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Model order reduction for bilinear control systems with inhomogeneous initial conditions

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ABSTRACT
This work focuses on the model order reduction problem for bilinear control systems with nonzero initial conditions. Based on the Volterra series analysis, the system response can be decomposed into three parts. The first two parts are the zero input response and zero initial condition response of the system. The third part describes the response which couples the effect of the nonzero initial condition and the nonzero input. The system corresponding to the third part is a bilinear control system with a special time-varying input coefficient matrix. We show that such a system is equivalent to a time-invariant bilinear control system, and conventional model reduction methods can be applied to reduce it. We propose to reduce each of the component responses independently and then combine them to approximate the full system response. This method is of high flexibility and shows promising results.

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Model order reduction; bilinear control system; inhomogeneous initial condition; kernel energy averaging; approximation of matrix exponential function

1. Introduction

In this paper, we focus on model order reduction for bilinear control systems with *nonzero* initial conditions. Firstly, we address the problem for single-input single-output (SISO) systems. Later, we generalise the ideas to multiple-input multiple-output (MIMO) systems. A SISO bilinear control system with nonzero initial conditions is represented in the state-space form as

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t), \\ y(t) = Cx(t),\ x(0) = x_0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, and $C \in \mathbb{R}^{1\times n}$. The signals $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the state vector, input and output of the system, respectively. The initial condition is denoted as $x_0 \in \mathbb{R}^n$. Throughout the paper, the system (1) is assumed to be *bounded-input bounded-output* (BIBO) stable. A sufficient condition for this to happen is the matrix A to be Hurwitz (the spectrum of A, $\Lambda(A) \in \mathbb{C}^-$) and the matrix N to be sufficiently bounded. We refer to Zhang and Lam (2002) for detailed explanations on BIBO stability.

The goal is to find a reduced-order bilinear control system having the state-space representation

$$\Sigma_r : \begin{cases} \dot{x}_r(t) = A_r x_r(t) + N_r x_r(t)u(t) + B_r u(t), \\ y_r(t) = C_r x_r(t),\ x_r(0) = x_{r0}, \end{cases} \tag{2}$$

where $A_r \in \mathbb{R}^{r\times r}$, $N_r \in \mathbb{R}^{r\times r}$, $B_r \in \mathbb{R}^r$ and $C_r \in \mathbb{R}^{1\times r}$ with a considerably smaller dimension $r \ll n$, such that the reduced-order system output $y_r(t)$ provides a good approximation of $y(t)$ without specifying the input $u(t)$ and the initial condition x_0.

For bilinear control systems with *zero* initial condition, i.e. $x_0 = 0$, a wide range of methods has been developed to construct accurate reduced-order models. There are singular value decomposition (SVD) based methods such as balanced truncation (Benner & Damm, 2011; Condon & Ivanov, 2004), truncated-Gramian balanced truncation (Al-Baiyat & Bettayeb, 1993; Benner et al., 2017) which extend the balanced truncation method (Moore, 1981) from linear time-invariant (LTI) systems to bilinear control systems. Interpolation-based methods also exist, for example Krylov subspace methods (Bai & Skoogh, 2006; Breiten & Damm, 2010; Feng & Benner, 2007; Phillips, 2003), interpolation of Volterra series (Flagg & Gugercin, 2015) and the bilinear iterative rational Krylov algorithm (BIRKA) (Benner & Breiten, 2012). Another category of the methods solves the bilinear $\mathcal{H}_2$ optimal model reduction problem by using Riemannian optimisation methods, for example the pioneering work of Zhang and Lam (2002) and the follow-up works in Benner et al. (2019) and Bruns (2015), which take advantage of the Grassmann manifold and develop several algorithms to construct the reduced-order models.

However, as long as the initial condition is *nonzero*, reduced-order models constructed by the aforementioned methods may cause large deviations in the transient simulation and sometimes even in the steady state. A natural idea to solve such a problem is to translate the state vector from $x(t)$ to $z(t) = x(t) - x_0$ (Baur et al., 2014). The resulting bilinear system has the state-space representation

$$\tilde{\Sigma} : \begin{cases} \dot{z}(t) = Az(t) + Nz(t)u(t) + \begin{pmatrix} B + Nx_0 & Ax_0 \end{pmatrix} \begin{pmatrix} u(t) \\ 1 \end{pmatrix}, \\ y(t) = Cz(t) + Cx_0, z(0) = 0. \end{cases} \tag{3}$$

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Then, model order reduction can be applied to $\tilde{\Sigma}$ in (3) rather than the original bilinear control system in (1). Such a method is quite restrictive since it requires the exact knowledge of the initial condition. When the initial condition changes, this method is not expected to provide a good approximation of the system anymore. Thus a new reduction is necessary. For multi-query applications, repetitively reducing the system can result in a large amount of computational effort. Simulation-based methods such as proper orthogonal decomposition (POD) (see e.g. Antoulas, 2005; Gubisch & Volkwein, 2017) suffer from the same problem.

To improve the approximation accuracy, a novel approach was proposed by Heinkenschloss et al. (2011) for LTI systems, which is based on the superposition principle. Instead of assuming the exact knowledge of x_0, the authors of Heinkenschloss et al. (2011) assumed that the initial condition lies in a certain n_0-dimensional subspace $\mathcal{X}_0 := \operatorname{span}\{X_0\}$ with $n_0 \ll n$. The initial condition hence can be expressed as $x_0 = X_0 u_0$. As a result, one only needs to augment the input coefficient matrix of the LTI system to $(B,\ X_0)$ and the input signal to $col(u(t), u_0)$.[1] Nevertheless, the augmentation of the input may increase the Hankel singular values significantly. Indeed, as shown in Beattie et al. (2017), if the initial condition contributes to a large amount of energy in the augmented system, the system can become very difficult to reduce as the Hankel singular values decay too slow. In this case, a reduced-order model with a higher order is usually required. Also, from the system theoretic perspective, the augmented input u_0 is not an $\mathcal{L}_2$ function as long as it is nonzero. To tackle these problems, Beattie et al. (2017) takes the full advantage of the superposition principle and decomposes the system into two systems, which correspond to the zero input response and the zero initial condition response, respectively. By reducing each of them independently and combining the reduced-order responses, the approximation accuracy can be improved, meanwhile the reduced orders can be controlled independently. Furthermore, independent reduction of each system offers great flexibility. The method can be easily implemented in a parallel computing fashion.

In this paper, we follow the strategy of Beattie et al. (2017). The main difficulty to extend the method to bilinear control systems is that the superposition principle is no longer applicable. However, one can still decompose the system by applying Volterra series analysis. We observed that the response of system (1) can be decomposed into three parts. The first and the second correspond to the responses of system (1) to $u(t) = 0$ and $x_0 = 0$, respectively. The third part describes the coupling effects of the nonzero initial condition and the input, which can be modelled by a bilinear control system with a time-varying input coefficient matrix. The existing model order reduction methods for bilinear control systems cannot be applied to this system because of its time-varying behaviour. We propose a novel method, which considers the averaged or total reachability and observability energy. Then the reduction of this part amounts to the reduction of a time-invariant bilinear control system and an LTI system independently. Hence, the reduction of the full system requires reduction of two LTI systems and reduction of two bilinear control systems. The reduction of each system is totally independent of the others. Hence the method offers a great flexibility, similar to the LTI context. It is worth mentioning that a similar notion of averaging in model reduction was also proposed in Nilsson and Rantzer (2009), but its application is not straightforward here.

