Model predictive control of hybrid systems : stability and robustness

DOI:
10.6100/IR612103

Document status and date:
Published: 01/01/2006

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Model Predictive Control of Hybrid Systems: Stability and Robustness

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de
Technische Universiteit Eindhoven,
op gezag van de Rector Magnificus, prof.dr.ir. C.J. van Duijn,
voor een commissie aangewezen door het College voor
Promoties in het openbaar te verdedigen op
donderdag 7 september 2006 om 16.00 uur

door

Mircea Lazar

geboren te Iaşi, Roemenië

Dit proefschrift is goedgekeurd door de promotor:

prof.dr.ir. P.P.J. van den Bosch

Copromotor:
dr.ir. W.P.M.H. Heemels

CIP-DATA LIBRARY TECHNISCHE UNIVERSITEIT EINDHOVEN

Lazar, Mircea

Model predictive control of hybrid systems : stability and robustness / by Mircea Lazar. – Eindhoven : Technische Universiteit Eindhoven, 2006.
Proefschrift. – ISBN-10: 90-386-1823-9
ISBN-13: 978-90-386-1823-4
NUR 959
Trefw.: regelsystemen ; parameterschatting / discrete systemen / nietlineaire systemen ; stabiliteit / robuuste regelsystemen.
Subject headings: predictive control / discrete time systems / Lyapunov methods / robust control.

părinţilor mei
to my parents

Eerste promotor: Prof.dr.ir. P.P.J. van den Bosch

Copromotor: Dr.ir. W.P.M.H. Heemels

Kerncommissie:

Prof.dr. H. Nijmeijer
Prof.dr. A.R. Teel
Dr.ir. A. Bemporad

The Ph.D. work is supported by the Dutch Technology Foundation STW grant DMR. 5675.

The Ph.D. work forms a part of the research program of the Dutch Institute of Systems and Control (DISC).

Contents

Acknowledgements

This thesis presents the results of the four years of research on the project "Model predictive control for hybrid systems," which was financially supported by the grant DMR.5675 of the Dutch Technology Foundation STW.

The completion of the research that led to the results published in this thesis would have not been possible without the constant support and advice received from my co-promotor, Dr. Maurice Heemels. I will always be grateful for his investment of patience, time and effort in my work. Especially, I appreciate very much the countless hours of discussions (sometimes even by phone, when I was at home in Romania), during which rough research ideas were brought closer to mathematical perfection. I will remember with pleasure the good moments that we had together during conferences and at the university. These moments made my PhD experience much more enjoyable. I am also grateful to Maurice for helping me with my career.

I would like to thank my promotor Prof. Paul van den Bosch for his helpful advice and for providing me with the best working environment that one could hope for. He has always been there for me when I needed his opinion and he supported me through my PhD journey. Thank you for trusting in me and for helping me with my career.

My gratitude goes to Dr. Siep Weiland for his careful and detailed comments on the articles that we have written together and for many helpful discussions on various research topics. Especially, if it wasn't for Siep, I probably would have never discovered the advantages of writing papers in Latex.

A special thanks goes to Dr. Alberto Bemporad who played a crucial role in my development as a researcher in model predictive control. Alberto, thank you for guiding me through some of the research topics I have been working on. Your advice had a great impact on the topics discussed in this thesis and has proven to be very rewarding. I am also grateful for the wonderful time I have spent in Siena during my visits there.

I would like to thank Prof. Henk Nijmeijer for being a member of the core committee of this thesis and for his very useful comments. I have always admired his work on nonlinear systems. Being taught the theory of nonlinear systems by professors Henk Nijmeijer and Arjan van der Schaft during the DISC (Dutch Institute of Systems and Control) course on nonlinear systems was an unique and very rewarding experience.

I would like to address a special word of gratitude to Prof. Andrew Teel, whose work had an important influence on some of the research presented in this thesis. I have learned many useful results while attending his course on hybrid systems in Bologna. Also, I am grateful for your very useful comments. Thank you for your willingness to be a member of the core committee of this thesis and for making the effort to travel to Eindhoven; your presence is very much appreciated.

I am grateful to Prof. Hans Schumacher, Prof. Arjan van der Schaft and Prof. Hans Hellendoorn for agreeing to be in the committee of this thesis and for being able to participate at the defence ceremony.

I am also very grateful to Prof. Ton Backx for his willingness to chair the committee of this thesis as the Dean of the Electrical Engineering Faculty.

A word of gratitude goes to Alessandro Alessio, Dr. David Muñoz de la Peña, Dr. Teodoro Alamo, Bas Roset, Prof. Alessandro Astolfi, Prof. Robin De Keyser and Prof. Octavian Pastravanu for their constructive cooperation on our papers. The results presented in Section 3.5.2 of this thesis were developed in close collaboration with Alessandro Alessio and Alberto Bemporad and they will also be a part of the PhD thesis of Alessandro Alessio.

During the four years I have worked on my PhD thesis I had the opportunity to spend enjoyable moments in the company of friends and colleagues. I would like to thank Aleksandar, Patricia, Mark, Andrej, Nelis, Maarten, John, Heico (Electrical Engineering, TUE); Apostolos, Bas, Nathan, Michiel (Mechanical Engineering, TUE); Agung (Twente); Daniele (TUD); Alessandro, Stefano, Filippo (Siena); David (Sevilla); Cristi (Zaragoza); Bert (Leuven) and all my colleagues from the Control Systems Group in the Electrical Engineering Faculty for the nice moments we spent together on various occasions.

I would like to thank Nicoleta for the beautiful moments that we shared together.

I am eternally indebted to my parents Roxana and Corneliu, my grandparents on the father side, Eleonora and Ilie, and my grandparents on the mother side, Magdalena and Florin, for all their support and love. This thesis is dedicated to my parents.

Mircea Lazar
Eindhoven, June, 2006

Summary

This thesis considers the stabilization and the robust stabilization of certain classes of hybrid systems using model predictive control. Hybrid systems represent a broad class of dynamical systems in which discrete behavior (usually described by a finite state machine) and continuous behavior (usually described by differential or difference equations) interact. Examples of hybrid dynamics can be found in many application domains and disciplines, such as embedded systems, process control, automated traffic-management systems, electrical circuits, mechanical and bio-mechanical systems, biological and bio-medical systems and economics. These systems are inherently nonlinear, discontinuous and multi-modal. As such, methodologies for stability analysis and (robust) stabilizing controller synthesis developed for linear or continuous nonlinear systems do not apply. This motivates the need for a new controller design methodology that is able to cope with discontinuous and multi-modal system dynamics, especially considering its wide practical applicability.

Model predictive control (MPC) (also referred to as receding horizon control) is a control strategy that offers attractive solutions, already successfully implemented in industry, for the regulation of constrained linear or nonlinear systems. In this thesis, the MPC controller design methodology will be employed for the regulation of constrained hybrid systems. One of the reasons for the success of MPC algorithms is their ability to handle hard constraints on states/outputs and inputs. Stability and robustness are probably the most studied properties of MPC controllers, as they are indispensable to practical implementation. A complete theory on (robust) stability of MPC has been developed for linear and continuous nonlinear systems. However, these results do not carry over to hybrid systems easily. These challenges will be taken up in this thesis.

As a starting point, in Chapter 2 of this thesis we build a theoretical framework on stability and input-to-state stability that allows for discontinuous and nonlinear system dynamics. These results act as the theoretical foundation of the thesis, enabling us to establish stability and robust stability results for hybrid systems in closed-loop with various model predictive control schemes.

The (nominal) stability problem of hybrid systems in closed-loop with MPC controllers is solved in its full generality in Chapter 3. The focus is

on a particular class of hybrid systems, namely piecewise affine (PWA) systems. This class of hybrid systems is very appealing as it provides a simple mathematical description on one hand, and a very high modeling power on the other hand. For particular choices of MPC cost functions and constrained PWA systems as prediction models, novel algorithms for computing a terminal cost and a local state-feedback controller that satisfy the developed stabilization conditions are presented. Algorithms for calculating low complexity piecewise polyhedral invariant sets for PWA systems are also developed. These positively invariant sets are either polyhedral, or consist of a union of a number of polyhedra that is equal to the number of affine subsystems of the PWA system. This is a significant reduction in complexity, compared to piecewise polyhedral invariant sets for PWA systems obtained via other existing algorithms. Hence, besides the study of the fundamental property of stability, the aim is to create control algorithms of low complexity to enable their on-line implementation.

Before addressing the robust stabilization of PWA systems using MPC in Chapter 5, two interesting examples are presented in Chapter 4. These examples feature two discontinuous PWA systems that both admit a discontinuous piecewise quadratic Lyapunov function and are exponentially stable. However, one of the PWA systems is non-robust to *arbitrarily small* perturbations, while the other one is globally input-to-state stable (ISS) with respect to disturbance inputs. This indicates that one should be careful in inferring robustness from nominal stability. Moreover, for the example that is robust, the input-to-state stability property cannot be proven via a *continuous* piecewise quadratic (PWQ) Lyapunov function. However, as ISS can be established via a discontinuous PWQ Lyapunov function, the conservatism of continuous PWQ Lyapunov functions is shown in this setting. Therefore, this thesis provides a theoretical framework that can be used to establish robustness in terms of ISS of discontinuous PWA systems via discontinuous ISS Lyapunov functions. The sufficient conditions for ISS of PWA systems are formulated as linear matrix inequalities, which can be solved efficiently via semi-definite programming. These sufficient conditions also serve as a tool for establishing robustness of nominally stable hybrid MPC controllers a posteriori, after the MPC control law has been calculated explicitly as a PWA state-feedback. Furthermore, we also present a technique based on linear matrix inequalities for synthesizing input-to-state stabilizing state-feedback controllers for PWA systems.

In Chapter 5, the problem of robust stabilization of PWA systems using MPC is considered. Previous solutions to this problem rely without exceptions on the assumption that the PWA system dynamics is a continuous function of the state. Clearly, this requirement is quite restrictive and artificial, as a continuous PWA system is in fact a Lipschitz continuous system. In Chapter 5 we present an input-to-state stabilizing MPC scheme for PWA systems based on tightened constraints that allows for discontinuous system dynamics and discontinuous MPC value functions. The advantage of this new approach, besides being the first robust stabilizing MPC scheme applicable to discontinuous PWA systems, is that the resulting MPC optimization problem can still be formulated as mixed integer linear programming problem, which is a standard optimization problem in hybrid MPC.

A min-max approach to the robust stabilization of perturbed nonlinear systems using MPC is presented in Chapter 6. Min-max MPC, although computationally more demanding, can provide feedback to the disturbance, resulting in better performance when the controlled system is affected by perturbations. We show that only *input-to-state practical stability* can be ensured in general for perturbed nonlinear systems in closed-loop with min-max MPC schemes. However, new sufficient conditions that guarantee *input-to-state stability* of the min-max MPC closed-loop system are derived, via a dual-mode approach. These conditions are formulated in terms of properties that the terminal cost and a local state-feedback controller must satisfy. New techniques for calculating the terminal cost and the local controller for perturbed linear and PWA systems are also presented in Chapter 6.

The final part of the thesis focuses on the design of robustly stabilizing, but computationally friendly, sub-optimal MPC algorithms for perturbed nonlinear systems and hybrid systems. This goal is achieved via new, simpler stabilizing constraints, that can be implemented as a finite number of linear inequalities. These algorithms are attractive for real-life implementation, when solvers usually provide a sub-optimal control action, rather than a globally optimal one. The potential for practical applications is illustrated via a case study on the control of DC-DC converters. Preliminary real-time computational results are encouraging, as the MPC control action is always computed within the allowed sampling interval, which is well below one millisecond for the considered Buck-Boost DC-DC converter.

In conclusion, *this thesis contains a complete framework on the synthesis of model predictive controllers for hybrid systems that guarantees stable and robust closed-loop systems*. The latter properties are indispensable for any application of these control algorithms in practice. In the set-ups of the MPC algorithms, a clear focus was also on keeping the on-line computational burden low via simpler stabilizing constraints. The example on the control of DC-DC converters showed that the application to (very) fast systems comes within reach. This opens up a completely new range of applications, next to the traditional process control for typically slow systems. Therefore, the developed theory represents a fertile ground for future practical applications and it opens many roads for future research in model predictive control and stability of hybrid systems as well.

Motto:

"Do not worry about your difficulties in mathematics; I can assure you that mine are still greater."

- *Albert Einstein*

1

Introduction

This thesis deals with the synthesis of stabilizing and robust controllers for constrained discrete-time hybrid systems. Hybrid systems describe processes that include both continuous and discrete dynamics. The large variety of practical situations where hybrid systems are encountered (for example, physical processes interacting with discrete embedded controllers) led over the years to an increasing interest in various aspects of hybrid systems. An appealing solution to the control of these systems is provided by the model predictive control methodology, due to its capability to a priori take into account constraints when computing the control action. Also, since the principles of model predictive control do not depend on the type of model applied for prediction, this methodology can be employed to formulate controller design set-ups for hybrid systems. However, the properties of such control schemes and the feasibility of their implementation have to be reconsidered in the hybrid context. In this thesis we focus in particular on stability and robustness.

1.1 Hybrid systems

Technological innovation pushes towards the consideration of systems of a mixed continuous and discrete nature, which are sometimes called hybrid systems. Hybrid systems arise, for instance, from the combination of an analog continuous-time process and a digital time asynchronous controller. Many consumer products (cars, micro-wave units, copy machines and so on) are controlled through embedded software, rendering the overall process a system with mixed dynamics. As a consequence, hybrid systems abound in our homes, probably more than we realize. Moreover, into the near future the number of computer-controlled products in our homes will grow even further. To support this evolution, new methodologies for the analysis and synthesis of hybrid systems are needed. To guarantee the safety and proper functionality, we have to improve our understanding of the interaction between physical processes, digital controllers and software, as all three parts influence the dynamic behavior of the overall process. Control theorists and computer scientists (and others) are joining forces to approach the huge challenges in this field (see Figure 1.1 for an illustrative picture).

Hybrid nature is not necessarily caused by human intervention in smooth systems. Although many examples originate from adding digital controllers to physical processes, the switching between dynamical regimes is naturally present in a variety of systems. For instance in mechanics, one encounters friction models that make a clear distinction between stick and slip phases. Other examples include models describing the evolution of rigid bodies. In this case the governing equations depend crucially on the fact whether a contact is active or not. Indeed, the dynamics of a robot arm moving freely in space is completely different from the situation in which it is striking the surface of an object. Backlash in gears and dead zones in cog wheels also result in multi-modal descriptions. It is not difficult to come up with interesting applications in the mechanical area: control of robotic manipulators driving nails or breaking objects (Brogliato et al., 1997), reduction of rattling in gear boxes of cars, drilling machines (Pfeiffer and Glocker, 1996), simulation of crash-tests, regulating landing maneuvers of aircraft, design of juggling robots (Brogliato and Rio, 2000), and so on. Examples are not only found in the mechanical domain. Nowadays switches like thyristors and diodes are used in electrical networks for a great variety of applications in both power engineering and signal processing. Examples include switched-capacitor filters, modulators, analog-to-digital converters, switching power converters, duty-ratio control, choppers, etc. In the ideal case, diodes are considered as elements with two (discrete) modes: the blocking mode and the conducting mode. Mode transitions for diodes or stick-slip friction are governed by state events (sometimes also called internally induced events), i.e., certain system variables (current or voltage) changing sign. In duty-ratio control the duration of a switch being open and closed (or the ratio between them in a fixed time interval) is determined by a control system and hence, the transitions are triggered by time events (externally induced events). Control design must take switching and impact phenomena into account such that a desirable behavior of the closed-loop system is realized. From the foregoing it is clear that systematic ways of designing controllers

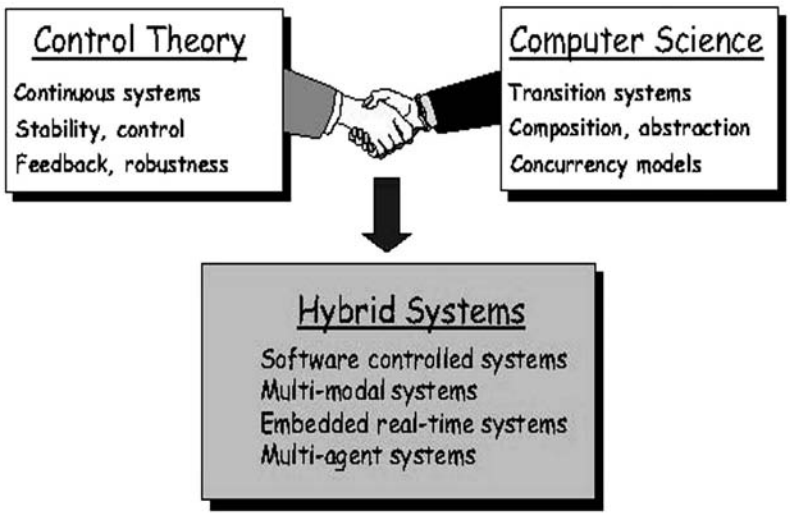

Figure 1.1: The control community shakes hands with the computer science community to better understand the role and the benefits of hybrid systems.

for these type of hybrid systems are needed.

The simplest definition of a hybrid system[1] is a system whose behavior is characterized by several modes of operation. In each mode the evolution of the continuous state of the system is described by its own difference or differential equation. The system switches between the various modes when a particular event occurs (see Figure 1.2 for an example, where the arrows depict the events). These events can have different origins: they can be caused by variables crossing specific thresholds (state events), by the elapsing of certain time periods (time events), or by external inputs (input events). At the switching time instant discontinuities may occur in the system variables, so there may be a reset of the state. *We will typically study hybrid systems in which the continuous input will be designed and the mode transitions are externally induced by state events.*

There are several approaches to model hybrid systems. Hybrid automata (Branicky et al., 1998) can model a large class of hybrid systems as they con-

[1]The term "hybrid systems" was originally introduced by Witsenhausen in 1966, for describing the combination of continuous dynamical and discrete event systems (Witsenhausen, 1966).

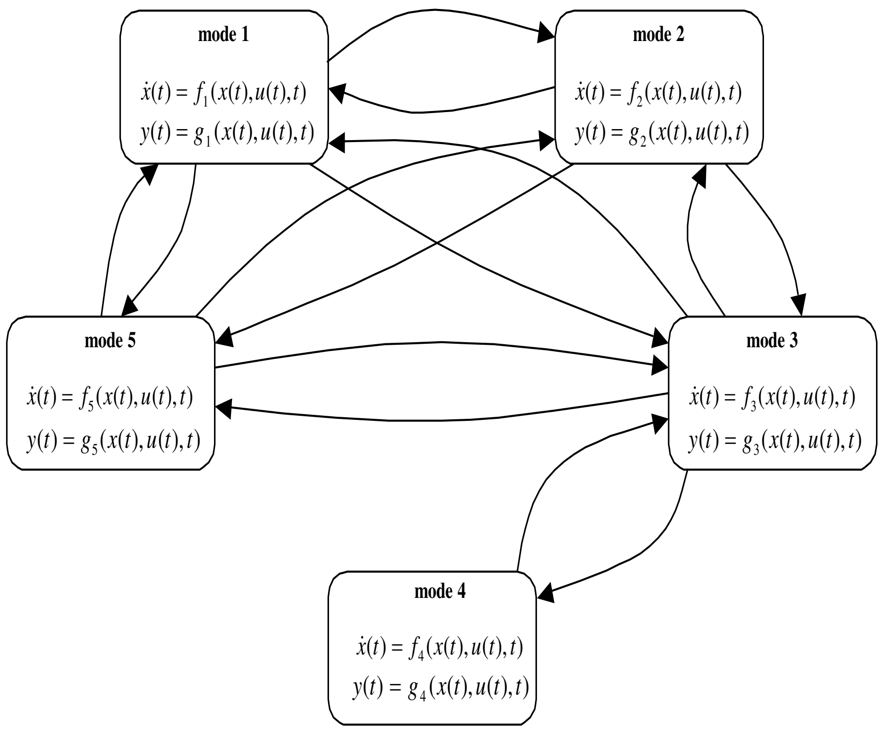

Figure 1.2: An example of a hybrid systems consisting of 5 *modes. Its behavior in each mode is described by a set of differential equations, while the mode transitions can be triggered by state, input or time dependent events.*

sider a discrete event system where the continuous dynamics in each discrete state are modeled by an arbitrary differential (difference) equation. Such models are used in (Branicky et al., 1998) to formulate a general stability analysis and controller synthesis framework for hybrid systems. Results for modeling and stability analysis of hybrid systems have also been presented in the more recent works (Lygeros et al., 2003), where dynamical properties of hybrid automata are investigated, and (Goebel and Teel, 2006), where a new formalism based on hybrid time domains is defined for hybrid systems and it is employed to derive results on stability. These results are very important because they provide a unified view on solution concepts and stability theory for general hybrid systems (see also (van der Schaft and Schumacher, 2000) for a comprehensive overview).

However, a general model of hybrid systems, although it can capture a lot of situations (high modeling power), usually leads to a high level of complexity with respect to analysis and controller design techniques. Indeed, for a particular instance, a high analytical and computational complexity originates from the fact that the structure of particular classes of hybrid systems is not exploited. Therefore, in each choice of modeling formalism there is always the trade-off between the modeling power and the complexity of the analysis. That is why the research in hybrid systems also focuses on particular subclasses that have a simpler representation and more structure, but still include a wide range of industrially relevant processes. The following classes of hybrid systems are part of this category:

- Piecewise affine (PWA) systems (Sontag, 1981, 1996; Johansson and Rantzer, 1998; Ferrari-Trecate et al., 2002);
- Switched systems (Liberzon, 2003);
- Mixed logical dynamical (MLD) systems (Bemporad and Morari, 1999; Bemporad, 2004);
- Linear complementarity (LC) systems (van der Schaft and Schumacher, 1998; Heemels et al., 2000);
- Discrete event systems extended to include time driven dynamics (Cassandras et al., 2001);
- Max-min-plus-scaling (MMPS) systems (De Schutter and van den Boom, 2001).

In particular, PWA systems have become popular due to their accessible mathematical description on one hand, and their ability to model a broad class of (hybrid) systems on the other hand: in (Heemels et al., 2001) it has been proven that PWA systems are equivalent under certain mild assumptions with other relevant classes of hybrid systems, such as MLD systems, LC systems and MMPS systems. Also, it is well known that PWA systems can approximate nonlinear systems arbitrarily well (Sontag, 1981) and they can arise from the interconnection of linear systems and automata (Sontag, 1996). The modeling power of PWA systems has already been shown in several applications, such as switched power converters (Leenaerts, 1996), optimal control of DC-DC converters and direct torque control of three-phase

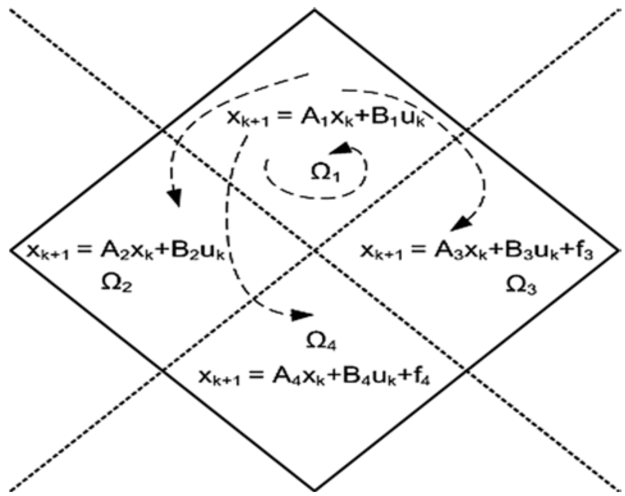

Figure 1.3: A constrained piecewise affine system: in each polyhedral region a different affine dynamics is active. The system dynamics changes when the state crosses the switching boundaries (denoted by dotted diagonal lines).

induction motors (Geyer et al., 2005), applications to automotive systems (Bemporad et al., 2003a; Borrelli, 2003; Vasak et al., 2006) and systems biology (Musters and van Riel, 2004; Drulhe et al., 2006), to mention just a few.

A significant part of this thesis focuses on discrete-time[2] *PWA systems of the form*:

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{if} \quad x_k \in \Omega_j, \quad j \in \mathcal{S},$$

where, $k \geq 0$ denotes the discrete-time instant, $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the state and the input, respectively, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$ with $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ a *finite set* of indices and s denotes the number of discrete modes or affine sub-systems. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of the state-space. A graphical description of a PWA system is given in Figure 1.3.

The main difficulties in modeling and controlling hybrid systems arise due to the interaction between discrete (discontinuous) actions (such as the

[2]We use the term "*discrete-time systems*" to denote systems described by difference equations and the term "*continuous-time systems*" to denote systems described by differential equations. This is different from the term "*continuous systems*" or "*continuous dynamics*", which refers to the continuity property of the function that describes the evolution in time (either discrete-time or continuous-time) of the system state.

ones triggered in the PWA system of Figure 1.3 by the state belonging to a certain region in the state-space) and continuous dynamics. As this field is still far from mature, mostly ad hoc and heuristic techniques of experienced operators and designers represent the fastest way to solve such problems in practice. However, this commonly leads to a major effort in tuning, prototyping and trouble-shooting, due to the lack of understanding and systematic synthesis methodologies that take into account both continuous and discrete aspects of hybrid systems. It is clear that techniques facilitating combined analysis and synthesis of both discontinuous and continuous parts are indispensable to build more efficient controller design methods for hybrid systems.

The aim of this thesis is to provide a general framework for designing stabilizing and robust controllers for practically relevant classes of discrete-time hybrid systems, with an emphasis on PWA systems. A major part of the proposed approach relies on the model predictive control methodology.

1.2 Model predictive control

Model predictive control (MPC) (also referred to as receding horizon control) is a control strategy that offers attractive solutions for the regulation of constrained linear or nonlinear systems and, more recently, also for the regulation of hybrid systems. Within a relatively short time, MPC has reached a certain maturity due to the continuously increasing interest shown for this distinctive part of control theory. This is illustrated by its successful implementation in industry and by many excellent articles and books as well. See, for example, (Garcia et al., 1989; Mayne et al., 2000; Qin and Badgwell, 2003; Findeisen et al., 2003; Camacho and Bordons, 2004) and the references therein.

The initial MPC algorithms utilized only linear input/output models. In this framework, several solutions have been proposed both in the industrial world and in the academic world: IDCOM - Identification and command (later MAC - Model algorithmic control) at ADERSA (Richalet et al., 1978) and DMC - Dynamic matrix control at Shell (Cutler and Ramaker, 1980), which use step and impulse response models, (the adaptive control branch) MUSMAR - Multistep multivariable adaptive regulator (Mosca et al., 1984) - the first MPC formulation that is based on state-space linear models, and EPSAC - Extend predictive self-adaptive control (De Keyser and van Cauwenberghe, 1985). Generalized frameworks for setting up MPC algorithms based on input/output models were also developed later on, from which the most significant ones are GPC - Generalized predictive control (Clarke et al., 1987) and UPC - Unified predictive control (Soeterboek, 1992). The next step of the academic community was to extend the MPC algorithms based on state-space models to continuous (smooth) nonlinear systems, which includes the following approaches: nonlinear MPC with zero state terminal equality constraint (Keerthi and Gilbert, 1988), dual-mode nonlinear MPC (Michalska and Mayne, 1993) and quasi-infinite horizon nonlinear MPC (Chen and Allgöwer, 1996). More recent general set-ups for synthesizing stabilizing MPC algorithms for smooth nonlinear systems can be found in (Magni et al., 2001; Grimm et al., 2005). The first MPC approach for the control of hybrid systems has been reported quite recently in (Bemporad and Morari, 1999) (more about MPC for hybrid systems can be found in the next section).

One of the reasons for the fruitful achievements of MPC algorithms consists in the intuitive way of addressing the control problem. In comparison with conventional control, which often uses a pre-computed state or output feedback control law, predictive control uses a discrete-time[3] model of the system to obtain an estimate (prediction) of its future behavior. This is done by applying a set of input sequences to a model, with the measured state/ouput as initial condition, while taking into account constraints. An optimization problem built around a performance oriented cost function is then solved to choose an optimal sequence of controls from all feasible sequences. The feedback control law is then obtained in a receding horizon manner by applying to the system only the first element of the computed sequence of optimal controls, and repeating the whole procedure at the next discrete-time step. Summarizing the above discussion, one can conclude that MPC is built around the following key principles:

- The explicit use of a process model for calculating predictions of the future plant behavior;
- The optimization of an objective function subject to constraints, which yields an optimal sequence of controls;
- The receding horizon strategy, according to which only the first element of the optimal sequence of controls is applied on-line.

The MPC methodology involves solving on-line an open-loop finite horizon optimal control problem subject to input, state and/or output constraints.

A graphical illustration of this concept is depicted in Figure 1.4.

[3]Although continuous-time models can also be employed, see (Mayne et al., 2000), MPC is mostly based on discrete-time models.

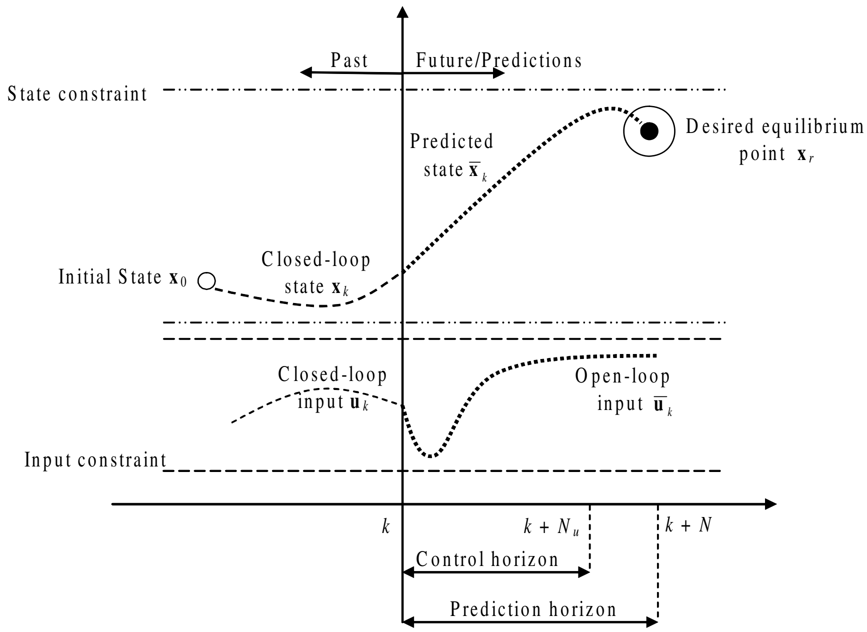

Figure 1.4: A graphical illustration of Model Predictive Control.

At each discrete-time instant k, the measured variables and the process model (linear, nonlinear or hybrid) are used to (predict) calculate the future behavior of the controlled plant over a specified time horizon, which is usually called the prediction horizon and is denoted by N. This is achieved by considering a future control scenario as the input sequence applied to the process model, which must be calculated such that certain desired constraints and objectives are fulfilled. To do that, a cost function is minimized subject to constraints, yielding an optimal sequence of controls over a specified time horizon, which is usually called control horizon and is denoted by N_u. According to the receding horizon control strategy, only the first element of the computed optimal sequence of controls is then applied to the plant and this sequence of steps is repeated at the next discrete-time instant, for the updated state.

The MPC methodology can be summarized formally as the following constrained optimization problem:

Problem 1.2.1 Let $N \geq 1$ be given and let $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ be sets that implement state and input constraints, respectively, and contain the origin in their interior. The prediction model is $x_{k+1} = g(x_k, u_k)$, $k \geq 0$, with $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a nonlinear, possibly discontinuous function with $g(0,0) = 0$. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0,0) = 0$ be known mappings. At every discrete-time instant $k \geq 0$ let $x_k \in \mathbb{X}$ be the measured state, let $x_{0|k} \triangleq x_k$ and minimize the cost function

$$J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}),$$

over all input sequences $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k})$ subject to the constraints:

$$\begin{aligned} & x_{i+1|k} \triangleq g(x_{i|k}, u_{i|k}), \quad i = 0, \ldots, N-1, \\ & x_{i|k} \in \mathbb{X}, \quad \text{for all} \quad i = 1, \ldots, N, \\ & u_{i|k} \in \mathbb{U}, \quad \text{for all} \quad i = 0, \ldots, N-1. \end{aligned}$$

In Problem 1.2.1, $F(\cdot)$, $L(\cdot,\cdot)$ and N denote the terminal cost, the stage cost and the prediction horizon, respectively. The term $x_{i|k}$ denotes the predicted state at future discrete-time instant $i \in [0, N]$, obtained at discrete-time instant $k \geq 0$ by applying the input sequence $\{u_{i|k}\}_{i=0,\ldots,N-1}$ to a model of the system, i.e. $x_{k+1} = g(x_k, u_k)$, with the measured state x_k as initial condition, i.e. $x_{0|k} = x_k$. The control actions in the sequence $\{u_{i|k}\}_{i=0,\ldots,N-1}$ constitute the optimization variables. Suppose that the above MPC optimization problem is solvable and let $\{u^*_{i|k}\}_{i=0,\ldots,N-1}$ denote an optimal solution. The MPC control action is obtained as follows:

$$u^{\text{MPC}}(x_k) \triangleq u^*_{0|k}; \quad k \geq 0.$$

One of the most studied research problems regarding MPC, which is also addressed in this thesis, consists in how to guarantee stability of a system in closed-loop with an MPC controller, e.g. obtained by solving Problem 1.2.1, as this is not automatically guaranteed and is indispensable for industrial applications. For linear and *continuous* nonlinear systems, many solutions to this problem have been developed, see the survey (Mayne et al., 2000) for a comprehensive and well documented overview. The most popular approach is the so-called *terminal cost and constraint set* method, which requires that the terminal predicted state, i.e. $x_{N|k}$, is constrained inside a terminal set that contains the origin (the equilibrium) in its interior. Then, under the assumption that *the system dynamics and the MPC value function corresponding to Problem 1.2.1 are continuous*, sufficient stabilization conditions, in terms of properties that a terminal cost $F(\cdot)$ and a terminal constraint set (usually denoted by $\mathbb{X}_T$) must satisfy, can be found in (Mayne et al., 2000).

1.3 Open problems in model predictive control of hybrid systems

Although the key principles of MPC are independent of the type of system, e.g. linear, nonlinear or hybrid, the computational complexity of the MPC constrained optimization problem, as well as the stability issues, strongly depend on the type of model used for prediction. For instance, assuming that the MPC cost is defined using *quadratic forms* (Hahn, 1967) and the constraint sets are polyhedra,

- Problem 1.2.1 is a *quadratic programming problem* if the model is linear;
- Problem 1.2.1 is a *nonlinear optimization problem* if the model is nonlinear;
- Problem 1.2.1 is a *mixed integer quadratic programming problem* (Bemporad and Morari, 1999) if the model is piecewise affine.

Therefore, depending on the utilized prediction model and MPC cost function, different tools are required for solving the MPC optimization problem.

Next to the practical motivation to develop systematic analysis and control design methods for hybrid systems, there is also a strong theoretical motivation for the work in this thesis, as fundamental properties like stability and robustness are unclear in the hybrid context. Indeed, the switching behavior of hybrid models, such as PWA ones, makes them inherently nonlinear and discontinuous. As such, the stabilization conditions for smooth nonlinear MPC, e.g. as the ones presented in (Mayne et al., 2000), do not apply when hybrid models are used for prediction. Also, the techniques developed for linear systems to compute the terminal cost and the terminal set such that these stabilization conditions are fulfilled do not work for PWA or other hybrid systems. In fact, in the excellent survey (Mayne et al., 2000), as a direction for future research, it was pointed out that all MPC stability notions should be reconsidered in the hybrid context.

The research on model predictive control of hybrid systems focuses on efficient ways to solve the corresponding finite horizon constrained optimization problems and on techniques to a priori guarantee stability of the controlled system. Fruitful results have already been obtained regarding computational aspects, starting with the seminal paper (Bemporad and Morari, 1999) that considers MPC costs defined using quadratic forms and mixed logical dynamical (MLD) prediction models. In this case the hybrid MPC optimization problem corresponding to Problem 1.2.1 is a mixed integer quadratic programming (MIQP) problem. Significant advances have been obtained for MPC costs defined using $1, \infty$-norms and MLD or PWA prediction models, see, for example, (Bemporad et al., 2000; Baotic et al., 2003; Borrelli, 2003). In this case, the hybrid MPC optimization problem corresponding to Problem 1.2.1 is a mixed integer linear programming (MILP) problem. Although the computational complexity of these types of optimization problems is non-polynomial, they can be solved explicitly for low dimensional[4] systems, via multi parametric programming (Bemporad et al., 2000; Borrelli, 2003). The most recent research on computational aspects of MPC for hybrid systems deals with methods for solving MIQP and MILP problems efficiently, e.g. see (Lazar and Heemels, 2003; Bemporad and Giorgetti, 2003; Grieder et al., 2005) and the references therein. Software tools that can be used to solve hybrid MPC optimization problems for MLD or PWA systems can be found in the following Matlab toolboxes: the Hybrid Toolbox (HT)(Bemporad, 2003) and the Multi Parametric Toolbox (MPT) (Kvasnica et al., 2004).

This thesis focuses mainly on the (robust) stability of hybrid systems in closed-loop with MPC controllers. Attractivity was proven for the equilibrium of the closed-loop system in (Bemporad and Morari, 1999; Borrelli, 2003). Besides attractivity, Lyapunov stability is also a desirable property from a practical point of view. This is because, if attractivity alone is guaranteed, then an arbitrarily small disturbance acting on the system close to the origin can cause the state to drift arbitrarily far away from the origin, before converging again. A general proof of Lyapunov stability is missing from the literature on MPC of hybrid systems.

Proofs of Lyapunov stability for particular types of MPC cost functions and hybrid prediction models have appeared recently, see, for example, (Kerrigan and Mayne, 2002; Mayne and Rakovic, 2003; Grieder et al., 2005). Still, note that (Kerrigan and Mayne, 2002) and (Mayne and Rakovic, 2003) consider *continuous* PWA systems, which are in fact *Lipschitz continuous* systems. Moreover, the Lyapunov stability result of Theorem 2 in (Kerrigan and Mayne, 2002) is obtained by forcing the MPC value function (based on

[4]In principle, these problems can be solved explicitly for systems of any dimension, but the computational burden limits this approach to systems of low dimension in practice.

$1, \infty$-norms) to be zero in the terminal set and by taking certain assumptions that are hard to be *a priori* guaranteed in general. The stability result of Theorem 5 in (Mayne and Rakovic, 2003) (MPC based on quadratic costs) uses the assumption that the origin is in the interior of one of the regions in the state-space partition of the PWA system, a case in which the stabilization conditions for linear MPC apply. Theorem 3.1 in (Grieder et al., 2005) addresses asymptotic stability of quadratic cost MPC of general PWA systems, but it relies on the survey (Mayne et al., 2000), where continuity of the MPC value function is assumed (see Section 3.2 of (Mayne et al., 2000)). Note that continuity of the MPC value function cannot be guaranteed when hybrid prediction models are used, as it is illustrated by many examples presented in this thesis (see also (Borrelli, 2003)).

