obtain

$$d_n + k_n(a_{n+1} - a_n) + |a_{n+1} - a_n| \le d_{n+1},$$
$$\Longrightarrow \quad d_n + k_n\tfrac{1}{2}(\tfrac{1}{n^2} + \tfrac{1}{(n+1)^2}) \le d_{n+1}. \tag{44}$$

Iterating (44), we arrive at

$$\begin{aligned} d_{n+1} &\ge d_1 + \sum_{i=1}^{n}[(k_1 - 2) + 2i]\tfrac{1}{2}(\tfrac{1}{i^2} + \tfrac{1}{(i+1)^2}) \\ &\ge d_1 - |k_1 - 2| \sum_{i=1}^{n} \tfrac{1}{2}(\tfrac{1}{i^2} + \tfrac{1}{(i+1)^2}) + \sum_{i=1}^{n} i\,(\tfrac{1}{i^2} + \tfrac{1}{(i+1)^2}) \ge d_1 - |k_1 - 2|\tfrac{\pi^2}{6} + \sum_{i=1}^{n}\tfrac{1}{i}. \end{aligned}$$

Thus, $(d_n)_n$ and $(k_n)_n$ tend to ∞ as n goes to infinity. Combining this with (42), it follows that $-u_2(s_2) = \infty$ for $s_2 \ge \frac{\pi^2}{6}$. □

Arguably, the above counterexample is rather artificial. In particular a crucial property is that the problem consists of countably many problems which are mutually not connected: there exist infinitely many disjoint intervals $I_n = (x_n, y_n], n \ge 1$ such that every $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ is concentrated on $\bigcup_n I_n \times I_n$. By Lemma 4.2 this is reflected in the prices of European calls by the property

$$\mathbb{E}_{\mathbb{Q}}[(S_1 - s)_+] = \mathbb{E}_{\mathbb{Q}}[(S_2 - s)_+]$$

whenever the strike s is the endpoint of some interval I_n.

Clearly it would be desirable to find conditions which guarantee that the dual supremum is attained. However we are not able to do so at the present stage.

4.4. **A c-convex approach.** In the dual part of usual transport problem it is suffices to maximize over all pairs of functions (u_1, u_2) where u_1 is the conjugate of u_2 with respect to Φ, i.e., satisfies

$$u_1(s_1) = \inf_{s_2} \Phi(s_1, s_2) - u_2(s_2).$$

(We refer the reader to [Vil03, Section 2.4], [Vil09, Chapter 5] for details on this topic.)

An analogous result holds true in the present martingale setup. Its relevance stems from the fact that it simplifies the construction of hedging strategies for options depending on two future time points. Unfortunately we are not aware of a generalization to the multi-period case.

Given a function $g : \mathbb{R} \to (-\infty, \infty]$, we write g^{**} for its convex envelope[11]. For $G\colon \mathbb{R}^2 \to \mathbb{R}$, let $G^{**}\colon \mathbb{R}^2 \to \mathbb{R}$ be the function satisfying

$$G^{**}(s_1, .) = (G(s_1, .))^{**}$$

for every $s_1 \in \mathbb{R}$. (It is straight forward to prove that G^{**} is Borel measurable resp. lower semi-continuous whenever G is.)

Proposition 4.4. *Let $\Phi : \mathbb{R}^2 \to (-\infty, \infty]$ be a lower semi-continuous function such that $\Phi(s_1, s_2) \geq -K(1 + |s_1| + |s_2|)$, $s_1, s_2 \in \mathbb{R}$ and assume that there is some $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ satisfying $\mathbb{E}_{\mathbb{Q}}[\Phi] < \infty$. Then*

$$P = \sup_{u_2\colon \mathbb{R}\to\mathbb{R}, \int |u_2|\, d\mu_2 < \infty} \mathbb{E}_{\mu_1}[(\Phi(S_1, S_1) - u_2(S_1))^{**}] + \mathbb{E}_{\mu_2}[u_2(S_2)]. \tag{45}$$

(In the course of the proof we will see that for every choice of u_2 the first integral in (45) is well defined, assuming possibly the value $-\infty$.)