The remaining parts of the paper are organised as follows. Section 2 first presents the preliminary results on the response decomposition and then discusses how to transform a bilinear control system with a particular time-varying input coefficient matrix into a time-invariant bilinear control system in the Volterra kernel energy sense. In Section 3 we discuss the model reduction strategies using balanced truncation and BIRKA, as well as the generalisation from SISO systems to MIMO systems. Numerical examples are shown in Section 4. Section 5 concludes the paper.

2. Preliminary results

2.1 System response decomposition

Consider the bilinear control system Σ in (1). The system is assumed to be BIBO stable, hence the Volterra series converges (see e.g. Siu & Schetzen, 1991; Zhang & Lam, 2002). The system output $y(t)$ can be explicitly expressed as (D'Alessandro et al., 1974)

$$y(t) = y_{\mathrm{x}}(t) + y_{\mathrm{u}}(t) + y_{\mathrm{xu}}(t),$$

where

$$y_{\mathrm{x}}(t) = Ce^{At}x_0 = Ce^{At}X_0u_0 \tag{4}$$

is the zero input response,

$$y_{\mathrm{u}}(t) = \sum_{i=1}^{\infty}\int_0^t\int_0^{\tau_i}\cdots\int_0^{\tau_2} Ce^{A(t-\tau_i)}Ne^{A(\tau_i-\tau_{i-1})}N\cdots \times Ne^{A(\tau_2-\tau_1)}Bu(\tau_1)\cdots u(\tau_{i-1})u(\tau_i)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{i-1}\mathrm{d}\tau_i \tag{5}$$

is the zero initial condition response, and

$$y_{\mathrm{xu}}(t) = \sum_{i=1}^{\infty}\int_0^t\int_0^{\tau_i}\cdots\int_0^{\tau_2} Ce^{A(t-\tau_i)}Ne^{A(\tau_i-\tau_{i-1})}N\cdots Ne^{A(\tau_2-\tau_1)} \times \tilde{B}(\tau_1)u(\tau_1)\cdots u(\tau_{i-1})u(\tau_i)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{i-1}\,\mathrm{d}\tau_i, \tag{6}$$

with $\tilde{B}(t) = Ne^{At}X_0u_0$, describes the coupling effects of input and the nonzero initial condition. A schematic plot of this decomposition is depicted in Figure 1.

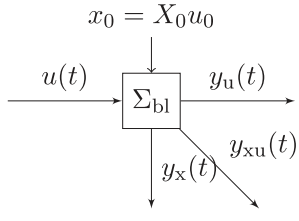

Figure 1. Schematic plot of system decomposition for bilinear control systems.

First, we notice that the response $y_x(t)$ in (4) is the output of the following LTI system

$$\Sigma_x : \begin{cases} \dot{w}_x(t) = Aw_x(t) + X_0 u_x(t), \\ y_x(t) = Cw_x(t), \ w_x(0) = 0, \end{cases} \tag{7}$$

with the input $u_x(t) = u_0\delta(t)$, i.e. the scaled Dirac function. Second, the response $y_u(t)$ is the response of system Σ in (1) with $x_0 = 0$, i.e. it is given by the bilinear control system having the state-space representation

$$\Sigma_u : \begin{cases} \dot{w}_u(t) = Aw_u(t) + Nw_u(t)u(t) + Bu(t), \\ y_u(t) = Cw_u(t), \ w_u(0) = 0. \end{cases} \tag{8}$$

Finally, the response $y_{xu}(t)$ is the output of a time-varying bilinear control system with the state-space representation

$$\Sigma_{xu} : \begin{cases} \dot{w}_{xu}(t) = Aw_{xu}(t) + Nw_{xu}(t)u(t) + B_{xu}(t)u_0 u(t), \\ y_{xu}(t) = Cw_{xu}(t), \ w_{xu}(0) = 0, \end{cases} \tag{9}$$

where $B_{xu}(t) = Ne^{At}X_0$.

Remark 2.1: There exists another method to decompose the system response, which does not require to carry out the Volterra series analysis of the system (1). Clearly we know that the input-to-state response of the system contains the zero input response $w_x(t) = e^{At}X_0u_0$, which is the response of the differential equation $\dot{w}_x(t) = Aw_x(t)$ with $w_x(0) = x_0$ and the zero initial condition response $w_u(t)$ which can be obtained from the state equation of (8). Then defining the remaining of the system state as $\epsilon(t) = x(t) - w_x(t) - w_u(t)$, one can derive that

$$\begin{aligned} \dot{\epsilon}(t) &= \dot{x}(t) - \dot{w}_x(t) - \dot{w}_u(t) \\ &= A\epsilon(t) + N\epsilon(t)u(t) + Nw_x(t)u(t) \\ &= A\epsilon(t) + N\epsilon(t)u(t) + Ne^{At}X_0u_0u(t). \end{aligned}$$

It is immediate that $\epsilon(t)$ solves the state equation in (9). Hence, $\epsilon(t) = w_{xu}(t)$.

As a consequence, a bilinear system Σ with nonzero initial conditions as in (1) can be decomposed into three different systems Σ_x, Σ_u and Σ_{xu} given by the equations (7), (8), and (9), respectively. Reduced-order models for systems Σ_x and Σ_u can be obtained by the conventional methods introduced in Section 1. However, due to its time-varying behaviour, model reduction of system Σ_{xu} (9) is not straightforward. In what follows, we propose the notion of kernel energy averaging, which will lead to a new procedure based on averaged Gramians to reduce systems of the form (9).

2.2 Kernel energy averaging

Although the system Σ_{xu} in (9) is time-varying, the reachability and observability concepts still follow the conventional definitions for bilinear control systems. We refer to D'Alessandro et al. (1974) and Dorissen (1989) for details. Let

$$\begin{aligned} &P_1(t, \tau_1) = e^{A(t-\tau_1)}B_{xu}(\tau_1), \\ &P_i(t, \tau_1, \ldots, \tau_i) = e^{A(t-\tau_i)}N \cdots Ne^{A(\tau_2-\tau_1)}B_{xu}(\tau_1), \ i \geq 2. \end{aligned} \tag{10}$$

Upon existence, for any $t \geq 0$ the time-varying reachability Gramian is defined as

$$R(t) = \sum_{i=1}^{\infty} \int_{\tau_i}^{t} \int_{\tau_{i-1}}^{\tau_i} \cdots \int_{-\infty}^{\tau_2} P_i P_i^{\top} \, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_{i-1} \, \mathrm{d}\tau_i. \tag{11}$$

Similarly, let t_{i-1} denote $\tau_i - \tau_{i-1}$, $i \geq 2$, but the final one t_i is defined as $t - \tau_i$ and let

$$\begin{aligned} &O_1(t_1) = Ce^{At_1}, \\ &O_i(t_1, \ldots, t_i) = Ce^{At_i}N \cdots Ne^{At_1}, \quad i \geq 2. \end{aligned} \tag{12}$$

Upon existence, the observability Gramian is defined as

$$Q = \sum_{i=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} O_i^{\top} O_i \, \mathrm{d}t_1 \cdots \mathrm{d}t_i. \tag{13}$$

Notice that only the reachability Gramian is time-dependent because only the matrix $B_{xu}(t)$ defining system (9) is time-varying. Both of the Gramians $R(t)$ and Q are symmetric and positive semi-definite. The sufficient conditions for existence of the Gramians can be found in Zhang and Lam (2002) and Benner et al. (2019).