Another option to guarantee stability for MPC of hybrid systems based on $1, \infty$-norms, presented in (Christophersen et al., 2004), is to perform an *a posteriori* check of stability, after computing the MPC control law as an explicit PWA function (Borrelli, 2003). However, in case such an a posteriori test fails, there is no systematic procedure available for re-tuning the parameters of the MPC algorithm such that stability is achieved.

Recently, results on robust MPC of hybrid systems have been presented in (Kerrigan and Mayne, 2002) and (Rakovic and Mayne, 2004), which deal with dynamic programming and tube based approaches, respectively, for solving feedback *min-max* MPC problems (Mayne, 2001) for *continuous PWA systems*, and also provide a robust stability guarantee. However, the resulting min-max MPC optimization problems usually have a much higher computational burden, compared to standard hybrid MPC MIQP or MILP problems. Also, the results of (Kerrigan and Mayne, 2002; Rakovic and Mayne, 2004) are not applicable to *discontinuous PWA systems*.

As discussed in this section, there is both a practical and theoretical need for a complete framework for setting up stabilizing and robust predictive controllers for hybrid systems. The results of this thesis, which are summarized in the next section, contribute to the development of such a framework.

1.4 Outline of the thesis

The remainder of this thesis is organized as follows.

Chapter 2, entitled *Classical stability results revisited*, provides general results on three well-known properties of discrete-time nonlinear systems:

- asymptotic stability in the Lyapunov sense (Kalman and Bertram, 1960b);

- input-to-state stability (Jiang and Wang, 2001);
- input-to-state practical stability (Jiang, 1993).

The focus is on discontinuous system dynamics and discontinuous candidate Lyapunov functions as this is the case in the hybrid context. It is shown that *continuity of the system dynamics and the Lyapunov function at the equilibrium point*, rather than *continuity on a neighborhood*[5] *of the equilibrium*, is sufficient to guarantee asymptotic stability in the Lyapunov sense or input-to-state stability for discrete-time, possibly discontinuous, nonlinear systems. The importance of these results for stability theory of hybrid systems cannot be overstated, as it is illustrated by the following elementary example of a PWA system:

$$x_{k+1} = \begin{cases} A_1 x_k & \text{if } [1\ 0]x_k \leq 1 \\ A_2 x_k & \text{if } [1\ 0]x_k > 1, \end{cases}$$

where

$$A_1 = \begin{bmatrix} 0.62 & -0.2848 \\ -0.0712 & 0.8336 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5125 & -0.2855 \\ 1.4277 & 0.5125 \end{bmatrix}.$$

This discontinuous PWA system is asymptotically stable in the Lyapunov sense, see Figure 1.5 for a the trajectory plot. However, one can easily check that the search for a *continuous* piecewise quadratic Lyapunov function[6] for the considered system fails, while a *discontinuous* piecewise quadratic Lyapunov function can be found[7]. Therefore, even in very simple hybrid systems as the one presented above, stability can often be established via a discontinuous (piecewise quadratic) Lyapunov function. In such cases, robust stability can no longer be established using classical arguments (see Chapter 4 for details), which rely on continuous Lyapunov functions. However, the above example is input-to-state stable, as it will be shown in chapter four. This property will be established via a *discontinuous* ISS Lyapunov function.

The results developed in Chapter 2, by allowing for discontinuous system dynamics and Lyapunov functions, can be employed throughout the thesis

[5]By continuity on a neighborhood of the equilibrium we mean continuity at all the points contained in this neighborhood.

[6]This search can be performed efficiently via semi-definite programming (Boyd et al., 1994), see Chapter 4 for details.

[7]Since the system is robustly stable, other types of continuous Lyapunov functions may exist for this system, as implied by the converse theorems presented in (Kellett and Teel, 2004). However, it is not clear how one could construct a continuous Lyapunov function in this case.

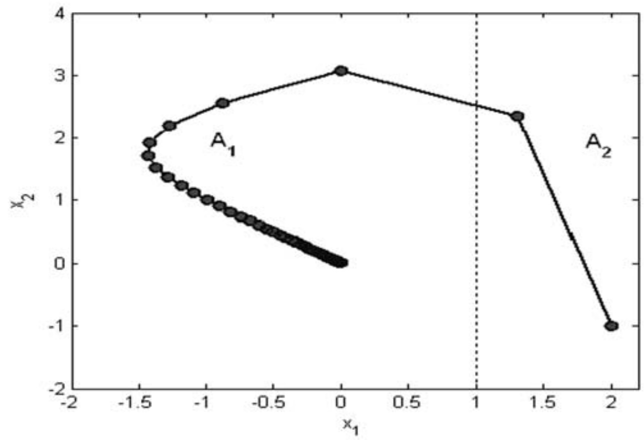

Figure 1.5: A trajectory plot for the considered PWA system.

to derive stability and robust stability results for piecewise affine systems or more general hybrid systems in closed-loop with MPC controllers. Additionally, in the theorems regarding input-to-state (practical) stability, a special type of $\mathcal{K}_\infty$-functions (Hahn, 1967) is employed, which makes it possible to derive explicit bounds on the evolution of the system state. These bounds are important because they provide an a priori guarantee of robust performance for the closed-loop system.

Chapter 3, entitled *Stabilizing model predictive control of hybrid systems*, addresses the stability problem of hybrid systems in closed-loop with MPC controllers in its full generality and presents *a priori* sufficient conditions for Lyapunov asymptotic stability and exponential stability of the closed-loop system. The contribution of this chapter is threefold. First, a general theorem on asymptotic stability in the Lyapunov sense that unifies most of the previous results on stability of MPC is presented. This theorem applies to a wide class of hybrid systems and MPC cost functions, and it does not require continuity of the MPC value function nor of the system dynamics. Second, for particular choices of MPC criteria and constrained PWA systems as prediction models, novel algorithms for computing a terminal cost and a local state-feedback controller that satisfy the developed stabilization conditions are presented. For quadratic MPC costs, the stabilization conditions translate into a linear matrix inequality while, for MPC costs based on $1, \infty$-norms, they are obtained as norm inequalities. Third, new ways for calculating positively invariant sets for feedback controlled PWA systems are also presented. The focus is on obtaining invariant sets for PWA systems that are:

- *piecewise polyhedral* - i.e. they consist of a union of a finite number of polyhedra,
- and have a *low complexity* - i.e. the number of polyhedra forming the union is as small as possible.

The developed algorithms yield positively invariant sets for PWA systems that are either polyhedral, or consist of a union of a number of polyhedra that is equal to the number of affine sub-systems of the PWA system. This is a significant reduction in complexity, compared to piecewise polyhedral invariant sets for PWA systems obtained via other algorithms from the literature, such as the one of (Rakovic et al., 2004). The low complexity piecewise polyhedral invariant sets can be used as terminal constraint sets in MPC schemes for PWA systems, resulting in a computationally more friendly MIQP or MILP optimization problem.

Chapter 4, entitled *Global input-to-state stability and stabilization of discrete-time PWA systems*, investigates robust stability of piecewise affine systems. We are motivated by the following insightful example:

$$x_{k+1} = A_j x_k + f_j \quad \text{when} \quad x_k \in \Omega_j,$$

with $j \in \mathcal{S} \triangleq \{1, \dots, 9\}$, $k \in \mathbb{Z}_+$, and where

$$A_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } j \neq 7;\ A_7 = \begin{bmatrix} 0.35 & 0.6062 \\ 0.0048 & -0.0072 \end{bmatrix};\ f_1 = -f_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix};$$

$$f_3 = f_4 = f_5 = f_6 = \begin{bmatrix} 0 \\ -1 \end{bmatrix};\ f_7 = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ f_8 = \begin{bmatrix} 0.4 \\ -0.1 \end{bmatrix};\ f_9 = \begin{bmatrix} -0.4 \\ -0.1 \end{bmatrix}.$$

The system state takes values in the set $\mathbb{X} \triangleq \cup_{j \in \mathcal{S}} \Omega_j$, where the regions Ω_j are polyhedra (not necessarily closed), as it can be seen in Figure 1.6. The state trajectories[8] of the system obtained for the initial states $x_0 = [0.2\ 3.6]^\top \in \Omega_2$ (square dotted line) and $x_0 = [0.2\ 3.601]^\top \in \Omega_1$ (circle dotted line) are plotted in Figure 1.6. The considered PWA system is asymptotically stable in the Lyapunov sense. In fact, in Chapter 4 it is proven that the system is even exponentially stable for all initial conditions in $\mathbb{X}$.

Note that, it is well-known that exponentially stable smooth nonlinear systems, usually also have some (inherent) robustness (this is in fact due to

[8]Note that the regions Ω_1 and Ω_2 are defined such that for all $x \in \partial\Omega_1 \cap \partial\Omega_2$ the dynamics $x_{k+1} = A_2 x_k + f_2$ is employed.

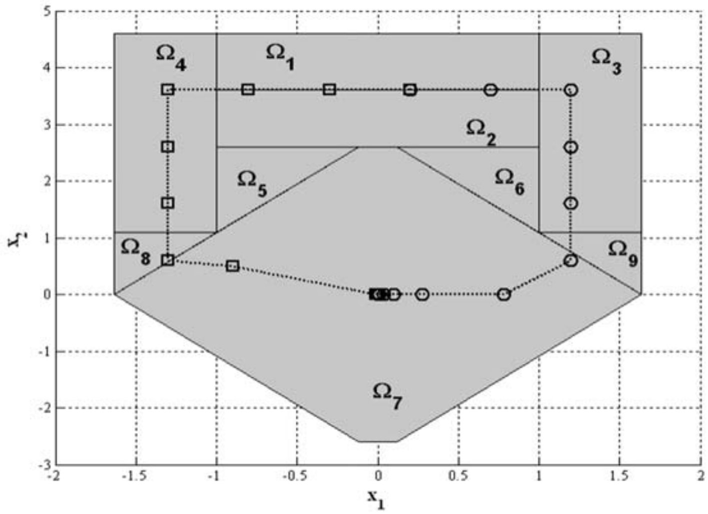

Figure 1.6: Unperturbed state trajectories (square and circle dotted lines).

the presence of a continuous Lyapunov function, see (Grimm et al., 2004)). However, in the fourth chapter it is also demonstrated that this PWA system has no robustness (i.e. it looses the asymptotic stability property) in the presence of *arbitrarily small* additive disturbances, a phenomenon that is called "zero robustness"[9]. This phenomenon, which is mainly due to the absence of a continuous Lyapunov function, issues a warning: *in discrete-time it is crucial that disturbances are taken into account when analyzing stability of PWA or hybrid systems, since nominal stability can be easily lost in the presence of arbitrarily small perturbations*. Therefore, sufficient conditions for global input-to-state (practical) stability and stabilization of discrete-time, possibly discontinuous, PWA systems are developed. Piecewise quadratic candidate input-to-state stable Lyapunov functions are employed for both analysis and synthesis purposes. This enables us to obtain sufficient conditions based on linear matrix inequalities, which can be solved efficiently.

Chapter 5, entitled *Robust stabilization of discontinuous piecewise affine systems using model predictive control*, employs the input-to-state stability framework to investigate the robustness of discrete-time (discontinuous) PWA systems in closed-loop with MPC, or hybrid MPC for short. The

[9]This phenomenon was originally reported for smooth nonlinear control systems in (Grimm et al., 2003a, 2004) and for discontinuous discrete-time nonlinear systems in (Kellett and Teel, 2004).

contribution of this chapter is threefold:

- it indicates that one should be cautious in inferring robust stability results from nominal stability even in the case of PWA systems in closed-loop with MPC. Indeed, we show via an example taken from literature that hybrid MPC can generate (discontinuous) MPC values functions *that are not input-to-state Lyapunov functions*;
- it provides an a posteriori test for checking robustness of nominally stabilizing hybrid MPC schemes, possibly with a discontinuous MPC value function;
- it presents a novel method based on tightened constraints for obtaining MPC schemes for hybrid systems with an a priori robust stability guarantee.

The advantage of this new approach is that the resulting MPC optimization problems can still be formulated as MILP or MIQP problems, which are standard for hybrid MPC.

Chapter 6, entitled *Input-to-state stabilizing min-max predictive controllers*, employs a min-max MPC set-up to synthesize robust predictive controllers for perturbed nonlinear systems. *Input-to-state practical stability* and *input-to-state stability* of the closed-loop system are investigated[10], with the aim of providing a priori sufficient conditions for input-to-state stability of min-max nonlinear MPC.

First, it is shown that only input-to-state practical stability can be ensured in general for perturbed nonlinear systems in closed-loop with min-max MPC schemes and explicit bounds on the evolution of the closed-loop system state are provided. Then, new sufficient conditions that guarantee input-to-state stability of the min-max MPC closed-loop system are derived, via a dual-mode approach. These conditions are formulated in terms of properties that the terminal cost and a local state-feedback controller must satisfy. New techniques that employ linear matrix inequalities in the case of quadratic MPC cost functions and, norm inequalities in the case of MPC cost functions based on $1, \infty$-norms, are developed for calculating the terminal cost and the local controller for perturbed linear and PWA systems.

Chapter 7, entitled *Robust sub-optimal predictive controllers*, focuses on the design of robustly stabilizing, but computationally friendly, sub-optimal MPC algorithms for perturbed nonlinear systems and hybrid systems. This

[10]Note that the input-to-state practical stability property does not imply asymptotic stability when the disturbance input vanishes, as it is implied by input-to-state stability.

goal is achieved via simpler stabilizing constraints, that can be implemented as a finite number of linear inequalities. Two sub-optimal nonlinear MPC algorithms are presented. The first one is based on a contraction argument, i.e. it is proven that, if the norm of the state of the nominal closed-loop system is sufficiently decreasing at each sampling instant, then input-to-state stability is guaranteed. The second MPC scheme resorts to an ∞-norm based artificial Lyapunov function, which only depends on the measured state and the first element of the sub-optimal sequence of predicted future inputs. Both these schemes have an input-to-state stability guarantee with respect to additive disturbance inputs.

For the class of PWA systems, it is shown how the sub-optimal MPC scheme based on an artificial Lyapunov function can be modified to ensure input-to-state stability with respect to measurement noise. This modified scheme is particularly relevant because it can be used in interconnection with an input-to-state stable observer, resulting in an asymptotically stable closed-loop system. Methods for computing such artificial Lyapunov functions off-line for linear or PWA models are also indicated.

A case study on the control of DC-DC converters that includes preliminary real-time computational results is included to illustrate the potential of the developed theory for practical applications. As the sampling period of the considered DC-DC converter is well below one millisecond, this indicates that the proposed MPC schemes are implementable for (very) fast systems, which opens up a whole new range of industrial applications in electrical, mechatronic and automotive systems.

Several concluding remarks and directions for future research are presented Chapter 8, entitled *Conclusions*.

1.5 Summary of publications

This thesis is mostly based on published or submitted articles.

Chapter 2 contains results presented in:

- (Lazar et al., 2005a): M. Lazar, W.P.M.H. Heemels, A. Bemporad and S. Weiland. On the stability and robustness of non-smooth nonlinear model predictive control. *International Workshop on Assessment and Future Directions of NMPC*, Freudenstadt-Lauterbad, Germany.
- (Lazar et al., 2006c): M. Lazar, D. Muñoz de la Peña, W.P.M.H. Heemels and T. Alamo. Min-max nonlinear model predictive control with guaranteed input-to-state stability. *17th Symposium on Mathematical Theory for Networks and Systems*, Kyoto, Japan.

- (Lazar et al., 2006d): M. Lazar, D. Muñoz de la Peña, W.P.M.H. Heemels and T. Alamo. On input-to-state stability of min-max nonlinear model predictive control. Submitted to a journal, March, 2006.

The results presented in Chapter 3 are published in:

- (Lazar et al., 2004b): M. Lazar, W.P.M.H. Heemels, S. Weiland and A. Bemporad. Stabilizing receding horizon control of PWL systems: An LMI approach. *16th Symposium on Mathematical Theory for Networks and Systems*, Leuven, Belgium.
- (Lazar et al., 2004a): M. Lazar, W.P.M.H. Heemels, S. Weiland and A. Bemporad. Stabilization conditions for model predictive control of constrained PWA systems. *43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas.
- (Lazar et al., 2005c): M. Lazar, W.P.M.H. Heemels, S. Weiland, A. Bemporad and O. Pastravanu. Infinity norms as Lyapunov functions for model predictive control of constrained PWA systems. *Hybrid Systems: Computation and Control*, Zürich, Switzerland. Lecture Notes in Computer Science, vol. 3414, Springer Verlag.
- (Lazar et al., 2005b): M. Lazar, W.P.M.H. Heemels, S. Weiland and A. Bemporad. On the stability of quadratic forms based model predictive control of constrained PWA systems. *24th American Control Conference*, Portland, Oregon.
- (Lazar et al., 2006a): M. Lazar, A. Alessio, A. Bemporad and W.P.M.H. Heemels. Squaring the circle: An algorithm for obtaining polyhedral invariant sets from ellipsoidal ones. *25th American Control Conference*, Minneapolis, Minnesota.
- (Alessio et al., 2006): A. Alessio, M. Lazar, A. Bemporad and W.P.M.H. Heemels. Squaring the circle: An algorithm for obtaining polyhedral invariant sets from ellipsoidal ones. Provisionally accepted for publication in Automatica, July, 2006.
- (Lazar et al., 2006b). M. Lazar, W.P.M.H. Heemels, S. Weiland and A. Bemporad. Stabilizing model predictive control of hybrid systems. To appear in *IEEE Transactions on Automatic Control*, 2006.

Chapter 4 is based on:

- (Lazar and Heemels, 2006a): M. Lazar and W.P.M.H. Heemels. Global input-to-state stability and stabilization of discrete-time piecewise affine systems. *2nd IFAC Conference on Analysis and Design of Hybrid Systems*, Alghero, Italy.
- (Lazar and Heemels, 2006b): M. Lazar and W.P.M.H. Heemels. Global input-to-state stability and stabilization of discrete-time piecewise affine systems. Submitted to a journal, 2006.

The results of Chapter 5 are presented in:

- (Lazar et al., 2005a): M. Lazar, W.P.M.H. Heemels, A. Bemporad and S. Weiland. On the stability and robustness of non-smooth nonlinear model predictive control. *International Workshop on Assessment and Future Directions of NMPC*, Freudenstadt-Lauterbad, Germany.
- (Lazar and Heemels, 2006c): M. Lazar and W.P.M.H. Heemels. A new dual-mode hybrid MPC algorithm with a robust stability guarantee. *2nd IFAC Conference on Analysis and Design of Hybrid Systems*, Alghero, Italy.
- (Lazar and Heemels, 2006d): M. Lazar and W.P.M.H. Heemels. Robust stabilization of discontinuous piecewise affine systems using model predictive control. Submitted to a journal, 2006.

Chapter 6 contains results presented in:

- (Lazar et al., 2006c): M. Lazar, D. Muñoz de la Peña, W.P.M.H. Heemels and T. Alamo. Min-max nonlinear model predictive control with guaranteed input-to-state stability. *17th Symposium on Mathematical Theory for Networks and Systems*, Kyoto, Japan.
- (Lazar et al., 2006e): M. Lazar, D. Muñoz de la Peña, W.P.M.H. Heemels and T. Alamo. Stabilizing feedback min-max linear model predictive control: New methods for computing the terminal cost. Submitted to a conference, 2006.
- (Lazar et al., 2006d): M. Lazar, D. Muñoz de la Peña, W.P.M.H. Heemels and T. Alamo. On input-to-state stability of min-max nonlinear model predictive control. Submitted to a journal, 2006.

Finally, Chapter 7 is based on:

- (Lazar et al., 2006f): M. Lazar, B.J.P. Roset, W.P.M.H. Heemels, H. Nijmeijer and P.P.J. van den Bosch. Input-to-state stabilizing sub-optimal nonlinear MPC algorithms with an application to DC-DC converters. *IFAC Workshop on Nonlinear Model Predictive Control for Fast Systems*, Grenoble, France.

Some other results on similar topics, which are not included in this thesis, can be found in (Lazar and Heemels, 2003), (Lazar and De Keyser, 2004), (Roset, Lazar, Nijmeijer, and Heemels, 2006a), (Roset, Lazar, Nijmeijer, and Heemels, 2006b) and (Corona, Lazar, De Schutter, and Heemels, 2006).

1.6 Basic mathematical notation and definitions

In this section, some basic mathematical notation and standard definitions are recalled to make the manuscript self-contained.

Sets and operations with sets:

- $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integers and the set of non-negative integers, respectively;
- $\mathbb{Z}_{\geq c_1}$ and $\mathbb{Z}_{(c_1,c_2]}$ denote the sets $\{k \in \mathbb{Z}_+ \mid k \geq c_1\}$ and $\{k \in \mathbb{Z}_+ \mid c_1 < k \leq c_2\}$, respectively, for some $c_1, c_2 \in \mathbb{Z}_+$;
- For a set $\mathbb{S} \subseteq \mathbb{R}^n$, $\mathbb{S}^N$ denotes the N-dimensional Cartesian product $\mathbb{S} \times \ldots \times \mathbb{S}$, for some $N \in \mathbb{Z}_{\geq 1}$;
- For a set $\mathcal{P} \subseteq \mathbb{R}^n$, $\partial\mathcal{P}$ denotes the boundary of $\mathcal{P}$, $\text{int}(\mathcal{P})$ denotes the interior of $\mathcal{P}$, $\text{cl}(\mathcal{P})$ denotes the closure of $\mathcal{P}$, $\text{card}(\mathcal{P})$ denotes the number of elements of $\mathcal{P}$ and $\text{Co}(\mathcal{P})$ denotes the convex hull of $\mathcal{P}$;
- For any real $\lambda \geq 0$ and set $\mathcal{P} \subseteq \mathbb{R}^n$, the set $\lambda\mathcal{P}$ is defined as

  $$\lambda\mathcal{P} \triangleq \{x \in \mathbb{R}^n \mid x = \lambda y \text{ for some } y \in \mathcal{P}\};$$

- For two arbitrary sets $\mathcal{P}_1 \subseteq \mathbb{R}^n$ and $\mathcal{P}_2 \subseteq \mathbb{R}^n$, $\mathcal{P}_1 \cup \mathcal{P}_2$ denotes their union, $\mathcal{P}_1 \cap \mathcal{P}_2$ denotes their intersection, $\mathcal{P}_1 \setminus \mathcal{P}_2$ denotes their set difference, $\mathcal{P}_1 \subset \mathcal{P}_2$ (or $\mathcal{P}_1 \subsetneq \mathcal{P}_2$) denotes "$\mathcal{P}_1$ is subset of, but not equal to, $\mathcal{P}_2$", $\mathcal{P}_1 \subseteq \mathcal{P}_2$ denotes "$\mathcal{P}_1$ is subset of, or equal to $\mathcal{P}_2$";
- For two arbitrary sets $\mathcal{P}_1 \subseteq \mathbb{R}^n$ and $\mathcal{P}_2 \subseteq \mathbb{R}^n$,

  $$\mathcal{P}_1 \sim \mathcal{P}_2 \triangleq \{x \in \mathbb{R}^n \mid x + \mathcal{P}_2 \subseteq \mathcal{P}_1\}$$

  denotes their Pontryagin difference and

  $$\mathcal{P}_1 \oplus \mathcal{P}_2 \triangleq \{x + y \mid x \in \mathcal{P}_1, y \in \mathcal{P}_2\}$$

  denotes their Minkowski sum;
- A convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a C-set;
- A polyhedron (or a polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces;
- A piecewise polyhedral set is a set obtained as the union of a finite number of polyhedral sets.

Vectors, matrices and norms:

- For a real number $a \in \mathbb{R}$, $|a|$ denotes its absolute value and $\lceil a \rceil$ denotes the smallest integer larger than a;
- For a sequence $\{z_j\}_{j\in\mathbb{Z}_+}$ with $z_j \in \mathbb{R}^l$, $z_{[k]}$ denotes the truncation of $\{z_j\}_{j\in\mathbb{Z}_+}$ at time $k \in \mathbb{Z}_+$, i.e. $z_{[k]} = \{z_j\}_{j\in\mathbb{Z}_{[0,k]}}$, and $z_{[k_1,k_2]}$ denotes the truncation of $\{z_j\}_{j\in\mathbb{Z}_+}$ at times $k_1 \in \mathbb{Z}_{\geq 1}$ and $k_2 \in \mathbb{Z}_{\geq k_1}$, i.e. $z_{[k_1,k_2]} = \{z_j\}_{j\in\mathbb{Z}_{[k_1,k_2]}}$;
- The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as:

  $$\|x\|_p \triangleq \begin{cases} (|x_1|^p + \ldots + |x_n|^p)^{\frac{1}{p}}, & p \in \mathbb{Z}_{[1,\infty)} \\ \max_{i=1,\ldots,n} |x_i|, & p = \infty, \end{cases}$$

  where x_i, $i = 1, \ldots, n$ is the i-th component of x, $\|x\|_2$ is also called the Euclidean norm and $\|x\|_\infty$ is also called the infinity (or the maximum) norm;
- Let $\|\cdot\|$ denote an arbitrary Hölder p-norm. For a sequence $\{z_j\}_{j\in\mathbb{Z}_+}$ with $z_j \in \mathbb{R}^n$,

  $$\|\{z_j\}_{j\in\mathbb{Z}_+}\| \triangleq \sup\{\|z_j\| \mid j \in \mathbb{Z}_+\};$$
- I_n denotes the identity matrix of dimension $n \times n$;
- For some matrices $L_1, \ldots, L_n$, $\mathrm{diag}([L_1, \ldots, L_n])$ denotes a diagonal matrix of appropriate dimensions with the matrices $L_1, \ldots, L_n$ on the main diagonal;
- For a matrix $Z \in \mathbb{R}^{m\times n}$ and $p \in \mathbb{Z}_{\geq 1}$ or $p = \infty$

  $$\|Z\|_p \triangleq \sup_{x\neq 0} \frac{\|Zx\|_p}{\|x\|_p},$$

  denotes its induced matrix norm. It is well known, see, for example, (Golub and Van Loan, 1989), that $\|Z\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |Z^{\{ij\}}|$, where $Z^{\{ij\}}$ is the ij-th entry of Z;
- For a matrix $Z \in \mathbb{R}^{m\times n}$, $Z^\top$ denotes its transpose and Z^{-1} denotes its inverse (if it exists);
- For a matrix $Z \in \mathbb{R}^{n\times n}$, $Z > 0$ denotes "Z is positive definite", i.e. for all $x \in \mathbb{R}^n \setminus \{0\}$ it holds that $x^\top Zx > 0$, and $Z = Z^\top$;

- For a matrix $Z \in \mathbb{R}^{m \times n}$ with full-column rank, $Z^{-L} \triangleq (Z^\top Z)^{-1} Z^\top$ denotes the Moore-Penrose inverse of Z, which satisfies $Z^{-L}Z = I_n$;
- For a positive definite and symmetric matrix Z, $Z^{\frac{1}{2}}$ denotes its Cholesky factor, which satisfies $(Z^{\frac{1}{2}})^\top Z^{\frac{1}{2}} = Z^{\frac{1}{2}}(Z^{\frac{1}{2}})^\top = Z$;
- For a positive definite matrix Z, $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ denote the smallest and the largest eigenvalue of Z, respectively.

2

Classical stability results revisited

In this chapter we recall classical stability properties for discrete-time nonlinear systems, such as Lyapunov stability and input-to-state stability. The aim is to derive sufficient stabilization conditions that allow for discontinuous system dynamics and discontinuous candidate Lyapunov functions, situations that were less of interest before the “hybrid era”.

2.1 Introduction

The Lyapunov stability of nonlinear systems, introduced in the fundamental work (Lyapunov, 1907), is one of the most important system properties studied in control systems theory. Practically, the goal of any controller design methodology is to obtain a closed-loop system which is (at least) Lyapunov stable. A lot of interest has been shown to stability theory of nonlinear systems, even in the early stages of control system theory, as it is demonstrated by the excellent survey on continuous-time results (Kalman and Bertram, 1960a). The sufficient conditions for Lyapunov stability presented in (Kalman and Bertram, 1960a) are based on the existence of a so-called “candidate Lyapunov function” that enjoys certain properties. In particular, continuity and differentiability of the candidate Lyapunov function and continuity of the system dynamics are required.

Extensions to discrete-time nonlinear systems, although not as numerous as the articles treating the continuous-time case, have soon followed, see the early works[1] (Li, 1934), (Hahn, 1958), whose results are summarized in the survey (Kalman and Bertram, 1960b). Later on, as discrete-time systems became more important in control applications, due to the introduction of digital computers, several works that also cover the discrete-time case have

[1] Copies of these two articles can be supplied upon request.

appeared, see, for example, (Freeman, 1965; Willems, 1970; LaSalle, 1976) and the more recent books (Vidyasagar, 1993; Khalil, 2002). In discrete-time the candidate Lyapunov function and the system dynamics do not necessarily have to be continuous, as it will be shown in this chapter. This is extremely interesting for hybrid systems, as in this case the system dynamics can be discontinuous. However, the above-cited discrete-time results cannot be directly employed to study stability of hybrid systems, as continuity of the system dynamics is still a standing assumption in most of the literature. Moreover, a complete formal proof is actually missing in the above references. In this chapter we restate classical stability results (with some minor relaxations) and we prove that continuity at the equilibrium point alone, rather than continuity on a neighborhood of the equilibrium suffices for *Lyapunov stability* in discrete-time.

For continuous-time nonlinear systems affected by external (disturbance) inputs, the input-to-state stability (ISS) framework, introduced in the fundamental works (Sontag, 1989, 1990; Sontag and Wang, 1995), has proven to be very successful, as it generalizes the Lyapunov stability concept to systems affected by perturbations. Extensions to the discrete-time case have recently been developed in (Jiang and Wang, 2001, 2002; Jiang et al., 2004). Similarly to the Lyapunov stability property, sufficient conditions for ISS are derived in terms of properties that a so-called "candidate ISS Lyapunov function" must satisfy. In this chapter we will consider a particular case of the more general sufficient conditions of (Jiang and Wang, 2001) to establish explicit bounds on the evolution of the perturbed system state and we will show that continuity at the equilibrium point alone, rather than continuity on a neighborhood of the equilibrium suffices also for ISS in discrete-time. The input-to-state practical stability (ISpS) property, introduced in (Jiang, 1993; Jiang et al., 1994, 1996), is also studied in this chapter, as this is relevant for systems which are not even continuous at the equilibrium point (see Chapter 4 for details) or for candidate ISS Lyapunov functions that are not even continuous at the equilibrium point (such functions arise in the context of min-max MPC, see Chapter 6 for details). A discrete-time version of the continuous-time result of (Jiang et al., 1996) is presented.

2.2 Lyapunov stability

Consider the time-invariant discrete-time autonomous nonlinear system described by

$$x_{k+1} = G(x_k), \tag{2.1}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary, *possibly discontinuous*, nonlinear function. A point $x^* \in \mathbb{R}^n$ is an equilibrium point of system (2.1), if $G(x^*) = x^*$. For convenience we recall the following definitions related to stability.

Definition 2.2.1 Let $x^* \in \mathbb{R}^n$ be an equilibrium point of system (2.1) and let $\mathbb{X} \subseteq \mathbb{R}^n$ be a set that contains an open neighborhood of x^*.

1. The equilibrium x^* is *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|x_0 - x^*\| \leq \delta \quad \Rightarrow \quad \|x_k - x^*\| \leq \varepsilon \quad \text{for all} \quad k \in \mathbb{Z}_+,$$

   where x_k is the state of system (2.1) at time $k \in \mathbb{Z}_+$ with initial state x_0 at time $k = 0$.

2. The equilibrium x^* is *attractive in* $\mathbb{X}$ if

$$\lim_{k \to \infty} \|x_k - x^*\| = 0, \quad \text{for all} \quad x_0 \in \mathbb{X}.$$

3. The equilibrium x^* is *locally attractive* if there exists a $\delta > 0$ such that

$$\|x_0 - x^*\| \leq \delta \quad \Rightarrow \quad \lim_{k \to \infty} \|x_k - x^*\| = 0.$$

4. The equilibrium x^* is *globally attractive* if it is attractive in $\mathbb{R}^n$.

5. The equilibrium x^* is *asymptotically stable in* $\mathbb{X}$ *in the Lyapunov sense* if it is both Lyapunov stable and attractive in $\mathbb{X}$.

6. The equilibrium x^* is *locally (globally) asymptotically stable in the Lyapunov sense* if it is both Lyapunov stable and locally (globally) attractive.

7. The equilibrium x^* is *exponentially stable in* $\mathbb{X}$ if there exist $\theta > 0$ and $\tau \in [0, 1)$ such that

$$\|x_k - x^*\| \leq \theta \|x_0 - x^*\| \tau^k, \quad \text{for all} \quad x_0 \in \mathbb{X} \quad \text{and for all} \quad k \in \mathbb{Z}_+.$$

8. The equilibrium x^* is *locally exponentially stable* if there exists a $\delta > 0$, $\theta > 0$ and $\tau \in [0, 1)$ such that

$$\|x_0 - x^*\| \leq \delta \quad \Rightarrow \quad \|x_k - x^*\| \leq \theta \|x_0 - x^*\| \tau^k, \quad \text{for all} \quad k \in \mathbb{Z}_+.$$

9. The equilibrium x^* is *globally exponentially stable* if it is exponentially stable in $\mathbb{R}^n$.

Definition 2.2.2 A real-valued scalar function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *belongs to class* $\mathcal{M}$ ($\varphi \in \mathcal{M}$) if it is non-decreasing, $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$. A real-valued scalar function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ ($\varphi \in \mathcal{K}$) if it is continuous, strictly increasing and $\varphi(0) = 0$. A real-valued scalar function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ ($\varphi \in \mathcal{K}_\infty$) if $\varphi \in \mathcal{K}$ and it is radially unbounded (i.e. $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$).

Definition 2.2.3 Let $0 \leq \lambda \leq 1$ be given. A set $\mathcal{P} \subseteq \mathbb{R}^n$ that contains the origin in its interior is called a λ-*contractive set* for system (2.1) if for all $x \in \mathcal{P}$ it holds that $G(x) \in \lambda\mathcal{P}$. For $\lambda = 1$ a λ-contractive set is called a *positively invariant set.*

In this section we formulate discrete-time stability results for the *discontinuous* autonomous nonlinear system (2.1). For simplicity of exposition we assume that $x^* = 0$ is an equilibrium point for system (2.1), i.e. $G(0) = 0$.

Theorem 2.2.4 *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a bounded positively invariant set for system* (2.1) *that contains a neighborhood $\mathcal{N}$ of the equilibrium $x^* = 0$ and let α_1, α_2 and α_3 be class $\mathcal{K}$-functions. Suppose there exists a function $V : \mathbb{X} \rightarrow \mathbb{R}_+$ with $V(0) = 0$ such that:*

$$V(x) \geq \alpha_1(\|x\|), \quad \forall x \in \mathbb{X}, \tag{2.2a}$$

$$V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathcal{N}, \tag{2.2b}$$

$$V(G(x)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{X}. \tag{2.2c}$$

Then the following results hold:

(i) *The origin of the nonlinear system* (2.1) *is asymptotically stable in the Lyapunov sense in $\mathbb{X}$.*

(ii) *If the inequalities in* (2.2) *hold with $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ for some constants $a, b, c, \lambda > 0$, then the origin of the nonlinear system* (2.1) *is locally exponentially stable. Moreover, if the inequality* (2.2b) *holds for $\mathcal{N} = \mathbb{X}$, then the origin of the nonlinear system* (2.1) *is exponentially stable in $\mathbb{X}$.*

Proof: *Stability*. Let x_k represent the solution of (2.1) at time $k \in \mathbb{Z}_+$, obtained from the initial condition x_0 at time $k = 0$. Take an $\eta > 0$ such that the ball $\mathcal{B}_\eta \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq \eta\}$ satisfies $\mathcal{B}_\eta \subseteq \mathcal{N}$. Since $\alpha_1, \alpha_2 \in \mathcal{K}$

we can choose for any $0 < \varepsilon \leq \eta$ a $\delta \in (0, \varepsilon)$ such that $\alpha_2(\delta) < \alpha_1(\varepsilon)$. For any $x_0 \in \mathcal{B}_\delta \subseteq \mathbb{X}$, due to positive invariance of $\mathbb{X}$, from (2.2b) it follows that

$$\ldots \leq V(x_{k+1}) \leq V(x_k) \leq \ldots \leq V(x_0) \leq \alpha_2(\|x_0\|) \leq \alpha_2(\delta) < \alpha_1(\varepsilon).$$

Since we have that $V(x) \geq \alpha_1(\varepsilon)$ for all $x \in \mathbb{X} \setminus \mathcal{B}_\varepsilon$ it follows that $x_k \in \mathcal{B}_\varepsilon$ for all $k \in \mathbb{Z}_+$. Hence, the origin of the nonlinear system (2.1) is *Lyapunov stable*.

Attractivity. Since $\Delta V(x_k) \triangleq V(x_{k+1}) - V(x_k) \leq 0$ and $V(\cdot)$ is lower bounded by zero, it follows that $\lim_{k\to\infty} V(x_k) = V_L \geq 0$ exists. Then,

$$\lim_{k\to\infty} \Delta V(x_k) = V_L - V_L = 0.$$

Since $0 \leq \alpha_3(\|x_k\|) \leq -\Delta V(x_k)$, it follows that $\lim_{k\to\infty} \alpha_3(\|x_k\|) = 0$. Assume by contradiction that $\|x_k\| \nrightarrow 0$ for $k \to \infty$. Then there exists a subsequence $\{x_{k_l}\}_{l\in\mathbb{Z}_+}$ such that $\|x_{k_l}\| > \mu > 0$ for all $l \geq 0$, which by monotonicity and positivity of α_3 implies that $\alpha_3(\|x_{k_l}\|) \geq \alpha_3(\mu) > 0$ for all $l \geq 0$. Hence, we reached a contradiction of convergence of $\alpha_3(\|x_k\|)$ to zero. Hence, $\lim_{k\to\infty} \|x_k\| = 0$ for all $x_0 \in \mathbb{X}$, which implies that the origin of the nonlinear system (2.1) is attractive in $\mathbb{X}$ and thus, we have *asymptotic stability in $\mathbb{X}$ in the Lyapunov sense*.