Proof. We start to show that the primal value P is greater or equal than the right hand side of (45). Let $u_2\colon \mathbb{R} \to \mathbb{R}$ be a μ_2-integrable function. For $\mathbb{Q} \in \mathcal{M}(\mu_1, \mu_2)$ satisfying $\mathbb{E}_{\mathbb{Q}}[\Phi] < \infty$ we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\Phi(S_1, S_2)] &= \mathbb{E}_{\mathbb{Q}}[\Phi(S_1, S_2) - u_2(S_2)] + \mathbb{E}_{\mu_2}[u_2(S_2)] \\
&\geq \mathbb{E}_{\mathbb{Q}}[(\Phi(S_1, S_2) - u_2(S_2))^{**}] + \mathbb{E}_{\mu_2}[u_2(S_2)] \\
&= \mathbb{E}_{\mu_1}[\mathbb{E}_{\mathbb{Q}}[(\Phi(S_1, S_2) - u_2(S_2))^{**}| S_1]] + \mathbb{E}_{\mu_2}[u_2(S_2)] && (46) \\
&\geq \mathbb{E}_{\mu_1}[(\Phi(S_1, \mathbb{E}_{\mathbb{Q}}[S_2| S_1]) - u_2(\mathbb{E}_{\mathbb{Q}}[S_2| S_1]))^{**}] + \mathbb{E}_{\mu_2}[u_2(S_2)] && (47) \\
&= \mathbb{E}_{\mu_1}[(\Phi(S_1, S_1) - u_2(S_1))^{**}] + \mathbb{E}_{\mu_2}[u_2(S_2)],
\end{aligned}$$

where the inequality between (46) and (47) holds due to Jensen's inequality. This proves the first inequality.

To establish the reverse inequality, we make a simple observation. Let $s_1 \in \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be some function. Suppose that for $u_1 \in \mathbb{R}$ there exists $\Delta \in \mathbb{R}$ such that

$$u_1 + \Delta \cdot (s_2 - s_1) \leq g(s_2)$$

for all $s_2 \in \mathbb{R}$. Then $u_1 \leq g^{**}(s_1)$.

Applying this for $s_1 \in \mathbb{R}$ to the function $s_2 \mapsto g(s_2) = \Phi(s_1, s_2) - u_2(s_2)$ we obtain

$$\sup_{u_2} \mathbb{E}_{\mu_1}[(\Phi(S_1, S_1) - u_2(S_1))^{**}] + \mathbb{E}_{\mu_2}[u_2(S_2)] \tag{48}$$

$$\geq \sup_{u_2} \sup_{u_1\,:\, \exists\Delta, u_1(s_1)+\Delta(s_1)(s_2-s_1)\leq\Phi(s_1,s_2)-u_2(s_2)} \mathbb{E}_{\mu_1}[u_1(S_1)] + \mathbb{E}_{\mu_2}[u_2(S_2)] \tag{49}$$

$$= \sup_{u_1,u_2\,:\,\exists\Delta,\, \Psi_{u_1,u_2,\Delta}\leq\Phi} \mathbb{E}_{\mu_1}[u_1(S_1)] + \mathbb{E}_{\mu_2}[u_2(S_2)] = D = P\,, \tag{50}$$

where we tacitly assumed that the suprema are taken over μ_i-integrable functions $u_i\colon \mathbb{R} \to \mathbb{R}, i = 1, 2$ and that $\Delta\colon \mathbb{R} \to \mathbb{R}$ is bounded measurable. ◻

[11] I.e. $g^{**} : \mathbb{R} \to \mathbb{R}$ is the largest convex function smaller than or equal to g.

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Summary

This paper focusses on robust pricing and hedging of exotic options written on one risky asset.

Given call prices at finitely many time points $t_1, \dots, t_n$ the set of martingale models calibrated to these prices leads to an interval of consistent prices of a pre-specified exotic option. Theorem 1 resp. Corollary 1.1 assert inter alia that every price outside this interval gives rise to a *model-independent* arbitrage opportunity. This arbitrage can be realized through a semi-static sub/super-hedging strategy consisting in dynamic trading in the underlying and a static portfolio of call options.

Our approach to these results is based on the duality theory of mass transport.

Appendix

As a special case of [Kel84, Theorem 2.14] we have the duality equation

$$P_{MK}(\Phi) = \inf\{I_\pi(\Phi) : \pi \in \Pi(\mu_1, \dots, \mu_n)\} = \sup\Big\{\sum_{i=1}^n \int u_i \, d\mu_i : u_1 \oplus \dots \oplus u_n \le \Phi,\ u_i \text{ is } \mu_i\text{-integrable}\Big\}$$

for every lower semi-continuous cost function $\Phi : \mathbb{R}^n \to [0, \infty]$. The main task in the subsequent proof of Proposition 2.1 is to show that the duality equation is obtained if one restricts to functions in the class $\mathcal{S}$ in the dual problem.