Theorem 2.1: *The time-varying reachability Gramian $R(t)$ given by* (11)*, if it exists, is the solution of the generalised differential Lyapunov equation:*

$$\begin{aligned} \dot{R}(t) = AR(t) + R(t)A^{\top} + NR(t)N^{\top} \\ + B_{xu}(t)B_{xu}^{\top}(t), \quad R(0) = 0. \end{aligned} \tag{14}$$

Similarly, the observability Gramian given by (13)*, if it exists, is the solution of the generalised Lyapunov equation:*

$$A^{\top}Q + QA + N^{\top}QN + C^{\top}C = 0. \tag{15}$$

Proof: Differentiating $R(t)$ in (11), applying the Leibniz integral rule and using the fact of asymptotic stability (A is a Hurwitz matrix), the differential Lyapunov equation (14) can be obtained. The condition $R(0) = 0$ is a consequence of the fact that if $t = 0$ in (11), we steer the system from zero state to zero state (initial state is always zero), hence no input energy is required. The observability part is a standard result from Al-Baiyat and Bettayeb (1993). ■

Noticing that if the time-varying matrix $B_{xu}(t)$ is only slowly time-varying, one may treat it as the constant matrix $B_{xu}(0) = NX_0$. Then Σ_{xu} is not considered as a time-varying system anymore, which means its reachability Gramian satisfies $R(t) \equiv R$ with R the solution of the generalised Lyapunov equation

$$AR + RA^{\top} + NRN^{\top} + B_{xu}(0)B_{xu}^{\top}(0) = 0.$$

Then the conventional model order reduction methods developed for bilinear dynamical systems such as balanced truncation (Benner et al., 2017), interpolation-based model reduction

(Benner & Breiten, 2012) and Riemannian optimisation based methods (Benner et al., 2019; Xu et al., 2017) can be applied to reduce it.

Recalling the system response $y_{\mathrm{xu}}(t)$ in (6), the system Σ_{xu} in (9) has the Volterra kernels:

$$\begin{aligned} h_{\mathrm{xu},1}(t,\tau_1) &= Ce^{A(t-\tau_1)}B_{\mathrm{xu}}(\tau_1),\\ h_{\mathrm{xu,i}}(t,\tau_1,\ldots,\tau_i) &= Ce^{A(t-\tau_i)}N\cdots Ne^{A(\tau_2-\tau_1)}B_{\mathrm{xu}}(\tau_1),\quad i\geq 2. \end{aligned} \tag{16}$$

We define the energy of the Volterra kernels as

$$\mathcal{E}_{\mathrm{xu}} := \sqrt{\sum_{i=1}^{\infty}\int_0^{\infty}\int_0^{t}\cdots\int_0^{\tau_2} h_{\mathrm{xu,i}}h_{\mathrm{xu,i}}^{\top}\mathrm{d}\tau_1\cdots\mathrm{d}\tau_i\,\mathrm{d}t}, \tag{17}$$

which can be seen as the averaged (or total) kernel energy for $t\in[0,\infty)$. Also, let us denote $\bar{R}$ to be the averaged reachability Gramian, satisfying $\bar{R}=\int_0^{\infty}R(t)\,\mathrm{d}t$. The following proposition holds.

Proposition 2.2: *Suppose that the matrix*

$$\mathcal{M} = A\otimes I + I\otimes A + N\otimes N \tag{18}$$

is Hurwitz. Then, the solution $R(t)$ of (14) is integrable in $[0,\infty)$ and satisfies $\lim_{t\to\infty}R(t)=0$.

Proof: By vectorising Equation (14), one obtains

$$\dot{r}(t) = \mathcal{M}r(t) + b(t),\quad r(0)=0,$$

where $r(t)=\mathrm{vec}\,(R(t))$ and $b(t)=\mathrm{vec}\left(B_{\mathrm{xu}}(t)B_{\mathrm{xu}}^{\top}(t)\right)$ and $B_{\mathrm{xu}}(t)=Ne^{At}X_0$. Since $\mathcal{M}$ is Hurwitz and $b(t)$ has an exponentially deceasing behaviour, there exist constants $C_{\mathrm{e}},C_{\mathrm{b}}>0$ and $\alpha_{\mathrm{e}},\alpha_{\mathrm{b}}<0$ such that $\|e^{\mathcal{M}t}\|\leq C_{\mathrm{e}}e^{\alpha_{\mathrm{e}}t}$ and $\|b(t)\|\leq C_{\mathrm{b}}e^{\alpha_{\mathrm{b}}t}$ for $t\geq 0$. Hence, using the variation of constant method, one has

$$r(t) = \int_0^t e^{\mathcal{M}(t-\tau)}b(\tau)\,\mathrm{d}\tau.$$

As a consequence,

$$\begin{aligned} \|r(t)\| &\leq \int_0^t \|e^{\mathcal{M}(t-\tau)}\|\|b(\tau)\|\,\mathrm{d}\tau\\ &\leq C_{\mathrm{e}}C_{\mathrm{b}}\int_0^t e^{\alpha_{\mathrm{e}}(t-\tau)}e^{\alpha_{\mathrm{b}}\tau}\,\mathrm{d}\tau = \frac{C_{\mathrm{e}}C_{\mathrm{b}}}{\alpha_{\mathrm{b}}-\alpha_{\mathrm{e}}}(e^{\alpha_{\mathrm{b}}t}-e^{\alpha_{\mathrm{e}}t}). \end{aligned}$$

From the above inequality, one obtains that $\|r(t)\|$ is integrable and $\|r(t)\|\to 0$, if $t\to\infty$, which proves the result. ■

From Proposition 2.2, one can conclude that the averaged reachability Gramian $\bar{R}$ is finite if $\mathcal{M}$ in (18) is Hurwitz. From now on, we assume that the matrix $\mathcal{M}$ is Hurwitz. In fact, a BIBO stable bilinear control system requires the matrix N to be sufficiently small in norm. Therefore, the assumption often holds for BIBO stable bilinear systems. The following theorem provides a computational framework of the averaged Volterra kernel energy and its connection with the averaged reachability Gramian.