Exponential stability. Take an $\eta > 0$ such that $\mathcal{B}_\eta \subseteq \mathcal{N}$ and select for any $0 < \varepsilon \leq \eta$ a $\delta \in (0, \varepsilon)$ such that $\alpha_2(\delta) < \alpha_1(\varepsilon)$, as in the proof Lyapunov stability. Suppose $x_0 \in \mathcal{B}_\delta$. Then $x_k \in \mathcal{B}_\varepsilon \subseteq \mathcal{N}$ for all $k \in \mathbb{Z}_+$. Therefore it holds that $V(x_k) \leq \alpha_2(\|x_k\|)$ and $V(x_{k+1}) - V(x_k) \leq -\alpha_3(\|x_k\|)$ for all $k \in \mathbb{Z}_+$. Then, we have that for all $k \in \mathbb{Z}_+$

$$V(G(x_k)) - V(x_k) \leq -c\|x_k\|^\lambda = -\frac{c}{b}\alpha_2(\|x_k\|) \leq -\frac{c}{b}V(x_k).$$

This implies that:

$$V(x_k) \leq (1 - \frac{c}{b})^k V(x_0) \quad \text{for all} \quad k \in \mathbb{Z}_+.$$

In order to show that $0 \leq 1 - \frac{c}{b} < 1$, we use the inequalities (2.2b) and (2.2c), which yield:

$$0 \leq V(G(x_k)) \leq V(x_k) - c\|x_k\|^\lambda \leq \alpha_2(\|x_k\|) - c\|x_k\|^\lambda = (b - c)\|x_k\|^\lambda.$$

Hence, it follows that $b \geq c > 0$. Then, we have that $\rho \triangleq 1 - \frac{c}{b} \in [0, 1)$. From (2.2a) and (2.2b) it follows that

$$a\|x_k\|^\lambda \leq V(x_k) \leq \rho^k V(x_0) \leq \rho^k b\|x_0\|^\lambda, \quad \text{for all} \quad k \in \mathbb{Z}_+.$$

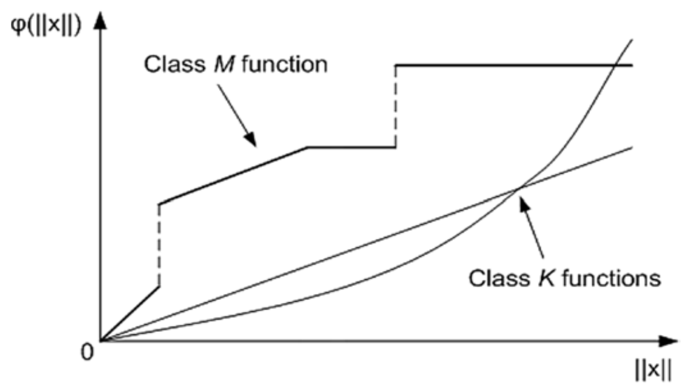

Figure 2.1: A graphical illustration of $\mathcal{M}$-functions versus $\mathcal{K}$-functions.

Hence, $\|x_k\| \leq \theta\|x_0\|\tau^k$ for all $x_0 \in \mathcal{B}_\delta$ and all $k \in \mathbb{Z}_+$, with $\theta \triangleq (\frac{b}{a})^{\frac{1}{\lambda}} > 0$ and $\tau \triangleq \rho^{\frac{1}{\lambda}} \in [0, 1)$. This means that the origin of the nonlinear system (2.1) is *locally exponentially stable*, i.e. in a ball $\mathcal{B}_\delta \subseteq \mathcal{N}$. Moreover, since $\mathbb{X}$ is a positively invariant set for system (2.1), if inequality (2.2b) holds for $\mathcal{N} = \mathbb{X}$ then, by applying the same reasoning as above, it follows that the origin of the nonlinear system (2.1) is *exponentially stable in* $\mathbb{X}$. ■

Definition 2.2.5 A function $V(\cdot)$ that satisfies the hypothesis of Theorem 2.2.4 is called a *Lyapunov function*.

Consider the following aspects regarding Theorem 2.2.4:

(i) The hypothesis of Theorem 2.2.4 allows that both $G(\cdot)$ and $V(\cdot)$ are discontinuous. It *only* implies continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$;

(ii) We only use $\alpha_1, \alpha_2 \in \mathcal{K}$ locally, in an arbitrary neighborhood of the origin. Outside this neighborhood it is sufficient that $\alpha_1, \alpha_2 \in \mathcal{M}$, see Figure 2.1 for an illustrative plot. Allowing for class $\mathcal{M}$ bounds, which can be discontinuous, might be convenient from a synthesis point of view, e.g. when dealing with hybrid systems. This is because a class $\mathcal{M}$ upper bound is less conservative and more easy to construct than a global class $\mathcal{K}$ upper bound, when dealing with piecewise continuous systems, such as piecewise affine or switched systems;

(iii) For $x \in \mathcal{B}_\delta \subseteq \mathcal{N}$ we have that $\|x\| \leq \delta$, which implies that for $x \in \mathbb{X} \setminus \mathcal{B}_\delta$, $\|x\| > \delta$. Then, from inequality (2.2a) it follows that there exists a lower bound on $V(\cdot)$ outside the ball $\mathcal{B}_\delta$, i.e. for $x \in \mathbb{X} \setminus \mathcal{B}_\delta$. This replaces the more common and somewhat more restrictive assumption that $V(\cdot)$ is radially unbounded, i.e. $V(x) \to \infty$ as $x \to \infty$;

(iv) The inequality (2.2c) implies that inequality (2.2a) is satisfied with $\alpha_1(s) \triangleq \alpha_3(s)$.

2.3 Input-to-state stability

Consider the discrete-time perturbed autonomous nonlinear system:

$$x_{k+1} = G(x_k, v_k), \quad k \in \mathbb{Z}_+, \tag{2.3}$$

where $x_k \in \mathbb{R}^n$ is the state, $v_k \in \mathbb{R}^{d_v}$ is an unknown disturbance input and $G : \mathbb{R}^n \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is a nonlinear, possibly discontinuous function. For simplicity of notation, we assume that the origin is an equilibrium in (2.3) for zero disturbance input, meaning that $G(0, 0) = 0$.

Definition 2.3.1 For a given $0 \leq \lambda \leq 1$, a set $\mathcal{P} \subseteq \mathbb{R}^n$ with $0 \in \text{int}(\mathcal{P})$ is called a *robust λ-contractive set (with respect to $\mathbb{V}$)* for system (2.3) if for all $x \in \mathcal{P}$ it holds that $G(x, v) \in \lambda\mathcal{P}$ for all $v \in \mathbb{V}$. For $\lambda = 1$ a robust λ-contractive set (with respect to $\mathbb{V}$) is called a *Robust Positively Invariant (RPI) set (with respect to $\mathbb{V}$)*.

Definition 2.3.2 A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class $\mathcal{KL}$* if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s, k) = 0$.

Next, we introduce the notion of global input-to-state stability, as defined in (Jiang and Wang, 2001), for the discrete-time nonlinear system (2.3).

Definition 2.3.3 The perturbed system (2.3) is *globally Input-to-State Stable (ISS)* if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each initial condition $x_0 \in \mathbb{R}^n$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{R}^{d_v}$ for all $j \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) \quad \text{for all} \quad k \in \mathbb{Z}_{\geq 1}.$$

In this thesis we will often deal with constrained systems. Therefore we also introduce the following local ISS notion.

Definition 2.3.4 Let $\mathbb{X}$ and $\mathbb{V}$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^{d_v}$, respectively, with $0 \in \text{int}(\mathbb{X})$. We call system (2.3) *ISS for initial conditions in $\mathbb{X}$ and disturbances in $\mathbb{V}$* if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ such that, for each $x_0 \in \mathbb{X}$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) \quad \text{for all} \quad k \in \mathbb{Z}_{\geq 1}.$$

For brevity, throughout the remainder of this thesis, we will use the term "input-to-state stable" or "ISS" to denote the local property of Definition 2.3.4, i.e. "ISS for initial conditions in $\mathbb{X}$ and disturbances in $\mathbb{V}$", unless explicitly specified otherwise. The term "global ISS" will be employed to denote the global property of Definition 2.3.3.

Note that, in the case when the disturbance input converges to zero, the input-to-state stability property implies asymptotic stability, i.e. the state also converges to zero.

We are now ready to state the main result of this section.

Theorem 2.3.5 *Let $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ for some $a, b, c, \lambda > 0$ and let $\sigma \in \mathcal{K}$. Let $\mathbb{V}$ be a subset of $\mathbb{R}^{d_v}$ that contains the origin. Let $\mathbb{X}$ with $0 \in \operatorname{int}(\mathbb{X})$ be a RPI set for system (2.3) and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function with $V(0) = 0$. Consider the following inequalities:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2.4a}$$

$$V(G(x, v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|). \tag{2.4b}$$

If inequalities (2.4) hold for all $x \in \mathbb{X}$ and all $v \in \mathbb{V}$, then system (2.3) is ISS for initial conditions in $\mathbb{X}$ and disturbances in $\mathbb{V}$. Moreover, the ISS property of Definition 2.3.4 holds with

$$\beta(s, k) \triangleq \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \tag{2.5}$$

where $\rho \triangleq 1 - \frac{c}{b} \in [0, 1)$.

Proof: From the hypothesis we have that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{X}$. For $x \in \mathbb{X} \setminus \{0\}$, due to $V(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{X}$ we obtain:

$$V(x) - \alpha_3(\|x\|) \leq \left(1 - \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)}\right) V(x) = \rho V(x), \quad \forall x \in \mathbb{X} \setminus \{0\},$$

where $\rho \triangleq 1 - \frac{c}{b}$. Next, we show that $\rho \in [0, 1)$. Since inequality (2.4) holds for $v = 0$ it follows that

$$0 \leq V(G(x, 0)) \leq V(x) - c\|x\|^\lambda \leq (b - c)\|x\|^\lambda.$$

Hence, it follows that $b \geq c > 0$ and thus $\rho \in [0,1)$. Since $V(0) - \alpha_3(\|0\|) = \rho V(0) = 0$, we have that $V(x) - \alpha_3(\|x\|) \leq \rho V(x)$ for all $x \in \mathbb{X}$. Then,

$$V(G(x_k, v_k)) = V(x_{k+1}) \leq \rho V(x_k) + \sigma(\|v_k\|), \quad \forall x_k \in \mathbb{X},\, v_k \in \mathbb{V}, k \in \mathbb{Z}_+.$$

Due to robust positive invariance of $\mathbb{X}$ we can apply the above inequality repetitively, which yields:

$$V(x_{k+1}) \leq \rho^{k+1} V(x_0) + \rho^k \sigma(\|v_0\|) + \rho^{k-1}\sigma(\|v_1\|) + \ldots + \sigma(\|v_k\|),$$

for all $x_0 \in \mathbb{X}$, $v_k \in \mathbb{V}$, $k \in \mathbb{Z}_+$. Then, it follows that:

$$\begin{aligned}\alpha_1(\|x_{k+1}\|) \leq V(x_{k+1}) &\leq \rho^{k+1}\alpha_2(\|x_0\|) + \sum_{i=0}^{k} \rho^i \sigma(\|w_{k-i}\|) \\ &\leq \rho^{k+1}\alpha_2(\|x_0\|) + \sigma(\|w_{[k]}\|)\frac{1}{1-\rho},\end{aligned}$$

for all $x_0 \in \mathbb{X}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. One can easily check that $\alpha_1 \in \mathcal{K}_\infty$ implies $\alpha_1^{-1} \in \mathcal{K}_\infty$. Then, taking into account that $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ we obtain:

$$\begin{aligned}\|x_{k+1}\| &\leq \alpha_1^{-1}\left(\rho^{k+1}\alpha_2(\|x_0\|) + \sigma(\|v_{[k]}\|)\frac{1}{1-\rho}\right) \\ &\leq \alpha_1^{-1}\left(2\max\left(\rho^{k+1}\alpha_2(\|x_0\|), \sigma(\|v_{[k]}\|)\frac{1}{1-\rho}\right)\right) \\ &\leq \alpha_1^{-1}(2\rho^{k+1}\alpha_2(\|x_0\|)) + \alpha_1^{-1}\left(2\sigma(\|v_{[k]}\|)\frac{1}{1-\rho}\right),\end{aligned}$$

for all $x_0 \in \mathbb{X}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Next, we consider two situations: $\rho = 0$ or $\rho \in (0,1)$. If $\rho = 0$ we have that

$$\begin{aligned}\|x_k\| \leq \alpha_1^{-1}(\sigma(\|v_{k-1}\|)) &\leq \beta(\|x_0\|, k) + \alpha_1^{-1}(\sigma(\|v_{[k-1]}\|)) \\ &\leq \beta(\|x_0\|, k) + \alpha_1^{-1}(2\sigma(\|v_{[k-1]}\|))\end{aligned}$$

for any $\beta \in \mathcal{KL}$, $k \in \mathbb{Z}_{\geq 1}$.

For $\rho \in (0,1)$, let $\beta(s,k) \triangleq \alpha_1^{-1}(2\rho^k\alpha_2(s))$. For a fixed $k \in \mathbb{Z}_+$, we have that $\beta(\cdot,k) \in \mathcal{K}$ due to $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\rho \in (0,1)$ For a fixed s, it follows that $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty}\beta(s,k) = 0$, due to $\rho \in (0,1)$ and $\alpha_1^{-1} \in \mathcal{K}_\infty$. Thus, it follows that $\beta \in \mathcal{KL}$.

Now let $\gamma(s) \triangleq \alpha_1^{-1}(2\sigma(s)\frac{1}{1-\rho})$. Since $\frac{1}{1-\rho} > 0$, it follows that $\gamma \in \mathcal{K}$ due to $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$.

Hence, the perturbed nonlinear system (2.3) is ISS in the sense of Definition 2.3.4 for all $x_0 \in \mathbb{X}$ and all disturbance inputs $v_k \in \mathbb{V}$, $k \in \mathbb{Z}_+$, with β and γ as in (2.5). ■

Definition 2.3.6 A function $V(\cdot)$ that satisfies the hypothesis of Theorem 2.3.5 is called an *ISS Lyapunov function*.

Consider the following aspects regarding Theorem 2.3.5:

(i) The hypothesis of Theorem 2.3.5 allows that both $G(\cdot,\cdot)$ and $V(\cdot)$ are discontinuous. It *only* implies continuity at the point $x = 0$, and *not* necessarily on a neighborhood of $x = 0$;

(ii) The proof of this theorem can also be based on the proof of Lemma 3.5 in (Jiang and Wang, 2001). Note that although continuity of the candidate ISS Lyapunov function $V(\cdot)$ is assumed in Lemma 3.5 of (Jiang and Wang, 2001), the continuity property is not actually used in the proof;

(iii) By considering a special type of $\mathcal{K}$-functions we are able to derive explicit bounds on the evolution of the state trajectory, i.e. via the relation (2.5). In (Jiang and Wang, 2001), the existence of suitable bounds is demonstrated, but an explicit relation is not derived;

(iv) These explicit bounds imply a stronger property for the system, than the one required by the results of (Jiang and Wang, 2001), i.e. they imply exponential stability in the absence of disturbances. However, they are not conservative for the control design methodologies presented in this thesis as the employed candidate ISS Lyapunov functions always satisfy the inequalities (2.4) with $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ for some $a, b, c, \lambda > 0$.

2.4 *Input-to-state practical stability*

Consider the discrete-time autonomous perturbed nonlinear system:

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{2.6}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state, unknown *time-varying parametric uncertainties* and other *disturbance inputs* (possibly additive). $G : \mathbb{R}^n \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, function. We assume that $\mathbb{W}$ is a known compact set and $\mathbb{V}$ is a known C-set.

Note that we distinguish between time-varying parametric uncertainties (*or parametric uncertainties for short*) and other disturbances, such as additive disturbances, because the results developed in this section will be employed in Chapter 6, where linear and nonlinear systems affected by both parametric and additive perturbations are considered.

Definition 2.4.1 For a given $0 \leq \lambda \leq 1$, a set $\mathcal{P} \subseteq \mathbb{R}^n$ that contains the origin in its interior is called a *robust λ-contractive set (with respect to $\mathbb{W}$ and $\mathbb{V}$)* for system (2.6) if for all $x \in \mathcal{P}$ it holds that $G(x, w, v) \in \lambda\mathcal{P}$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. For $\lambda = 1$ a robust λ-contractive set (with respect to $\mathbb{W}$ and $\mathbb{V}$) is called a *robust positively invariant set (with respect to $\mathbb{W}$ and $\mathbb{V}$)*.

Next, we define the notions of global and local Input-to-State practical Stability (ISpS) (Jiang, 1993; Jiang et al., 1994, 1996) for the discrete-time nonlinear system (2.6).

Definition 2.4.2 The system (2.6) is said to be *globally ISpS* if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a non-negative number d such that, for each $x_0 \in \mathbb{R}^n$, all $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{R}^{d_w}$ for all $j \in \mathbb{Z}_+$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{R}^{d_v}$ for all $j \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{2.7}$$

Definition 2.4.3 The system (2.6) is said to be *ISpS* for initial conditions in a set $\mathbb{X} \subseteq \mathbb{R}^n$ if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a non-negative number d such that, for each $x_0 \in \mathbb{X}$, all $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, it holds that the corresponding state trajectory satisfies

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|) + d, \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{2.8}$$

If the origin is an equilibrium in (2.6) for zero disturbance input v (i.e $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$), $\mathbb{X}$ contains the origin in its interior and inequality (2.8) is satisfied for $d = 0$, the system (2.6) is said to be *ISS* for initial conditions in $\mathbb{X}$.

Note that in the above definition we have extended the notion of input-to-state stability, introduced in the previous subsection for nonlinear systems affected by disturbance inputs only, to *nonlinear systems affected by both time-varying parametric uncertainties and (other) disturbance inputs*. The relevance of considering this case will become clear in Chapter 6.

In what follows we state a *discrete-time* version of the *continuous-time* ISpS sufficient conditions of Proposition 2.1 of (Jiang et al., 1996).

Theorem 2.4.4 *Let d_1, d_2 be non-negative numbers and let a, b, c, λ be positive numbers with $c \leq b$. Let $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$ and let $\sigma \in \mathcal{K}$. Furthermore, let $\mathbb{X}$ be a RPI set for system (2.6) and let $V : \mathbb{X} \to \mathbb{R}_+$ be a function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + d_1 \tag{2.9a}$$

$$V(G(x, w, v)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2 \tag{2.9b}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Then it holds that:

(i) *The system (2.6) is ISpS for initial conditions in $\mathbb{X}$ and the ISpS property (2.8) of Definition 2.4.3 holds for*

$$\beta(s,k) \triangleq \alpha_1^{-1}(4\rho^k \alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \quad d \triangleq \alpha_1^{-1}(4\xi), \tag{2.10}$$

where $\xi \triangleq d_1 + \frac{d_2}{1-\rho}$ and $\rho \triangleq 1 - \frac{c}{b} \in [0, 1)$.

(ii) *If $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$, $0 \in \operatorname{int}(\mathbb{X})$ and the inequalities (2.9) hold for $d_1 = d_2 = 0$, the system (2.6) is ISS for initial conditions in $\mathbb{X}$ and the ISS property (2.8) of Definition 2.4.3 (i.e. when $d = 0$) holds for*

$$\beta(s,k) \triangleq \alpha_1^{-1}(2\rho^k \alpha_2(s)), \quad \gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right), \tag{2.11}$$

where $\rho \triangleq 1 - \frac{c}{b} \in [0, 1)$.

Proof: (i) From $V(x) \leq \alpha_2(\|x\|) + d_1$ for all $x \in \mathbb{X}$, we have that for any $x \in \mathbb{X} \setminus \{0\}$ it holds:

$$V(x) - \alpha_3(\|x\|) \leq V(x) - \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)}(V(x) - d_1) = \rho V(x) + (1-\rho)d_1,$$

where $\rho \triangleq 1 - \frac{c}{b} \in [0, 1)$. In fact, the above inequality holds for all $x \in \mathbb{X}$, since $V(0) - \alpha_3(0) = V(0) = \rho V(0) + (1-\rho)V(0) \leq \rho V(0) + (1-\rho)d_1$. Then, inequality (2.9b) becomes

$$V(G(x, w, v)) \leq \rho V(x) + \sigma(\|v\|) + (1-\rho)d_1 + d_2, \tag{2.12}$$

for all $x \in \mathbb{X}$, $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Due to robust positive invariance of $\mathbb{X}$, inequality (2.12) yields repetitively:

$$V(x_{k+1}) \leq \rho^{k+1} V(x_0) + \sum_{i=0}^{k} \rho^i (\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2)$$

for all $x_0 \in \mathbb{X}$, $w_{[k]} \in \mathbb{W}^{k+1}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Then, taking (2.9a) into account and using the property $\sigma(\|v_i\|) \leq \sigma(\|v_{[k]}\|)$ for all $i \leq k$ and the identity $\sum_{i=0}^{k} \rho^i = \frac{1-\rho^{k+1}}{1-\rho}$, the following inequalities hold:

$$
\begin{aligned}
V(x_{k+1}) &\leq \rho^{k+1}\alpha_2(\|x_0\|) + \rho^{k+1}d_1 + \sum_{i=0}^{k} \rho^i(\sigma(\|v_{k-i}\|) + (1-\rho)d_1 + d_2) \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \rho^{k+1}d_1 + (\sigma(\|v_{[k]}\|) + (1-\rho)d_1 + d_2)\sum_{i=0}^{k} \rho^i \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \frac{1-\rho^{k+1}}{1-\rho}\sigma(\|v_{[k]}\|) + d_1 + \frac{1-\rho^{k+1}}{1-\rho}d_2 \\
&\leq \rho^{k+1}\alpha_2(\|x_0\|) + \frac{1}{1-\rho}\sigma(\|v_{[k]}\|) + d_1 + \frac{1}{1-\rho}d_2,
\end{aligned}
$$

for all $x_0 \in \mathbb{X}$, $w_{[k]} \in \mathbb{W}^{k+1}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$. Let $\xi \triangleq d_1 + \frac{d_2}{1-\rho}$. Taking (2.9a) into account and letting α_1^{-1} denote the inverse of α_1, we obtain:

$$
\|x_{k+1}\| \leq \alpha_1^{-1}(V(x_{k+1})) \leq \alpha_1^{-1}\left(\rho^{k+1}\alpha_2(\|x_0\|) + \xi + \frac{\sigma(\|v_{[k]}\|)}{1-\rho}\right). \quad (2.13)
$$

Applying the following inequality,

$$
\alpha_1^{-1}(z+y) \leq \alpha_1^{-1}(2\max(z,y)) \leq \alpha_1^{-1}(2z) + \alpha_1^{-1}(2y), \quad (2.14)
$$

it is easy to see that

$$
\|x_{k+1}\| \leq \alpha_1^{-1}(4\rho^{k+1}\alpha_2(\|x_0\|)) + \alpha_1^{-1}(2\frac{\sigma(\|v_{[k]}\|)}{1-\rho}) + \alpha_1^{-1}(4\xi),
$$

for all $x_0 \in \mathbb{X}$, $w_{[k]} \in \mathbb{W}^{k+1}$, $v_{[k]} \in \mathbb{V}^{k+1}$, $k \in \mathbb{Z}_+$.

We distinguish between two cases: $\rho \neq 0$ and $\rho = 0$. First, suppose $\rho \in (0,1)$ and let $\beta(s,k) \triangleq \alpha_1^{-1}(4\rho^k\alpha_2(s))$. For a fixed $k \in \mathbb{Z}_+$, we have that $\beta(\cdot,k) \in \mathcal{K}$ due to $\alpha_2 \in \mathcal{K}_\infty$, $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\rho \in (0,1)$. For a fixed s, it follows that $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty}\beta(s,k) = 0$, due to $\rho \in (0,1)$ and $\alpha_1^{-1} \in \mathcal{K}_\infty$. Thus, it follows that $\beta \in \mathcal{KL}$.

Now let $\gamma(s) \triangleq \alpha_1^{-1}(\frac{2\sigma(s)}{1-\rho})$. Since $\frac{1}{1-\rho} > 0$, it follows that $\gamma \in \mathcal{K}$ due to $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$.

Finally, let $d \triangleq \alpha_1^{-1}(4\xi)$. Since $\rho \in (0,1)$ and $d_1, d_2 \geq 0$, we have that $d \geq 0$.

In case when $\rho = 0$ we have that

$$\begin{aligned}\|x_k\| &\leq \alpha_1^{-1}(2\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(2\xi)\\ &\leq \beta(\|x_0\|, k) + \alpha_1^{-1}(2\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(2\xi)\\ &\leq \beta(\|x_0\|, k) + \alpha_1^{-1}(2\sigma(\|v_{[k-1]}\|)) + \alpha_1^{-1}(4\xi)\end{aligned}$$

for any $\beta \in \mathcal{KL}$, $\xi \triangleq d_1 + d_2$ and $k \in \mathbb{Z}_{\geq 1}$.

Hence, the perturbed system (2.6) is ISpS in the sense of Definition 2.4.3 for initial conditions in $\mathbb{X}$ and property (2.8) is satisfied with the functions given in (2.10).

(ii) Note that, following the proof of statement (i), it is trivial to observe that when the sufficient conditions (2.9) are satisfied for $d_1 = d_2 = 0$, then ISS is achieved, since $d = \alpha_1^{-1}(4\xi) = \alpha_1^{-1}(0) = 0$. From (2.13) and (2.14) and following the reasoning used in the proof of Theorem 2.3.5 it can be easily shown that the ISS property of Definition 2.3.4 actually holds with the functions given in (2.11). ■

Definition 2.4.5 A function $V(\cdot)$ that satisfies the hypothesis of Theorem 2.4.4 including part (i) is called an *ISpS Lyapunov function*.

Note that the hypothesis of Theorem 2.4.4 including part (i) (but excluding part (ii)) does not require continuity of $G(\cdot, \cdot, \cdot)$ or $V(\cdot)$, nor that $G(0, w, 0) = 0$ or $V(0) = 0$. This makes the ISpS framework suitable for analyzing stability of nonlinear systems in closed-loop with min-max MPC controllers, since the min-max MPC value function is not zero at zero in general (see Chapter 6 for details).

2.5 Conclusions

In this chapter we have presented general theorems on asymptotic stability in the Lyapunov sense, input-to-state stability and input-to-state practical stability, respectively. Regarding the assumptions on the system dynamics and the candidate (ISS) Lyapunov function, it has been shown that continuity at the equilibrium point, rather than continuity on a neighborhood of the equilibrium point is sufficient for Lyapunov stability or ISS. These results will open up the establishment of Lyapunov stability, ISS or ISpS for the particular cases of PWA systems and MPC closed-loop systems, as these systems are typically not continuous on a neighborhood of the origin.

3

Stabilizing model predictive control of hybrid systems

In this chapter we present a complete framework for the synthesis of stabilizing model predictive controllers, based on both quadratic and $1, \infty$-norms costs, for hybrid systems. This includes new techniques for computing stabilizing state-feedback controllers and low complexity piecewise polyhedral positively invariant sets for the class of PWA systems.

3.1 Introduction

One of the problems in model predictive control (MPC) that has received an increased attention over the years consists in guaranteeing closed-loop stability for the controlled system. The usual approach to ensure stability in MPC is to consider the value function of the MPC cost as a candidate Lyapunov function. Then, if the system dynamics is continuous, the classical Lyapunov stability theory (Kalman and Bertram, 1960b) can be used to prove that the MPC control law is stabilizing (Keerthi and Gilbert, 1988). For a comprehensive overview on stability of receding horizon control in discrete-time we refer the reader to (Mayne et al., 2000) and the references therein.

The recent development of MPC for hybrid systems, which are inherently discontinuous and nonlinear, requires a reconsideration of the stability results, as it was also pointed out in the excellent survey (Mayne et al., 2000).

Attractivity was proven for hybrid systems in closed-loop with model predictive controllers in (Bemporad and Morari, 1999; Borrelli, 2003). However, proofs of Lyapunov stability only appeared in the literature recently, for particular classes of hybrid systems and MPC cost functions, see, for example, (Kerrigan and Mayne, 2002; Mayne and Rakovic, 2003; Grieder et al., 2005). In these works, either *continuous* piecewise affine (PWA) systems are considered, (Kerrigan and Mayne, 2002; Mayne and Rakovic, 2003), which are in fact *Lipschitz continuous* systems or, in (Grieder et al., 2005), asymptotic stability is established via the results of (Mayne et al., 2000), where continuity of the MPC value function is assumed. Note that this property does not hold in general for the MPC value function, when hybrid systems, such as PWA systems, are employed as prediction models, as it is illustrated by many examples presented in this thesis (see also (Borrelli, 2003)).

In this chapter we present a priori verifiable conditions that guarantee stability of discrete-time nonlinear, *possibly discontinuous*, systems in closed-loop with MPC controllers. We develop a general theorem on asymptotic stability in the Lyapunov sense that unifies most of the previous results on stability of MPC. This theorem applies to a wide class of hybrid systems and MPC cost functions, and it does not require continuity of the MPC value function nor of the system dynamics. Efficient methods for calculating the terminal cost, for both quadratic and $1, \infty$-norm MPC costs, and the terminal constraint set are developed for the class of *discontinuous* PWA systems, with the origin not necessarily in the interior of one of the regions in the state-space partition. New algorithms for calculating *low complexity piecewise polyhedral* positively invariant sets for PWA systems are also presented.

3.2 Setting up the MPC optimization problem

Consider the following time-invariant discrete-time nonlinear system

$$x_{k+1} = g(x_k, u_k), \tag{3.1}$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input at the discrete-time instant $k \in \mathbb{Z}_+$ and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary, *possibly discontinuous*, nonlinear function. The sets $\mathbb{X}$ and $\mathbb{U}$ specify state and input constraints and it is assumed that they are compact polyhedral sets that contain the origin in their interior. We assume for simplicity that the origin is an equilibrium state for (3.1) with $u = 0$, meaning that $g(0, 0) = 0$. Note that the class of nonlinear dynamical systems (3.1) contains certain classes of hybrid systems, such as PWA systems, due to the fact that $g(\cdot,\cdot)$ may be discontinuous.

For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by system (3.1) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a desired target set that contains the origin. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$ is

$$\mathcal{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\, x_{N|k} \in \mathbb{X}_T\}.$$

The MPC optimization problem can now be formulated as follows.

Problem 3.2.1 Let the target set $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given and let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0,0) = 0$ be mappings. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost function

$$J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}),$$

with prediction model (3.1), over all input sequences $\mathbf{u}_k \in \mathcal{U}_N(x_k)$.

In the MPC literature, $F(\cdot)$, $L(\cdot,\cdot)$ and N are called the terminal cost, the stage cost and the prediction horizon, respectively. We call an initial state $x_0 \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_0) \neq \emptyset$. Similarly, Problem 3.2.1 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to Problem 3.2.1 and let

$$V_{\mathrm{MPC}} : \mathbb{X}_f(N) \to \mathbb{R}_+, \quad V_{\mathrm{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k) \tag{3.2}$$

denote the MPC value function corresponding to Problem 3.2.1. We assume[1] that there exists an optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ for Problem 3.2.1 and any state $x_k \in \mathbb{X}_f(N)$. Hence, the infimum in (3.2) is a minimum and $V_{\mathrm{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, *the MPC control law* is defined as

$$u^{\mathrm{MPC}}(x_k) \triangleq u_{0|k}^*; \quad k \in \mathbb{Z}_+. \tag{3.3}$$

The stability results presented in this chapter also hold when the optimum is not unique in Problem 3.2.1, i.e. all results apply irrespective of which optimal sequence is selected.

[1] This assumption is satisfied for PWA prediction models and quadratic or $1, \infty$-norm MPC costs (see Section 3.3 for details).

3.3 Stabilization conditions for hybrid MPC

In this section we investigate the MPC stabilization of the *discontinuous* nonlinear system (3.1), which also includes certain relevant classes of hybrid systems. We will employ *terminal cost and constraint set* and *terminal equality constraint* methods, as the ones used for *smooth* nonlinear systems in (Mayne et al., 2000) to guarantee stability for the closed-loop system (3.1)-(3.3). Typically, these methods rely on continuity of $V_{\text{MPC}}(\cdot)$ and of the system dynamics (e.g., see Section 3.2 of (Mayne et al., 2000) or Theorem 4.4.2 of (Goodwin et al., 2005)). This property is no longer guaranteed in the case of discontinuous dynamical systems, such as hybrid systems. Actually, in the survey (Mayne et al., 2000) it was pointed out that all the concepts and ideas used in MPC should be reconsidered in the hybrid context.

3.3.1 Main results

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ denote an arbitrary, possibly discontinuous, nonlinear function with $h(0) = 0$ and let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$ denote the safe set with respect to *state and input* constraints for $h(\cdot)$.

The following theorem was obtained as a kind of general and unifying result by combining previous results on stability of discrete-time nonlinear MPC.

Assumption 3.3.1 *Terminal cost and constraint set*: There exist $\alpha_1, \alpha_2 \in \mathcal{K}$, a neighborhood of the origin $\mathcal{N} \subseteq \mathbb{X}_f(N)$ and a feedback control law $h(\cdot)$ such that $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$, with $0 \in \text{int}(\mathbb{X}_T)$, is a positively invariant set for system (3.1) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$,

$$L(x,u) \geq \alpha_1(\|x\|) \quad \text{for all} \quad x \in \mathbb{X}_f(N) \quad \text{and all} \quad u \in \mathbb{U}, \tag{3.4a}$$

$$F(x) \leq \alpha_2(\|x\|) \quad \text{for all} \quad x \in \mathcal{N} \quad \text{and} \tag{3.4b}$$

$$F(g(x,h(x))) - F(x) + L(x,h(x)) \leq 0 \quad \text{for all} \quad x \in \mathbb{X}_T. \tag{3.4c}$$

Assumption 3.3.2 *Terminal equality constraint*: $\mathbb{X}_T = \{0\}$, $F(x) = 0$ for all $x \in \mathbb{X}$ and there exist $\alpha_1, \tilde{\alpha}_2 \in \mathcal{K}$ and a neighborhood of the origin $\mathcal{N} \subseteq \mathbb{X}_f(N)$ such that $L(x,u) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}_f(N)$ and all $u \in \mathbb{U}$ and $L(x^*_{i|k}, u^*_{i|k}) \leq \tilde{\alpha}_2(\|x_k\|)$, for any optimal $\mathbf{u}^*_k \in \mathcal{U}_N(x_k)$, initial state $x_k \triangleq x^*_{0|k} \in \mathcal{N}$ and $i = 0, \ldots, N-1$, where $(x^*_{1|k}, \ldots, x^*_{N|k}) \triangleq \mathbf{x}_k(x_k, \mathbf{u}^*_k)$.

Theorem 3.3.3 *Fix $N \in \mathbb{Z}_{\geq 1}$ and suppose that either Assumption 3.3.1 holds, or Assumption 3.3.2 holds. Then:*

(i) *If Problem 3.2.1 is feasible at time* $k \in \mathbb{Z}_+$ *for state* $x_k \in \mathbb{X}$*, Problem 3.2.1 is feasible at time* $k+1$ *for state* $x_{k+1} = g(x_k, u^{MPC}(x_k))$*. Moreover,* $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$*;*

(ii) *The origin of the MPC closed-loop system* (3.1)-(3.3) *is asymptotically stable in the Lyapunov sense for initial conditions in* $\mathbb{X}_f(N)$*;*

(iii) *If Assumption 3.3.1 holds with* $\alpha_1(s) \triangleq as^\lambda$*,* $\alpha_2(s) \triangleq bs^\lambda$ *for some constants* $a, b, \lambda > 0$*, the origin of the MPC closed-loop system* (3.1)-(3.3) *is exponentially stable in* $\mathbb{X}_f(N)$*.*

Proof: First, we address the terminal cost and constraint case, i.e. when Assumption 3.3.1 holds. Consider an optimal sequence of controls

$$\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$$

obtained by solving Problem 3.2.1 and the shifted sequence of controls

$$\mathbf{u}_{k+1} \triangleq (u_{1|k}^*, \ldots, u_{N-1|k}^*, h(x_{N-1|k+1})), \tag{3.5}$$

where $x_{N-1|k+1}$ is the state at prediction time $N-1$, obtained at discrete-time $k+1$ by applying the input sequence $u_{1|k}^*, \ldots, u_{N-1|k}^*$ to system (3.1) with initial condition $x_{0|k+1} \triangleq x_{1|k}^* = x_{k+1} = g(x_k, u^{\mathrm{MPC}}(x_k))$.

(i) If Problem 3.2.1 is feasible at time $k \in \mathbb{Z}_+$ for state $x_k \in \mathbb{X}$ then there exists $\mathbf{u}_k^* \in \mathcal{U}_N(x_k)$ that solves Problem 3.2.1. Then, it follows that $x_{N-1|k+1} \in \mathbb{X}_T$. Due to positive invariance of $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$ it holds that $x_{N|k+1} \in \mathbb{X}_T$ and thus, $\mathbf{u}_{k+1} \in \mathcal{U}_N(x_{k+1})$. This implies that Problem 3.2.1 is feasible for state $x_{k+1} = g(x_k, u^{\mathrm{MPC}}(x_k))$. Moreover, all states in the set $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$ are feasible with respect to Problem 3.2.1, as the feedback $u_k \triangleq h(x_k)$ can be applied for any $x_k \in \mathbb{X}_T$ and any $k \in \mathbb{Z}_+$. This implies that $\mathbb{X}_T \subseteq \mathbb{X}_f(N)$.

(ii) By Assumption 3.3.1, inequality (3.4a), we have that

$$V_{\mathrm{MPC}}(x) \geq L(x, u^{\mathrm{MPC}}(x)) \geq \alpha_1(\|x\|), \quad \forall x \in \mathbb{X}_f(N). \tag{3.6}$$

Let $\tilde{\mathbf{x}} \triangleq (\tilde{x}_{1|k}, \ldots, \tilde{x}_{N|k})$ denote the state sequence generated by the "local" dynamics $x_{k+1} = g(x_k, h(x_k))$ from initial state $\tilde{x}_{0|k} \triangleq x \in \mathbb{X}_T$. Since $\tilde{\mathbf{x}} \in \mathbb{X}_T^N$, inequality (3.4c) holds for all elements of the sequence $\tilde{\mathbf{x}}$, yielding:

$$\begin{aligned}
&F(\tilde{x}_{1|k}) - F(\tilde{x}_{0|k}) + L(\tilde{x}_{0|k}, h(\tilde{x}_{0|k})) \leq 0,\\
&F(\tilde{x}_{2|k}) - F(\tilde{x}_{1|k}) + L(\tilde{x}_{1|k}, h(\tilde{x}_{1|k})) \leq 0,\\
&\ldots, \quad F(\tilde{x}_{N|k}) - F(\tilde{x}_{N-1|k}) + L(\tilde{x}_{N-1|k}, h(\tilde{x}_{N-1|k})) \leq 0.
\end{aligned}$$

From the above inequalities, by optimality and by Assumption 3.3.1, inequality (3.4b), it follows that

$$V_{\text{MPC}}(x) \leq J(x, \tilde{\mathbf{u}}) \leq F(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}_T, \tag{3.7}$$

where $\tilde{\mathbf{u}} \triangleq (h(\tilde{x}_{0|k}), \ldots, h(\tilde{x}_{N-1|k}))$. By optimality, we observe that for all $x_k \in \mathbb{X}_f(N)$ it holds that:

$$\begin{aligned} V_{\text{MPC}}(x_{k+1}) - V_{\text{MPC}}(x_k) &= J(x_{k+1}, \mathbf{u}^*_{k+1}) - J(x_k, \mathbf{u}^*_k) \\ &\leq J(x_{k+1}, \mathbf{u}_{k+1}) - J(x_k, \mathbf{u}^*_k) \\ &= -L(x_k, u^{\text{MPC}}(x_k)) + F(x_{N|k+1}) \\ &\quad - F(x^*_{N|k}) + L(x^*_{N|k}, h(x^*_{N|k})). \end{aligned} \tag{3.8}$$

By the hypothesis (3.4c), from $x^*_{N|k} \in \mathbb{X}_T$ and using Assumption 3.3.1 it follows that

$$\begin{aligned} V_{\text{MPC}}(g(x, u^{\text{MPC}}(x)) - V_{\text{MPC}}(x) &\leq -L(x, u^{\text{MPC}}(x)) \\ &\leq -\alpha_1(\|x\|), \ \forall x \in \mathbb{X}_f(N). \end{aligned} \tag{3.9}$$

Hence, V_{MPC} is a Lyapunov function. The second statement of Theorem 3.3.3 then follows from Theorem 2.2.4 part (i).