Proof of Proposition 2.1. As in the proof of Theorem 1, it is sufficient to prove the duality equation in the case $\Phi \ge 0$.

Given a bounded continuous function f and $\varepsilon > 0$, then for every $i = 1, \dots, n$ there is some $u \in \mathcal{S}$ such that $f \ge u$ and $\int f - u \, d\mu_i < \varepsilon$. Therefore we may change the class of admissible functions from $\mathcal{S}$ to $C_b(\mathbb{R})$, i.e. it suffices to prove

$$P_{MK}(\Phi) = \sup\Big\{\sum_{i=1}^n \int u_i \, d\mu_i : u_1 \oplus \dots \oplus u_n \le \Phi,\ u_i \in C_b(\mathbb{R})\Big\}. \tag{51}$$

We will first show this under the additional assumption that $\Phi \in C_c(\mathbb{R}^n)$. By [Kel84, Theorem 2.14] we have that for each $\eta > 0$ there exist μ_i-integrable functions u_i, $i = 1, \dots, n$ such that

$$P_{MK}(\Phi) - \sum_{i=1}^n \int u_i \, d\mu_i \le \eta$$

and $u_1 \oplus \dots \oplus u_n \le \Phi$. Note that the latter inequality implies that $u_1, \dots, u_n$ are uniformly bounded since Φ is uniformly bounded from above.

To replace u_1 by a function in C_b we consider $H = \Phi - (u_1 \oplus \dots \oplus u_n)$ and define

$$\tilde{u}_1(x_1) := \inf_{x_2, \dots, x_n \in \mathbb{R}} H(x_1, \dots, x_n) \tag{52}$$

for $x_1 \in \mathbb{R}$. We claim that $\tilde{u}_1$ is (uniformly) continuous. Indeed, as Φ is uniformly continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, x' \in \mathbb{R}$, $|x - x'| < \delta$, then

$$|H(x, x_2, \dots, x_n) - H(x', x_2, \dots, x_n)| = |\Phi(x, x_2, \dots, x_n) - \Phi(x', x_2, \dots, x_n)| < \varepsilon.$$

Thus we obtain

$$|\tilde{u}_1(x) - \tilde{u}_1(x')| = \Big| \inf_{x_2, \dots, x_n \in \mathbb{R}} H(x, x_2, \dots, x_n) - \inf_{x_2, \dots, x_n \in \mathbb{R}} H(x', x_2, \dots, x_n) \Big| \le \varepsilon$$

whenever $|x - x'| < \delta$. By definition $\tilde{u}_1$ is also bounded from below and satisfies $\tilde{u}_1 \geq u_1$ as well as

$$\tilde{u}_1 \oplus u_2 \oplus \ldots \oplus u_n \leq \Phi.$$

Iteratively replacing the functions $u_2, \ldots, u_n$ in the same fashion, we obtain (51) in the case $\Phi \in C_c(\mathbb{R}^n)$.

Using precisely the same argument as in the proof of Theorem 1, we obtain the duality relation in the case of a general, lower semi-continuous function $\Phi : \mathbb{R}^n \to [0, \infty]$.

□

References

[AH96] D.R. Adams and L.I. Hedberg. *Function spaces and potential theory*, volume 314 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996.

[AMS08] H. Albrecher, P.A. Mayer, and W. Schoutens. General lower bounds for arithmetic Asian option prices. *Appl. Math. Finance*, 15(1-2):123–149, 2008.

[AP03] L. Ambrosio and A. Pratelli. Existence and stability results in the L^1 theory of optimal transportation. In *Optimal transportation and applications (Martina Franca, 2001)*, volume 1813 of *Lecture Notes in Math.*, pages 123–160. Springer, Berlin, 2003.

[Ber05] L. Bergomi. Smile dynamics ii. *Risk*, 18(10):67–73, 2005.

[BHR01a] H. Brown, D. Hobson, and L.C.G. Rogers. The maximum maximum of a martingale constrained by an intermediate law. *Probab. Theory Related Fields*, 119(4):558–578, 2001.

[BHR01b] H. Brown, D. Hobson, and L.C.G. Rogers. Robust hedging of barrier options. *Math. Finance*, 11(3):285–314, 2001.

[BJ12] M. Beiglböck and N. Juillet. On a problem of martingale transport under marginal constraints. Preprint, 2012.