Theorem 2.3: *The Volterra kernel energy given by (17) satisfies*

$$\mathcal{E}_{\mathrm{xu}} = \sqrt{C\bar{R}C^{\top}} = \sqrt{\mathrm{trace}\left(X_0^{\top}\bar{Q}X_0\right)}, \tag{19}$$

where $\bar{R}$ is the solution of

$$A\bar{R} + \bar{R}A^{\top} + N\bar{R}N^{\top} + NPN^{\top} = 0, \tag{20}$$

with

$$AP + PA^{\top} + X_0X_0^{\top} = 0, \tag{21}$$

and $\bar{Q}$ is the solution of

$$A^{\top}\bar{Q} + \bar{Q}A + N^{\top}QN = 0, \tag{22}$$

with Q the solution of (15).

Proof: The kernel energy $\mathcal{E}_{\mathrm{xu}}$ defined by (17) can be written as

$$\mathcal{E}_{\mathrm{xu}} = \sqrt{C\int_0^{\infty}R(t)\,\mathrm{d}tC^{\top}} = \sqrt{\mathrm{trace}\left(\int_0^{\infty}B_{\mathrm{xu}}^{\top}(t)QB_{\mathrm{xu}}(t)\,\mathrm{d}t\right)}.$$

Hence, by integrating (14), we have

$$\begin{aligned} 0 &= R(\infty) - R(0)\\ &= A\bar{R} + \bar{R}A^{\top} + N\bar{R}N^{\top} + \int_0^{\infty}B_{\mathrm{xu}}(t)B_{\mathrm{xu}}^{\top}(t)\,\mathrm{d}t, \end{aligned}$$

since $R(\infty)=0=R(0)$. The integral term $\int_0^{\infty}B_{\mathrm{xu}}(t)B_{\mathrm{xu}}^{\top}(t)\,\mathrm{d}t$ is computed as

$$\int_0^{\infty}B_{\mathrm{xu}}(t)B_{\mathrm{xu}}^{\top}(t)\,\mathrm{d}t = N\int_0^{\infty}e^{At}X_0X_0^{\top}e^{A^{\top}t}\,\mathrm{d}tN^{\top} = NPN^{\top},$$

where P is the solution of (21). Similarly, another integral term $\int_0^{\infty}B_{\mathrm{xu}}^{\top}(t)QB_{\mathrm{xu}}(t)\,\mathrm{d}t$ is computed as

$$\int_0^{\infty}B_{\mathrm{xu}}^{\top}(t)QB_{\mathrm{xu}}(t)\,\mathrm{d}t = X_0^{\top}\int_0^{\infty}e^{A^{\top}t}N^{\top}QNe^{At}\,\mathrm{d}tX_0 = X_0^{\top}\bar{Q}X_0,$$

where $\bar{Q}$ is certainly the solution of (22). ■

Corollary 2.4: *The Volterra kernel energy $\mathcal{E}_{\mathrm{xu}}$ in (17) can also be computed from*

$$\mathcal{E}_{\mathrm{xu}} = \sqrt{\mathrm{trace}\left(NPN^{\top}Q\right)}, \tag{23}$$

where P and Q are solutions of (21) and (15), respectively.

Proof: The kernel energy $\mathcal{E}_{\mathrm{xu}}$ can be computed as

$$\begin{aligned} \mathcal{E}_{\mathrm{xu}} &= \sqrt{\mathrm{trace}\left(\int_0^{\infty}B_{\mathrm{xu}}^{\top}(t)QB_{\mathrm{xu}}(t)\,\mathrm{d}t\right)}\\ &= \sqrt{\mathrm{trace}\left(\int_0^{\infty}B_{\mathrm{xu}}(t)B_{\mathrm{xu}}^{\top}(t)\,\mathrm{d}tQ\right)} = \sqrt{\mathrm{trace}\left(NPN^{\top}Q\right)}. \end{aligned}$$

■

The Gramians $\bar{R}$ and $\bar{Q}$ quantify the averaged reachability and observability for $t\in[0,\infty)$, respectively. Another merit of

considering the averaged kernel energy rather than the time-varying one is that the time-varying bilinear control system Σ_{xu} given by (9) can be transformed into a time-invariant bilinear control system assuming that Σ_x is reachable.

Theorem 2.5: *Let the solution of* (21) *be* $P = LL^\top$ *positive definite and* $L \in \mathbb{R}^{n\times n}$ *be the corresponding Cholesky factor. The Volterra kernel energy of the system*

$$\bar{\Sigma}_{xu}: \begin{cases} \dot{\bar{w}}_{xu}(t) = A\bar{w}_{xu}(t) + N\bar{w}_{xu}(t)u(t) + NL\bar{u}_{xu}(t), \\ \bar{y}_{xu}(t) = C\bar{w}_{xu}(t),\ \bar{w}_{xu}(0) = 0. \end{cases} \tag{24}$$

with $\bar{u}_{xu}(t) = w(t)u_0u(t)$ *for an unknown matrix function* $w(t) \in \mathbb{R}^{n\times n_0}$, *equals* $\mathcal{E}_{xu}$ *in* (17). *That is, the two systems* Σ_{xu} *and* $\bar{\Sigma}_{xu}$ *are equivalent in the sense of the averaged Volterra kernel energy.*

Proof: It can be checked that the Volterra kernel energy of system $\bar{\Sigma}_{xu}$ is

$$\bar{\mathcal{E}}_{xu} = \sqrt{\operatorname{trace}\left(L^\top N^\top QNL\right)} = \sqrt{\operatorname{trace}\left(C\check{R}C^\top\right)},$$

where $\check{R}$ is the solution of

$$A\check{R} + \check{R}A^\top + N\check{R}N^\top + NLL^\top N^\top = 0.$$

Noting that $P = LL^\top$ and applying the properties of the trace operation, we have $\check{R} = \bar{R}$ and $\bar{\mathcal{E}}_{xu} = \mathcal{E}_{xu}$. ■

When considering $\mathcal{H}_2$ norm based model reduction methods for system Σ_{xu}, one can construct the projection matrices for system $\bar{\Sigma}_{xu}$, completely avoiding the time-varying matrix $B_{xu}(t)$. Another advantage is that if a Lyapunov-type approximation method is applied to system Σ_x in (7), the matrix P or its Cholesky factor L is immediately available. Since the Cholesky factor L often has low numerical rank, one can approximate P by $P \approx \hat{L}\hat{L}^\top$ with $\hat{L} \in \mathbb{R}^{n\times \hat{m}_{xu}}$ and $\hat{m}_{xu} \ll n$.

3. Model order reduction

In Section 2, we showed that the response of a bilinear control system with inhomogeneous initial conditions can be decomposed as the responses of three different systems with zero initial condition. As a result, model order reduction can be applied to those three systems independently.