(iii) From the proof of part (ii), under the additional hypothesis it follows that $V_{\text{MPC}}(\cdot)$ satisfies the hypothesis of Theorem 2.2.4 part (ii), which is sufficient for local exponential stability. This is because the class $\mathcal{K}$ upper bound, i.e. $\alpha_2(\cdot)$, on $V_{\text{MPC}}(\cdot)$ only holds for the states in $\mathbb{X}_T$. However, in (Limon et al., 2006) it is shown how a class $\mathcal{K}$ upper bound on V_{MPC} for all states in $\mathbb{X}_f(N)$ can be obtained from the local one, under the assumption of compactness of $\mathbb{X}$ and $\mathbb{U}$, and boundedness of the functions $F(\cdot)$ and $L(\cdot, \cdot)$, but without requiring continuity of $V_{\text{MPC}}(\cdot)$. Using the technique of (Limon et al., 2006), exponential stability in $\mathbb{X}_f(N)$ for the hybrid MPC closed-loop system is readily proven.

Next, we consider the terminal equality constraint case, i.e. when Assumption 3.3.2 holds.

The proof of the first statement provided above for the terminal cost and constraint set approach also applies for the terminal equality constraint case, since $h(x) = 0$ for all $x \in \mathbb{X}$ and $\mathbb{X}_T = \{0\}$ is positively invariant. By Assumption 3.3.2, inequality (3.6) holds. Since $\mathbb{X}_T = \{0\}$, $F(x) = 0$ and $h(x) = 0$ for all $x \in \mathbb{X}$ the inequality (3.4c) holds. However, note that the terminal cost no longer provides a suitable upper bound for the MPC value function. Letting $x^*_{0|k} \triangleq x_k$, by Assumption 3.3.2 we have that

$$V_{\mathrm{MPC}}(x_k) = J(x_k, \mathbf{u}^*_k) = \sum_{i=0}^{N-1} L(x^*_{i|k}, u^*_{i|k}) \leq N\tilde{\alpha}_2(\|x_k\|), \quad \forall x_k \in \mathcal{N}. \tag{3.10}$$

Therefore, $V_{\mathrm{MPC}}(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathcal{N}$, with $\alpha_2(s) \triangleq N\tilde{\alpha}_2(s) \in \mathcal{K}$. Using the same reasoning as the one employed for the terminal cost and constraint set approach, it can be shown that the statements follow from Theorem 2.2.4. ■

3.3.2 The class of PWA systems

Throughout the rest of this chapter we focus on the class of time-invariant discrete-time piecewise affine (PWA) systems (Sontag, 1981, 1996) described by equations of the form

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \Omega_j, \quad j \in \mathcal{S}, \tag{3.11}$$

which is a sub-class of the discontinuous nonlinear system (3.1). Also, we take the nonlinear function $h(\cdot)$ as a piecewise linear (PWL) state-feedback control law, i.e.

$$h(x) \triangleq K_j x \quad \text{when} \quad x_k \in \Omega_j, \quad j \in \mathcal{S}. \tag{3.12}$$

Here, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$ with $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ a *finite set* of indices and s denotes the number of discrete modes. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\mathrm{int}(\Omega_i) \cap \mathrm{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 \triangleq \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 \triangleq \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (3.11) with $u = 0$ and we require that

$$f_j = 0 \text{ for all } j \in \mathcal{S}_0. \tag{3.13}$$

The class of hybrid systems described by (3.11)-(3.13) contains PWA systems which *may be discontinuous over the boundaries* and which are piecewise linear, instead of PWA, in the state space region $\cup_{j\in\mathcal{S}_0}\Omega_j$. This class of PWA systems is more general than the one considered in (Kerrigan and Mayne, 2002; Mayne and Rakovic, 2003), where continuity of the system dynamics is assumed. Also, system (3.11)-(3.13) includes the case when the origin lies on the boundaries of multiple regions Ω_j. This is a case of interest, since many physical systems exhibit a change in their dynamic behavior when certain states change their sign.

If the PWA system (3.11) is used as prediction model, the optimization problem corresponding to Problem 3.2.1 is a MIQP problem in case quadratic costs are used, and a MILP problem in case $1, \infty$-norm costs are employed. These problems can be solved using the tools of Hybrid Toolbox (Bemporad, 2003) or Multi Parametric Toolbox (Kvasnica et al., 2004). Note that if a MIQP (MILP) problem is feasible, the global optimum is attained because, in principle, a MIQP (MILP) consists of a finite number of QP (LP) problems, see, for example (Borrelli, 2003). Then, due to the fact that each QP (LP) (with bounded feasible set) attains its optimum, the existence of an optimum for the MIQP (MILP) problem is guaranteed (although it may not be unique). Hence, the standing assumption employed in Section 3.2 on existence of optimal control sequences holds for PWA prediction models and the result of Theorem 3.3.3 applies.

Although we focus on PWA systems of the form (3.11), the results developed in the remainder of this chapter have a wider applicability since it is known (Heemels et al., 2001) that PWA systems are equivalent under certain mild assumptions with other relevant classes of hybrid systems, such as mixed logical dynamical systems (Bemporad and Morari, 1999) and linear complementarity systems (van der Schaft and Schumacher, 1998). Also, it is well known that PWA systems can approximate nonlinear systems arbitrarily well (Sontag, 1981) and they arise from the interconnection of linear systems and automata (Sontag, 1996).

3.3.3 The problem statement reconsidered

For a given stage cost $L(\cdot, \cdot)$, the fundamental stability results for MPC of hybrid systems provided in this section come down to computing a terminal cost $F(\cdot)$, a nonlinear function $h(\cdot)$ and a terminal set $\mathbb{X}_T$ such that Assumption 3.3.1 holds (the terminal equality constraint situation is treated separately in Section 3.6). This is a non-trivial problem, which depends on the type of system dynamics and MPC cost.

For example, in the particular case of PWA systems and quadratic MPC costs this problem has only been solved partially, in (Grieder et al., 2005), i.e. by employing a common quadratic Lyapunov function for system (3.11) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, with $h(\cdot)$ defined in (3.12). This is known to be conservative (see, for example, (Ferrari-Trecate et al., 2002)) because there are PWA systems which only admit a piecewise quadratic (PWQ) Lyapunov function (see also Section 3.7, for an example where the method of (Grieder et al., 2005) fails). In the case of MPC costs based on $1, \infty$-norms, to the author's knowledge, there is no systematic method available for solving this problem. A complete solution to the problem of calculating the terminal cost is presented in Section 3.4.

The problem of computing the terminal set $\mathbb{X}_T$ boils down to computing positively invariant sets for PWA systems, which is a notoriously difficult problem. An algorithm for calculating the *maximal* positively invariant set for PWA systems was recently presented in (Rakovic et al., 2004). However, it is known that the maximal positively invariant set inside a given compact set is a piecewise polyhedral set for PWA systems, which can be very complex (i.e. it may consist of the union of a very large number of polyhedra, which in principle can be infinite, if the algorithm does not converge). This in turn influences the computational complexity of the MIQP (MILP) MPC optimization problem. Hence, it would be desirable to obtain a trade-off between the size of the terminal constraint set and its complexity. Section 3.5 deals with this issue.

Note that once a quadratic or $1, \infty$-norm terminal cost $F(\cdot)$, a nonlinear function $h(\cdot)$ of the form (3.12) and a piecewise polyhedral terminal set $\mathbb{X}_T$ that satisfy Assumption 3.3.1 for system (3.11) have been calculated, it is well-known (Borrelli, 2003) that the set of feasible states $\mathbb{X}_f(N)$ can be obtained explicitly for a fixed value of the prediction horizon $N \in \mathbb{Z}_{\geq 1}$ using either the HT Bemporad (2003) or the MPT Kvasnica et al. (2004). This may help in selecting a suitable prediction horizon N.

3.4 Computation of the terminal cost

In this section we provide solutions to the problem of computing a terminal cost $F(\cdot)$ and a function $h(\cdot)$ of the form (3.12) that satisfy Assumption 3.3.1 for both quadratic and $1, \infty$-norm MPC costs.

3.4.1 Quadratic MPC costs

Consider the case when quadratic forms are used to define the cost function, i.e.

$$F(x) = \|P_j^{\frac{1}{2}} x\|_2^2 = x^\top P_j x \text{ when } x \in \mathbb{X}_T \cap \Omega_j$$

and

$$L(x, u) = \|Q^{\frac{1}{2}} x\|_2^2 + \|R^{\frac{1}{2}} u\|_2^2 = x^\top Q x + u^\top R u.$$

Without any significant loss of generality, in this subsection we assume that $\mathbb{X}_T \subseteq \cup_{j \in \mathcal{S}_0} \Omega_j$. Let s_0 denote the number of elements of $\mathcal{S}_0$. In this case $P_j, Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are assumed to be positive definite and symmetric matrices. For the above stage and terminal costs it holds that

$$L(x,u) \geq x^\top Q x \geq \lambda_{\min}(Q)\|x\|_2^2$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, and that

$$F(x) \leq \max_{j\in\mathcal{S}_0} \lambda_{\max}(P_j)\|x\|_2^2$$

for all $x \in \mathbb{R}^n$. Therefore, a part of Assumption 3.3.1 is trivially satisfied with $\alpha_1(\|x\|) \triangleq \lambda_{\min}(Q)\|x\|_2^2$ and $\alpha_2(\|x\|) \triangleq \max_{j\in\mathcal{S}_0} \lambda_{\max}(P_j)\|x\|_2^2$.

Next, we provide methods for calculating the matrices $\{(P_j, K_j) \mid j \in \mathcal{S}_0\}$ such that inequality (3.4) is satisfied for the PWA system (3.11).

Let $\mathcal{Q}_{ji} \triangleq \{x \in \Omega_j \mid \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0$ and let $\mathcal{S}_{t0} \triangleq \{(j,i) \in \mathcal{S}_0 \times \mathcal{S}_0 \mid \mathcal{Q}_{ji} \neq \emptyset\}$. The set of pairs of indices $\mathcal{S}_{t0}$ can be easily determined off-line by solving s_0^2 linear programs. Consider now the PWL sub-system of the PWA system (3.11), i.e.

$$x_{k+1} = A_j x_k + B_j u_k, \quad \text{when} \quad x_k \in \mathbb{X}_T \cap \Omega_j, \quad j \in \mathcal{S}_0. \tag{3.14}$$

The set $\mathcal{S}_{t0}$ contains all discrete mode transitions that can occur in system (3.14), i.e. a transition from Ω_j to Ω_i can occur if and only if $(j,i) \in \mathcal{S}_{t0}$. Letting u_k be the control law (3.12) in (3.14) and substituting the resulting closed-loop system and $F(\cdot)$ in (3.4) yields that it is sufficient to find (P_j, K_j) with $P_j > 0$ for all $j \in \mathcal{S}_0$ that satisfy the matrix inequality

$$P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j > 0, \quad \forall (j,i) \in \mathcal{S}_{t0}, \tag{3.15}$$

for (3.4) to be satisfied with strict inequality. Next, we present three methods that can be used to solve the nonlinear matrix inequality (3.15) efficiently using semi-definite programming.

Lemma 3.4.1 *Let $\{(P_j, K_j, Z_j, Y_j, G_j) \mid j \in \mathcal{S}_0\}$ with Z_j,P_j positive definite and G_j invertible for all $j \in \mathcal{S}_0$ denote unknown variables that are related according to $Z_j = P_j^{-1}$, $Y_j = K_j P_j^{-1}$ and $K_j = Y_j G_j^{-1}$, $j \in \mathcal{S}_0$. Then the following matrix inequalities are equivalent:*

$$\begin{pmatrix} P_i & 0 \\ 0 & P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j \end{pmatrix} > 0,\ (j,i) \in \mathcal{S}_{t0}; \tag{3.16}$$

$$\begin{pmatrix} Z_j & Z_j & Y_j^\top & (A_jZ_j+B_jY_j)^\top \\ Z_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_jZ_j+B_jY_j) & 0 & 0 & Z_i \end{pmatrix} > 0,\ (j,i)\in\mathcal{S}_{t0}; \quad (3.17)$$

$$\begin{pmatrix} Z_j & (A_jZ_j+B_jY_j)^\top & (R^{\frac{1}{2}}Y_j)^\top & (Q^{\frac{1}{2}}Z_j)^\top \\ (A_jZ_j+B_jY_j) & Z_i & 0 & 0 \\ R^{\frac{1}{2}}Y_j & 0 & I_n & 0 \\ Q^{\frac{1}{2}}Z_j & 0 & 0 & I_n \end{pmatrix} > 0,\ (j,i)\in\mathcal{S}_{t0}; \quad (3.18)$$

$$\begin{pmatrix} G_j+G_j^\top-Z_j & G_j^\top & Y_j^\top & (A_jG_j+B_jY_j)^\top \\ G_j & Q^{-1} & 0 & 0 \\ Y_j & 0 & R^{-1} & 0 \\ (A_jG_j+B_jY_j) & 0 & 0 & Z_i \end{pmatrix} > 0,\ (j,i)\in\mathcal{S}_{t0}. \quad (3.19)$$

Proof: First we prove that the matrix inequality (3.16) and the LMI (3.17) are equivalent. We start by applying the Schur complement to (3.17), which yields:

$$Z_j - \begin{pmatrix} Z_j & Y_j^\top & (A_jZ_j+B_jY_j)^\top \end{pmatrix} \begin{pmatrix} Q & 0 & 0 \\ 0 & R & \\ 0 & 0 & Z_i^{-1} \end{pmatrix} \begin{pmatrix} Z_j \\ Y_j \\ (A_jZ_j+B_jY_j) \end{pmatrix} > 0$$

and $\begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & R^{-1} & 0 \\ 0 & 0 & Z_i \end{pmatrix} > 0$ for all $(j,i)\in\mathcal{S}_{t0}$. Since $Q>0$ and $R>0$ it follows that

$$\begin{pmatrix} Z_i & 0 \\ 0 & Z_j - Z_jQZ_j - Y_j^\top RY_j - (A_jZ_j+B_jY_j)^\top Z_i^{-1}(A_jZ_j+B_jY_j) \end{pmatrix} > 0.$$

Substituting $Z_j \triangleq P_j^{-1}$, $Z_i \triangleq P_i^{-1}$ and $Y_j \triangleq K_jP_j^{-1}$ in the above matrix inequality and pre-multiplying and post-multiplying with $\begin{pmatrix} P_i & 0 \\ 0 & P_j \end{pmatrix} > 0$ yields the equivalent matrix inequality (3.16).

The proof that (3.16) and the LMI (3.18) are equivalent is analogue to the proof of the equivalence (3.16) $\Leftrightarrow$ (3.17) (see also (Kothare et al., 1996) for a proof of (3.16) $\Leftrightarrow$ (3.18) for the particular case when a common terminal weight P and a linear feedback K are used). Finally, it can be proven that (3.16) and the LMI (3.19) are equivalent by applying the Schur complement to (3.19) in the same way as it was applied to (3.17) and exploiting the inequality $G_j^\top Z_j^{-1} G_j \geq G_j + G_j^\top - Z_j$ for all $j \in \mathcal{S}_0$ (see also the proof of Theorem 2 of (Daafouz et al., 2002), which deals with the stability of feedback controlled switched linear systems, for insight). ■

After solving any of the above LMIs, the terminal weights P_j and the feedbacks K_j are simply recovered as $P_j \triangleq Z_j^{-1}$ and $K_j \triangleq Y_j Z_j^{-1}$, $j \in \mathcal{S}_0$ for (3.17) and (3.18) and as $P_j \triangleq Z_j^{-1}$ and $K_j \triangleq Y_j G_j^{-1}$, $j \in \mathcal{S}_0$ for (3.19).

If any of the above LMIs is feasible for $P_j = P$ for all $j \in \mathcal{S}_0$ implies that $F(x) = x^\top P x$ is a local *common quadratic Lyapunov function* for system (3.14) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, with $h(\cdot)$ as defined in (3.12). Letting $P_j \neq P_i$ for $i \neq j$, $(i,j) \in \mathcal{S}_{t0}$ implies a relaxation in the sense that solving any of the above LMIs now amounts to searching for a *PWQ Lyapunov function* (Johansson and Rantzer, 1998; Ferrari-Trecate et al., 2002).

Next, we employ an S-procedure technique with respect to the matrix inequality (3.15), as done in (Johansson and Rantzer, 1998), to further reduce conservativeness, i.e., for all $(j,i) \in \mathcal{S}_{t0}$ we consider the inequality

$$P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j - E_{ji}^\top U_{ji} E_{ji} > 0 \quad (3.20)$$

in the unknowns (P_j, K_j, U_{ji}), where the matrices P_j are the terminal weights, the matrices U_{ji} have all entries non-negative and the matrices E_{ji} define the cones $\mathcal{C}_{ji}$, which are such that $\mathcal{C}_{ji} \triangleq \{x \in \mathbb{R}^n \mid E_{ji} x \geq 0\}$ and $\mathcal{Q}_{ji} \subseteq \mathcal{C}_{ji}$ for all $(j,i) \in \mathcal{S}_{t0}$. Note that if (P_j, K_j, U_{ji}) with $P_j > 0$ and U_{ji} with all entries non-negative for all $(j,i) \in \mathcal{S}_{t0}$ satisfy (3.20), then it follows that

$$\begin{aligned} &x^\top (P_j - (A_j + B_j K_j)^\top P_i (A_j + B_j K_j) - Q - K_j^\top R K_j) x \\ &\qquad \geq x^\top (E_{ji}^\top U_{ji} E_{ji}) x \geq 0 \end{aligned} \quad (3.21)$$

whenever $x \in \mathcal{Q}_{ji} \subseteq \mathcal{C}_{ji}$, $(j,i) \in \mathcal{S}_{t0}$. Hence, (3.4) is satisfied and conservativeness is reduced when comparing to the matrix inequality (3.15). However, the techniques used in the proof of Lemma 3.4.1 can not be used to transform (3.20) into an LMI, as this would require the matrices U_{ji} to be positive definite, which increases conservativeness.

We therefore develop an alternative method for finding a solution to the matrix inequality (3.20). This method is based on solving a sequence of LMIs that is obtained by fixing a suitable basis of the state space and successively selecting tuning parameters. Consider the eigenvalue decomposition $P_j = V_j \Sigma_j V_j^\top$, $j \in \mathcal{S}_0$ where $\Sigma_j = \operatorname{diag}(\sigma_{1j}, \ldots, \sigma_{nj})$, $\sigma_{1j} \geq \ldots \geq \sigma_{nj}$ and $V_j^\top = V_j^{-1}$. Assume that the orthonormal matrices $\{V_j \mid j \in \mathcal{S}_0\}$ are known and let $\Gamma_j \triangleq \mathrm{diag}(\gamma_{1j}, \ldots, \gamma_{nj}), j \in \mathcal{S}_0$ denote an arbitrary diagonal matrix. For any $(j, i) \in \mathcal{S}_{t0}$, consider now the following LMI:

$$\Delta_{ji} \triangleq \begin{pmatrix} V_j \Sigma_j V_j^\top - Q - E_{ji}^\top U_{ji} E_{ji} & (A_j + B_j K_j)^\top V_i & K_j^\top \\ V_i^\top (A_j + B_j K_j) & \Gamma_i & 0 \\ K_j & 0 & R^{-1} \end{pmatrix} > 0, \quad (3.22)$$

in the unknowns $\{(\sigma_{1j}, \ldots, \sigma_{nj}), (\gamma_{1i}, \ldots, \gamma_{ni}), K_j, U_{ji} \mid (j, i) \in \mathcal{S}_{t0}\}$. In addition to (3.22) we require that the linear scalar inequalities

$$\sigma_{1j} \geq \ldots \geq \sigma_{nj} > 0, \quad \gamma_{nj} \geq \ldots \geq \gamma_{1j} > 0, \quad (3.23a)$$

$$\frac{1}{\epsilon_{lj}} - \sigma_{lj} \geq 0, \quad \epsilon_{lj} - \gamma_{lj} \geq 0, \quad l = 1, \ldots, n, \quad (3.23b)$$

with ϵ_{lj} fixed constants (scaling factors) in $(0, 1]$, are satisfied for all $j \in \mathcal{S}_0$ and that

$$U_{ji} \text{ has all entries non-negative }, \forall (j, i) \in \mathcal{S}_{t0}. \quad (3.24)$$

Note that the scaling factors $\epsilon_{lj} \in (0, 1]$ are assumed to be known in (3.23) and that condition (3.24) can be easily written as an LMI. Hence, the conditions (3.22)-(3.23)-(3.24) are in the LMI form.

Theorem 3.4.2 *Choose the orthonormal matrices* V_j *and the scaling factors* $\epsilon_{lj} \in (0, 1]$, $l = 1, \ldots, n$, $j \in \mathcal{S}_0$ *such that the LMI* (3.22)-(3.23)-(3.24) *is feasible. Let* $(\sigma_{1j}, \ldots, \sigma_{nj})$, $(\gamma_{1i}, \ldots, \gamma_{ni})$, K_j, U_{ji} *be a solution. Then* (P_j, K_j, U_{ji}) *with* $P_j = V_j \,\mathrm{diag}(\sigma_{1j}, \ldots, \sigma_{nj}) V_j^\top > 0$ *is a solution of the matrix inequality* (3.20).

Proof: Since $\{(\sigma_{1j}, \ldots, \sigma_{nj}), (\gamma_{1i}, \ldots, \gamma_{ni}), K_j, U_{ji} \mid (j, i) \in \mathcal{S}_{t0}\}$ satisfy the LMI (3.22)-(3.23)-(3.24) we can apply the Schur complement to (3.22), which yields

$$V_j \Sigma_j V_j^\top - (A_j + B_j K_j)^\top V_i \Gamma_i^{-1} V_i^\top (A_j + B_j K_j) - Q - K_j^\top R K_j - E_{ji}^\top U_{ji} E_{ji} > 0.$$

By adding and subtracting $(A_j + B_j K_j)^\top V_i \Sigma_i V_i^\top (A_j + B_j K_j)$ in the above inequality we obtain the equivalent

$$\begin{aligned} V_j \Sigma_j V_j^\top &- (A_j + B_j K_j)^\top V_i \Sigma_i V_i^\top (A_j + B_j K_j) \\ &- Q - K_j^\top R K_j - E_{ji}^\top U_{ji} E_{ji} \\ &> (A_j + B_j K_j)^\top V_i \Gamma_i^{-1} V_i^\top (A_j + B_j K_j) \\ &- (A_j + B_j K_j)^\top V_i \Sigma_i V_i^\top (A_j + B_j K_j). \end{aligned} \quad (3.25)$$

From (3.23b) we have that $1 - \sigma_{lj}\gamma_{lj} \geq 0$ for all $l = 1, \ldots, n$ and all $j \in \mathcal{S}_0$. Then, the inequality

$$\Gamma_i^{-1} - \Sigma_i = \begin{pmatrix} \frac{1-\gamma_{1i}\sigma_{1i}}{\gamma_{1i}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1-\gamma_{ni}\sigma_{ni}}{\gamma_{ni}} \end{pmatrix} \geq 0$$

holds for all $i \in \mathcal{S}_0$ and from (3.25) it follows that the inequality

$$V_j\Sigma_jV_j^\top - (A_j + B_jK_j)^\top V_i\Sigma_iV_i^\top(A_j + B_jK_j) - Q - K_j^\top RK_j - E_{ji}^\top U_{ji}E_{ji} > 0$$

is satisfied for all $(j, i) \in \mathcal{S}_{t0}$. The matrix inequality (3.20) is obtained by letting $P_j = V_j\Sigma_jV_j^\top > 0$ for all $j \in \mathcal{S}_0$ in the above inequality. ■

Note that solving the LMI (3.22)-(3.23)-(3.24) hinges on the fact that the orthonormal matrices V_j and the scaling factors $\epsilon_{lj} \in (0, 1]$, $l = 1, \ldots, n$, $j \in \mathcal{S}_0$ must be chosen a priori. This is not a problem with respect to the scaling factors, which can be chosen arbitrarily small. However, when it comes to fixing the matrices V_j, it is interesting to find out how they should be chosen such that by varying $\sigma_{1j}, \ldots, \sigma_{nj}$ a sufficiently wide range of P_j matrices is covered. An answer to this question can be obtained for the two dimensional case, where all orthonormal matrices can be parameterized as follows:

$$V_j \triangleq \begin{pmatrix} -\sin\theta_j & \cos\theta_j \\ \cos\theta_j & \sin\theta_j \end{pmatrix}, \tag{3.26}$$

where $0 \leq \theta_j \leq \pi$. In this way, multiple solutions of the LMI (3.22)-(3.23)-(3.24) can be obtained by varying θ_j, as will be illustrated in Section 3.7. A similar explicit form of V_j can be specified also in the three dimensional case, by using two angles, i.e., θ_{1j} and θ_{2j}. However, these expressions get more complicated in higher dimensional spaces.

3.4.2 MPC costs based on $1, \infty$-norms

Consider the case when $1, \infty$-norms are used to define the cost function, i.e.

$$F(x) = \|P_jx\| \quad \text{when} \quad x \in \mathbb{X}_T \cap \Omega_j$$

and

$$L(x, u) = \|Qx\| + \|Ru\|,$$

where $\|\cdot\|$ denotes the 1-norm or the ∞-norm, for brevity of notation. Here $P_j \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{r\times n}$ are assumed to be matrices that have full-column rank.

In this setting, contrary to a quadratic MPC cost, we no longer require that $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$. Also, we consider the PWA system (3.11):

$$x_{k+1} = A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \mathbb{X}_T \cap \Omega_j, \quad j \in \mathcal{S}, \tag{3.27}$$

instead of the PWL sub-system (3.14).

Since Q has full-column rank there always exists a positive number γ such that $\|Qx\| \geq \gamma\|x\|$ for all $x \in \mathbb{R}^n$. Then it follows that $L(x,u) \geq \|Qx\| \geq \gamma\|x\|$, for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$, and that $F(x) \leq \max_{j\in\mathcal{S}} \|P_j\|\|x\|$ for all $x \in \mathbb{R}^n$. Hence, a part of Assumption 3.3.1 is trivially satisfied with $\alpha_1(\|x\|) \triangleq \gamma\|x\|$ and $\alpha_2(\|x\|) \triangleq \max_{j\in\mathcal{S}} \|P_j\|\|x\|$.

Next, we provide methods for calculating the matrices $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ such that inequality (3.4) is satisfied for the PWA system (3.11).

Let $\mathcal{Q}_{ji} \triangleq \{x \in \Omega_j \mid \exists u \in \mathbb{U} : A_j x + B_j u + f_j \in \Omega_i\}$, $(j,i) \in \mathcal{S} \times \mathcal{S}$ and let $\mathcal{S}_t \triangleq \{(j,i) \in \mathcal{S} \times \mathcal{S} \mid \mathcal{Q}_{ji} \neq \emptyset\}$. Note that the set $\mathcal{S}_t$ defined here differs from the set $\mathcal{S}_{t0}$, since it also incorporates the indices $j \in \mathcal{S}_1$, i.e. $\mathcal{S}_{t0} = \mathcal{S}_t \cap \{\mathcal{S}_0 \times \mathcal{S}_0\}$. The set of pairs of indices $\mathcal{S}_t$ can be easily determined off-line by solving s^2 linear programs. The set $\mathcal{S}_t$ contains all discrete mode transitions that can occur in the PWA system (3.27), i.e. if $(j,i) \in \mathcal{S}_t$ then a transition from Ω_j to Ω_i can occur.

Substituting (3.27) and $F(\cdot)$ in (3.4) yields that, for (3.4) to be satisfied, it is sufficient to find $\{(P_j, K_j) \mid j \in \mathcal{S}\}$ that satisfy:

$$\|P_i((A_j + B_j K_j)x + f_j)\| - \|P_j x\| + \|Qx\| + \|RK_j x\| \leq 0, \quad \forall x \in \mathbb{X}_T, \tag{3.28}$$

for all $(j,i) \in \mathcal{S}_t$. Now consider the following norm inequalities:

$$\|P_i(A_j + B_j K_j)P_j^{-L}\| + \|QP_j^{-L}\| + \|RK_j P_j^{-L}\| \leq 1 - \gamma_{ji}, \quad (j,i) \in \mathcal{S}_t \tag{3.29}$$

and

$$\|P_i f_j\| \leq \gamma_{ji}\|P_j x\|, \quad \forall x \in \mathbb{X}_T \cap \Omega_j, \quad (j,i) \in \mathcal{S}_t, \tag{3.30}$$

where $\gamma_{ji} \in [0,1)$, $(j,i) \in \mathcal{S}_t$. Note that, because of (3.13), (3.30) trivially holds if $\mathcal{S} = \mathcal{S}_0$.

Theorem 3.4.3 *Suppose* (3.29)-(3.30) *is solvable in* (P_j, K_j, γ_{ji}) *where* P_j *has full-column rank and* $\gamma_{ji} \in [0,1)$ *for* $(j,i) \in \mathcal{S}_t$. *Then* (P_j, K_j) *with* $j \in \mathcal{S}$ *is a solution of the norm inequality* (3.28).

Proof: Since $\{(P_j, K_j, \gamma_{ji}) \mid (j,i) \in \mathcal{S}_t\}$ satisfy (3.29) it follows that

$$\|P_i(A_j + B_j K_j)P_j^{-L}\| + \|QP_j^{-L}\| + \|RK_j P_j^{-L}\| + \gamma_{ji} - 1 \leq 0, \quad (j,i) \in \mathcal{S}_t. \tag{3.31}$$

Right multiplying the inequality (3.31) with $\|P_j x\|$ and using the inequality (3.30) yields:

$$\begin{aligned}0 &\geq \|P_i(A_j + B_jK_j)P_j^{-L}\|\|P_jx\| + \|QP_j^{-L}\|\|P_jx\| \\ &+ \gamma_{ji}\|P_jx\| + \|RK_jP_j^{-L}\|\|P_jx\| - \|P_jx\| \geq \\ &\geq \|P_i(A_j + B_jK_j)P_j^{-L}P_jx\| + \|QP_j^{-L}P_jx\| \\ &+ \|P_if_j\| + \|RK_jP_j^{-L}P_jx\| - \|P_jx\| \geq \\ &\geq \|P_i(A_j + B_jK_j)x + P_if_j\| + \|RK_jx\| + \|Qx\| - \|P_jx\|. \end{aligned} \tag{3.32}$$

Hence, inequality (3.28) holds. ■

A way to solve the norm inequalities (3.29) is to minimize the cost

$$\begin{aligned} &J_1\left(\{P_j, K_j \mid j \in \mathcal{S}\}\right) \triangleq \\ &\max_{(j,i)\in\mathcal{S}_t} \left(\|P_i(A_j + B_jK_j)P_j^{-L}\| + \|QP_j^{-L}\| + \|RK_jP_j^{-L}\|\right), \end{aligned}$$

if the resulting value function is less than 1. This is a non-convex nonlinear optimization problem, which can be solved using black-box optimization solvers, such as *fmincon* and *fminunc* of Matlab. In order to ensure that the matrices P_j, $j \in \mathcal{S}$ have full-column rank one can either impose the constraints $\det(P_j^\top P_j) > 0$ using *fmincon* or, the *fminunc* solver can be used in combination with appropriately shifting the initial condition when the optimization algorithm converges to parameters for which $P_j^\top P_j$ is not invertible. Alternatively, one can solve an optimization problem with a zero cost subject to the nonlinear constraint $J_1\left(\{P_j, K_j \mid j \in \mathcal{S}\}\right) < 1$. The nonlinear nature of these optimization problems is not critical for on-line implementation, since they are solved off-line.

Once the matrices P_j and the numbers γ_{ji} satisfying (3.29) have been found, one still has to check that they also satisfy inequality (3.30), provided that $\mathcal{S} \neq \mathcal{S}_0$. For example, this can be verified by checking the inequality

$$\|P_if_j\| \leq \gamma_{ji} \min_{x\in\mathbb{X}_T\cap\Omega_j} \|P_jx\|, \quad (j,i) \in \mathcal{S}_t(\mathbb{X}_T),$$

where $\mathcal{S}_t(\mathbb{X}_T) \triangleq \{(j,i) \mid \mathbb{X}_T \cap \Omega_j \neq \emptyset\} \cap \mathcal{S}_1$. In order to overcome the difficulty of solving (3.29)-(3.30) simultaneously, one can require that $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ is a positively invariant set only for the PWL sub-system (3.14), as done for quadratic MPC costs.

Note that the auxiliary control action (3.12) defines now a local state feedback, instead of a global state feedback, as in Theorem 3.4.3. In this case Theorem 3.4.3 can be reformulated as follows.

Corollary 3.4.4 *Suppose that the inequality*

$$\|P_i(A_j + B_jK_j)P_j^{-L}\| + \|QP_j^{-L}\| + \|RK_jP_j^{-L}\| \leq 1 \qquad (3.33)$$

is solvable in (P_j, K_j) *for* P_j *with full-column rank for* $(j,i) \in \mathcal{S}_{t0}$ *and that* $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$. *Then* (P_j, K_j) *with* $j \in \mathcal{S}_0$ *is a solution of the norm inequality* (3.28).

Proof: Since $\mathbb{X}_T \subseteq \cup_{j\in\mathcal{S}_0}\Omega_j$ it follows that the inequality (3.28) only needs to be satisfied for $(j,i) \in \mathcal{S}_{t0}$. From (3.13) we have that $f_j = 0$ for all $j \in \mathcal{S}_0$ and thus, inequality (3.30) is directly satisfied with equality for $\gamma_{ji} = 0$ and for all $(j,i) \in \mathcal{S}_{t0}$. Then the result follows from Theorem 3.4.3. ■

3.5 Computation of the terminal set: Low complexity invariant sets for PWA systems

In this section we address the problem of computing a terminal constraint set that satisfies Assumption 3.3.1 for the class of PWA systems. More specifically, we present three methods for computing low complexity *piecewise polyhedral* positively invariant sets for PWA systems.

3.5.1 Polyhedral invariant sets

Consider the closed-loop system (3.14) with the feedback gains calculated using one of the methods from Section 3.4, i.e.

$$x_{k+1} = (A_j + B_jK_j)x_k \triangleq A_j^{cl}x_k \quad \text{when} \quad x_k \in \Omega_j, \quad j \in \mathcal{S}_0. \qquad (3.34)$$

The first method deals with the computation of a *polyhedral* positively invariant set for the PWL system (3.34). To do so, we consider the autonomous switched linear system corresponding to (3.34), i.e.

$$x_{k+1} = A_j^{cl}x_k, \quad j \in \mathcal{S}_0. \qquad (3.35)$$

Note that we removed the switching rule from (3.34), turning the PWL system (3.34) into a switched linear system (3.35) with arbitrary switching.

Definition 3.5.1 Let $0 \leq \lambda \leq 1$ be given. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a λ-*contractive set* for system (3.35) with arbitrary switching if for all $x \in \mathcal{P}$ *and all* $j \in \mathcal{S}_0$ it holds that $A_j^{cl}x \in \lambda\mathcal{P}$. For $\lambda = 1$, $\mathcal{P}$ is called a *positively invariant set* for system (3.35) with arbitrary switching.

We make use of the following result.

Lemma 3.5.2 *A set which is positively invariant (λ-contractive) for the switched linear system* (3.35) *under arbitrary switching is also a positively invariant (λ-contractive) set for the PWL system* (3.34).

Proof: This follows directly from the fact that, for the PWL system (3.34), the update of the state is equal to $x_{k+1} = A_j^{cl} x_k$ for at least one $j \in \mathcal{S}_0$ at any discrete-time instant $k \in \mathbb{Z}_+$. ■

Since we require that $\mathbb{X}_T \subseteq \mathbb{X}_\mathbb{U} \cap \{\cup_{j \in \mathcal{S}_0} \Omega_j\}$ and $\mathbb{X}_\mathbb{U}$ is not convex in general, we consider in the following a new safe set, $\widetilde{\mathbb{X}}_\mathbb{U}$, taken as a reasonably large compact polyhedral set (that contains the origin in its interior) inside $\mathbb{X}_\mathbb{U} \cap \{\cup_{j \in \mathcal{S}_0} \Omega_j\}$. For instance, if $\mathbb{X}_\mathbb{U} \subseteq \cup_{j \in \mathcal{S}_0} \Omega_j$ is a polyhedron, we set $\widetilde{\mathbb{X}}_\mathbb{U} = \mathbb{X}_\mathbb{U}$ or, if $\cup_{j \in \mathcal{S}_0} \Omega_j$ is a polyhedron we could set $\widetilde{\mathbb{X}}_\mathbb{U} = \{x \in \cup_{j \in \mathcal{S}_0} \Omega_j \mid K_j x \in \mathbb{U}, \forall j \in \mathcal{S}_0\}$. For an arbitrary target set $\mathbb{X}$ we define $\mathcal{Q}_j^1(\mathbb{X}) \triangleq \{x \in \mathbb{R}^n \mid A_j^{cl} x \in \mathbb{X}\}$. Note that if $\mathbb{X}$ is a polyhedron that contains the origin, then $\mathcal{Q}_j^1(\mathbb{X})$ has the same properties and, if $\mathbb{X}$ is compact, then $\mathcal{Q}_j^1(\mathbb{X})$ is closed (see (Blanchini, 1994) for proofs).

Consider now the following sequence of sets:

$$\mathbb{X}_0 = \widetilde{\mathbb{X}}_\mathbb{U},\ \mathbb{X}_i = \bigcap_{j \in \mathcal{S}_0} \mathbb{X}_i^j,\ i = 1, 2, \ldots, \tag{3.36}$$

where $\mathbb{X}_i^j \triangleq \mathcal{Q}_j^1(\mathbb{X}_{i-1}) \bigcap \mathbb{X}_{i-1},\ i = 1, 2, \ldots$.

Theorem 3.5.3 *The following properties hold with respect to the sequence of sets* (3.36)*:*

(i) *The maximal positively invariant set contained in the safe set* $\widetilde{\mathbb{X}}_\mathbb{U}$ *for system* (3.35) *with arbitrary switching is a convex set that contains the origin and is given by*

$$\mathcal{P} = \bigcap_{i=0}^{\infty} \mathbb{X}_i = \lim_{i \to \infty} \mathbb{X}_i; \tag{3.37}$$

(ii) *If an algorithm based on the recurrent sequence of sets* (3.36) *terminates in a finite number of iterations then the set* $\mathcal{P}$ *defined as in* (3.37) *is a polyhedral set;*

(iii) *If there exists a λ-contractive set with* $0 < \lambda < 1$ *for system* (3.35) *under arbitrary switching that contains the origin in its interior, then an algorithm based on the recurrent sequence of sets* (3.36) *terminates in a finite number of iterations and* $\mathcal{P}$ *contains the origin in its interior;*

(iv) *The set* $\mathcal{P}$ *defined in* (3.37) *is a positively invariant set for the PWL system* (3.34).