[BP02] D. Bertsimas and I. Popescu. On the relation between option and stock prices: a convex optimization approach. *Oper. Res.*, 50(2):358–374, 2002.

[CDDV08] X. Chen, G. Deelstra, J. Dhaene, and M. Vanmaele. Static super-replicating strategies for a class of exotic options. *Insurance Math. Econom.*, 42(3):1067–1085, 2008.

[CL10] P. Carr and R. Lee. Hedging variance options on continuous semimartingales. *Finance Stoch.*, 14(2):179–207, 2010.

[CO09] A.M.G. Cox and J. Obłój. Robust pricing and hedging of double no-touch options. *ArXiv e-prints*, January 2009.

[CO11] A.M.G. Cox and J. Obłój. Robust hedging of double touch barrier options. *SIAM J. Financial Math.*, 2:141–182, 2011.

[Cou07] L. Cousot. Conditions on option prices for absence of arbitrage and exact calibration. *Journal of Banking & Finance*, 31(11):3377–3397, 2007. Risk Management and Quantitative Approaches in Finance.

[CW11] A. M. G. Cox and J. Wang. Root's Barrier: Construction, Optimality and Applications to Variance Options. *ArXiv e-prints*, April 2011.

[DH07] M.H.A. Davis and D. Hobson. The range of traded option prices. *Math. Finance*, 17(1):1–14, 2007.

[Dup94] B. Dupire. Pricing with a smile. *Risk*, 7(1):18–20, 1994.

[GHLT11] A. Galichon, P. Henry-Labordère, and N. Touzi. A Stochastic Control Approach to No-Arbitrage Bounds Given Marginals, with an Application to Lookback Options. *SSRN eLibrary*, 2011.

[HK12] D. Hobson and M. Klimmek. Model independent hedging strategies for variance swaps. *Finance and Stochastics*, to appear, April 2012.

[HL09] P. Henry-Labordère. Calibration of local stochastic volatility models to market smiles. *Risk*, September 2009.

[HL11] P. Henry-Labordère. Automated option pricing: Numerical methods. *SSRN eLibrary*, 2011.

[HLT12] P. Henry-Labordère and N. Touzi. Maximum maximum of martingales given marginals. *SSRN eLibrary*, 2012.

[HLW05a] D. Hobson, P. Laurence, and T.H. Wang. Static-arbitrage optimal subreplicating strategies for basket options. *Insurance Math. Econom.*, 37(3):553–572, 2005.

[HLW05b] D. Hobson, P. Laurence, and T.H. Wang. Static-arbitrage upper bounds for the prices of basket options. *Quant. Finance*, 5(4):329–342, 2005.

[HN12] D. Hobson and A. Neuberger. Robust bounds for forward start options. *Mathematical Finance*, 22(1):31–56, December 2012.

[Hob98] D. Hobson. Robust hedging of the lookback option. *Finance and Stochastics*, 2:329–347, 1998. 10.1007/s007800050044.

[Hob11] D. Hobson. The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, Berlin, 2011.

[Hob12] D. Hobson. Personal communication, February 2012.

[HP02] D. Hobson and J.L. Pedersen. The minimum maximum of a continuous martingale with given initial and terminal laws. *Ann. Probab.*, 30(2):978–999, 2002.

[Kel84] H.G. Kellerer. Duality theorems for marginal problems. *Z. Wahrsch. Verw. Gebiete*, 67(4):399–432, 1984.

[LW04] P. Laurence and T.H. Wang. What's a basket worth? February 2004.

[LW05] P. Laurence and T.H. Wang. Sharp upper and lower bounds for basket options. *Applied Mathematical Finance*, 12(3):253–282, 2005.

[MY02] D.B. Madan and M. Yor. Making Markov martingales meet marginals: with explicit constructions. *Bernoulli*, 8(4):509–536, 2002.

[NH00] A. Neuberger and S.D. Hodges. Rational bounds on the prices of exotic options. *SSRN eLibrary*, 2000.

[PTVF07] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery. *Numerical recipes*. Cambridge University Press, Cambridge, third edition, 2007. The art of scientific computing.

[Str65] V. Strassen. The existence of probability measures with given marginals. *Ann. Math. Statist.*, 36:423–439, 1965.

[Str85] H. Strasser. *Mathematical theory of statistics*, volume 7 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1985. Statistical experiments and asymptotic decision theory.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Vil09] C. Villani. *Optimal Transport. Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.