The reduction of systems Σ_x in (7) and Σ_u in (8) follows the conventional model reduction methods for LTI systems and bilinear control systems, respectively. To reduce system Σ_x, one can apply the balanced truncation method, which provides an error bound in terms of energy-to-energy gain ($\mathcal{H}_\infty$ norm) or the iterative rational Krylov algorithm (IRKA) for an $\mathcal{H}_2$ norm (energy-to-peak gain) optimal approximation, for example see Antoulas (2005) and Gugercin et al. (2008). If balanced truncation is applied, the reachability Gramian P can also be used in the transformation as well as reduction of system Σ_{xu}. IRKA seems a better choice here because only the impulse response of Σ_x in (7) is of interest. The $\mathcal{H}_2$ optimal approximation minimises the impulse response energy (Gugercin et al., 2008). To reduce the bilinear control system Σ_u in (8), balanced truncation, truncated-Gramian balanced truncation, BIRKA and truncated-BIRKA (Flagg & Gugercin, 2015) can be applied. Again, if balanced truncation is applied, the observability Gramian Q can be used to reduce system Σ_{xu} as well. In the remaining part of this section, we concentrate on the reduction of system Σ_{xu}.

3.1 Model order reduction of system Σ_{xu}

Consider system Σ_{xu} given by (9). After applying the projection-based model reduction, one obtains the system matrices

$$\begin{aligned} \hat{A}_{xu} = W_{xu}^\top AV_{xu},\ \hat{N}_{xu} = W_{xu}^\top NV_{xu}, \\ \hat{B}_{xu}(t) = W_{xu}^\top B_{xu}(t),\ \hat{C}_{xu} = CV_{xu}. \end{aligned} \tag{25}$$

The matrices V_{xu} and W_{xu} are the bi-orthogonal projection matrices and hence, they satisfy $W_{xu}^\top V_{xu} = I_r$. The matrix $\hat{B}_{xu}(t)$ is

$$\hat{B}_{xu}(t) = W_{xu}^\top B_{xu}(t) = W_{xu}^\top Ne^{At}X_0, \tag{26}$$

where the full-order matrix exponential application $e^{At}X_0$ still needs to be evaluated in the simulation of the reduced-order model. To overcome such a difficulty, we propose to reduce the computational complexity of $B_{xu}(t)$ by reducing an LTI system, which has the state-space representation

$$\Sigma_B: \begin{cases} \dot{w}_B(t) = Aw_B(t) + X_0u_B(t), \\ B_{xu}(t) = Nw_B(t),\ w_B(0) = 0, \end{cases} \tag{27}$$

where $u_B(t) = I_{n_0}\delta(t)$. The reachability Gramian of system Σ_B is again P, i.e. the solution of (21). Hence, if balanced truncation is applied to reduce system Σ_B, one only needs to compute the observability Gramian by solving

$$A^\top Q_B + Q_BA + N^\top N = 0,$$

which saves half of the computational effort. The difference between $B_{xu}(t)$ and its balanced truncation approximation $\hat{B}_{xu}(t)$ is Antoulas (2005)

$$\|B_{xu} - \hat{B}_{xu}\|_{L_2} \le \left(2\sum_{i=r_B+1}^{n} \sigma_{B,i}\right)\|u_B\|_{L_2},$$

where $\sigma_{B,i}$, $i = 1, 2, \ldots, n$ are the Hankel singular values of system Σ_B. In applications, r_B is usually much smaller than n, specially in the case where N has low rank.

One may notice that again we are only interested in the impulse responses of system Σ_B. Hence, IRKA can be more effective than balanced truncation. Although there is no prior knowledge of the approximation error, IRKA provides an optimal approximation of system Σ_B in the sense of impulse response energy. The second advantage of IRKA is that the reduced-order system only depends on the subspace spanned by the projection matrices. Hence, whenever N is low rank, then one can first construct a full rank matrix $L_N \in \mathbb{R}^{n_N\times n}$ with $\operatorname{span}\{L_N^\top\} = \operatorname{span}\{N^\top\}$ and then use L_N as the output coefficient matrix to construct the projection matrices.

To reduce the system Σ_{xu} given by (9), two methods are proposed. The first one is a balanced truncation type method.

Instead of considering the original time-varying reachability Gramian $R(t)$ in (11) and the observability Gramian Q in (13), we consider the averaged Gramians $\bar{R}$ and $\bar{Q}$. Although two more Lyapunov equations need to be solved, we avoid solving the generalised differential Lyapunov equation (14), which is a far more complicated and numerically expensive task. By balancing the averaged Gramians $\bar{R}$ and $\bar{Q}$, the projection matrices can be obtained by using the conventional algorithms which are designed for LTI systems, e.g. the square-root method (see for example Antoulas, 2005).

The second method is a BIRKA-type method (Benner & Breiten, 2012). As it is only applicable to time-invariant bilinear control systems, we propose to reduce the system in (24) rather than system $\Sigma_{\rm xu}$ itself. Notice that in (24), the system has two types of inputs, the first is the original input $u(t)$ and the second is $\bar{u}_{\rm xu}(t)$. To apply BIRKA, we treat them as two independent inputs and augment the state equation as

$$\dot{\bar{w}}_{\rm xu}(t) = A\bar{w}_{\rm xu}(t) + \sum_{k=1}^{1+\hat{m}_{\rm xu}} N_k \bar{w}_{\rm xu}\bar{u}_i(t) + \bar{B}_{\rm xu}\bar{u}(t),$$

where $N_1 = N$ and $N_k = 0_{n\times n}$, $k = 2, 3, \ldots, \hat{m}_{\rm xu}+1$, $\bar{B}_{\rm xu} = (0_{n\times 1}, N\hat{L})$ and $\bar{u} = col(u, \bar{u}_{\rm xu})$. In the augmentation, we use the low rank factor $\hat{L}$ of P. The Volterra kernel energy is expected to be quite close to the averaged kernel energy $\mathcal{E}_{\rm xu}$. And a tall matrix $\bar{B}_{\rm xu}$ may reduce the computational time of BIRKA.

In conclusion, the model reduction problem for the bilinear control system Σ in (1) with inhomogeneous initial conditions can be recast as reducing systems $\Sigma_{\rm x}$, Σ_u, $\Sigma_{\rm xu}$ (or $\bar{\Sigma}_{\rm xu}$) and $\Sigma_{\rm B}$ independently. Up to now, all the discussions were carried out for SISO bilinear control systems. In fact, all results can be easily generalised to MIMO systems. In the following subsection, we briefly explain this generalisation.