Proof: (i) If $x \in \mathcal{P}$ then $x \in \mathbb{X}_i$ for all i. Hence, we have that $A_j^{cl} x \in \mathbb{X}_{i-1}$ for all $j \in \mathcal{S}_0$ and all $i \geq 1$. Then $A_j^{cl} x \in \mathcal{P}$ for all $j \in \mathcal{S}_0$. So, $\mathcal{P}$ is a positively invariant set for system (3.35) with arbitrary switching.

In order to prove that the set $\mathcal{P}$ is maximal let $\widetilde{\mathcal{P}} \subseteq \widetilde{\mathbb{X}}_{\mathbb{U}} = \mathbb{X}_0$ be a positively invariant set for system (3.35) with arbitrary switching. To use induction, we assume that $\widetilde{\mathcal{P}} \subseteq \mathbb{X}_i$ for some i (note that this holds for $i = 0$). Due to the positive invariance of $\widetilde{\mathcal{P}}$, for any $x \in \widetilde{\mathcal{P}}$ we have that $A_j^{cl} x \in \widetilde{\mathcal{P}} \subseteq \mathbb{X}_i$ for all $j \in \mathcal{S}_0$. Hence, $x \in \mathbb{X}_{i+1}$. Thus, $\widetilde{\mathcal{P}} \subset \mathbb{X}_{i+1}$ and by induction $\widetilde{\mathcal{P}} \subset \mathbb{X}_i$ for all i, which yields $\widetilde{\mathcal{P}} \subset \bigcap_{i=0}^{\infty} \mathbb{X}_i = \mathcal{P}$.

Now we prove that $\mathcal{P}$ is a convex set. Assume that $\mathcal{P}$ is the maximal positively invariant set for system (3.35) with arbitrary switching. Then we have that $\mathcal{P}$ is a positively invariant set for any linear subsystem in (3.35) and thus, it follows from (Gutman and Cwikel, 1987) that the convex hull of $\mathcal{P}$ is also a positively invariant set for any linear system in (3.35). Hence, the convex hull of $\mathcal{P}$ is a positively invariant set for system (3.35) under arbitrary switching. Since $\widetilde{\mathbb{X}}_{\mathbb{U}}$ is a convex set, it follows that the convex hull of $\mathcal{P}$ is included in $\widetilde{\mathbb{X}}_{\mathbb{U}}$. By maximality, the convex hull of $\mathcal{P}$ is also included in $\mathcal{P}$ and thus, $\mathcal{P}$ is convex.

As the origin is an equilibrium for $x_{k+1} = A_j^{cl} x, \forall j \in \mathcal{S}_0$, $\mathcal{P}$ contains the origin.

(ii) Assume that the algorithm (3.36) terminates in i^* steps. Then, it follows directly from $\mathbb{X}_i \subseteq \mathbb{X}_{i-1}$ for all $i > 0$ that $\mathbb{X}_i = \mathbb{X}_{i^*}$ for all $i \geq i^*$ and $\mathcal{P} = \mathbb{X}_{i^*}$. Since $\widetilde{\mathbb{X}}_{\mathbb{U}}$ is a polyhedral set and from the fact that the intersection of polyhedra produces polyhedra, it follows that the sets $\mathbb{X}_0^j \triangleq \mathcal{Q}_j^1(\widetilde{\mathbb{X}}_{\mathbb{U}}) \bigcap \widetilde{\mathbb{X}}_{\mathbb{U}}$ are polyhedra for all $j \in \mathcal{S}_0$. Then it follows that the set $\mathbb{X}_1$ is a polyhedral set and, for the same reason, $\mathbb{X}_i$, $i = 2, 3, \ldots$, are polyhedral sets. Then, it follows that $\mathcal{P}$ is also a polyhedral set.

(iii) The proof is essentially due to (Kolmanovsky and Gilbert, 1998). Let $\mathcal{E}$ denote a λ-contractive set with $0 < \lambda < 1$ for system (3.35) under arbitrary switching that contains the origin in its interior. Then there exist $c_2 > c_1 > 0$ such that $c_1\mathcal{E} \subsetneq \widetilde{\mathbb{X}}_{\mathbb{U}} \subsetneq c_2\mathcal{E}$. Since $c_2\mathcal{E}$ is λ-contractive, we have that any state trajectory starting on the boundary or in the interior of $c_2\mathcal{E}$ reaches in i discrete-time steps the set $\lambda^i c_2\mathcal{E}$. Hence, there exists an i^* such that all the state trajectories starting inside $\widetilde{\mathbb{X}}_{\mathbb{U}} \subsetneq c_2\mathcal{E}$ lie in $c_1\mathcal{E}$ within i^* discrete-time steps. Since $c_1\mathcal{E}$ is λ-contractive and thus, positively invariant, it follows that if a state trajectory stays i^* discrete-time steps inside $\widetilde{\mathbb{X}}_{\mathbb{U}}$, then it stays in forever. Hence, $\mathbb{X}_{i^*} \subseteq \mathcal{P}$ and thus, $\mathbb{X}_{i^*} = \mathcal{P}$. As $c_1\mathcal{E} \subseteq \mathcal{P}$, $\mathcal{P}$ contains the origin in its interior.

(iv) This follows directly from (i). ■

Remark 3.5.4 If an algorithm based on (3.36) is used to calculate a positively invariant for system (3.34), then a number of s_0 one-step controllable sets $\mathcal{Q}_j^1(\mathbb{X}_{i-1})$ must be computed at each iteration. Alternatively, if the maximal positively invariant set is computed with the algorithm of (Rakovic et al., 2004), then s_0^i one-step controllable sets must be calculated at the i-th iteration, which yields a combinatorial explosion.

Therefore, the computation of the convex positively invariant set $\mathcal{P}$ defined in (3.37) is computationally more friendly than the computation of the maximal positively invariant set.

3.5.2 Squaring the circle

In this subsection we provide a second method for computing positively invariant sets for PWL systems. In fact, we present a new approach to the computation of piecewise polyhedral positively invariant sets for general perturbed nonlinear systems. As this method also applies to piecewise linear systems defined on conical regions in the state-space systems, it can be employed to obtain a terminal constraint set that satisfies Assumption 3.3.1.

Given a β-contractive ellipsoidal set $\mathcal{E}$, e.g. obtained as the sublevel set of a quadratic Lyapunov function, the key idea of this method is to construct a polyhedral set that lies between the ellipsoidal sets $\beta\mathcal{E}$ and $\mathcal{E}$. We prove that the resulting polyhedral set is positively invariant (and contractive if an additional requirement is satisfied).

The problem of fitting a polyhedral set between two ellipsoidal sets (with one ellipsoidal set contained in the interior of the other ellipsoidal set) is solved by treating the ellipsoidal sets as sublevel sets of quadratic functions and constructing a PieceWise Affine (PWA) function that approximates the "outer" quadratic function well enough, i.e. so that its graph lies between the graphs of the two quadratic functions. A solution to the original problem is then obtained by retrieving a suitable sublevel set of the resulting PWA function.

One of the advantages of the proposed algorithm is that it requires the solution of a finite number of QP problems and its computational complexity is bounded. This bound guarantees that the algorithm has finite termination. Also, due to its unique geometrical approach, which is independent of the system dynamics, the method is applicable to a wide class of systems, including linear systems affected by disturbances or subject to input saturation, switched linear systems under arbitrary switching and piecewise linear systems defined on conical regions in the state-space, as already mentioned.

Preliminaries

Given $(n+1)$ *affinely independent*[2] points $(\theta_0, \ldots, \theta_n)$ of $\mathbb{R}^n$, we define a simplex S as

$$S \triangleq \text{Co}(\theta_0, \ldots, \theta_n) \triangleq \{x \in \mathbb{R}^n \mid x = \sum_{l=0}^{n} \mu_l \theta_l, \sum_{l=0}^{n} \mu_l = 1, \\ \mu_l \geq 0 \text{ for } l = 0, 1, \ldots, n\}.$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic function* if $f(x) \triangleq x^\top P x + Cx + \alpha$ for some $P \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{1 \times n}$ and $\alpha \in \mathbb{R}$. A quadratic function f is *strictly convex* if and only if $P > 0$. An ellipsoid (or an ellipsoidal set) $\mathcal{E}$ is a sublevel set (corresponding to some constant level $f_0 \in \mathbb{R}_+$) of a strictly convex quadratic function, i.e. $\mathcal{E} \triangleq \{x \in \mathbb{R}^n \mid f(x) \leq f_0\}$. Let $\Omega_1, \ldots, \Omega_N$ denote a polyhedral partition of $\mathbb{R}^n$, i.e. Ω_i is a polyhedron (not necessarily closed) for all $i = 1, \ldots, N$, $\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \emptyset$ for $i \neq j$ and $\cup_{i=1,\ldots,N} \Omega_i = \mathbb{R}^n$.

Definition 3.5.5 A function $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^\top P_i x + C_i x + \alpha_i$ when $x \in \Omega_i$, $i = 1, \ldots, N$ is called a *PieceWise Quadratic (PWQ) function*. A function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ with $\bar{f}(x) = H_i x + a_i$ when $x \in \Omega_i$, for some $H_i \in \mathbb{R}^{1 \times n}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, N$ is called a *PieceWise Affine (PWA) function*.

A piecewise ellipsoidal set is a sublevel set of a piecewise quadratic function with matrices $P_i > 0$ for all $i = 1, \ldots, N$.

General problem statement and proposed solution

Consider the discrete-time perturbed nonlinear system:

$$x_{k+1} = G(x_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{3.38}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state and an unknown *parametric uncertainty* and *disturbance input*, respectively, and $\mathbb{W}$ and $\mathbb{V}$ are known, bounded sets. $G : \mathbb{R}^n \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary, possibly discontinuous, nonlinear function. For simplicity, we assume that the origin is an equilibrium in (3.38) for zero disturbance input, meaning that $G(0, w, 0) = 0$ for all $w \in \mathbb{W}$.

[2] By this we mean that $(1\ \theta_0^\top)^\top, \ldots, (1\ \theta_n^\top)^\top$ are linearly independent in $\mathbb{R}^{n+1}$.

Definition 3.5.6 For a given $0 \leq \lambda \leq 1$, a set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a *(robust) λ-contractive set* for system (3.38) if for all $x \in \mathcal{P}$ it holds that $G(x, w, v) \in \lambda\mathcal{P}$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. For $\lambda = 1$ a (robust) λ-contractive set is called a *(robust) positively invariant set.*

For a set $\mathcal{P} \subseteq \mathbb{R}^n$, let $\mathcal{Q}_1(\mathcal{P}) \triangleq \{x \in \mathbb{R}^n \mid G(x, w, v) \in \mathcal{P}, \forall w \in \mathbb{W}, \forall v \in \mathbb{V}\}$ denote the (robust) one-step controllable set for system (3.38), with respect to $\mathcal{P}$. We are interested in the problem of computing polyhedral Positively Invariant (PI) sets and polyhedral contractive sets for system (3.38), which includes PWA systems as a particular case.

Problem 3.5.7 Suppose that a (piecewise) ellipsoidal β-contractive set with $\beta \in [0,\ 1)$ is known for system (3.38). (i) Construct a (piecewise) polyhedral PI set for system (3.38); (ii) Construct a (piecewise) polyhedral λ-contractive set with $\lambda \in [0, 1)$ for system (3.38).

Note that systematic solutions to obtain β-contractive (piecewise) *ellipsoidal* sets are available in the literature for many relevant subclasses of (3.38), such as linear systems subject to input saturation (Hu et al., 2002), linear systems affected by parametric uncertainties (Kothare et al., 1996) and/or additive disturbances (Kolmanovsky and Gilbert, 1998), piecewise affine systems (Ferrari-Trecate et al., 2002). Typically, they are obtained as sublevel sets of quadratic (PWQ) Lyapunov functions, which can be calculated efficiently via semi-definite programming.

Most of the existent methods for solving Problem 3.5.7 are based on recursive algorithms that compute one-step controllable or one-step reachable sets (Blanchini, 1994) and they are applicable to *perturbed linear systems*. For example, see the *forward procedure* presented in (Blanchini, 1994) (extensions of this method to piecewise affine systems were proposed in (Rakovic et al., 2004; Lazar et al., 2004a)), the *backward procedure* introduced in (Blanchini et al., 1995) or the reachability based algorithm given in (Kolmanovsky and Gilbert, 1998). Although these algorithms do not require that an ellipsoidal contractive set is known, existence of a quadratic Lyapunov function (and thus, existence of an ellipsoidal contractive set) can be used to prove finite termination for the forward procedure, e.g. see (Rakovic et al., 2004). In the sequel we generalize results from (Blanchini, 1995) to obtain a novel solution to Problem 3.5.7. In (Blanchini, 1995) (see Lemma 4.1 and Lemma 4.2), where perturbed linear systems are considered, it was shown that a polyhedral set contained in between two *convex* sublevel sets of a Lyapunov function is positively invariant and λ-contractive. The result of

(Blanchini, 1995) is extended in the theorem presented below to a wide class of systems, which includes, for example, any PWQ stabilizable system.

Theorem 3.5.8 *Consider system* (3.38) *and let* $\mathcal{E} \subseteq \mathbb{R}^n$ *be a* β*-contractive set for system* (3.38)*, for some* $\beta \in (0, 1)$*, that contains the origin in its interior.*

(i) *Suppose there exists a set* $\mathcal{P} \subseteq \mathbb{R}^n$ *that satisfies* $\beta\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$*. Then,* $\mathcal{P}$ *is a PI set for system* (3.38) *and* $0 \in \text{int}(\mathcal{P})$*;*

(ii) *Let* $\beta\mathcal{E} \subset \lambda\mathcal{P} \subset \mathcal{P} \subset \mathcal{E}$ *for some*[3] $\lambda \in (0, 1)$*. Then,* $\mathcal{P}$ *is a* λ*-contractive set for system* (3.38) *and* $0 \in \text{int}(\mathcal{P})$*. Moreover,* $\mathcal{Q}_1(\lambda\mathcal{P})$ *is a* λ*-contractive set for system* (3.38) *and* $\mathcal{E} \subset \mathcal{Q}_1(\lambda\mathcal{P})$*.*

Proof: (i) For any $x \in \mathcal{P} \subset \mathcal{E}$ it follows that $G(x, w, v) \in \beta\mathcal{E} \subset \mathcal{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$ due to the fact that $\mathcal{E}$ is a β-contractive set for system (3.38). Hence, $\mathcal{P}$ is a PI set for system (3.38). Since $\mathcal{E}$ contains the origin in its interior, $\beta\mathcal{E}$ contains the origin in its interior and thus, $0 \in \text{int}(\mathcal{P})$;

(ii) Applying the same reasoning as above we have that for any $x \in \mathcal{P} \subset \mathcal{E}$ it follows that $G(x, w, v) \in \beta\mathcal{E} \subset \lambda\mathcal{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$ due to the fact that $\mathcal{E}$ is a β-contractive set for system (3.38). Hence, $\mathcal{P}$ is a λ-contractive set for system (3.38) and $0 \in \text{int}(\mathcal{P})$;

Moreover, from the fact that for any $x \in \mathcal{E}$ it holds that $G(x, w, v) \in \beta\mathcal{E} \subset \lambda\mathcal{P}$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$, it follows that $\mathcal{E} \subset \mathcal{Q}_1(\lambda\mathcal{P})$. Since $\mathcal{P} \subset \mathcal{E}$, we have that $\mathcal{P} \subset \mathcal{Q}_1(\lambda\mathcal{P})$ and thus, $\lambda\mathcal{P} \subset \lambda\mathcal{Q}_1(\lambda\mathcal{P})$. Then, for any $x \in \mathcal{Q}_1(\lambda\mathcal{P})$ we have that $G(x, w, v) \in \lambda\mathcal{P} \subset \lambda\mathcal{Q}_1(\lambda\mathcal{P})$ for any $w \in \mathbb{W}$ and any $v \in \mathbb{V}$. Hence, $\mathcal{Q}_1(\lambda\mathcal{P})$ is a λ-contractive set for system (3.38) and $\mathcal{E} \subset \mathcal{Q}_1(\lambda\mathcal{P})$. ■

Note that the results of Theorem 3.5.8 also apply to certain types of *non-convex* sets $\mathcal{E}$ and $\mathcal{P}$, i.e. piecewise ellipsoidal and piecewise polyhedral sets, respectively (see Section 3.7 for an illustrative example). Also, a λ-contractive polyhedral set $\mathcal{P}$ can be obtained without the additional hypothesis of Theorem 3.5.8-(ii). Indeed, if $\mathcal{E}$ is β-contractive with $\beta \in (0, 1)$ we can solve the "tighter" inclusion $\sqrt{\beta}\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$. Then, we obtain

$$\beta\mathcal{E} \subset \sqrt{\beta}\mathcal{P} \subset \mathcal{P} \subset \mathcal{E},$$

which is the hypothesis of Theorem 3.5.8-(ii) with $\lambda = \sqrt{\beta}$.

The case of interest in this paper is, as stated in Problem 3.5.7, when $\mathcal{E}$ is a piecewise *ellipsoidal* set and $\mathcal{P}$ is a piecewise *polyhedral* set. By

[3] Note that the result also holds when $\beta = 0$ and $\lambda = 0$ but in this case $\mathcal{P}$ does not necessarily contain the origin in its interior.

Theorem 3.5.8, it is sufficient to construct a piecewise polyhedral set $\mathcal{P}$ that lies between the piecewise ellipsoidal sets $\beta\mathcal{E}$ and $\mathcal{E}$ to obtain a solution to Problem 3.5.7. In the next section we present an algorithm for solving this problem of computational geometry.

Remark 3.5.9 The result $\mathcal{E} \subset \mathcal{Q}_1(\lambda\mathcal{P})$ (of Theorem 3.5.8-(ii)) is relevant when the state of system (3.38) is constrained in a compact polyhedral set $\mathbb{X} \subset \mathbb{R}^n$ with $0 \in \text{int}(\mathbb{X})$. Then, given the largest β-contractive piecewise *ellipsoidal* set contained in $\mathbb{X}$, a larger, piecewise *polyhedral*, λ-contractive set can be simply obtained by computing the set $\mathcal{Q}_1(\lambda\mathcal{P}) \cap \mathbb{X}$.

Construction of the desired polyhedral set

Next, we present a solution to the problem of fitting a piecewise polyhedral set $\mathcal{P}$ between two piecewise ellipsoidal sets where one is contained in the interior of the other, i.e. $\beta\mathcal{E} \subsetneq \mathcal{E}$, with β a number[4] in $(0, 1)$. In case $\mathcal{E}$ is an ellipsoid, the main idea is to treat the sets $\mathcal{E}$ and $\beta\mathcal{E}$ as sublevel sets of two *quadratic* functions $f_{\mathcal{E}}(x)$ and $f_{\beta\mathcal{E}}(x)$, respectively, that correspond to a certain constant (level) $f_0 \in \mathbb{R}_+$, i.e. $\mathcal{E} \triangleq \{x \in \mathbb{R}^n \mid f_{\mathcal{E}}(x) \leq f_0\}$ and $\beta\mathcal{E} \triangleq \{x \in \mathbb{R}^n \mid f_{\beta\mathcal{E}}(x) \leq f_0\}$. Then, we compute a PWA function $\bar{f}$ that satisfies $f_{\beta\mathcal{E}}(x) > \bar{f}(x) \geq f_{\mathcal{E}}(x)$ for all $x \in \mathbb{R}^n$. The desired polyhedral set is obtained as $\mathcal{P} \triangleq \{x \in \mathbb{R}^n \mid \bar{f}(x) \leq f_0\}$.

In the *piecewise quadratic* case we assume that the polyhedral partitioning $\{\Omega_j \mid j \in \mathcal{S}\}$ ($\mathcal{S}$ is a finite set of indexes) consists of cones, which ensures that $\beta\Omega_j \subseteq \Omega_j$. We write $\mathcal{E}$ as:

$$\mathcal{E} = \bigcup_{j\in\mathcal{S}} (\mathcal{E}_j \cap \Omega_j) \text{ with } \mathcal{E}_j \triangleq \{x \in \mathbb{R}^n \mid f_{\mathcal{E}_j}(x) \leq f_0\},$$

where $f_{\mathcal{E}_j} \triangleq x^\top P_j x + C_j x_+ \alpha_j$ is a strictly convex quadratic function for all $j \in \mathcal{S}$. Then, we construct a PWA function $\bar{f}_j(x)$, as in the quadratic case mentioned above, such that $f_{\beta\mathcal{E}_j}(x) > \bar{f}_j(x) \geq f_{\mathcal{E}_j}(x)$ for all $x \in \mathbb{R}^n$ and for all $j \in \mathcal{S}$. Then, a piecewise polyhedral set $\mathcal{P}$ that satisfies $\beta\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$ is simply obtained as

$$\mathcal{P} = \bigcup_{j\in\mathcal{S}} (\mathcal{P}_j \cap \Omega_j) \text{ with } \mathcal{P}_j \triangleq \{x \in \mathbb{R}^n \mid \bar{f}_j(x) \leq f_0\}.$$

[4]The case $\beta = 0$ is trivial: any $\mathcal{P} \subset \mathcal{E}$ with $0 \in \text{int}(\mathcal{P})$ works.

Indeed, as $\mathcal{P}_j$ is a polyhedral set that satisfies $\beta\mathcal{E}_j \subset \mathcal{P}_j \subset \mathcal{E}_j$, $j \in \mathcal{S}$, we obtain

$$\mathcal{P} = \bigcup_{j\in\mathcal{S}} (\mathcal{P}_j \cap \Omega_j) \subset \bigcup_{j\in\mathcal{S}} (\mathcal{E}_j \cap \Omega_j) = \mathcal{E}.$$

Since $\beta\mathcal{E}_j \subset \mathcal{P}_j$ and $\beta\Omega_j \subseteq \Omega_j$ for all $j \in \mathcal{S}$, we have that:

$$\begin{aligned}\beta\mathcal{E} = \beta\left(\bigcup_{j\in\mathcal{S}} (\mathcal{E}_j \cap \Omega_j)\right) &= \bigcup_{j\in\mathcal{S}} \beta(\mathcal{E}_j \cap \Omega_j) = \\ = \bigcup_{j\in\mathcal{S}} (\beta\mathcal{E}_j \cap \beta\Omega_j) &\subseteq \bigcup_{j\in\mathcal{S}} (\mathcal{P}_j \cap \Omega_j) = \mathcal{P}.\end{aligned}$$

As the PWQ case can be split into a finite number of quadratic instances of the problem, in the following we consider only the quadratic case, i.e. when the set $\mathcal{E}$ is a sublevel set of a strictly convex quadratic function $f_{\mathcal{E}}$.

Next, choose $P \in \mathbb{R}^{n\times n}$ (with $P > 0$) and $f_0, \alpha_{\mathcal{E}} \in \mathbb{R}$ (with $f_0 > \alpha_{\mathcal{E}}$) such that $\mathcal{E}$ is the sublevel set of $f_{\mathcal{E}}(x) \triangleq x^\top Px + \alpha_{\mathcal{E}}$, corresponding to the level f_0. Then, we have that $\beta\mathcal{E}$ is the sublevel set of $f_{\beta\mathcal{E}}(x) \triangleq x^\top Px + \alpha_{\beta\mathcal{E}}$, corresponding to the level f_0, where $\alpha_{\beta\mathcal{E}} \triangleq (1-\beta^2)f_0 + \beta^2\alpha_{\mathcal{E}} > \alpha_{\mathcal{E}}$. Consider now an initial polyhedron $\mathcal{P}_0 \subset \mathbb{R}^n$ that contains $\mathcal{E}$. Let $(\tilde{\theta}_0, .., \tilde{\theta}_m)$, with $m \geq n$, be the vertices of $\mathcal{P}_0$. An initial set of simpleces $S_1^0, \ldots, S_{l_0}^0$ that contains these points is determined by Delaunay triangulation (Yepremyan and Falk, 2005). Then, for every simplex $S_i^0 \triangleq \mathrm{Co}(\theta_{0i}^0, \ldots, \theta_{ni}^0)$, $i = 1, \ldots, l_0$, the following operations are performed.

Algorithm 3.5.10 1. Let $k = 0$.

2. For every simplex S_i^k, $i = 1, \ldots, l_k$, construct the matrix
$$M_i^k \triangleq \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \theta_{0i}^k & \theta_{1i}^k & \ldots & \theta_{ni}^k \end{bmatrix}.$$
3. Set $v_i^k \triangleq [\ f_{\mathcal{E}}(\theta_{0i}^k)\ f_{\mathcal{E}}(\theta_{1i}^k)\ \ldots\ f_{\mathcal{E}}(\theta_{ni}^k)\ ]^\top$ and construct the function
$$\bar{f}_i^k(x) \triangleq (v_i^k)^\top (M_i^k)^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}.$$
4. Solve the QP problem:
$$J_i^{k*} \triangleq \min_{x\in S_i^k} \left\{ J_i^k(x) \triangleq f_{\beta\mathcal{E}}(x) - \bar{f}_i^k(x) \right\}, \tag{3.39}$$
and let $x_i^{k*} \triangleq \arg\min_{x\in S_i^k} J_i^k(x)$.

5. If $J_i^{k*} > 0$ for all $i = 1, \ldots, l_k$, then Stop. Otherwise, for all S_i^k, $i = 1, \ldots, l_k$, for which $J_i^{k*} \leq 0$ build new simpleces $\overline{S}_0^i, \overline{S}_1^i, \ldots, \overline{S}_n^i$ defined by the vertices $(x_i^{k*}, \theta_{1i}^k, \ldots, \theta_{ni}^k)$, $(\theta_{0i}^k, x_i^{k*}, \ldots, \theta_{ni}^k)$, $\ldots$ and $(\theta_{0i}^k, \ldots, \theta_{ni}^k, x_i^{k*})$, respectively. Increment k by one, add the new simpleces $\overline{S}_0^i, \overline{S}_1^i, \ldots, \overline{S}_n^i$ to the set of simpleces $\{S_i^k\}_{i=1,\ldots,l_k}$ and repeat the algorithm recursively from Step 2.

Algorithm 3.5.10 computes a simplicial partition of a given initial polyhedral set $\mathcal{P}_0$ that contains the ellipsoidal set $\mathcal{E}$, by splitting a single simplex S_i^k into $n+1$ simpleces. This is done by fixing a new vertex x_i^{k*} which is obtained by solving the QP problem (3.39), and by calculating a new PWA approximation over the new set of simpleces. The steps of Algorithm 3.5.10 are repeated for all resulting simpleces, until $J_i^{k*} > 0$ for all simpleces. At every iteration k, a tighter PWA approximation of the quadratic function $f_{\mathcal{E}}$ is obtained. Algorithm 3.5.10 proceeds in a typical branch & bound way, i.e. *branching* on a new vertex x_i^{k*}, and *bounding* whenever it finds a simplex S_i^k for which it holds that $J_i^{k*} > 0$.

Suppose Algorithm 3.5.10 stops. At the $\bar{k}$-th iteration[5] for some $\bar{k} \in \mathbb{Z}_+$, the following PWA function is generated:

$$\begin{aligned} \bar{f}(x) &\triangleq \bar{f}_i^{\bar{k}}(x) \text{ when } x \in S_i^{\bar{k}}, i = 1, \ldots, l_{\bar{k}} \\ &\triangleq H_i^{\bar{k}} x + a_i^{\bar{k}} \text{ when } x \in S_i^{\bar{k}}, i = 1, \ldots, l_{\bar{k}}, \end{aligned} \tag{3.40}$$

where $l_{\bar{k}}$ is the number of simpleces obtained at the end of Algorithm 3.5.10 and $H_i^{\bar{k}} x + a_i^{\bar{k}} = (v_i^{\bar{k}})^\top (M_i^{\bar{k}})^{-1} \begin{bmatrix} 1 \\ x \end{bmatrix}$. The PWA function $\bar{f}$ constructed via Algorithm 3.5.10 is a continuous function. Moreover, for $x = \sum_{j=0}^n \mu_j \theta_{ji}$ with $\sum_{j=0}^n \mu_j = 1$, the corresponding functions $\bar{f}_i^{\bar{k}}$ satisfy:

$$\bar{f}_i^{\bar{k}}(x) = \bar{f}_i^{\bar{k}} \left(\sum_{j=0}^n \mu_j \theta_{ji} \right) = \sum_{j=0}^n \mu_j f_{\mathcal{E}}(\theta_{ji}),$$

which, by strict convexity of $f_{\mathcal{E}}$, implies that $\bar{f}_i^{\bar{k}}(x) \geq f_{\mathcal{E}}(x)$ for all $x \in S_i^{\bar{k}}$ and all $i = 1, \ldots, l_{\bar{k}}$. Hence, $\bar{f}(x) \geq f_{\mathcal{E}}(x)$ for all $x \in \mathcal{P}_0$. Since the stopping criterion defined in Step 4 of Algorithm 3.5.10 assures that at the end of the entire procedure the optimal value $J_i^{\bar{k}*}$ of the QP problem (3.39) will be greater than zero in every simplex $S_i^{\bar{k}}$, $i = 1, \ldots, l_{\bar{k}}$, it follows that

$$f_{\mathcal{E}}(x) \leq \bar{f}(x) < f_{\beta\mathcal{E}}(x), \quad \forall x \in \cup_{i=1,\ldots,l_{\bar{k}}} S_i^{\bar{k}}.$$

[5]The existence of a finite $\bar{k}$ will be proven in the sequel.

Then, the sublevel set of $\bar{f}$ given by

$$\overline{\mathcal{P}} \triangleq \bigcup_{i=1,\ldots,l_{\bar{k}}} \{x \in S_i^{\bar{k}} \mid H_i^{\bar{k}} x + a_i^{\bar{k}} \leq f_0\}$$

satisfies $\beta\mathcal{E} \subset \overline{\mathcal{P}} \subset \mathcal{E}$. Indeed, note that for $x \in \overline{\mathcal{P}}$ it holds that

$$\bar{f}(x) \leq f_0 \Rightarrow f_{\mathcal{E}}(x) \leq \bar{f}(x) \leq f_0 \Rightarrow x \in \mathcal{E},$$

and for $x \in \beta\mathcal{E}$ it holds that

$$f_{\beta\mathcal{E}}(x) \leq f_0 \Rightarrow \bar{f}(x) < f_{\beta\mathcal{E}}(x) \leq f_0 \Rightarrow x \in \overline{\mathcal{P}}.$$

The desired polyhedral set $\mathcal{P}$ (see Figure 3.1) satisfying $\beta\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$, is obtained as the convex hull of the vertices of $\overline{\mathcal{P}}$. Indeed,

$$\beta\mathcal{E} \subset \overline{\mathcal{P}} \subset \mathcal{P} \Rightarrow \beta\mathcal{E} \subset \mathcal{P}$$

and, by the convexity of $\mathcal{E}$, it holds that

$$\mathcal{P} \triangleq \text{Co}(\overline{\mathcal{P}}) \subseteq \text{Co}(\mathcal{E}) = \mathcal{E}.$$

Note that the computation of the vertices of $\overline{\mathcal{P}}$ and of their convex hull can be performed efficiently using, for instance, the Geometric Bounding Toolbox (GBT) (Veres, 1995). Also, an ellipsoidal β-contractive set with $\beta \in (0,1)$ as small as possible is desirable, as this will result in a polyhedral positively invariant (λ-contractive) set of lower complexity.

Remark 3.5.11 The approximation error

$$\bar{\varepsilon} \triangleq \max_{x \in \mathcal{P}_0} [\bar{f}(x) - f_{\mathcal{E}}(x)]$$

obtained at the end of Algorithm 3.5.10 is upper bounded by the allowed maximum error defined as $\varepsilon_{\max} \triangleq \min_{x \in \mathcal{P}_0} [f_{\beta\mathcal{E}}(x) - f_{\mathcal{E}}(x)] > 0$. Thus, the Stop criterion of Algorithm 3.5.10 can be set as $J_i^{k*} > \delta$ for some $\delta \in (0, \varepsilon_{\max})$, instead of just $J_i^{k*} > 0$, to create a gap between $\mathcal{P}$ and $\beta\mathcal{E}$. A larger δ will result in a smaller $\lambda \in (0,1)$ for which it holds that $\beta\mathcal{E} \subset \lambda\mathcal{P} \subset \mathcal{P} \subset \mathcal{E}$. The number of vertices of $\mathcal{P}$ tends to infinity, $\mathcal{P}$ recovers the ellipsoidal set $\mathcal{E}$ and λ tends to β when δ tends to $\varepsilon_{\max}$.

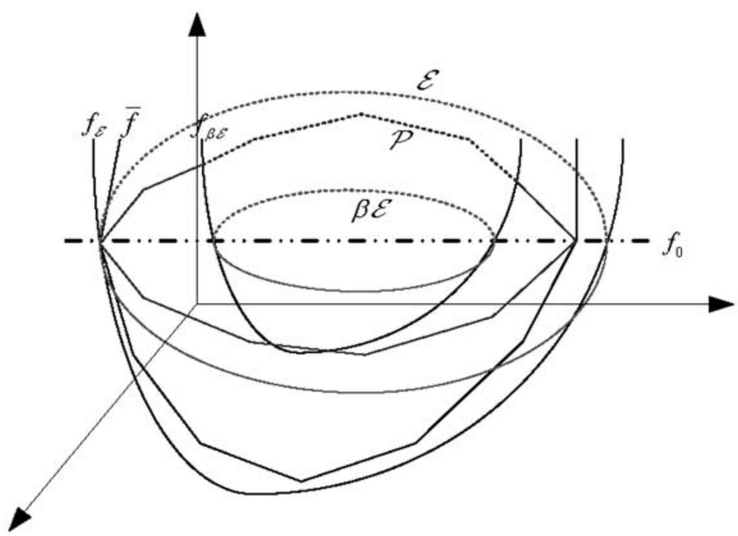

Figure 3.1: Illustration of the proposed solution for constructing the polyhedral invariant set $\mathcal{P}$.

An estimate of the computational complexity

Algorithm 3.5.10 computes at every iteration k a tighter PWA approximation $\bar{f}^k$ of the given strictly convex quadratic function $f_{\mathcal{E}}$. It stops when the approximation error obtained at the k-th iteration of the algorithm satisfies

$$\varepsilon_k \triangleq \max_{x \in \mathcal{P}_0} [\bar{f}^k(x) - f_{\mathcal{E}}(x)] \leq \varepsilon_{\max}, \quad k \in \mathbb{Z}_+.$$

Indeed, if the above inequality holds for some finite $\bar{k} \in \mathbb{Z}_+$, then for all $x \in \mathcal{P}_0$ it holds that $\bar{f}^{\bar{k}}(x) - f_{\mathcal{E}}(x) < \varepsilon_{\max}$, which implies that $\bar{f}^{\bar{k}}(x) < f_{\mathcal{E}}(x) + \varepsilon_{\max} \leq f_{\beta\mathcal{E}}(x)$. Consider now the following assumption.

Assumption 3.5.12 The optimum x_i^{k*} obtained in Step 4 of Algorithm 3.5.10 for every simplex S_i^k, $i = 1, \ldots, l_k$, $k \in \mathbb{Z}_+$, satisfies $x_i^{k*} \in \text{int}(S_i^k)$.

In (Alessio et al., 2005), the authors proved that under Assumption 3.5.12 the error ε_k committed at the k-th iteration of the algorithm is such that

$$\frac{\varepsilon_{k-1}}{4} \leq \varepsilon_k \leq \frac{\varepsilon_{k-1}}{2}, \quad \forall k \in \mathbb{Z}_+ \setminus \{0\}.$$

The algorithm builds recursively a tree where in each node it stores the vertices of the current simplex S_i^k and the pairs (H_i^k, a_i^k) such that $\bar{f}^k(x) =$

$H_i^k x + a_i^k$, for all $x \in S_i^k$, $i \geq 1$, $k \in \mathbb{Z}_+$. If the value of J_i^{k*} for the current simplex is less than zero, then Algorithm 3.5.10 splits S_i^k in $n+1$ simpleces and adds a new level to the tree. The height of the tree can be easily computed once the values of the initial error $\varepsilon_0 \triangleq \max_{x \in \mathcal{P}_0}[\bar{f}^0(x) - f_{\mathcal{E}}(x)]$ and of the allowed maximum error ε_{max} are known, which yields the following upper bound on the complexity of Algorithm 3.5.10.

Theorem 3.5.13 *Suppose that the initial polyhedral set $\mathcal{P}_0$, the initial error ε_0 and the desired final approximation error $\varepsilon_{\max}$ are known. Furthermore, suppose Assumption 3.5.12 holds*[6]*. Then, the following bound holds on the height ξ_T of the tree generated by Algorithm 3.5.10:*

$$\xi_T \leq \left\lceil \log_2 \frac{\varepsilon_0}{\varepsilon_{max}} \right\rceil .$$

Note that the height ξ_T of the tree and the number of nodes give the number of simpleces for which the steps of Algorithm 3.5.10 have to be performed. This in turn yields the number of QP problems that have to be solved, which is of order $l_0(n+1)^{\left\lceil \log_2 \frac{\varepsilon_0}{\varepsilon_{\max}} \right\rceil}$, where l_0 is the initial number of simpleces and n is the dimension of the state-space. Hence, Algorithm 3.5.10 always terminates in finite time.

Examples

We present two examples to illustrate the effectiveness of the new algorithm for computing polyhedral positively invariant sets.

Linear systems subject to input saturation: Consider the following linear system subject to input saturation (Hu et al., 2002):

$$x_{k+1} = Ax_k + B\,\mathrm{sat}(u_k), \quad k \in \mathbb{Z}_+, \tag{3.41}$$

where $A = \begin{bmatrix} 0.8876 & -0.5555 \\ 0.5555 & 1.5542 \end{bmatrix}$, $B = \begin{bmatrix} -0.1124 \\ 0.5555 \end{bmatrix}$ and $\mathrm{sat}(u_k) \triangleq \mathrm{sgn}(u_k)\min(1, |u_k|)$. In (Hu et al., 2002), a quadratically stabilizing state-feedback control law for system (3.41), i.e. $u_k = Fx_k$, with $F = [-0.7651 \ \ -2.0299]$, and a quadratic Lyapunov function $V(x) = x^\top P x$ with $P = \begin{bmatrix} 5.0127 & -0.6475 \\ -0.6475 & 4.2135 \end{bmatrix}$ were calculated. Since this control law does not take into account the input saturation, the maximal feasible domain of attraction for the closed-loop system is given by the ellipsoidal sublevel set $\mathcal{E}_F$ of V, corresponding to the level $f_0 = 0.8237$,

[6] If x_i^{k*} lies on a facet of S_i^k for some $i \geq 1$, $k \in \mathbb{Z}_+$, the same result holds with some minor modifications to the splitting strategy.