3.2 Generalisation to MIMO systems

In the MIMO context, a bilinear control system with nonzero initial conditions is represented in state-space form as

$$\Sigma_{\rm MIMO}: \begin{cases} \dot{x}(t) = Ax(t) + \sum_{k=1}^{m} N_k x(t)u_k(t) + Bu(t), \\ y(t) = Cx(t),\ x(0) = x_0, \end{cases} \tag{28}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $x_0 = X_0 u_0 \in \mathbb{R}^n$ and $N_k \in \mathbb{R}^{n\times n}$ for $k = 1, \ldots, m$. By similar reasoning, the output $y(t)$ from (28) can also be decomposed as the sum $y(t) = y_{\rm x}(t) + y_{\rm u}(t) + y_{\rm xu}(t)$, where $y_{\rm x}(t)$ is the response of the same LTI system $\Sigma_{\rm x}$ in (7). The response $y_{\rm u}(t)$ comes from the system Σ_u given by

$$\Sigma_{\rm u}: \begin{cases} \dot{w}_{\rm u}(t) = Aw_{\rm u}(t) + \sum_{k=1}^{m} N_k w_{\rm u}(t)u_k(t) + Bu(t), \\ y_{\rm u}(t) = Cw_{\rm u}(t),\ w_{\rm u}(0) = 0. \end{cases} \tag{29}$$

The response $y_{\rm xu}(t)$ again is the solution of a time-varying bilinear control system with the state-space representation

$$\Sigma_{\rm xu}: \begin{cases} \dot{w}_{\rm xu}(t) = Aw_{\rm xu}(t) + \sum_{k=1}^{m} N_k w_{\rm xu}(t)u_k(t) \\ + \sum_{k=1}^{m} B_{\rm xu,k}(t)u_0 u_k(t), \\ y_{\rm xu}(t) = Cw_{\rm xu}(t),\ w_{\rm xu}(0) = 0, \end{cases} \tag{30}$$

where $B_{\rm xu,k}(t) = N_k e^{At}X_0$ for $k = 1, \ldots, m$. Hence, one can reduce the homogeneous systems $\Sigma_{\rm x}$, $\Sigma_{\rm u}$ and $\Sigma_{\rm xu}$ instead of reducing the bilinear system with nonzero initial consition (28). In this context, all of the results derived in Section 2 can be easily generalised to MIMO systems. Moreover, if $P = LL^\top$ is the solution of (21) and $L \in \mathbb{R}^{n\times n}$, the system

$$\bar{\Sigma}_{\rm xu}: \begin{cases} \dot{\bar{w}}_{\rm xu}(t) = A\bar{w}_{\rm xu}(t) + \sum_{k=1}^{m} N_k \bar{w}_{\rm xu}(t)u_k(t) \\ + \sum_{k=1}^{m} N_k L\bar{u}_{\rm xu,k}(t), \\ \bar{y}_{\rm xu}(t) = C\bar{w}_{\rm xu}(t),\ \bar{w}_{\rm xu}(0) = 0. \end{cases} \tag{31}$$

with $\bar{u}_{\rm xu,k}(t) = w(t)u_0 u_k(t)$ for an unknown matrix function $w(t) \in \mathbb{R}^{n\times n_0}$, is equivalent to $\Sigma_{\rm xu}$ in the sense of the averaged Volterra kernel energy. Hence, the MIMO system (28) will be associated to the averaged Gramians $\bar{R}$ and $\bar{Q}$ satisfying

$$A\bar{R} + \bar{R}A^\top + \sum_{k=1}^{m} N_k \bar{R} N_k^\top + NPN^\top = 0,$$

$$A^\top \bar{Q} + \bar{Q}A + \sum_{k=1}^{m} N_k^\top Q N_k = 0,$$

where P and Q are the solutions of

$$AP + PA^\top + X_0 X_0^\top = 0,$$

$$A^\top Q + QA + \sum_{k=1}^{m} N_k^\top Q N_k + C^\top C = 0.$$

Finally, the reduction of systems $\Sigma_{\rm x}$ and $\Sigma_{\rm u}$ can be done by conventional model reduction methods for LTI systems and bilinear control systems, respectively. The reduction of system $\Sigma_{\rm xu}$ can be done in two different ways: (i) by a balanced truncation-type method using the averaged Gramians $\bar{R}$ and $\bar{Q}$; (ii) by a BIRKA-type method using the time-invariant bilinear system $\bar{\Sigma}_{\rm xu}$ instead of $\Sigma_{\rm xu}$.

4. Numerical examples

The methods suggested in this paper is demonstrated by two numerical examples, a nonlinear RC circuit and a 2D heat transfer model. The full-order model is denoted by 'FOM'. For each system, we apply

- reduction without considering the initial condition effect, denoted by 'ROM-BT' and 'ROM-H2' for balanced truncation and (B)IRKA, respectively;
- balanced truncation by assuming the exact knowledge of the initial condition and applying the method of Baur et al. (2014), denoted by 'ROM-AugBT';
- reduction by applying balanced truncation to all the decomposed systems as proposed in this paper, denoted by 'InhROM-BT';
- reduction by applying (B)IRKA to all the decomposed systems as proposed in this paper, denoted by 'InhROM-H2'.

4.1 RC circuit

The first example considered is a nonlinear RC circuit. The circuit has nonlinear resistors (or diodes) which show exponential

behaviour. In Bai and Skoogh (2006), the Carleman bilinearisation is applied to the nonlinear model to approximate the system by a bilinear control system. The system has a current source at the first node as the input and the output is the voltage of the first node. The system tested has 10 nodes, which means the state space has dimension $n = 110$ after bilinearisation. The detailed model can be found in Bai and Skoogh (2006).

For testing purposes, we choose X_0 as $X_0 = (\mathbf{e}_{10}, \mathbf{e}_{110})$, i.e. the 10th and 110th unit vectors in $\mathbb{R}^{110}$. Physically, it means that the 10th node has nonzero initial voltage and its squared value should be nonzero as well. Hence, the vector u_0 must be chosen as $col(\alpha, \alpha^2)$ for some $\alpha \in \mathbb{R}$. Here we assume that initially we know that the value of α can be 1, so the method in Baur et al. (2014) is applicable. We investigate the Hankel singular values[2] of Σ_x and Σ_B and generalised/averaged Hankel singular values of Σ and Σ_{xu} and set the truncation threshold as 10^{-6} uniformly for all the systems. The resulting reduced-order models thus have dimensions $r_x = 8$, $r_B = 17$, $r = r_u = 17$ and $r_{xu} = 20$. For $\mathcal{H}_2$ optimal approximations, we use the same reduced-order dimensions and set the maximal iteration number as 200 and the convergence tolerance as 10^{-8}. It turns out both IRKA and BIRKA converge within 70 iterations.

After reduction, the input is fixed as a sinusoidal function $u(t) = 5\cos(\pi/5t) + 5$. We first fix the initial condition by fixing $\alpha = 1$. The time domain responses and the absolute response errors are shown in Figures 2 and 3. It can be seen that all the reduced-order models can capture the steady state of the full-order model very well. However, without considering the effect of the initial condition, the reduced-order models ROM-BT and ROM-H2 cannot approximate the full-order model well in the transient process. In this case, the augmented method also performs well because the initial condition is fixed. However, the generalised Hankel singular values of the augmented system show that if we set the truncation threshold at 10^{-6}, the required dimension is $r_{aug} = 31$, which is larger than the reduced dimensions of the decomposed systems.

To show that the augmented method may over emphasise the effect of the initial condition, we change the initial condition by setting $\alpha = 2$. The absolute response errors in Figure 4 show that in the transient simulation, the augmented method results in even larger deviations than the method assuming zero initial condition. Since the initial condition is still in the subspace spanned by X_0, the approximation accuracy of the proposed method is higher than the others.

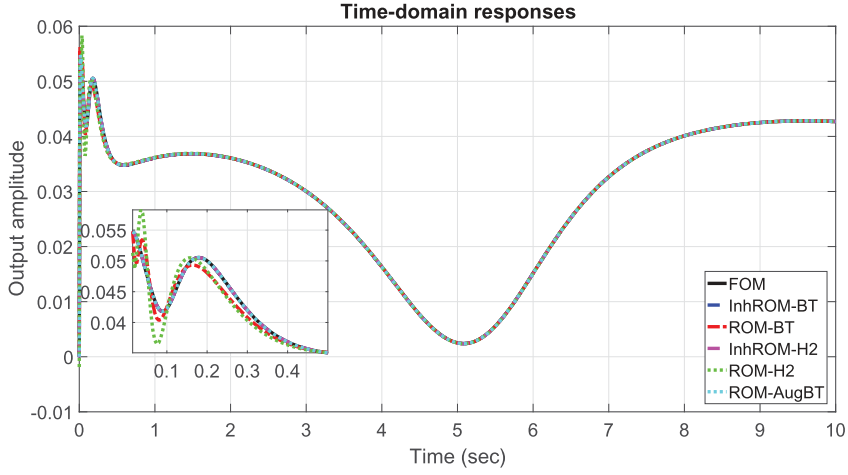

Figure 2. Time domain responses of full-order and reduced-order models of the nonlinear RC circuit with a fixed initial condition ($\alpha = 1$).

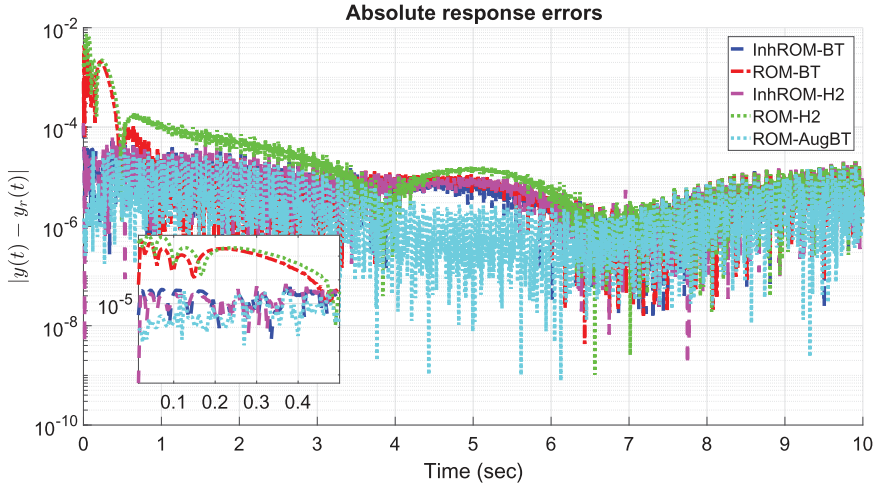

Figure 3. Absolute response errors of the reduced-order models of the nonlinear RC circuit with a fixed initial condition ($\alpha = 1$).

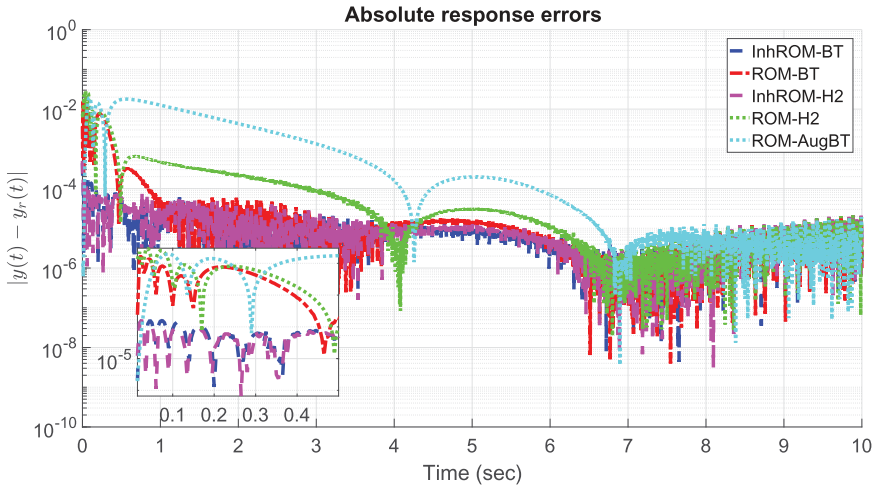

Figure 4. Absolute response errors of the reduced-order models of the nonlinear RC circuit with a possible inhomogeneous initial condition ($\alpha = 2$).

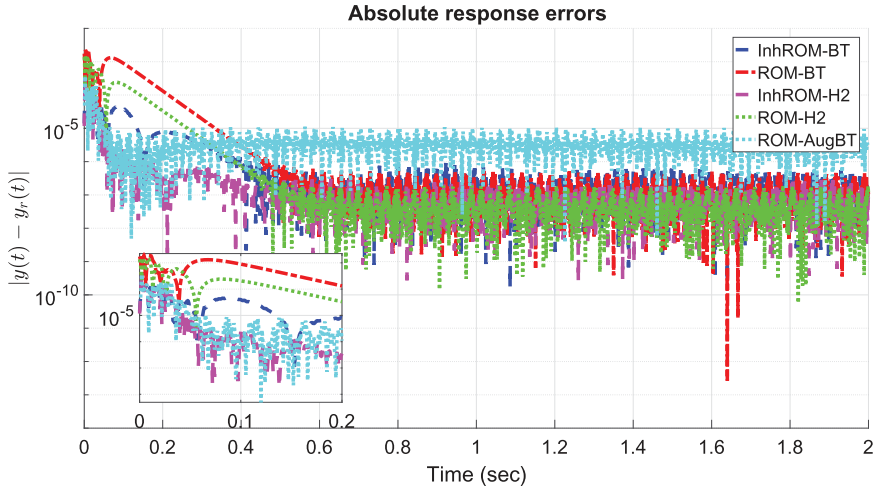

Figure 5. Absolute response errors of the reduced-order models of the 2D heat transfer system with a fixed initial condition (initial temperature is 1).

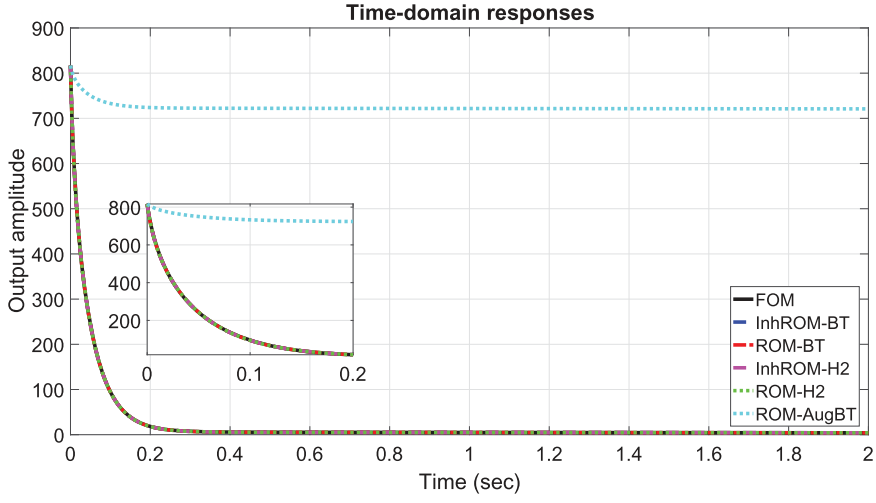

Figure 6. Time-domain responses of full-order and reduced-order models of the 2D heat transfer system with a random initial condition.