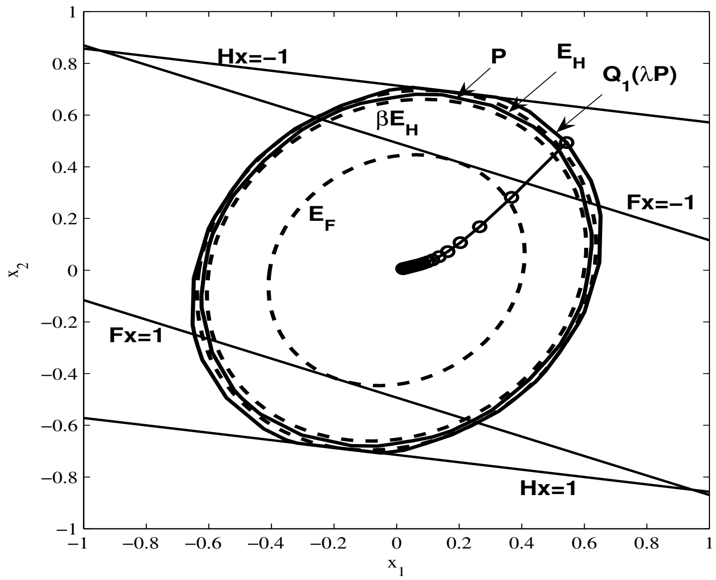

Figure 3.2: Polyhedral (solid line) and ellipsoidal (dashed line) invariant sets.

see Figure 3.2. To obtain a larger ellipsoidal domain of attraction for the feedback F, we employed the LMI technique of (Hu et al., 2002), which yielded the new feedback matrix $H = [-0.2 \ -1.4]$ that takes into account the effect of saturation and establishes the enlarged ellipsoidal domain of attraction $\mathcal{E}_H$ (i.e. the sublevel set of V, corresponding to the level $f_0 = 2$) for system (3.41) in closed-loop with $u_k = Fx_k$.

Next, we employed the method developed in this paper in order to calculate a polyhedral set $\mathcal{P}$ such that $\beta\mathcal{E}_H \subset \mathcal{P} \subset \mathcal{E}_H$, where $\beta = 0.95$ is the contraction factor of $\mathcal{E}_H$. The resulting polyhedron is λ-contractive with $\lambda = 0.98$ and has 65 vertices. The set $\mathcal{P}$ and the enlarged polyhedral domain of attraction $\mathcal{Q}_1(\lambda\mathcal{P})$, which contains the (ellipsoidal) domain of attraction $\mathcal{E}_H$, are plotted in Figure 3.2 together with the closed-loop state trajectory for the initial state $x_0 = [0.5434 \ 0.4938]^\top$.

Linear systems subject to additive disturbances: Consider the following discrete-time triple integrator affected by additive disturbances:

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{Z}_+, \tag{3.42}$$

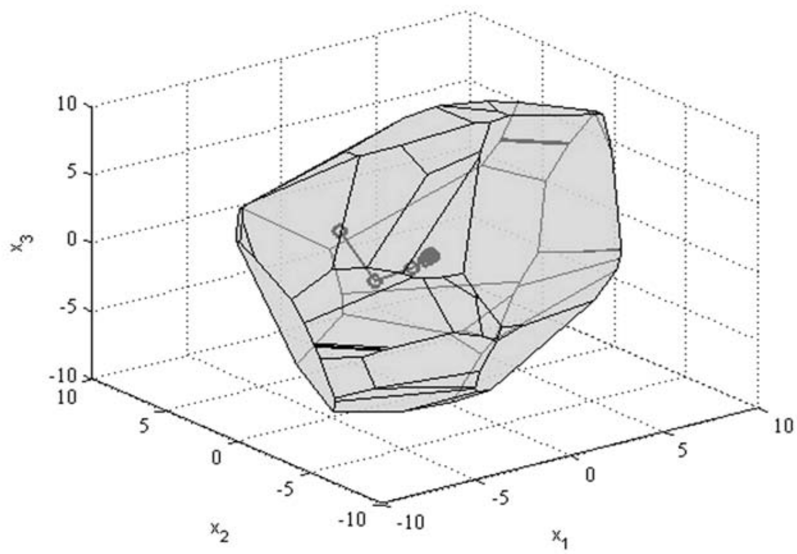

Figure 3.3: Polyhedral invariant set and state trajectory for system (3.42) *in closed-loop with $u = Kx$ and randomly generated disturbances v in $\mathbb{V}$.*

where $A = \begin{bmatrix} 1 & T_s & \frac{T_s^2}{2} \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{T_s^3}{3!} \\ \frac{T_s^2}{2} \\ T_s \end{bmatrix}$, $T_s = 0.8$, $v_k \in \mathbb{V}$ is the additive disturbance input, and $\mathbb{V} = [-0.1,\ 0.1] \times [-0.1,\ 0.1] \times [-0.1,\ 0.1]$. We calculated a robust stabilizing state-feedback control law for system (3.42), i.e. $u_k = Kx_k$, with $K = [-1.1739\ \ -2.4071\ \ -2.0888]$, together with a robust quadratic Lyapunov function $V(x) = x^\top Px$ with $P = \begin{bmatrix} 14.4684 & 13.5850 & 4.0221 \\ 13.5850 & 17.4375 & 5.4581 \\ 4.0221 & 5.4581 & 2.5328 \end{bmatrix}$. The procedure presented in this section was employed to calculate a polyhedral set $\mathcal{P}$ such that $\beta\mathcal{E} \subset \mathcal{P} \subset \mathcal{E}$, where $\mathcal{E}$ is the sublevel set of V, corresponding to the level $f_0 = 20$, and the contraction factor is $\beta = 0.8$. The resulting set $\mathcal{P}$ is λ-contractive with $\lambda = 0.9$ and has 56 vertices. A plot of $\mathcal{P}$ is given in Figure 3.3 together with a plot of the closed-loop system state trajectory obtained for $x_0 = [-3\ 2\ 2]^\top$ and randomly generated additive disturbances.

For an example of a PWA system we refer the reader to Section 3.7, which also illustrates the design of stabilizing terminal cost and constraint set MPC schemes for PWA systems.

3.5.3 Norms as Lyapunov functions

The third method for computing low complexity *piecewise polyhedral* positively invariant sets for PWA systems relies on the result of Theorem 3.4.3. We define

$$\mathcal{P} \triangleq \cup_{j\in\mathcal{S}}\{x \in \Omega_j \mid \|P_j x\| \leq c\}, \tag{3.43}$$

for some $c > 0$, where $\{P_j \mid j \in \mathcal{S}\}$ are the weights of the terminal cost and $\|\cdot\|$ denotes the either the 1-norm or the ∞-norm. For example, if $\mathcal{P}$ is used as the terminal set in Problem 3.2.1, to satisfy Assumption 3.3.1, c must be taken less than or equal to $\sup\{\mu > 0 \mid \{x \in \Omega_j \mid \|P_j x\| \leq \mu\} \subseteq \mathbb{X}_{\mathbb{U}}\}$.

Lemma 3.5.14 *Suppose that the hypothesis of Theorem 3.4.3 is satisfied. Then, the piecewise polyhedral set $\mathcal{P}$ defined in* (3.43) *is a positively invariant set for the PWA system* (3.27) *in closed-loop with $u_k = h(x_k) = K_j x_k$ when $x_k \in \Omega_j$, $j \in \mathcal{S}$, $k \in \mathbb{Z}_+$.*

Proof: By the hypothesis of Theorem 3.4.3, inequality (3.28) holds, i.e.

$$\|P_i((A_j+B_jK_j)x+f_j)\|-\|P_jx\|+\|Qx\|+\|RK_jx\| \leq 0, \quad \forall x \in \mathbb{R}^n, (j,i) \in \mathcal{S}_t.$$

Suppose that $x \in \mathcal{P} \cap \Omega_j$. Then, from the above inequality we have that

$$\|P_i((A_j+B_jK_j)x+f_j)\| \leq \|P_jx\|-\|Qx\| \leq \|P_jx\| \leq c, \quad \forall x \in \mathcal{P}, (j,i) \in \mathcal{S}_t.$$

Then, it follows that $(A_j + B_jK_j)x + f_j \in \mathcal{P}$ for all $x \in \mathcal{P} \cap \Omega_j$ and all $j \in \mathcal{S}$ and thus, $\mathcal{P}$ is a positively invariant set for the PWA system (3.27) in closed-loop with $u_k = h(x_k) = K_j x_k$ when $x_k \in \Omega_j$, $j \in \mathcal{S}$, $k \in \mathbb{Z}_+$. ■

Note that the set $\mathcal{P}$ defined in (3.43) is a sublevel set of the terminal cost $F(x) = \|P_j x\|$ when $x \in \Omega_j$ and it consists of the union of s polyhedra (this is because the regions Ω_j are assumed to be polyhedra). If a common terminal weight is used, i.e. $P_j = P$ for all $j \in \mathcal{S}$, then the set $\mathcal{P}$ defined in (3.43) is a polyhedral set.

3.6 Terminal equality constraint

In this section we consider the case when a terminal equality constraint is employed to guarantee stability. In this setting the terminal cost $F(x)$ is set equal to zero for all x and the terminal constraint set is taken as $\mathbb{X}_T = \{0\}$ in Problem 3.2.1. This implies that the terminal constraint now becomes $x_{N|k} = 0$. On one hand, this method has the advantage that the computation of a terminal cost and terminal set that satisfy Assumption 3.3.1 is no longer necessary. On the other hand, the terminal equality constraint method usually requires a larger prediction horizon for feasibility of the Problem 3.2.1, which increases the computational complexity of the MPC algorithm.

Note that the terminal equality constraint method, although it has been used since the early stages of hybrid MPC (Bemporad and Morari, 1999) has only been proven to guarantee attractivity for the closed-loop system (e.g. see Theorem 1 of (Bemporad and Morari, 1999)). We show that under suitable assumptions Lyapunov stability can also be achieved in this setting using the general theory developed in Section 3.3, i.e. we demonstrate that Assumption 3.3.2 is satisfied.

Consider an optimal control sequence obtained by solving Problem 3.2.1 at time $k \in \mathbb{Z}_+$, i.e. $\mathbf{u}_k^* = (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ and let $\mathbf{x}_k^*(x_k, \mathbf{u}_k^*) \triangleq (x_{1|k}^*, \ldots, x_{N|k}^*)$ denote the state sequence generated by system (3.38) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k^*$. Note that $x_{N|k}^* = 0$. Let $\|\cdot\|$ denote either the 1-norm or the ∞-norm, and consider the following assumption.

Assumption 3.6.1 There exist positive numbers τ_i such that $\|u_{i|k}^*\| \leq \tau_i \|x_k\|$ for all $x_k \in \mathbb{X}_f(N)$, and all $i = 0, \ldots, N-1$.

The above assumption requires that all the controls in the optimal sequence satisfy a regularity property, i.e. $\|u_{i|k}^*\| \leq \tau_i \|x_k\|$ for all $x_k \in \mathbb{X}_f(N)$, and all $i = 0, \ldots, N-1$. To a priori guarantee that Assumption 3.6.1 holds, one can add the regularity constraints to the MPC optimization problem explicitly, for some fixed values of the parameters τ_i, $i = 0, \ldots, N-1$.

We will make use of the following result.

Lemma 3.6.2 *Under Assumption 3.6.1 there exist some* $\eta_i > 0$ *such that*

$$\|x_{i|k}^*\| \leq \eta_i \|x_k\|, \quad \textit{for all} \quad x_k \in \mathbb{X}_f(N) \quad \textit{and all} \quad i = 0, \ldots, N-1. \tag{3.44}$$

Proof: We will use induction to prove Lemma 3.6.2. For $i = 0$, the inequality $\|x_{i|k}^*\| \leq \eta_i \|x_k\|$ holds for any $\eta_0 \geq 1$. Suppose $\|x_{i|k}^*\| \leq \eta_i \|x_k\|$ holds for some $0 \leq i \leq N-2$. Now we will prove that it holds for $i+1$. We have that

$$\|x_{i+1|k}^*\| = \|A_j x_{i|k}^* + B_j u_{i|k}^* + f_j\| \quad \text{when} \quad x_{i|k}^* \in \mathbb{X}_f(N) \cap \Omega_j, \quad j \in \mathcal{S}.$$

Since there exists a positive number μ such that $\|x\| \geq \mu$ for all $x \in \cup_{j \in \mathcal{S}_1} \Omega_j$ and $f_j = 0$ for $j \in \mathcal{S}_0$, it follows that there exists a positive number θ such that $\|f_j\| \leq \theta\|x\|$ for all $x \in \mathbb{R}^n$ and all $j \in \mathcal{S}$. Then, by Assumption 3.6.1 it follows that

$$\begin{aligned}\|x^*_{i+1|k}\| &\leq \|A_j\|\|x^*_{i|k}\| + \|B_j\|\|u^*_{i|k}\| + \|f_j\| \leq \\ &\leq \max_{j\in\mathcal{S}}(\|A_j\| + \tau_i\|B_j\| + \theta)\|x^*_{i|k}\|. \end{aligned} \tag{3.45}$$

Hence, by the induction hypothesis it follows that

$$\|x^*_{i+1|k}\| \leq \eta_{i+1}\|x_k\|,$$

for $\eta_{i+1} \triangleq \max_{j\in\mathcal{S}}(\|A_j\| + \tau_i\|B_j\| + \theta)\eta_i > 0$. ■

In the sequel we will show that the stage cost $L(\cdot,\cdot)$ satisfies Assumption 3.3.2 for both the quadratic forms case and the $1,\infty$-norms case.

Theorem 3.6.3 *Suppose that Assumption 3.6.1 holds and $L(x,u) \triangleq x^\top Qx + u^\top Ru$ or $L(x,u) \triangleq \|Qx\| + \|Ru\|$. Then the stage cost $L(\cdot,\cdot)$ satisfies Assumption 3.3.2.*

Proof: We have already proven in Section 3.4 that $L(\cdot,\cdot)$ satisfies part of Assumption 3.3.2 for $\alpha_1(\|x\|) = \lambda_{\min}(Q)\|x\|_2^2$ in the quadratic forms case and $\alpha_1(\|x\|) = \gamma\|x\|$ in the $1,\infty$-norms case. Now we prove that the second part of Assumption 3.3.2 is satisfied. Consider the quadratic stage cost, i.e. $L(x,u) = x^\top Qx + u^\top Ru$. From Lemma 3.6.2 and by Assumption 3.6.1 it follows that:

$$\begin{aligned}L(x^*_{i|k}, u^*_{i|k}) &\leq \lambda_{\max}(Q)\|x^*_{i|k}\|_2^2 + \lambda_{\max}(R)\|u^*_{i|k}\|_2^2 \leq \\ &\leq (\eta_i^2\lambda_{\max}(Q) + \tau_i^2\lambda_{\max}(R))\|x_k\|_2^2 \triangleq c_i\|x_k\|_2^2, \\ &\forall x_k \in \mathbb{X}_f(N),\ i = 0,\ldots,N-1,\end{aligned} \tag{3.46}$$

where $c_i > 0$ for all $i = 0,\ldots,N-1$. Applying the same reasoning for a $1,\infty$-norms stage cost, i.e. $L(x,u) = \|Qx\| + \|Ru\|$, it follows that:

$$\begin{aligned}L(x^*_{i|k}, u^*_{i|k}) &\leq \|Q\|\|x^*_{i|k}\| + \|R\|\|u^*_{i|k}\| \leq \\ &\leq (\eta_i\|Q\| + \tau_i\|R\|)\|x_k\| \triangleq a_i\|x_k\|, \\ &\forall x_k \in \mathbb{X}_f(N),\ i = 0,\ldots,N-1,\end{aligned} \tag{3.47}$$

where $a_i > 0$ for all $i = 0,\ldots,N-1$.

Hence, the stage cost $L(x,u)$ satisfies Assumption 3.3.2 for $\alpha_1(\|x\|) = \lambda_{\min}(Q)\|x\|_2^2$ and $\alpha_2(\|x\|) = \max_{i=0,\ldots,N-1} c_i\|x\|_2^2$, for the quadratic forms case, and $\alpha_1(\|x\|) = \gamma\|x\|$ and $\alpha_2(\|x\|) = \max_{i=0,\ldots,N-1} a_i\|x\|$ for the $1,\infty$-norms case. ■

We have shown that Assumption 3.3.2 holds for MPC costs based on both quadratic forms and $1, \infty$-norms. Hence, it follows from Theorem 3.3.3 that Lyapunov stability can be achieved for *terminal equality constraint* MPC of hybrid systems.

3.7 Illustrative examples

In this section we illustrate the various methods for setting up MPC schemes for PWA systems with an a priori stability guarantee by means of simulated examples.

3.7.1 Example 1

In this example we illustrate the approach of Lemma 3.4.1 for computing a terminal cost and the method of Section 3.5.1 for computing the terminal set as a polyhedral invariant set. Consider the system used in (Bemporad and Morari, 1999):

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad [1\ 0]x_k \geq 0 \\ A_2 x_k + B u_k & \text{if} \quad [1\ 0]x_k < 0 \end{cases} \tag{3.48}$$

subject to the constraints $x_k \in \mathbb{X} = [-5,5] \times [-5,5]$ and $u_k \in \mathbb{U} = [-1,1]$, where

$$A_1 = \begin{bmatrix} 0.35 & -0.602 \\ 0.6062 & 0.35 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.35 & 0.6062 \\ -0.6062 & 0.35 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The LMI (3.17) has been solved for $Z_1 = Z_2 = Z$, Y_1, Y_2 and for the weights $Q = I_2$, $R = 0.4$. We have obtained the terminal weight matrix $P = \text{diag}([1.4876\ 2.2434])$ and the feedback gains $K_1 = [-0.611\ -0.3572]$, $K_2 = [0.611\ -0.3572]$. We take the safe set with respect to state and input constraints as $\widetilde{\mathbb{X}}_{\mathbb{U}} = \{x \in \mathbb{X} \mid |K_j x| \leq 1,\ j = 1,2\}$. The polyhedral positively invariant set obtained with an algorithm based on the recurrent sequence of sets (3.36) is

$$\mathbb{X}_T = \left\{ x \in \widetilde{\mathbb{X}}_{\mathbb{U}} \mid \begin{bmatrix} -0.2121 & 0.373 \\ 0.2121 & -0.373 \\ 0.2121 & 0.373 \\ -0.2121 & -0.373 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \tag{3.49}$$

For system (3.48) and the terminal set (3.49), a prediction horizon of $N = 4$ ensures that $\mathbb{X} \subseteq \mathbb{X}_f(N)$. The set of feasible states for $N = 4$ (obtained using the MPT (Kvasnica et al., 2004)) is plotted in Figure 3.4. The simulation

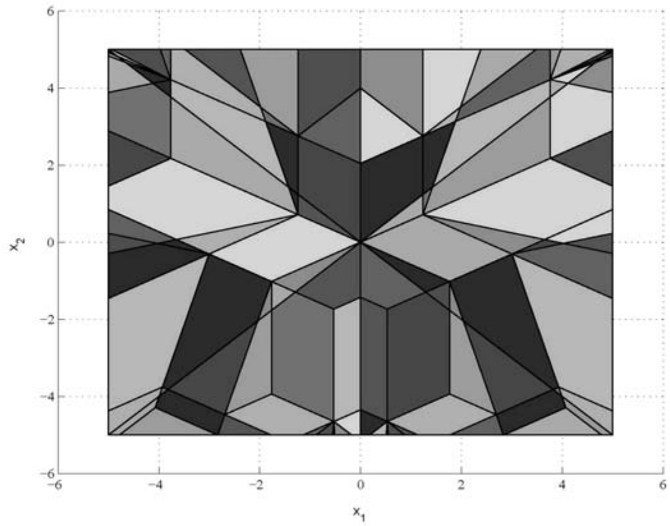

Figure 3.4: Example 1. The set of feasible states obtained for $N = 4$.

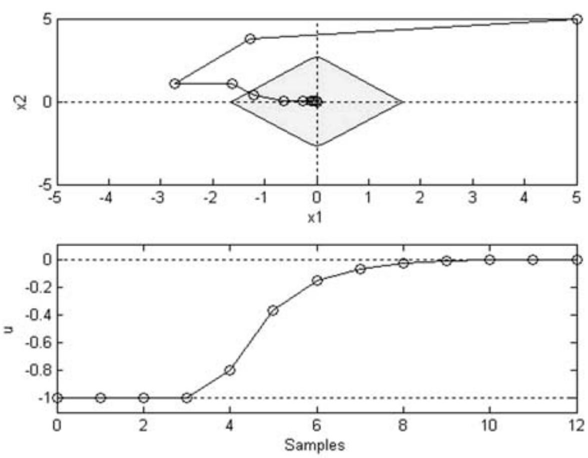

Figure 3.5: Example 1. Up: State trajectory - circle line, $\mathbb{X}_T$ - grey polyhedron; Down: Input history - circle line.

results are plotted in Figure 3.5 for system (3.48) with initial state $x_0 = [5\ 5]^\top$ in closed-loop with the MPC control (3.3) calculated for $N = 4$ using the Hybrid Toolbox (Bemporad, 2003), together with a plot of the terminal constraint set.

3.7.2 Example 2

In this example we illustrate the S-procedure approach of Theorem 3.4.2 for computing a terminal cost and the "squaring the circle" method of Section 3.5.2 for computing the terminal set as a piecewise polyhedral invariant set. Consider the following *open-loop unstable* 2D PWL system:

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad E_1 x_k > 0 \\ A_2 x_k + B u_k & \text{if} \quad E_2 x_k \geq 0 \\ A_3 x_k + B u_k & \text{if} \quad E_3 x_k > 0 \\ A_4 x_k + B u_k & \text{if} \quad E_4 x_k \geq 0 \end{cases} \tag{3.50}$$

subject to the constraints $x_k \in \mathbb{X} = [-10, 10] \times [-10, 10]$, $u_k \in \mathbb{U} = [-1, 1]$, where $A_1 = \left[\begin{smallmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, $A_3 = A_1$ and $A_4 = A_2$. The partitioning of the system is given by $E_1 = -E_3 = \left[\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$, $E_2 = -E_4 = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. The weights of the MPC cost are $Q = 10^{-4} I_2$ and $R = 10^{-3}$ and the cost is defined using quadratic forms. For system (3.50) the LMIs of Lemma 3.4.1 and the LMI proposed in (Grieder et al., 2005) turn out to be infeasible. With the S-procedure approach of Theorem 3.4.2 we obtained the following solution by solving the LMI (3.22)-(3.23)-(3.24) for the tuning factors $\epsilon_{11} = 0.04, \epsilon_{21} = 0.3, \epsilon_{12} = 0.08, \epsilon_{22} = 1$ and for the orthonormal matrices V_1, V_2 defined as in (3.26) for $\theta_1 = 2.4$ and $\theta_2 = 0.9$:

$$\begin{aligned}
&P_1 = \begin{bmatrix} 12.9707 & 10.9974 \\ 10.9974 & 14.9026 \end{bmatrix}, \; P_2 = \begin{bmatrix} 7.9915 & -5.5898 \\ -5.5898 & 5.3833 \end{bmatrix}, \\
&P_3 = P_1, \quad P_4 = P_2, \\
&K_1 = \begin{bmatrix} -0.7757 & -1.0299 \end{bmatrix}, \; K_2 = \begin{bmatrix} 0.6788 & -0.4302 \end{bmatrix}, \\
&K_3 = K_1, \quad K_4 = K_2, \\
&U_{11} = \begin{bmatrix} 0.4596 & 1.9626 \\ 1.9626 & 0.0198 \end{bmatrix}, \; U_{12} = \begin{bmatrix} 0.4545 & 2.0034 \\ 2.0034 & 0.0250 \end{bmatrix}, \\
&U_{21} = \begin{bmatrix} 0.0542 & 0.0841 \\ 0.0841 & 0.0506 \end{bmatrix}, \; U_{22} = \begin{bmatrix} 0.0599 & 0.0914 \\ 0.0914 & 0.0565 \end{bmatrix}, \\
&\sigma_{11} = 24.9765, \quad \sigma_{21} = 2.8969, \quad \sigma_{12} = 12.4273, \; \sigma_{22} = 0.9475, \\
&\gamma_{11} = 0.0395, \quad \gamma_{21} = 0.2954, \; \gamma_{12} = 0.0791, \quad \gamma_{22} = 0.9675.
\end{aligned} \tag{3.51}$$

The function $V(x) = x^\top P_j x$ when $x \in \Omega_j$, $j = 1, 2, 3, 4$, is a PWQ Lyapunov function for system (3.50) in closed-loop with $u_k = h(x_k) \triangleq K_j x_k$ when $x_k \in \Omega_j$, with the feedback gains given above. Its level sets are piecewise ellipsoidal β-contractive sets with a contraction factor $\beta = 0.9378$.

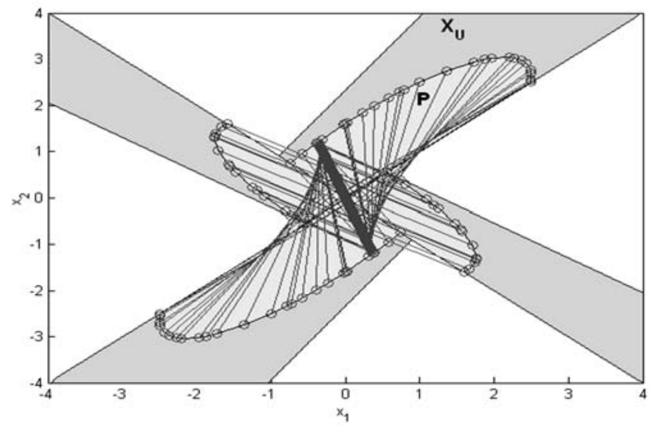

Figure 3.6: Example 2. Piecewise polyhedral invariant set $\mathcal{P}$ (light grey) to be employed as terminal set in the MPC scheme, $\mathbb{X}_{\mathbb{U}}$ (dark grey and light grey) and state-trajectory for system (3.50) *in closed-loop with $u_k = h(x_k)$.*

Therefore, we can employ the "squaring the circle" algorithm of Section 3.5 for computing a terminal constraint set that satisfy Assumption 3.3.1. A contractive piecewise polyhedral set $\mathcal{P}$ was computed for system (3.50) in closed-loop with $h(\cdot)$ with the feedbacks given in (3.51) using the approach of Theorem 3.5.8 and Algorithm 3.5.10 for the sublevel sets $\mathcal{E} \triangleq \{x \in \mathbb{X} \mid V(x) \leq 14\} \subseteq \mathbb{X}_{\mathbb{U}}$ and $\beta\mathcal{E}$. The resulting set $\mathcal{P}$ is the union of four polyhedra and it is a λ-contractive set with $\lambda = 0.9286$. The closed-loop state trajectories for system (3.50) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, with the vertices of $\mathcal{P}$ as initial conditions are plotted in Figure 3.7 together with a plot of the safe set $\mathbb{X}_{\mathbb{U}}$. The state trajectory of system (3.50) with initial state $x_0 = [-5 \ \ -3.8]^\top$ and in closed-loop with the MPC control (3.3) calculated for $N = 4$ using the Hybrid Toolbox (Bemporad, 2003) is plotted in Figure 3.7. The MPC controller successfully stabilizes the open-loop unstable system (3.50). Moreover, the MPC controller is able to stabilize a state that is far outside the safe set $\mathbb{X}_{\mathbb{U}}$, while satisfying the state and input constraint. This illustrates the advantage of MPC over the local stabilizing state-feedback control law $h(\cdot)$.

3.7.3 Example 3

In this example we illustrate the approach of Section 3.4.2 for computing a terminal cost and the method of Section 3.5.3 for computing the terminal set.

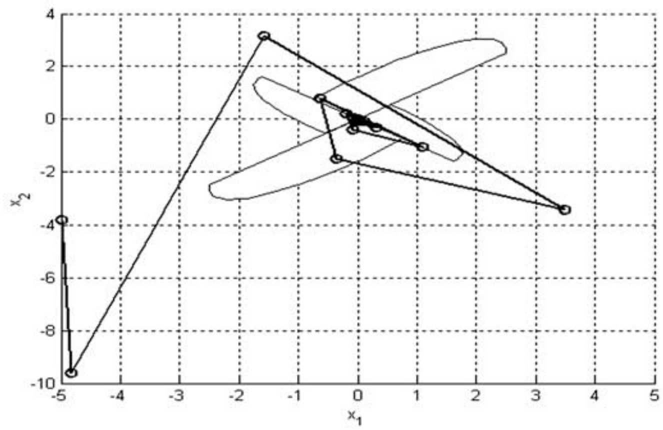

Figure 3.7: Example 2. State trajectory for the MPC closed-loop system - circle line; $\mathbb{X}_T$ - non-convex union of white polyhedra.

Consider the following *open-loop unstable* 3D PWA system with 4 modes:

$$x_{k+1} = \begin{cases} A_1 x_k + B_1 u_k & \text{if } [0\ 1\ 0]x_k > 0\ ,\ [0\ 0\ 1]x_k > 0 \\ A_2 x_k + B_2 u_k & \text{if } [0\ 1\ 0]x_k \geq 0\ ,\ [0\ 0\ 1]x_k \leq 0 \\ A_3 x_k + B_3 u_k & \text{if } [0\ 1\ 0]x_k < 0\ ,\ [0\ 0\ 1]x_k < 0 \\ A_4 x_k + B_4 u_k & \text{if } [0\ 1\ 0]x_k \leq 0\ ,\ [0\ 0\ 1]x_k \geq 0 \end{cases} \tag{3.52}$$

subject to the constraints $x_k \in \mathbb{X} = [-5, 5] \times [-5, 5] \times [-5, 5]$ and $u_k \in \mathbb{U} = [-2.5, 2.5]$, where

$$A_1 = \begin{bmatrix} -0.2523 & 0.4856 & 0.6467 \\ 0.5290 & -0.2616 & 0.3128 \\ -0.4415 & -0.2713 & -0.6967 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.5656 \\ 0.5460 \\ 0.9389 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.0647 & 0.1729 & -0.6542 \\ -0.3131 & -0.6691 & -0.6516 \\ -0.3085 & 0.0613 & 0.0099 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.6543 \\ 0.5266 \\ -0.0558 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.6402 & -0.5409 & -0.5629 \\ -0.6693 & -0.6874 & 0.1748 \\ -0.2812 & 0.4898 & -0.3526 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.7580 \\ -0.8050 \\ -0.4059 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -0.3501 & 0.2590 & 0.6695 \\ -0.4808 & 0.1905 & 0.3865 \\ -0.1217 & -0.2631 & -0.0013 \end{bmatrix},\ B_4 = \begin{bmatrix} 0.6961 \\ -0.7619 \\ -0.2590 \end{bmatrix}.$$

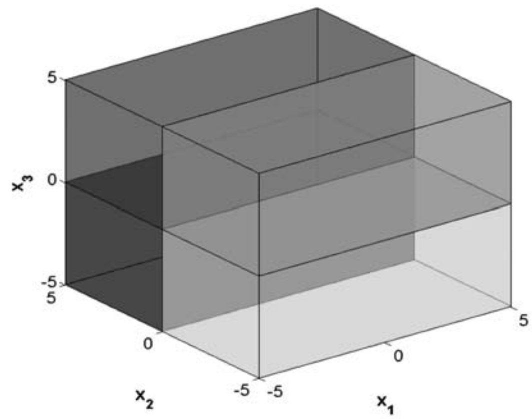

Figure 3.8: Example 3. State-space partition.

The state-space partition corresponding to system (3.52) is plotted in Figure 3.8. The weights of the MPC cost are $Q = 0.02I_3$ and $R = 0.01$ and the cost is defined using ∞-norms. The following solution to the inequality (3.29) (with ∞-norms) was found using a min-max formulation and the Matlab `fminunc` solver:

$$\begin{aligned}
P &= \begin{bmatrix} 0.7029 & 3.8486 & 1.1501 \\ 4.1796 & 0.5642 & 1.6656 \\ -1.4275 & 1.5026 & 5.3197 \\ -1.3717 & 2.5343 & -1.5468 \end{bmatrix}, \\
K_1 &= \begin{bmatrix} 0.4699 & 0.1750 & 0.1591 \end{bmatrix}, \; K_2 = \begin{bmatrix} 0.4039 & 0.4239 & 1.1529 \end{bmatrix}, \\
K_3 &= \begin{bmatrix} -0.7742 & -0.1436 & -0.1603 \end{bmatrix}, \\
K_4 &= \begin{bmatrix} -0.0800 & -0.0405 & -0.2867 \end{bmatrix}.
\end{aligned}$$

The terminal set (see Figure 3.9 for a plot) has been obtained as in (3.43) for $c = 4$. The resulting terminal set satisfies $\mathbb{X}_T \subset \mathbb{X}_{\mathbb{U}} = \cup_{j \in \mathcal{S}} \{x \in \Omega_j \mid K_j x \in \mathbb{U}\}$ for the gains given above. The simulation results are plotted in Figure 3.9 for system (3.52) with initial state $x_0 = [3.6 \; 2 \; 1]^\top$ and in closed-loop with the MPC control (3.3) calculated for the matrices P, Q and R given above, $N = 3$ and with $\mathbb{X}_T$ as the terminal set. As guaranteed by Theorem 3.4.3 and Theorem 3.3.3, the MPC control law (3.3) stabilizes the open-loop unstable system (3.52) while satisfying the state and input constraints.

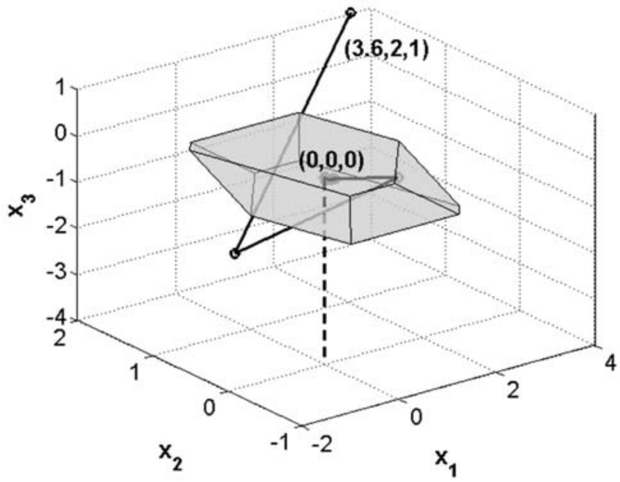

Figure 3.9: Example 3. MPC closed-loop trajectory - circle line and terminal constraint set $\mathbb{X}_T$.

3.8 Conclusions

In this chapter we have derived *a priori verifiable* sufficient conditions for Lyapunov asymptotic stability and exponential stability of model predictive control of hybrid systems. We developed a general theory which shows that Lyapunov stability can be achieved for both terminal cost and constraint set and terminal equality constraint MPC, even if the considered Lyapunov function and the system dynamics are discontinuous. In the particular case of constrained PWA systems and quadratic forms or $1, \infty$-norms based cost functions, new procedures for calculating the terminal cost and the terminal constraint set have been developed. If the MPC cost is defined using quadratic forms, then the terminal cost is calculated via semi-definite programming. For an $1, \infty$-norm based cost, the terminal cost is obtained by solving off-line an optimization problem. Novel algorithms for calculating low complexity piecewise polyhedral positively invariant sets for PWA systems have also been developed. Also, besides the advantages brought to the on-line computation of the MPC control action, the off-line computation of these positively invariant sets is numerically more friendly in comparison with the computation of the maximal positively invariant set. Several examples illustrated the presented theory.

4

Global input-to-state stability and stabilization of discrete-time piecewise affine systems

This chapter is concerned with inherent robustness of nominally stable discrete-time PWA and hybrid systems. An example shows that globally exponentially stable PWA systems can have zero robustness to arbitrarily small additive disturbances. This motivates the need for a methodology that enables robust stability analysis and robust controller synthesis for discrete-time PWA systems. An approach based on input-to-state stability (ISS) is employed in this chapter to develop such a framework.

4.1 Introduction

In the linear and the continuous nonlinear case, nominally stable systems generically have some robustness properties. To give an insight into this general statement, consider the following nominal and perturbed nonlinear systems:

$$\begin{aligned} x_{k+1} &= G(x_k) \\ x_{k+1} &= H(x_k, v_k) \triangleq G(x_k) + v_k, \end{aligned}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions and x_k, v_k are the state and an unknown additive disturbance input at discrete-time instant $k \in \mathbb{Z}_+$. As done classically, if the nominal dynamics $G(\cdot)$ enjoys a *continuous* (or even stronger, *Lipschitz continuous*) Lyapunov function $V(\cdot)$ (i.e. a function that satisfies the hypothesis of Theorem 2.2.4), then it is

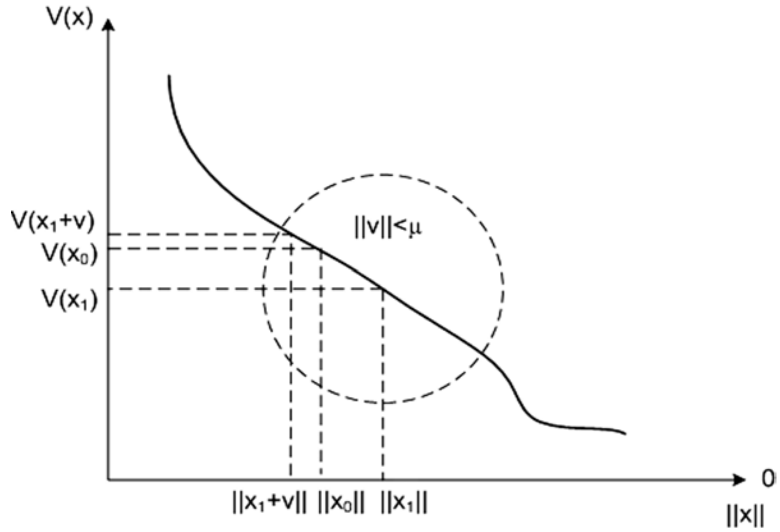

Figure 4.1: Continuous Lyapunov function.

easy to prove that this is also an ISS Lyapunov function (i.e. it satisfies the hypothesis of Theorem 2.3.5), which is sufficient for ensuring input-to-state stability. Indeed, continuity implies that for any compact subset $\mathcal{P}$ of $\mathbb{R}^n$ there exists a $\mathcal{K}$-function σ such that for any $x, y \in \mathcal{P}$ it holds that $|V(y) - V(x)| \leq \sigma(\|y - x\|)$. Hence, we have that

$$\begin{aligned} V(H(x,v)) - V(x) = V(G(x)+v) - V(x) &\leq V(G(x)) + \sigma(\|v\|) - V(x) \\ &\leq -\alpha_3(\|x\|) + \sigma(\|v\|) \end{aligned}$$

and thus, the continuous Lyapunov function $V(\cdot)$ is an *ISS Lyapunov* function. The graphical illustration of the above reasoning is given in Figure 4.1 (note that the origin is on the right side of the horizontal axis). The nominal state trajectory starting from the initial state x_0 reaches x_1 in one discrete-time step and the Lyapunov function $V(\cdot)$ is decreasing. If a disturbance acts additively on the system state at time $k = 1$, this may cause a jump (increase) in the value of $V(\cdot)$. However, due to continuity of $V(\cdot)$, this jump can be upper bounded by a $\mathcal{K}$-function of the disturbance input, as explained above. Thus, inherent robustness is established. For more general robust stability results that use *continuous* candidate Lyapunov functions we refer the reader to (Grimm et al., 2004; Messina et al., 2005).

Clearly, the above continuity based robustness (ISS) argument no longer holds if the Lyapunov function $V(\cdot)$ is discontinuous at some points. This is

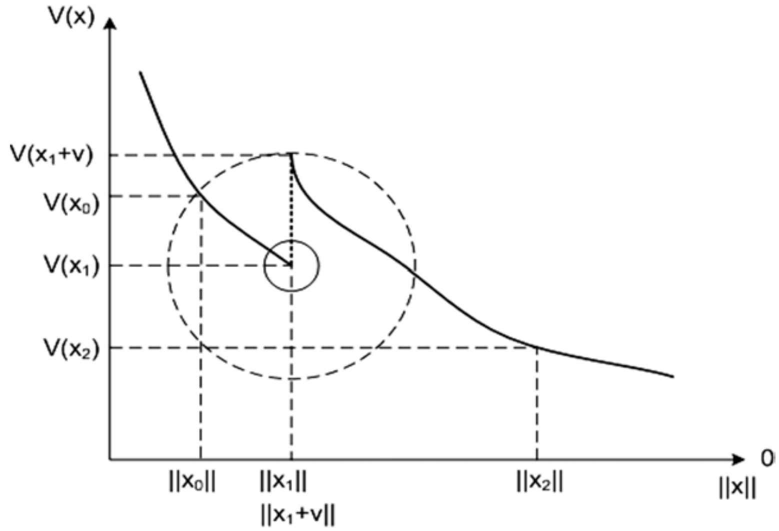

Figure 4.2: Discontinuous Lyapunov function.

illustrated graphically in Figure 4.2. The nominal state trajectory starting from the initial state x_0 reaches x_1 in one discrete-time instant and x_2 at the second discrete-time instant. The discontinuous Lyapunov function $V(\cdot)$ is decreasing. If an *arbitrarily small* disturbance acts additively on the system state at time $k = 1$ (note that $V(\cdot)$ is discontinuous at $x = x_1$) this may cause a jump in the value of $V(\cdot)$ by a fixed amount, which cannot be bounded locally by a $\mathcal{K}$-function. Therefore, in the case when stability is established via *discontinuous* Lyapunov functions, the classical way of proving (inherent) robustness fails and special precautions must be taken when establishing robustness from nominal stability.

The above observation is extremely interesting for discrete-time PWA and hybrid systems. This is due to the fact that the existing results on analysis of nominal stability of discrete-time PWA systems available in the literature, e.g. see (Mignone et al., 2000a; Ferrari-Trecate et al., 2002; Feng, 2002; Daafouz et al., 2002), rely on piecewise quadratic (PWQ), possibly discontinuous, candidate Lyapunov functions. Of course, the trivial solution for ensuring inherent robustness for PWA systems would be to search for a *continuous* Lyapunov function. However, this is known to be very conservative, as one can easily construct an example of a stable PWA system that admits a discontinuous PWQ Lyapunov function, but it does not admit a continuous PWQ Lyapunov function.

To illustrate this, consider the following piecewise linear system:

$$x_{k+1} = \begin{cases} A_1 x_k & \text{if } [1\ 0]x_k \leq 1 \\ A_2 x_k & \text{if } [1\ 0]x_k > 1, \end{cases} \tag{4.1}$$

where

$$A_1 = \begin{bmatrix} 0.62 & -0.2848 \\ -0.0712 & 0.8336 \end{bmatrix}, A_2 = \begin{bmatrix} 0.5125 & -0.2855 \\ 1.4277 & 0.5125 \end{bmatrix}.$$

The Linear Matrix Inequalities (LMI) technique of (Ferrari-Trecate et al., 2002) was employed for this system to compute a PWQ Lyapunov function. This led to the following LMIs in the unknowns P_1, P_2,

$$\begin{aligned} &P_1, P_2 > 0 \\ &P_1 - A_1^\top P_1 A_1 > 0,\ P_1 - A_1^\top P_2 A_1 > 0 \\ &P_2 - A_2^\top P_2 A_2 > 0,\ P_2 - A_2^\top P_1 A_2 > 0, \end{aligned} \tag{4.2}$$

which yielded a PWQ Lyapunov function, i.e.:

$$V(x) = \begin{cases} x^\top P_1 x & \text{if } [1\ 0]x_k \leq 1 \\ x^\top P_2 x & \text{if } [1\ 0]x_k > 1, \end{cases}$$

with

$$P_1 = \begin{bmatrix} 1.8306 & -0.2545 \\ -0.2545 & 1.2859 \end{bmatrix}, P_2 = \begin{bmatrix} 3.6652 & 0.3130 \\ 0.3130 & 0.9356 \end{bmatrix}.$$

This establishes nominal asymptotic stability in the Lyapunov sense for system (4.1), as it is also illustrated by the trajectory plot given in Figure 4.3. Note that the LMI (4.2) with the additional constraint $P_1 = P_2$ is infeasible, which implies that system (4.1) does not admit a common (and thus continuous) quadratic Lyapunov function. Moreover, the system (4.1) does not admit even a *continuous* PWQ Lyapunov function, as the corresponding condition for continuity over the switching hyperplane is equivalent with $P_1 = P_2$. The computed PWQ Lyapunov function is discontinuous. For example consider the point $x^* = [1\ 0.3]^\top$, for which it holds that $(x^*)^\top P_1 x^* = 1.7936 \neq 3.9372 = (x^*)^\top P_2 x^*$.

This example confirms that there exist nominally asymptotically stable PWA systems that admit a discontinuous PWQ Lyapunov function, but not a continuous PWQ one. In such cases, robustness can no longer be established in the classical way, which relies on continuous Lyapunov functions.

However, this example is input-to-state stable, as it can be easily verified (via the LMI sufficient condition for ISS presented later in this chapter, Section 4.4) that the *discontinuous* Lyapunov function computed above is also a *discontinuous* ISS Lyapunov function.

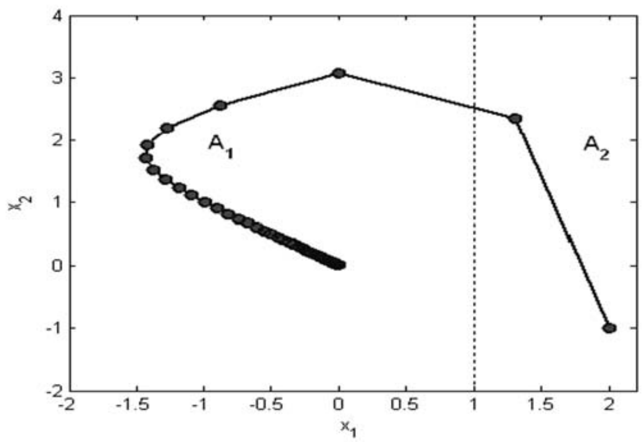

Figure 4.3: A trajectory plot for system (4.1).

4.2 An example of zero robustness in PWA systems

Although the absence of a *continuous* Lyapunov function does not necessarily imply that input-to-state stability is lost in the presence of perturbations, in (Grimm et al., 2004) and (Kellett and Teel, 2004) it was shown that it can lead to the so-called "zero robustness" phenomenon. More specifically, the authors of (Grimm et al., 2004) present several examples of smooth nonlinear control systems, while the authors of (Kellett and Teel, 2004) present an example of a discrete-time discontinuous nonlinear system, that are (is) nominally exponentially stable, but have (has) zero robustness to *arbitrarily small* disturbances. In other words, zero robustness means that the asymptotic stability property is completely lost in the presence of arbitrarily small perturbations. This phenomenon is mainly due to the absence of a continuous Lyapunov function.

As in the previous section we pointed out that in PWA systems nominal stability is often established via discontinuous Lyapunov functions, it is relevant to investigate whether or not the zero robustness phenomenon can occur in PWA systems as well. An answer to this question cannot be obtained directly from the examples of (Grimm et al., 2004; Kellett and Teel, 2004), as the dynamics (vectorfield) of a PWA system is affine (or linear) in any particular region of the state-space. The zero robustness phenomenon could only occur in piecewise affine systems for initial conditions in the vicinity of a switching hyperplane (i.e. at the boundaries of some regions in the state-space partition) at which the dynamics is discontinuous. Still, for such initial conditions, the affine (linear) vectorfield may cause the state to drift away from the switching hyperplane by a fixed distance. Then, stability would be preserved in the presence of *sufficiently small* perturbations. However, if the vectorfield in some state-space region yields a trajectory that is parallel to a "discontinuous" switching hyperplane, then zero robustness can occur for initial conditions belonging to the same switching hyperplane, as it is illustrated in the sequel.

Consider the discontinuous nominal and perturbed PWA systems

$$x_{k+1} = A_j x_k + f_j \quad \text{when} \quad x_k \in \Omega_j, \tag{4.3a}$$
$$\tilde{x}_{k+1} = A_j \tilde{x}_k + f_j + v_k \quad \text{when} \quad x_k \in \Omega_j, \tag{4.3b}$$

with $v_k \in \mathbb{V}_\varepsilon \triangleq \{v \in \mathbb{R}^2 \mid \|v\|_\infty \leq \varepsilon\}$ for some $\varepsilon > 0$, $j \in \mathcal{S} \triangleq \{1, \ldots, 9\}$, $k \in \mathbb{Z}_+$, and where

$$A_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } j \neq 7;\ A_7 = \begin{bmatrix} 0.35 & 0.6062 \\ 0.0048 & -0.0072 \end{bmatrix};\ f_1 = -f_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix};$$
$$f_3 = f_4 = f_5 = f_6 = \begin{bmatrix} 0 \\ -1 \end{bmatrix};\ f_7 = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ f_8 = \begin{bmatrix} 0.4 \\ -0.1 \end{bmatrix};\ f_9 = \begin{bmatrix} -0.4 \\ -0.1 \end{bmatrix}.$$

The system state takes values in the set $\mathbb{X} \triangleq \cup_{j \in \mathcal{S}} \Omega_j$, where the regions Ω_j are polyhedra (not necessarily closed), as shown in Figure 4.4. The state trajectories[1] of system (4.3) obtained for the initial states $x_0 = [0.2\ 3.6]^\top \in \Omega_2$ (square dotted line) and $x_0 = [0.2\ 3.601]^\top \in \Omega_1$ (circle dotted line) are plotted in Figure 4.4.

In the theorem below we derive some properties for this system.

Theorem 4.2.1 *The following statements hold:*

(i) *Let $P \in \mathbb{R}^{2\times 2}$ be the solution of the discrete-time Lyapunov equation $A_7^\top P A_7 - P = -Q$ obtained for some $Q > 0$. For example, $P = \left[\begin{smallmatrix} 1.1406 & 0.2420 \\ 0.2420 & 1.4171 \end{smallmatrix}\right]$ for $Q = I_2$. Then, the function*

$$V(x) \triangleq x_{10}^\top P x_{10} + \sum_{i=0}^{9} x_i^\top Q x_i, \tag{4.4}$$

where x_i is the solution of system (4.3a) *obtained at time $i \in \mathbb{Z}_{[0,10]}$ from the initial condition $x_0 \triangleq x \in \mathbb{X}$, is a discontinuous PWQ Lyapunov function for system* (4.3a)*;*

[1] Note that the regions Ω_1 and Ω_2 are defined such that for all $x \in \partial\Omega_1 \cap \partial\Omega_2$ the dynamics $x_{k+1} = A_2 x_k + f_2$ is employed.

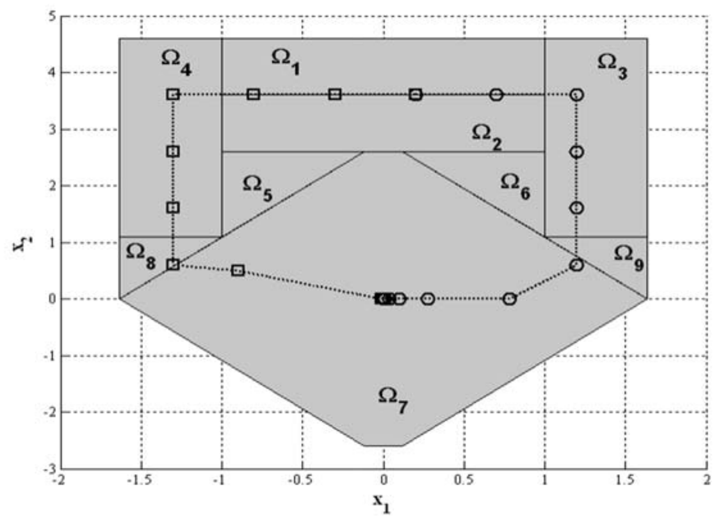

Figure 4.4: Unperturbed state trajectories (square and circle dotted lines) for system (4.3a).

(ii) *The PWA system* (4.3a) *is exponentially stable in* $\mathbb{X}$*;*

(iii) *For any* $\varepsilon > 0$ *the PWA system* (4.3b) *is not input-to-state stable for initial conditions in* $\partial\Omega_1 \cap \partial\Omega_2$ *and disturbances in* $\mathbb{V}_\varepsilon$*.*

Proof: (i) The following properties hold for the PWA system (4.3a), as it can be seen by inspection of the dynamics:

- (P1) $\|x_{k+1}\|_\infty \le \|x_k\|_\infty$ for all $x_k \in \mathbb{X}$, $k \in \mathbb{Z}_+$;
- (P2) For any initial state $x_0 \in \mathbb{X}$ the state trajectory satisfies $x_k \in \Omega_7$ for all $k \in \mathbb{Z}_{\ge 10}$;
- (P3) $\|A_7\|_\infty < 1$;
- (P4) Ω_7 is a Positively Invariant (PI) set for the dynamics $x_{k+1} = A_7 x_k + f_7$;
- (P5) $\mathbb{X}$ is a PI set for the PWA system (4.3).

By property (P5), $\mathbb{X}$ is a PI set for the PWA system (4.3). Therefore, we only need to prove that the PWQ function $V(\cdot)$ defined in (4.4) satisfies the hypothesis of Theorem 2.2.4. Note that the inequality

$$\alpha_1(\|x\|_2) \le V(x) \le \alpha_2(\|x\|_2)$$

is trivially satisfied for all $x \in \mathbb{X}$ with $\alpha_1(\|x\|_2) \triangleq \lambda_{\min}(Q)\|x\|_2^2$ and $\alpha_2(\|x\|_2) \triangleq \max_{j_0,\ldots,j_9}(\lambda_{\max}(P_Q))\,\|x\|_2^2$, where

$$P_Q \triangleq Q + \sum_{i=1}^{9}\left(\left(\prod_{p=0}^{i-1} A_{j_p}\right)^\top Q \left(\prod_{p=0}^{i-1} A_{j_p}\right)\right) + \left(\prod_{p=0}^{9} A_{j_p}\right)^\top P \left(\prod_{p=0}^{9} A_{j_p}\right)$$

and $j_p \in \mathcal{S}$ for all $p \in \mathbb{Z}_{[0,9]}$. Finally, for any $x \in \Omega_j$ and any $j \in \mathcal{S}$, by property (P2) it holds that

$$\begin{aligned} V(A_j x + f_j) - V(x) &= -x^\top Q x + x_{10}^\top(A_7^\top P A_7 - P + Q)x_{10} \\ &= -x^\top Q x \leq -\lambda_{\min}(Q)\|x\|_2^2 \triangleq -\alpha_3(\|x\|_2), \end{aligned}$$

where x_{10} is the solution of system (4.3) obtained from initial condition $x_0 = x$. One can easily check that $V(\cdot)$ defined in (4.4) is discontinuous, for example, at $x^* = [0.2\ 3.6]^\top \in \Omega_2$.

(ii) The statement follows directly from the result of part (i), via Theorem 2.2.4, part (ii);

(iii) In order to illustrate the zero robustness phenomenon for the perturbed PWA system (4.3b) we constructed an additive disturbance v_k, which at times $k = 0, 2, 4, \ldots$ is equal to $[0\ \varepsilon]^\top$ and at times $k = 1, 3, 5, \ldots$ is equal to $[0\ -\varepsilon]^\top$, where $\varepsilon > 0$ can be taken arbitrarily small. The system trajectory with initial state $\tilde{x}_0 = [0.2\ 3.6]^\top \in \partial\Omega_2 \cap \partial\Omega_1$ is given by $\tilde{x}_k = [0.2\ 3.6]^\top$, if $k = 0, 2, 4, \ldots$ and $\tilde{x}_{k+1} = [-0.3\ 3.6 + \varepsilon]^\top$, if $k = 1, 3, 5, \ldots$. This is a limit cycle with period 2 and $\|\tilde{x}_k\|_\infty \geq 3.6$ for all $k \in \mathbb{Z}_+$. Since for any $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ we can take ε arbitrarily small and $k^* \in \mathbb{Z}_+$ large enough such that $\beta(k, \|\tilde{x}_0\|_\infty) + \gamma(\|w_{[k-1]}\|_\infty) < 3.6 \leq \|\tilde{x}_k\|_\infty$ for all $k \geq k^*$, for any $\varepsilon > 0$, the PWA system (4.3b) is not ISS for initial conditions in $\partial\Omega_1 \cap \partial\Omega_2$ and disturbances in $\mathbb{V}_\varepsilon$. ■

The above example illustrated that nominally exponentially stable PWA systems (even with the origin in the interior of one of the state-space regions Ω_j) can have zero robustness to additive disturbances. Moreover, by taking any finite polyhedral partition of $\mathbb{R}^2 \setminus \mathbb{X}$, defining the dynamics in each polyhedral region of this partition to be $x_{k+1} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] x_k + \left[\begin{smallmatrix} 0.1 \\ -0.1 \end{smallmatrix}\right]$, $k \in \mathbb{Z}_+$, and adding these affine subsystems to the PWA system (4.3a), one obtains a PWA system that is *globally* exponentially stable, but it has zero robustness to arbitrarily small additive disturbances. Taking into account the fact that $\mathbb{R}^2$ is a robust positively invariant set for the so-obtained PWA system, one can easily prove via contradiction that this system does not admit a *continuous* Lyapunov function, using the reasoning presented in Section 4.1 and the result of Theorem 4.2.1 part (iii).

4.3 Problem statement

As shown in the previous section, nominally exponentially stable discrete-time PWA systems can have zero robustness to arbitrarily small additive disturbances, mainly due to the absence of a continuous Lyapunov function. Therefore, in discrete-time, it is important that disturbances are taken into account when synthesizing stabilizing controllers for PWA systems, which will be applied in practice.

Robust stability results for discrete-time PWA systems were presented in (Ferrari-Trecate et al., 2002, Section 3), which deals with LMI based l_2-gain analysis for PWA systems; and in (Grieder, 2004, Chp. 8.5), where it was observed that, if a robust positively invariant set can be calculated for a nominally asymptotically stable PWA system, then *local* robust convergence is ensured. For *continuous-time* input-to-state stability results for switched systems and hybrid systems we refer the reader to the recent works (Vu et al., 2005; Hespanha et al., 2005; Cai and Teel, 2005). However, to the best of the author's knowledge, a global robust stability analysis methodology for *discrete-time* PWA systems that can be used for both analysis and synthesis purposes is missing from the literature.

As such, in the remainder of this chapter we consider discrete-time PWA systems subject to *unbounded* additive disturbance inputs and we employ the input-to-state (practical) stability (ISpS) framework introduced in the second chapter to obtain *global* robust stability results. For simplicity and clarity of exposition, only PWQ candidate ISpS (ISS) Lyapunov functions are considered, but the results can also be extended *mutatis mutandis* to piecewise polynomial (Prajna and Papachristodoulou, 2003) or piecewise affine candidate functions. The sufficient conditions for ISpS (ISS) are expressed in terms of LMIs, which can be solved efficiently (Boyd et al., 1994). One of the advantages of using the ISpS (ISS) framework for studying robust stability of discrete-time PWA systems is that the results apply to PWA systems in their full generality, i.e. non-zero affine terms are allowed in the regions in the state-space partition whose closure contains the origin. Note that this situation is often excluded in other works. In the sequel we present a new LMI technique for dealing with non-zero affine terms, which does not rely on a system transformation and the S-procedure, e.g. as done in (Ferrari-Trecate et al., 2002, Remark 3). This new technique makes it possible to obtain LMI based sufficient conditions for input-to-state stabilizing controllers synthesis as well, and not just for analysis.

The rest of this chapter focuses on perturbed discrete-time, possibly discontinuous, PWA systems of the form

$$x_{k+1} = G(x_k, v_k) \triangleq A_j x_k + f_j + D_j v_k \quad \text{if} \quad x_k \in \Omega_j, \tag{4.5}$$

where $A_j \in \mathbb{R}^{n\times n}$, $f_j \in \mathbb{R}^n$, $D_j \in \mathbb{R}^{n\times d_v}$ for all $j \in \mathcal{S}$ and $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ is a *finite set* of indexes. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{R}^n$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{R}^n$ and $\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron. Let $\mathcal{S}_0 \triangleq \{j \in \mathcal{S} \mid 0 \in \text{cl}(\Omega_j)\}$, $\mathcal{S}_1 \triangleq \{j \in \mathcal{S} \mid 0 \notin \text{cl}(\Omega_j)\}$ and let $\mathcal{S}_{\text{aff}} \triangleq \{j \in \mathcal{S} \mid f_j \neq 0\}$, $\mathcal{S}_{\text{lin}} \triangleq \{j \in \mathcal{S} \mid f_j = 0\}$, so that $\mathcal{S}_0 \cup \mathcal{S}_1 = \mathcal{S}_{\text{aff}} \cup \mathcal{S}_{\text{lin}} = \mathcal{S}$.

The goal is to derive sufficient conditions for global ISpS and global ISS, respectively, of system (4.5). To do so, we consider PWQ candidate ISpS (ISS) Lyapunov functions of the form

$$V : \mathbb{R}^n \to \mathbb{R}_+,\ V(x) = x^\top P_j x \ \text{ if } \ x \in \Omega_j, \tag{4.6}$$

where P_j, $j \in \mathcal{S}$, are positive definite and symmetric matrices. For the remainder of this chapter we use $\|\cdot\|$ to denote the Euclidean norm for shortness. It is easy to observe that $V(\cdot)$ satisfies condition (2.9a) for $\alpha_1(\|x\|) \triangleq \min_{j\in\mathcal{S}} \lambda_{\min}(P_j)\|x\|^2$, $\alpha_2(\|x\|) \triangleq \max_{j\in\mathcal{S}} \lambda_{\max}(P_j)\|x\|^2$ and $d_1 = 0$.

4.4 Analysis

In this section we present LMI-based sufficient conditions for global ISpS (ISS) of system (4.5). Let Q be a known positive definite and symmetric matrix and let γ_1, γ_2 be known positive numbers with $\gamma_1\gamma_2 > 1$. For any $(j, i) \in \mathcal{S} \times \mathcal{S}$ consider now the following LMI:

$$\Delta_{ji} \triangleq \begin{pmatrix} \Xi_{ji} & -A_j^\top P_i & -A_j^\top P_i \\ -P_i A_j & \gamma_1 P_i & -P_i \\ -P_i A_j & -P_i & \gamma_2 P_i \end{pmatrix} > 0, \tag{4.7}$$

where

$$\Xi_{ji} \triangleq P_j - A_j^\top P_i A_j - E_j^\top U_{ji} E_j - Q - M_{ji}.$$

The matrix E_j, $j \in \mathcal{S}$, defines the cone $\mathcal{C}_j \triangleq \{x \in \mathbb{R}^n \mid E_j x \geq 0\}$ that is chosen such that $\Omega_j \subseteq \mathcal{C}_j$. The role of these matrices is to introduce an S-procedure relaxation (Johansson and Rantzer, 1998). The unknown variables in (4.7) are the matrices P_j, $j \in \mathcal{S}$, which are required to be positive definite and symmetric, the matrices U_{ji}, $(j, i) \in \mathcal{S} \times \mathcal{S}$, which are required to have non-negative elements, and the matrices M_{ji}, $(j, i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$, which are required to be positive definite and symmetric. For all $(j,i) \in \mathcal{S}_{\text{lin}} \times \mathcal{S}$ we take $M_{ji} = 0$. For any $(j,i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$, define

$$\mathcal{E}_{ji} \triangleq \{x \in \mathbb{R}^n \mid x^\top M_{ji} x < (1+\gamma_1) f_j^\top P_i f_j\}.$$

Theorem 4.4.1 *Let system* (4.5)*, the matrix $Q > 0$ and the numbers $\gamma_1, \gamma_2 > 0$ with $\gamma_1\gamma_2 > 1$ be given. Suppose that the LMIs*

$$\Delta_{ji} > 0, \quad (j,i) \in \mathcal{S} \times \mathcal{S} \tag{4.8}$$

are feasible. Then, it holds that:

(i) *The system* (4.5) *is globally ISpS;*

(ii) *If*[2] $(\cup_{i\in\mathcal{S}} \mathcal{E}_{ji}) \cap \Omega_j = \emptyset$ *for all $j \in \mathcal{S}_{\text{aff}}$, then system* (4.5) *is globally ISS;*

(iii) *If system* (4.5) *is piecewise Linear (PWL), i.e. $\mathcal{S}_{lin} = \mathcal{S}$, then system* (4.5) *is globally ISS.*

Proof: The proof consists in showing that $V(\cdot)$, as defined in (4.6), is an ISpS (ISS) Lyapunov function.

(i) By the hypothesis $\Delta_{ji} > 0$ for all $(j,i) \in \mathcal{S} \times \mathcal{S}$. Then, it follows that:

$$\begin{pmatrix} x^\top & f_j^\top & (D_j v)^\top \end{pmatrix} \Delta_{ji} \begin{pmatrix} x \\ f_j \\ D_j v \end{pmatrix} \geq 0,$$

for all $x \in \Omega_j$, $(j,i) \in \mathcal{S} \times \mathcal{S}$ and all $v \in \mathbb{R}^{d_v}$. The above inequality yields:

$$\begin{aligned}
&(A_j x + f_j + D_j v)^\top P_i (A_j x + f_j + D_j v) - x^\top P_j x \\
&\qquad \leq -x^\top Q x + (1+\gamma_2)(D_j v)^\top P_i (D_j v) - x^\top E_j^\top U_{ji} E_j x + \\
&\qquad\quad (1+\gamma_1) f_j^\top P_i f_j - x^\top M_{ji} x \\
&\qquad \leq -\lambda_{\min}(Q)\|x\|^2 + (1+\gamma_2) \max_{i\in\mathcal{S}} \lambda_{\max}(P_i) \max_{j\in\mathcal{S}} \|D_j\|^2 \|v\|^2 + \\
&\qquad\quad (1+\gamma_1) \max_{i\in\mathcal{S}} \lambda_{\max}(P_i) \max_{j\in\mathcal{S}} \|f_j\|^2.
\end{aligned} \tag{4.9}$$

Hence, $V(A_j x + f_j + D_j v) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + d_2$ for all $x \in \Omega_j$, $(j,i) \in \mathcal{S} \times \mathcal{S}$ and all $v \in \mathbb{R}^{d_v}$, where

$$\begin{aligned}
\alpha_3(\|x\|) &\triangleq \lambda_{\min}(Q)\|x\|^2, \\
\sigma(\|v\|) &\triangleq (1+\gamma_2) \max_{i\in\mathcal{S}} \lambda_{\max}(P_i) \max_{j\in\mathcal{S}} \|D_j\|^2 \|v\|^2, \\
d_2 &\triangleq (1+\gamma_1) \max_{i\in\mathcal{S}} \lambda_{\max}(P_i) \max_{j\in\mathcal{S}} \|f_j\|^2.
\end{aligned}$$

[2] Note that this implies $\mathcal{S}_0 \subseteq \mathcal{S}_{\text{lin}}$.

From (4.7) we also have that for all $(j,i) \in \mathcal{S} \times \mathcal{S}$,

$$\Delta_{ji} > 0 \Rightarrow \Xi_{ji} > 0 \Rightarrow x^\top (P_j - Q)x \geq 0 \quad \text{for all} \quad x \in \Omega_j.$$

Then, it follows that for all $j \in \mathcal{S}$ and all $x \in \Omega_j$:

$$\lambda_{\min}(Q)\|x\|^2 \leq x^\top Q x \leq x^\top P_j x \leq \max_{j\in\mathcal{S}} \lambda_{\max}(P_j)\|x\|^2,$$

which yields $\lambda_{\min}(Q) \triangleq c \leq b \triangleq \max_{j\in\mathcal{S}} \lambda_{\max}(P_j)$. Hence, the function $V(\cdot)$ defined in (4.6) satisfies the hypothesis of Theorem 2.4.4 with $d_1 = 0$ and $d_2 = (1+\gamma_1)\max_{i\in\mathcal{S}} \lambda_{\max}(P_i)\max_{j\in\mathcal{S}} \|f_j\|^2$. Then, the statement follows from Theorem 2.4.4.

(ii) To establish global ISS, we need to prove that in the above setting, we obtain $d_2 = 0$ under the additional hypothesis. For $j \in \mathcal{S}_{\text{lin}}$, if $x \in \Omega_j$ we obtain $d_2 = 0$ due to $f_j = 0$. For any $j \in \mathcal{S}_{\text{aff}}$, if $x \in \Omega_j$ it holds that $x \notin \cup_{i\in\mathcal{S}}\mathcal{E}_{ji}$. This yields:

$$(1+\gamma_1)f_j^\top P_i f_j - x^\top M_{ji} x \leq 0,$$

and thus, from the first inequality in (4.9) it follows that the function $V(\cdot)$ defined in (4.6) satisfies the hypothesis of Theorem 2.4.4 with $d_1 = d_2 = 0$. Then, the statement follows from Theorem 2.4.4.

(iii) This is a special case of part (ii). ■

The matrix Q gives the gain of the $\mathcal{K}$-function $\alpha_3(\cdot)$ and is related to the decrease of the state norm, and hence, to the transient behavior (see the result of Theorem 2.3.5). If ISpS (ISS) is the only goal, Q can be chosen less positive definite to reduce conservativeness of the LMI (4.8). The numbers γ_1, γ_2 and the matrices $\{P_j \mid j \in \mathcal{S}\}$ yield the constant $d_2 = (1+\gamma_1)\max_{i\in\mathcal{S}} \lambda_{\max}(P_i)\max_{j\in\mathcal{S}} \|f_j\|^2$ and the gain of the $\mathcal{K}$-function $\sigma(s) = (1+\gamma_2)\max_{i\in\mathcal{S}} \lambda_{\max}(P_i)\max_{j\in\mathcal{S}} \|D_j\|^2 s^2$. Note that a necessary condition for feasibility of the LMI (4.8) is $\gamma_1\gamma_2 > 1$. As it would be desirable to obtain a constant d_2 and gain of the function $\sigma(\cdot)$ as small as possible, one has to make a trade-off in choosing γ_1 and γ_2. One could add a cost criterion to (4.8) and specify γ_1, γ_2 as unknown variables in the resulting optimization problem, which might solve the trade-off. Although in this case (4.8) is a bilinear matrix inequality (i.e. due to $\gamma_1 P_i$, $\gamma_2 P_i$), since the unknowns γ_1, γ_2 are scalars, this problem can be solved efficiently via semi-definite programming solvers (software), e.g. (Sturm, 2001), (Löfberg, 2002), by setting lower and upper bounds for γ_1, γ_2 and doing bisections.

Note that, since we take $M_{ji} = 0$ for all $(j,i) \in \mathcal{S}_{\text{lin}} \times \mathcal{S}$, the LMI-based sufficient conditions for ISS (4.7) recover the LMI-based sufficient conditions for asymptotic stability presented in (Ferrari-Trecate et al., 2002) (for PWL systems) in the absence of disturbances (i.e. when $v_k = 0$ for all $k \in \mathbb{Z}_+$).

Consider now the following definition.

Definition 4.4.2 Let $\mathbb{V}$ be a compact and convex subset of $\mathbb{R}^{d_v}$. The system (4.5) is said to be *Globally Ultimately Bounded (GUB)* in a set $\mathcal{P} \subset \mathbb{R}^n$ if for all $x_0 \in \mathbb{R}^n$ and all $\{v_p\}_{p\in\mathbb{Z}_+}$ with $v_p \in \mathbb{V}$ for all $p \in \mathbb{Z}_+$, there exists an $i \in \mathbb{Z}_+$ such that the corresponding state trajectory satisfies $x_k \in \mathcal{P}$ for all $k \in \mathbb{Z}_{\geq i}$.

If the disturbance inputs are bounded, which is a reasonable assumption in practice, we show next that ISpS implies Global Ultimate Boundedness (GUB). This means that the ISpS property also implies the usual robust stability (convergence) property, e.g. as the one defined in (Grieder, 2004, Chp. 8.5), while the result of Theorem 4.4.1 part (i) applies to a more general class of PWA systems.

Assume that the additive disturbance input v_k takes values at all times $k \in \mathbb{Z}_+$ in a compact and convex subset $\mathbb{V}$ of $\mathbb{R}^n$ and, for any $(j,i) \in \mathcal{S}\times\mathcal{S}$, let

$$\xi_{ji} \triangleq (1+\gamma_1) f_j^\top P_i f_j + \max_{v\in\mathbb{V}} (1+\gamma_2) v^\top D_j^\top P_i D_j v.$$

Note that ξ_{ji} exists due to compactness of $\mathbb{V}$. For all $j \in \mathcal{S}$ define the set $\mathcal{M}_j \triangleq \cup_{i\in\mathcal{S}}\{x \in \Omega_j \mid x^\top M_{ji} x < \xi_{ji}\}$. Let $\mathcal{P} \triangleq \cup_{j\in\mathcal{S}}\mathcal{M}_j$ and let $\mathcal{R}_1(\mathcal{P}) \triangleq \{G(x,v) \mid x \in \mathcal{P},\ v \in \mathbb{V}\}$ be a robust one-step reachable set (Blanchini, 1994) for the PWA system (4.5).

Theorem 4.4.3 *Let* $\Upsilon \triangleq \max_{x\in\mathcal{R}_1(\mathcal{P})\cup\mathcal{P}} V(x)$, *let* $\mathcal{V}_\Upsilon \triangleq \{x \in \mathbb{R}^n \mid V(x) \leq \Upsilon\}$ *and suppose that the hypothesis of Theorem 4.4.1 including part (i) holds. Then, system* (4.5) *is GUB in the set* $\mathcal{V}_\Upsilon$.

Proof: As shown in the proof of Theorem 4.4.1, equation (4.9), for all $x \in \Omega_j$, $(j,i) \in \mathcal{S}\times\mathcal{S}$ we have

$$V(A_j x + f_j + D_j v) - V(x) \leq -\alpha_3(\|x\|) + (1+\gamma_2)(D_j v)^\top P_i (D_j v) + (1+\gamma_1) f_j^\top P_i f_j - x^\top M_{ji} x.$$

Then, for all $x \in \{\mathbb{R}^n \setminus \mathcal{P}\} \cap \Omega_j$ it holds that

$$V(A_j x + f_j + D_j v) - V(x) \leq -\alpha_3(\|x\|). \tag{4.10}$$

Now let $x_0 \notin \mathcal{V}_\Upsilon$. Since $V(x) \leq \Upsilon$ for all $x \in \mathcal{P}$ implies $\mathcal{P} \subseteq \mathcal{V}_\Upsilon$, it follows that $x_0 \notin \mathcal{P}$. We also have that $V(\cdot)$ satisfies condition (2.9a) with $d_1 = 0$, which, together with (4.10), implies via Theorem 2.4.4 part (ii) that there exists a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ such that the trajectory of system (4.5) satisfies $\|x_k\| \leq \beta(\|x_0\|, k)$ as long as $x_0, \ldots, x_k \notin \mathcal{P}$. Hence, there exists a finite $i \in \mathbb{Z}_+$ such that $x_i \in \mathcal{V}_\Upsilon$ for any $x_0 \notin \mathcal{V}_\Upsilon$. Next, we prove that $\mathcal{V}_\Upsilon$ is a Robust Positively Invariant (RPI) set for system (4.5). For any $x \in \mathcal{V}_\Upsilon \setminus \mathcal{P}$ it holds that

$$V(G(x,v)) \leq V(x) - \alpha_3(\|x\|) \leq V(x) \leq \Upsilon,$$

for all $v \in \mathbb{V}$. We also have that for any $x \in \mathcal{P}$, $G(x,v) \in \mathcal{R}_1(\mathcal{P})$ for all $v \in \mathbb{V}$, which yields $V(G(x,v)) \leq \Upsilon$. Thus, for any $x \in \mathcal{V}_\Upsilon$, it holds that $G(x,v) \in \mathcal{V}_\Upsilon$ for all $v \in \mathbb{V}$, which implies that $\mathcal{V}_\Upsilon$ is a RPI set for the PWA system (4.5) for all disturbances $v \in \mathbb{V}$. Hence, the PWA system (4.5) is GUB in the set $\mathcal{V}_\Upsilon$. ■

4.5 Synthesis

In this section we address the problem of input-to-state (practically) stabilizing controllers synthesis for perturbed discrete-time non-autonomous PWA systems:

$$x_{k+1} = g(x_k, u_k, v_k) \triangleq A_j x_k + B_j u_k + f_j + D_j v_k \quad \text{if} \quad x_k \in \Omega_j, \tag{4.11}$$

where $u_k \in \mathbb{R}^m$ is the input and $B_j \in \mathbb{R}^{n\times m}$ for all $j \in \mathcal{S}$. The nomenclature in (4.11) is similar with the one used in Section 4.4 for system (4.5).

In this section we take the control input as a PWL state-feedback control law of the form:

$$u_k \triangleq h(x_k) \triangleq K_j x_k \quad \text{if} \quad x_k \in \Omega_j, \tag{4.12}$$

where $K_j \in \mathbb{R}^{m\times n}$ for all $j \in \mathcal{S}$. The aim is to calculate the feedback gains $\{K_j \mid j \in \mathcal{S}\}$ such that the PWA closed-loop system (4.11)-(4.12) is globally ISpS and ISS, respectively. For this purpose we make use again of PWQ candidate ISpS (ISS) Lyapunov functions of the form (4.6).