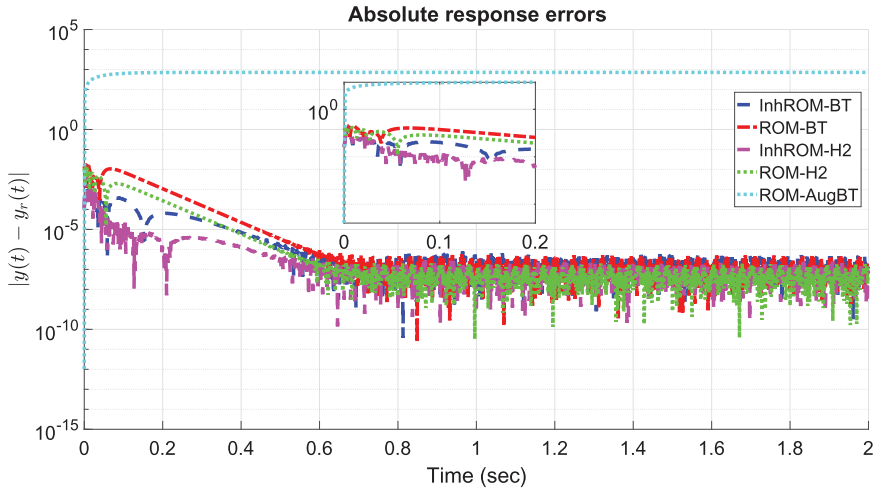

Figure 7. Absolute response errors of the reduced-order models of the 2D heat transfer system with a random initial condition.

4.2 2D heat transfer model

The second example that we study is a boundary controlled heat transfer model, which is modified from the one in Benner and Breiten (2012). The system is described by a 2D heat equation with Dirichlet and Robin boundary conditions as

$$\begin{aligned} &\partial_t x = \Delta x, && \text{in } (0,1)\times(0,1),\\ &x = 0, && \text{on } \Gamma_1, \Gamma_2, \Gamma_3,\\ &\mathbf{n}\cdot\nabla x = 0.25u(x-1), && \text{on } \Gamma_4, \end{aligned}$$

where $\Gamma_{1,2,3,4}$ are the left, top, right and bottom boundaries, respectively. The source term is u, which in this case is only applied at the bottom boundary. A spatial discretisation using $k = 10$ grid points results in a bilinear control system of dimension $n = k^2$. As stated, the system has only 1 input u and the output we considered is the sum of the temperature of all the grid points. Hence, matrix C is $C = \mathbb{1}_{1\times n}$.

Assuming that the initial temperature profile is homogeneously distributed, but nonzero, then we have $X_0 = C^\top$. The input is selected as $u(t) = 10e^{-0.2t}$. To apply the augmented method, we assume that the initial temperature is 1. By truncating the tails of the (generalised/averaged) Hankel singular values of the systems in (1), (7), (9) and (27), the reduced-order dimensions are $r_{\mathrm{x}} = 6, r_{\mathrm{B}} = 14, r = r_{\mathrm{u}} = r_{\mathrm{xu}} = 15$. The truncation thresholds are selected independently and they are all less than 10^{-8}. However, the generalised Hankel singular values of the augmented system flatten out after $r_{\mathrm{aug}} = 17$ and the value is around 6×10^{-7}. As a consequence, we expect that balanced truncation of the augmented system will not perform well even for the fixed initial condition. For $\mathcal{H}_2$ optimal approximations, we select the same reduced-order dimensions and set the maximal iteration number as 200 and the convergence tolerance as 10^{-8} again. In the test, only the system Σ_{B} does not converge in 200 iterations, but the tolerance reaches 10^{-3}. The absolute response errors are depicted in Figure 5. In this case, although balanced truncation models considering the initial condition effects are not as accurate as the $\mathcal{H}_2$ optimal approximations, they still outperform the method without considering the effect of initial conditions. In addition, the augmented method results in a lager deviation than the other methods at the steady state because of the relatively large truncation error. This problem cannot be fixed by increasing the reduced-order dimension because the generalised Hankel singular values flatten out after $r_{\mathrm{aug}} = 17$.

Now suppose that we only know the initial temperature profile is homogeneously distributed but the exact value is unknown. To demonstrate the performance of the proposed method, the initial temperature is randomly taken as 8.1681. Figures 6 and 7 show the time domain responses and the absolute response errors, respectively.

Clearly in this example, the augmented method over emphasises the effect of the initial condition and even cannot approximate the steady state of the system. Balanced truncation for this example is slightly worse than the $\mathcal{H}_2$ optimal approximation in the transient, but the proposed method certainly outperforms the method without considering the initial condition and also the augmented method. In addition, from the selection of the reduced-order dimensions, one can set the truncation threshold for each of the decomposed systems independently. We should also note that the balanced truncation variant does not suffer from the BIRKA convergence problems. Though it worked here, it cannot be guaranteed that one gets a good approximation when (B)IRKA does not converge.

5. Conclusions and outlook

In this work, we studied the model order reduction problem for bilinear control systems with inhomogeneous initial conditions. Based on the Volterra series analysis, we showed that the response of a bilinear control system with nonzero initial conditions can be decomposed as responses of an LTI system and two bilinear control systems and one of which is time-varying. By considering the time-averaged Volterra kernel energy, one can either use the averaged Gramians to perform balanced truncation or transform the time-varying bilinear control system into a time-invariant one and then perform the $\mathcal{H}_2$ optimal approximation techniques. We also proposed to reduce another LTI system to reduce the computational complexity of a matrix exponential function application. The resulting model reduction technique yields reductions of four independent systems, which are easily parallelizable. Numerical examples show

that the proposed method always outperforms the conventional methods assuming zero initial condition.

Bilinearisation of a nonlinear dynamical system is only a preliminary step of simplifying nonlinear systems. In recent years, other nonlinear systems such as quadratic-bilinear and polynomial systems attracted lots of attention. These systems can often be used formally to decrease the complexity of a nonlinear system, meanwhile providing the exact solution of the original system. Therefore, a future topic of the authors' interest is to extend the proposed method to other nonlinear dynamical systems such as quadratic-bilinear systems and polynomial systems. This extension would be beneficial for the simulation and control of complex nonlinear dynamical systems with non-homogeneous initial conditions as well as other multi-query applications of these systems, such as design optimisation.

Notes

1. $col(\cdot,\cdot,\cdot,\ldots)$ stands for column concatenation of scalars, vectors or matrices.
2. Note that in this paper when we refer to Hankel singular values, it means *relative* Hankel singular values. That is, all the Hankel singular values are divided by the largest Hankel singular value.

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