For any $(j,i) \in \mathcal{S} \times \mathcal{S}$, consider now the following LMI:

$$\Delta_{ji} \triangleq \begin{pmatrix} \Delta_{ji}^{11} & \Delta_{ji}^{12} \\ \Delta_{ji}^{21} & \Delta_{ji}^{22} \end{pmatrix} > 0, \tag{4.13}$$

where

$$\Delta_{ji}^{11} \triangleq \begin{pmatrix} Z_j & * & * \\ -(A_j Z_j + B_j Y_j) & \gamma_1 Z_i & -Z_i \\ -(A_j Z_j + B_j Y_j) & -Z_i & \gamma_2 Z_i \end{pmatrix},$$

the term $*$ denotes $-(A_jZ_j + B_jY_j)^\top$ and, for $j \in \mathcal{S}_{\text{aff}}$

$$\Delta_{ji}^{22} \triangleq \text{diag}\left(\left[\begin{pmatrix} Z_i & 0 & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}, \begin{pmatrix} N_{ji} & 0 & 0 \\ 0 & N_{ji} & 0 \\ 0 & 0 & N_{ji} \end{pmatrix}\right]\right)$$

$$\Delta_{ji}^{12} = \Delta_{ji}^{21\top} \triangleq \begin{pmatrix} (A_jZ_j + B_jY_j)^\top & 0 & 0 & Z_j & 0 & 0 & Z_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

while for $j \in \mathcal{S}_{\text{lin}}$,

$$\Delta_{ji}^{22} \triangleq \text{diag}\left(\left[\begin{pmatrix} Z_i & 0 & 0 \\ 0 & Z_i & 0 \\ 0 & 0 & Z_i \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}\right]\right)$$

$$\Delta_{ji}^{12} = \Delta_{ji}^{21\top} \triangleq \begin{pmatrix} (A_jZ_j + B_jY_j)^\top & 0 & 0 & Z_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The operator $\text{diag}([L_1, \ldots, L_n])$ denotes a diagonal matrix of appropriate dimensions with the matrices $L_1, \ldots, L_n$ on the main diagonal, and the element 0 denotes everywhere a zero matrix of appropriate dimensions. The unknown variables in (4.13) are the matrices $Z_j \in \mathbb{R}^{n\times n}$, $j \in \mathcal{S}$, which are required to be positive definite and symmetric, the matrices $Y_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$, and the matrices N_{ji}, $(j,i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$, which are required to be positive definite and symmetric. The matrix Q is a known positive definite and symmetric matrix and the numbers $\gamma_1, \gamma_2 > 0$ with $\gamma_1\gamma_2 > 1$ play the same role as described in Section 4.4. For any $(j,i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$, define

$$\mathcal{E}_{ji} \triangleq \{x \in \mathbb{R}^n \mid x^\top N_{ji}^{-1} x < (1+\gamma_1) f_j^\top P_i f_j\}.$$

Theorem 4.5.1 *Let system (4.11), the matrix $Q > 0$ and the numbers $\gamma_1, \gamma_2 > 0$ with $\gamma_1\gamma_2 > 1$ be given. Suppose that the LMIs*

$$\Delta_{ji} > 0, \quad (j,i) \in \mathcal{S} \times \mathcal{S} \tag{4.14}$$

are feasible and let $\{Z_j, Y_j \mid j \in \mathcal{S}\}$ and $\{N_{ji} \mid (j,i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}\}$ be a solution. For all $j \in \mathcal{S}$ let $P_j \triangleq Z_j^{-1}$ and let $K_j \triangleq Y_jZ_j^{-1}$. For all $(j,i) \in \mathcal{S}_{\text{lin}} \times \mathcal{S}$ take $M_{ji} = 0$. For all $(j,i) \in \mathcal{S}_{\text{aff}} \times \mathcal{S}$ take $M_{ji} = N_{ji}^{-1}$. Then, it holds that:

(i) *The closed-loop system* (4.11)-(4.12) *is globally ISpS;*

(ii) *If* $(\cup_{i\in\mathcal{S}}\mathcal{E}_{ji})\cap\Omega_j=\emptyset$ *for all* $j\in\mathcal{S}_{\text{aff}}$*, then the closed-loop system* (4.11)-(4.12) *is globally ISS;*

(iii) *If system* (4.11) *is PWL, i.e.* $\mathcal{S}_{lin}=\mathcal{S}$*, then the closed-loop system* (4.11)-(4.12) *is globally ISS.*

Proof: By applying the Schur complement (Boyd et al., 1994) to (4.14), for any $(j,i)\in\mathcal{S}\times\mathcal{S}$ we obtain

$$\Delta_{ji}^{11}-\Delta_{ji}^{21\top}\Delta_{ji}^{22-1}\Delta_{ji}^{21}>0,$$

which yields the equivalent matrix inequality:

$$\Phi_{ji}\triangleq\begin{pmatrix}\Gamma_{ji} & * & * \\ -(A_jZ_j+B_jY_j) & \gamma_1 Z_i & -Z_i \\ -(A_jZ_j+B_jY_j) & -Z_i & \gamma_2 Z_i\end{pmatrix}>0, \tag{4.15}$$

where the term $*$ denotes $-(A_jZ_j+B_jY_j)^\top$ and

$$\begin{aligned}\Gamma_{ji}\triangleq{}& Z_j-(A_jZ_j+B_jY_j)^\top Z_i^{-1}(A_jZ_j+B_jY_j)\\ &-Z_jQZ_j-Z_jN_{ji}^{-1}Z_j.\end{aligned}$$

By pre- and post-multiplying (4.15) with $\begin{pmatrix}Z_j^{-1} & 0 & 0\\ 0 & Z_i^{-1} & 0\\ 0 & 0 & Z_i^{-1}\end{pmatrix}$ and by substituting Z_j^{-1} with P_j, $Y_jZ_j^{-1}$ with K_j and N_{ji}^{-1} with M_{ji} turns inequality (4.15) into the equivalent matrix inequality:

$$\begin{pmatrix}\Xi_{ji} & * & * \\ -P_i(A_j+B_jK_j) & \gamma_1 P_i & -P_i \\ -P_i(A_j+B_jK_j) & -P_i & \gamma_2 P_i\end{pmatrix}>0,$$

for all $(j,i)\in\mathcal{S}\times\mathcal{S}$, where the term $*$ denotes $-(A_j+B_jK_j)^\top P_i$ and

$$\Xi_{ji}\triangleq P_j-(A_j+B_jK_j)^\top P_i(A_j+B_jK_j)-Q-M_{ji}.$$

Then, it follows that the LMI (4.8) is feasible for the closed-loop system (4.11)-(4.12) for all $(j,i)\in\mathcal{S}\times\mathcal{S}$. The rest of the proof is analogous to the proof of Theorem 4.4.1. ■

Note that using the same reasoning employed in the proof of Theorem 4.4.3, it can be proven that if the hypothesis of Theorem 4.5.1 part (i) is satisfied, then the closed-loop system (4.11)-(4.12) is GUB.

4.6 Illustrative examples

In this section we illustrate the theoretical results presented in Section 4.4 and Section 4.5 by means of two simulated examples.

4.6.1 Example 1

In this example we illustrate the S-procedure relaxation and the result of Theorem 4.4.1 part (iii). Consider the following perturbed PWL system:

$$x_{k+1} = \begin{cases} A_1 x_k + v_k & \text{if} \quad E_1 x_k > 0 \\ A_2 x_k + v_k & \text{if} \quad E_2 x_k \geq 0 \\ A_3 x_k + v_k & \text{if} \quad E_3 x_k > 0 \\ A_4 x_k + v_k & \text{if} \quad E_4 x_k \geq 0, \end{cases} \tag{4.16}$$

where all inequalities hold componentwise, $A_1 = \begin{bmatrix} 0.5 & 0.61 \\ 0.9 & 1.345 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.92 & 0.644 \\ 0.758 & -0.71 \end{bmatrix}$, $A_3 = A_1$ and $A_4 = A_2$. The state-space partition of system (4.16) is given by $E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ and $E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. Searching for a common quadratic or a PWQ without the S-relaxation ISS Lyapunov function did not succeed for system (4.16). However, by solving the LMI (4.7) for $Q = 10^{-4} I_2$, $\gamma_1 = 100$ and $\gamma_2 = 11$ we obtained the following PWQ with an S-relaxation ISS Lyapunov function $V(x) = x^\top P_j x$ when $x \in \Omega_j$, $j = 1, 2, 3, 4$:

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.1845 & 0.0494 \\ 0.0494 & 0.0335 \end{bmatrix}, \ P_2 = \begin{bmatrix} 0.0851 & -0.0110 \\ -0.0110 & 0.0336 \end{bmatrix}, \\ P_3 &= P_1, \quad P_4 = P_2, \\ U_{11} &= \begin{bmatrix} 0.0119 & 0.0519 \\ 0.0519 & 0.0223 \end{bmatrix}, \ U_{12} = \begin{bmatrix} 0.0120 & 0.0540 \\ 0.0540 & 0.0053 \end{bmatrix}, \\ U_{21} &= \begin{bmatrix} 0.0035 & 0.0048 \\ 0.0048 & 0.0041 \end{bmatrix}, \ U_{22} = 10^{-3} \begin{bmatrix} 0.1185 & 0.2265 \\ 0.2265 & 0.3749 \end{bmatrix}. \end{aligned}$$

States trajectories for system (4.16) with initial state $x_0 = [-10 \ 10]^\top$ are plotted in Figure 4.5 together with the additive disturbance inputs history. The disturbance inputs were randomly generated in the interval [0 1] until sampling time 70 and then they were set equal to zero. The gain of the function $\sigma(\cdot)$ corresponding to $\gamma_2 = 11$ is 2.3911. This yields an ISS gain equal to 15.4243 for system (4.16) via the relation $\gamma(s) \triangleq \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right)$ established in Chapter 2. As guaranteed by Theorem 4.4.1, system (4.16) is globally ISS, which ensures Lyapunov asymptotic stability when the disturbance inputs

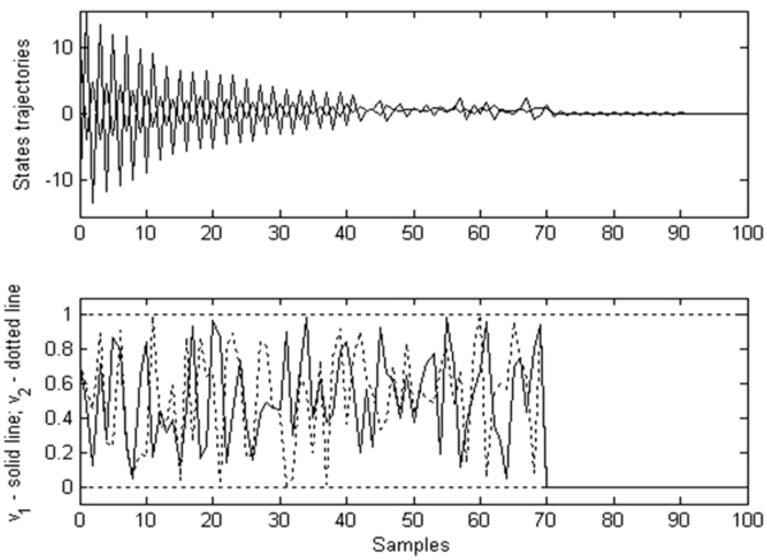

Figure 4.5: States trajectories and disturbances histories for system (4.16).

converge to zero, as it can be observed from the states trajectories plot given in Figure 4.5.

4.6.2 Example 2

In this example we illustrate the result of Theorem 4.5.1 part (ii). Let

$$A(T_s) \triangleq \begin{pmatrix} 1 & T_s & \frac{T_s^2}{2!} & \frac{T_s^3}{3!} \\ 0 & 1 & T_s & \frac{T_s^2}{2!} \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; B(T_s) \triangleq \begin{pmatrix} \frac{T_s^4}{4!} \\ \frac{T_s^3}{3!} \\ \frac{T_s^2}{2!} \\ T_s \end{pmatrix}$$

denote the dynamics corresponding to a discrete-time quadruple integrator, i.e. $x_{k+1} = A(T_s)x_k + B(T_s)u_k$, obtained from a continuous-time quadruple integrator via a sampled-and-hold device with sampling period $T_s > 0$. Let x_i, $i = 1, 2, 3, 4$, denote the i-th component of the state vector. Let $\mathbb{X} \triangleq \{x \in \mathbb{R}^4 \mid -2 < x_4 < 2\}$, let $\Omega_1 \triangleq \{x \in \mathbb{R}^4 \mid x_4 \geq 2\}$ and let $\Omega_4 \triangleq \{x \in \mathbb{R}^4 \mid x_4 \leq -2\}$. Let $\Omega_2 \triangleq \{x \in \mathbb{X} \mid x_4 \geq 0\}$ and $\Omega_3 \triangleq \{x \in \mathbb{X} \mid x_4 < 0\}$. Consider now the following perturbed piecewise affine system:

$$x_{k+1} = \begin{cases} A_1 x_k + B_1 u_k + f_1 + D_1 v_k & \text{if} \quad x_k \in \Omega_1 \\ A_2 x_k + B_2 u_k + f_2 + D_2 v_k & \text{if} \quad x_k \in \Omega_2 \\ A_3 x_k + B_3 u_k + f_3 + D_3 v_k & \text{if} \quad x_k \in \Omega_3 \\ A_4 x_k + B_4 u_k + f_4 + D_4 v_k & \text{if} \quad x_k \in \Omega_4, \end{cases} \tag{4.17}$$

where $A_1 = A_4 = A(1.2), B_1 = B_4 = B(1.2)$, $A_2 = A(0.9), B_2 = B(0.9)$, $A_3 = A(0.8), B_3 = B(0.8)$, $f_2 = f_3 = 0$, $f_1 = f_4 = [0.1\,0.1\,0.1\,0.1]^\top$ and $D_1 = D_2 = D_3 = D_4 = [1\,1\,1\,1]^\top$. The LMIs (4.14) were solved[3] for $Q = 0.01 I_4$, $\gamma_1 = 2$ and $\gamma_2 = 4$, yielding the following weights of the PWQ ISS Lyapunov function $V(x) = x^\top P_j x$ if $x \in \Omega_j$, $j = 1, 2, 3, 4$, feedbacks $\{K_j \mid j = 1, 2, 3, 4\}$ and matrix M:

$$P_1 = P_4 = \begin{bmatrix} 0.3866 & 0.7019 & 0.5532 & 0.1903 \\ 0.7019 & 1.5632 & 1.3131 & 0.4688 \\ 0.5532 & 1.3131 & 1.2255 & 0.4552 \\ 0.1903 & 0.4688 & 0.4552 & 0.1955 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.3574 & 0.6052 & 0.4420 & 0.1407 \\ 0.6052 & 1.2725 & 0.9894 & 0.3278 \\ 0.4420 & 0.9894 & 0.8812 & 0.3046 \\ 0.1407 & 0.3278 & 0.3046 & 0.1328 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.3779 & 0.6410 & 0.4597 & 0.1453 \\ 0.6410 & 1.3414 & 1.0298 & 0.3390 \\ 0.4597 & 1.0298 & 0.9007 & 0.3118 \\ 0.1453 & 0.3390 & 0.3118 & 0.1334 \end{bmatrix},$$

$$K_1 = K_4 = \begin{bmatrix} -0.3393 & -1.1789 & -1.8520 & -1.7028 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.5584 & -1.7607 & -2.4729 & -2.0012 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.6814 & -2.0895 & -2.8249 & -2.1705 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.0156 & 0.0075 & 0.0023 & 0.0005 \\ 0.0075 & 0.0212 & 0.0082 & 0.0016 \\ 0.0023 & 0.0082 & 0.0146 & 0.0044 \\ 0.0005 & 0.0016 & 0.0044 & 0.0081 \end{bmatrix}.$$

One can easily establish that the hypothesis of Theorem 4.5.1 part (ii) is satisfied, i.e. $\mathcal{E}_{1i} \cap \Omega_1 = \emptyset$ and $\mathcal{E}_{4i} \cap \Omega_4 = \emptyset$ for all $i = 1, 2, 3, 4$, by observing

[3] For simplicity we used a common matrix N for all possible mode transitions that can occur when the state is in mode one or mode four, i.e. $N = N_{11} = N_{12} = N_{13} = N_{44} = N_{42} = N_{43}$, which yields $M = N^{-1}$.

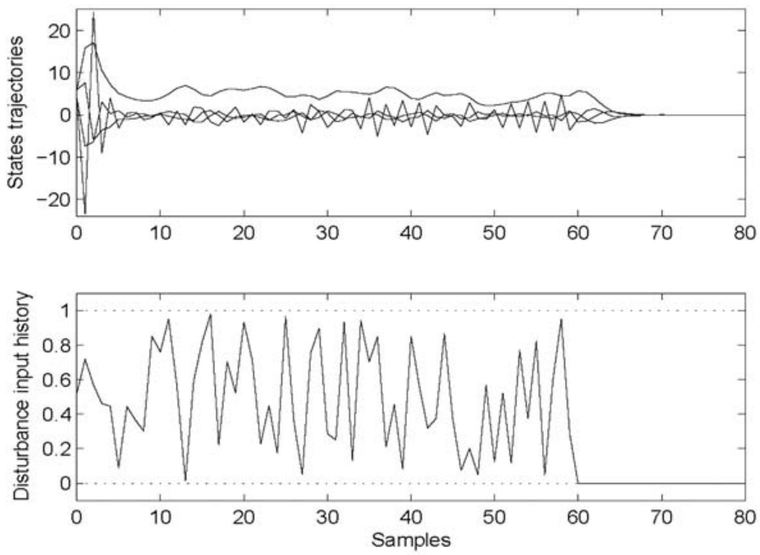

Figure 4.6: States trajectories and disturbance input histories for the closed-loop system (4.17)-(4.12).

that

$$\begin{aligned}\min_{x\in\Omega_1} x^\top M x &= \min_{x\in\Omega_4} x^\top M x \\ &= 0.4340 > 0.3221 = \max_{i=1,2,3,4}(1+\gamma_1) f_1^\top P_i f_1 = \max_{i=1,2,3,4}(1+\gamma_1) f_4^\top P_i f_4.\end{aligned}$$

Hence, system (4.17) in closed-loop with (4.12) is globally ISS. The gain of the function $\sigma(\cdot)$ corresponding to $\gamma_2 = 4$ is 15.8772. This yields an ISS gain equal to 42.52 for system (4.17)-(4.12) via the relation $\gamma(s) = \alpha_1^{-1}\left(\frac{2\sigma(s)}{1-\rho}\right) = 42.52s$ established in chapter two. The closed-loop states trajectories obtained for initial state $x_0 = [6\ 6\ 4\ 4]^\top$ are plotted in Figure 4.6 together with the additive disturbance input history. The disturbance input was randomly generated in the interval [0 1] until sampling time 60 and then it was set equal to zero. As guaranteed by Theorem 4.5.1, the closed-loop system (4.17)-(4.12) is globally ISS, which ensures asymptotic stability in the Lyapunov sense when the disturbance inputs converges to zero, as it can be observed in Figure 4.6.

4.7 Concluding remarks

In this chapter we presented LMI based sufficient conditions for global input-to-state (practical) stability and stabilization of discrete-time perturbed, possibly discontinuous, PWA systems. The importance of these results cannot be overstated since we have showed that nominally exponentially stable discrete-time PWA systems can have zero robustness to arbitrarily small additive disturbances and hence, special precautions must be taken when implementing stabilizing controllers for PWA systems in practice. The developed methodology has a wide applicability, including, besides the class of PWA systems, any hybrid system that can be transformed into an equivalent PWA form, e.g. mixed logical dynamical systems or linear complementarity systems.

State and input constraints have not been considered in order to obtain global ISpS (ISS) results. However, the usual LMI techniques (Boyd et al., 1994) for specifying state and/or input constraints can be added to the sufficient conditions presented in this paper, resulting in local ISpS (ISS) of constrained PWA systems. Also, a local (i.e. in some subset of $\cup_{j\in\mathcal{S}_0}\Omega_j$) ISS result is obtained under the hypothesis of Theorem 4.4.1 (Theorem 4.5.1) part (ii), in the case when $\mathcal{E}_{ji}\cap\Omega_j\neq\emptyset$ for some $(j,i)\in\mathcal{S}_{\text{aff}}\times\mathcal{S}$.

For simplicity and clarity of exposition we employed PWQ (with an S-procedure relaxation for analysis) candidate ISpS (ISS) Lyapunov functions of the form (4.6). However, the results can be extended to piecewise polynomial or piecewise affine candidate ISpS (ISS) Lyapunov functions. The future work deals with extensions to PWA systems affected by parametric uncertainties and the use of norm based candidate ISS Lyapunov functions.

5

Robust stabilization of discontinuous piecewise affine systems using model predictive control

This chapter employs the input-to-state stability (ISS) framework to investigate the robustness of discrete-time (discontinuous) piecewise affine (PWA) systems in closed-loop with stabilizing model predictive controllers (MPC). The importance of these issues cannot be overstated, since nominally stabilizing controllers are always affected by perturbations when applied in practice.

5.1 Introduction

A certain maturity was reached in the field of MPC for hybrid systems, regarding computational and nominal stability aspects. This is illustrated by the existing tools for solving hybrid MPC optimization problems, the Hybrid Toolbox (HT)(Bemporad, 2003) and the Multi Parametric Toolbox (MPT) (Kvasnica et al., 2004), and by the stability results published in the literature (see Chapter 3 for details). In this chapter we focus on *inherent robustness* of discrete-time PWA systems in closed-loop with MPC controllers (or hybrid MPC for short), which is a problem that was not addressed before in the literature. By the inherent robustness property we mean that a nominally stabilizing controller has some robustness in the presence of perturbations. Its importance cannot be overstated, since all controllers designed to be nominally stable are affected by perturbations when applied in practice. The goal is to provide an answer to the following important questions:

- Is it possible that *nominally stabilizing* MPC can generate MPC value functions that are *not* ISS Lyapunov functions?

- If the answer is yes, how can one test whether or not a specific nominally stabilizing hybrid MPC scheme, possibly with a discontinuous value function, has some robustness (i.e. how can one test if the MPC value function is a ISS Lyapunov function, despite its discontinuity)?

- In case such a test fails, how can one design MPC schemes for hybrid systems with an a priori robust stability guarantee?

Inherent robustness has been studied in MPC for linear and smooth nonlinear systems. In (Grimm et al., 2004) the authors proved that linear systems in closed-loop with stabilizing MPC are inherently robust due to the presence of a *continuous MPC value function*, which is an ISS Lyapunov function in this case. However, they also shown via examples that continuous and necessarily nonlinear systems in closed-loop with MPC can actually have zero robustness to arbitrarily small disturbances, in the absence of a continuous MPC value function[1]. The first contribution of this chapter is to issue a warning by presenting an example of a PWA system in closed-loop with a stabilizing MPC controller which generates an MPC value function that is *discontinuous* and, more importantly, it is *not* an ISS Lyapunov function. This indicates that the natural way to ensure ISS (robustness) in MPC fails for PWA systems. This brings us to the second and third questions above.

Several solutions that rely on *continuous* (or even *Lipschitz continuous*) system dynamics and/or MPC value functions were proposed for smooth nonlinear systems in the literature, e.g. see (Scokaert et al., 1997; Limon et al., 2002a; Grimm et al., 2003b, 2004), regarding the second and the third question posed here. Since hybrid systems are inherently nonlinear and discontinuous, and hybrid MPC value functions are discontinuous in general, these results are not applicable in hybrid MPC. Moreover, there is no systematic method in MPC in general for achieving inherent robustness, while allowing for discontinuous system dynamics and/or MPC value functions. Therefore, the second contribution of this chapter is an a posteriori test that can be used to check if a given nominally stabilizing hybrid MPC scheme, possibly with a discontinuous value function, is inherently robust. This test consists in solving a finite number of Linear Programming (LP) problems and can be performed once the explicit form of the MPC controller, its fea-

[1] The value function corresponding to the MPC cost is usually used as the candidate Lyapunov function to prove nominal stability.

sible set and the MPC value function are computed (see (Borrelli, 2003; Bemporad, 2003; Kvasnica et al., 2004) for details and software tools).

If the a posteriori robustness test fails for a specific hybrid MPC closed-loop system, there are no systematic ways available for discontinuous PWA systems that can be used to modify nominally stabilizing MPC schemes so that robustness is ensured. The third contribution of this chapter is a design method for setting up hybrid MPC schemes with an a priori ISS guarantee with respect to bounded additive disturbance inputs. This method restricts the predicted future states to a tightened, possibly disconnected subset of the state-space, and does not require continuity of the MPC value function, nor of the PWA system dynamics. Note that tightened constraints were used before in order to *ensure robust feasibility only*, in smooth nonlinear MPC (Limon et al., 2002a). In this chapter, however, an extension of this technique is employed for discontinuous PWA systems to achieve *both robust feasibility and ISS* (and thus, robustness to additive disturbance inputs).

A remark is dedicated to the results of (Kerrigan and Mayne, 2002) and (Rakovic and Mayne, 2004) that deal with dynamic programming and tube based, respectively, approaches for solving feedback *min-max* MPC problems for *continuous* PWA systems, and also provide a robust stability guarantee. However, these results are not really appealing for hybrid systems, as *continuous PWA systems* are *Lipschitz continuous systems* and, in this case, the previous results of (Scokaert et al., 1997; Limon et al., 2002a; Grimm et al., 2003b, 2004) also apply. In this chapter we use a different approach that does not resort to min-max formulations, to develop robust MPC schemes for *discontinuous* PWA systems, which is relevant for hybrid systems.

5.2 Preliminaries

We consider nominal and perturbed discrete-time PWA systems of the form:

$$x_{k+1} = g(x_k, u_k) \triangleq A_j x_k + B_j u_k + f_j \quad \text{when} \quad x_k \in \Omega_j, \tag{5.1a}$$

$$\tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, u_k, w_k) \triangleq A_j \tilde{x}_k + B_j u_k + f_j + w_k \quad \text{when} \quad \tilde{x}_k \in \Omega_j, \tag{5.1b}$$

where $w_k \in \mathbb{W} \subset \mathbb{R}^n$, $k \in \mathbb{Z}_+$ denotes an unknown *additive* disturbance input, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$ and $f_j \in \mathbb{R}^n$, $j \in \mathcal{S}$ with $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ a *finite set* of indexes. We assume that $\mathbb{W}$ is a bounded polyhedral set that contains the origin, and the state and the input are constrained in some polyhedral C-sets $\mathbb{X}$ and $\mathbb{U}$. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\text{int}(\Omega_i) \cap \text{int}(\Omega_j) = \emptyset$ for

$i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 \triangleq \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 \triangleq \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (5.1) with $u = 0$ and therefore we require that $f_j = 0$ for all $j \in \mathcal{S}_0$. Note that this includes PWA systems that are *discontinuous over the boundaries* of the state-space regions Ω_j, $j \in \mathcal{S}$.

Although we focus on PWA systems of the form (5.1), the results developed in this chapter have a wider applicability since it is known (Heemels et al., 2001) that PWA systems are equivalent under certain mild assumptions with other relevant classes of hybrid systems, such as mixed logical dynamical systems (Bemporad and Morari, 1999) and linear complementarity systems (van der Schaft and Schumacher, 1998).

Next, consider the case when the MPC methodology is used to generate the control input u_k, $k \in \mathbb{Z}_+$, in (5.1). For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by the nominal PWA system (5.1a) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$, where $\mathbb{U}^N \triangleq \mathbb{U} \times \ldots \times \mathbb{U}$. Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ denote a desired polyhedral target set. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$, $k \in \mathbb{Z}_+$, is

$$\mathcal{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid \mathbf{x}_k(x_k, \mathbf{u}_k) \in \mathbb{X}^N,\ x_{N|k} \in \mathbb{X}_T\}.$$

For the remainder of this chapter *let* $\|\cdot\|$ *denote the* ∞*-norm* for shortness. Consider now the functions $F(x) \triangleq \|P_j x\|$ when $x \in \Omega_j$ and $L(x, u) \triangleq \|Qx\| + \|Ru\|$, where $P_j \in \mathbb{R}^{p_j \times n}$, $j \in \mathcal{S}$, $Q \in \mathbb{R}^{q \times n}$ and $R \in \mathbb{R}^{r \times n}$ are assumed to be known matrices that have full-column rank.

Problem 5.2.1 Let $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost

$$J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}),$$

with prediction model (5.1a), over all sequences $\mathbf{u}_k$ in $\mathcal{U}_N(x_k)$.

We call an initial state $x_0 \in \mathbb{X}$ *feasible* if $\mathcal{U}_N(x_0) \neq \emptyset$. Similarly, Problem 5.2.1 is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to Problem 5.2.1 and let $\widehat{V} : \mathbb{X}_f(N) \rightarrow \mathbb{R}_+$, $\widehat{V}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k)$ denote the MPC value function corresponding to Problem 5.2.1. Suppose there exists[2] an optimal

[2]As explained in Section 3.3.2, this assumption is satisfied for PWA prediction models.

sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ for Problem 5.2.1 and any state $x_k \in \mathbb{X}_f(N)$. Then, $\widehat{V}(x_k) = J(x_k, \mathbf{u}_k^*)$ and the *MPC control law* is obtained as

$$\hat{u}(x_k) \triangleq u_{0|k}^*; \quad k \in \mathbb{Z}_+. \tag{5.2}$$

Consider an auxiliary state feedback control law $h_{\text{aux}} : \mathbb{R}^n \to \mathbb{R}^m$ with $h_{\text{aux}}(0) = 0$, which is usually employed in proving stability of *terminal cost and constraint set* MPC. In the PWA setting we take this state feedback PWL, i.e. $h_{\text{aux}}(x) \triangleq K_j x$ when $x \in \Omega_j$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$. Let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid h_{\text{aux}}(x) \in \mathbb{U}\}$ and let $\mathbb{X}_{\text{PI}}$ with $0 \in \text{int}(\mathbb{X}_{\text{PI}})$ be a Positively Invariant (PI) set for system (5.1a) in closed-loop with h_{aux} that is contained in $\mathbb{X}_{\mathbb{U}}$. Consider now the following assumption.

Assumption 5.2.2 There exist $\{P_j, K_j \mid j \in \mathcal{S}\}$ such that

$$\|P_i(A_j + B_jK_j)x + P_i f_j\| - \|P_j x\| + \|Qx\| + \|RK_j x\| \leq 0, \tag{5.3}$$

for all $x \in \mathbb{X}_{\text{PI}}$ and all $(j, i) \in \mathcal{S} \times \mathcal{S}$.

The requirement (5.3) is obtained from[3] Assumption 3.3.1, inequality (3.4c), for the particular case of a ∞-norm MPC cost and $h(x) = K_j x$ if $x \in \Omega_j$. Note that Assumption 5.2.2 implies that the origin of the PWA system (5.1a) is stabilizable.

Theorem 5.2.3 *Suppose that Assumption 5.2.2 holds and take $\mathbb{X}_T = \mathbb{X}_{PI}$. Then, the PWA system* (5.1a) *in closed-loop with the MPC controller* (5.2) *is asymptotically stable in the Lyapunov sense for initial conditions in $\mathbb{X}_f(N)$.*

The proof of Theorem 5.2.3, which can be obtained as a particular case of the proof of Theorem 3.3.3, relies on the fact that Assumption 5.2.2 is equivalent to

$$F(g(x, h_{\text{aux}}(x))) - F(x) + L(x, h_{\text{aux}}(x)) \leq 0, \quad \forall x \in \mathbb{X}_T.$$

This in turn ensures that the hybrid MPC value function $\widehat{V}(\cdot)$ is a *Lyapunov function* for the closed-loop system (5.1a)-(5.2), i.e. there exist $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\alpha_3(s) \triangleq cs^\lambda$, with $a, b, c, \lambda > 0$, such that $\alpha_1(\|x\|) \leq \widehat{V}(x) \leq \alpha_2(\|x\|)$ and $\widehat{V}(g(x, \hat{u}(x))) - \widehat{V}(x) \leq -\alpha_3(\|x\|)$ for all $x \in \mathbb{X}_f(N)$. See, for example, Chapter 3 for more details.

[3] This is the general assumption of Chapter 3 that contains the sufficient conditions for asymptotic stability in the Lyapunov sense of MPC of hybrid systems.

In the above setting, Theorem 8.4 of (Borrelli, 2003) states that the MPC control law $\hat{u}(\cdot)$ defined in (5.2) is a PWA state-feedback. Hence, the resulting *hybrid MPC closed-loop system* is a PWA system, i.e.

$$\begin{aligned} x_{k+1} = g(x_k, \hat{u}(x_k)) &= A_j x_k + B_j \hat{u}(x_k) + f_j, \\ \hat{u}(x_k) &= L_i x_k + l_i \quad \text{when} \quad x_k \in \Omega_j \cap \overline{\Omega}_i, \end{aligned} \tag{5.4a}$$

$$\begin{aligned} \tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, \hat{u}(\tilde{x}_k), w_k) &= A_j \tilde{x}_k + B_j \hat{u}(\tilde{x}_k) + f_j + w_k, \\ \hat{u}(\tilde{x}_k) &= L_i \tilde{x}_k + l_i \quad \text{when} \quad \tilde{x}_k \in \Omega_j \cap \overline{\Omega}_i, \end{aligned} \tag{5.4b}$$

with $(j,i) \in \mathcal{S} \times \overline{\mathcal{S}}$ ($\overline{\mathcal{S}}$ is a finite set of indexes), $k \in \mathbb{Z}_+$, where $L_i \in \mathbb{R}^{m\times n}$, $l_i \in \mathbb{R}^m$, and $\cup_{i\in\overline{\mathcal{S}}}\overline{\Omega}_i = \mathbb{X}_f(N)$ (with $\text{int}(\overline{\Omega}_i) \cap \text{int}(\overline{\Omega}_j) = \emptyset$ for $i \neq j$) is a new partition corresponding to the explicit MPC control law. Moreover, the MPC value function $\widehat{V}(\cdot)$ is a PWA function (recall that $\|\cdot\|$ denotes the ∞-norm), i.e.

$$\widehat{V}(x) \triangleq E_j x + e_j \text{ when } x \in \widehat{\Omega}_j, \tag{5.5}$$

where $E_j \in \mathbb{R}^{1\times n}$, $e_j \in \mathbb{R}$, j takes values in some finite set of indexes $\widehat{\mathcal{S}}$, and $\cup_{j\in\widehat{\mathcal{S}}}\widehat{\Omega}_j = \mathbb{X}_f(N)$ (with $\text{int}(\widehat{\Omega}_i) \cap \text{int}(\widehat{\Omega}_j) = \emptyset$ for $i \neq j$) is a new partition corresponding to the MPC value function.

5.3 A motivating example

As already mentioned in Chapter 4, in the linear and continuous nonlinear case, nominally stable systems generically have some robustness properties. This is usually due to the presence of a continuous Lyapunov function, which is also an ISS Lyapunov function in this case. Unfortunately, the classical continuity based inherent robustness (ISS) argument no longer holds if a Lyapunov function $V(\cdot)$ is discontinuous at some points.

Note that discontinuity of the candidate Lyapunov function $V(\cdot)$ does not necessarily obstruct the sufficient conditions for ISS established in Chapter 2 to hold. However, we show via an example from literature that stabilizing hybrid MPC can generate discontinuous value functions that are not ISS Lyapunov functions for perturbed systems of the form (5.4b). As in (5.4) and (5.5), the following notation will be used: for $i \in \overline{\mathcal{S}}$ and $j \in \widehat{\mathcal{S}}$, $\hat{u}_i(x) \triangleq L_i x + l_i$ and $\widehat{V}_j(x) \triangleq E_j x + e_j$ for any $x \in \mathbb{X}$.

Consider the following discontinuous PWA system, taken from (Mignone et al., 2000b):

$$x_{k+1} = \begin{cases} A_1 x_k + B u_k & \text{if} \quad D_1 x_k > 0 \\ A_2 x_k + B u_k & \text{if} \quad D_2 x_k \geq 0 \\ A_3 x_k + B u_k & \text{if} \quad D_3 x_k > 0 \\ A_4 x_k + B u_k & \text{if} \quad D_4 x_k \geq 0 \end{cases} \tag{5.6}$$

where $A_1 = \left[\begin{smallmatrix} -0.04 & -0.461 \\ -0.139 & 0.341 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0.936 & 0.323 \\ 0.788 & -0.049 \end{smallmatrix}\right]$, $A_3 = \left[\begin{smallmatrix} -0.857 & 0.815 \\ 0.491 & 0.62 \end{smallmatrix}\right]$, $A_4 = \left[\begin{smallmatrix} -0.022 & 0.644 \\ 0.758 & 0.271 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]^\top$, $D_1 = -D_3 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $D_2 = -D_4 = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and all inequalities hold componentwise. The state and the input of system (5.6) are constrained at all times in the sets $\mathbb{X} = [-10, 10] \times [-10, 10]$ and $\mathbb{U} = [-1, 1]$, respectively. The method presented in chapter three, Section 3.4.2, was employed to compute a common terminal weight matrix $P = P_1 = P_2 = P_3 = P_4$ and feedbacks $\{K_j \mid j = 1, \ldots, 4\}$ such that inequality (5.3) of Assumption 5.2.2 holds for the stage cost weights $Q = \text{diag}([1\ 1])$ and $R = 0.1$. The following matrices were obtained:

$$\begin{aligned} P &= \begin{bmatrix} 6.7001 & 3.1290 \\ -2.1107 & 4.1998 \end{bmatrix}, \ K_1 = \begin{bmatrix} 0.2703 & -0.1136 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -0.8042 & -0.2560 \end{bmatrix}, K_3 = \begin{bmatrix} 1.0122 & -0.7513 \end{bmatrix}, \\ K_4 &= \begin{bmatrix} -0.5548 & -1.1228 \end{bmatrix}. \end{aligned} \tag{5.7}$$

Then, we used the MPT (Kvasnica et al., 2004), which implements the algorithm of (Rakovic et al., 2004), to calculate the terminal constraint set $\mathbb{X}_T$ as the maximal positively invariant set contained in $\mathbb{X}_\mathbb{U}$ for system (5.6) in closed-loop with $u_k(x_k) = h_{\text{aux}}(x_k)$, $k \in \mathbb{Z}_+$, with the feedbacks given in (5.7), and where $\mathbb{X}_\mathbb{U} = \cup_{j=1,\ldots,4}\{x \in \Omega_j \mid K_j x \in \mathbb{U}\}$. By Theorem 5.2.3, this is sufficient to guarantee that the MPC closed-loop system (5.6)-(5.2) is asymptotically stable in the Lyapunov sense for all $x \in \mathbb{X}_f(N)$, $N \in \mathbb{Z}_{\geq 1}$. Then, the MPT was used to calculate the MPC control law (5.2) for $N = 1$ as an explicit PWA state-feedback, and to simulate the resulting PWA MPC closed-loop system (5.4a). The explicit MPC controller is defined over 86 state-space regions $\overline{\Omega}_i$, $i \in \overline{\mathcal{S}} \triangleq \{1, \ldots, 86\}$ that satisfy $\cup_{i \in \overline{\mathcal{S}}} \overline{\Omega}_i = \mathbb{X}_f(1)$. The set of feasible states $\mathbb{X}_f(1)$ is plotted in Figure 5.1 together with the partition corresponding to the explicit MPC control law.

Lemma 5.3.1 *For the MPC closed-loop system* (5.4) *corresponding to system* (5.6) *it holds that:*

(i) *The value function* $\widehat{V}(\cdot)$ *and the closed-loop dynamics* (5.4a) are not continuous*;*

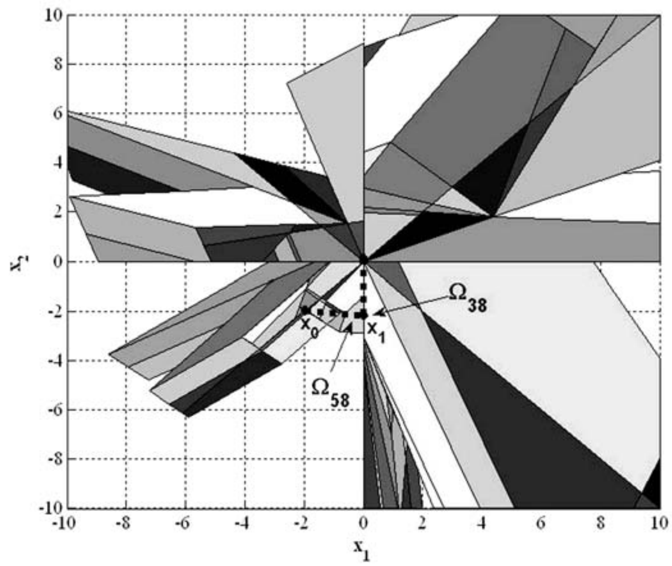

Figure 5.1: The feasible set $\mathbb{X}_f(1)$ *and state trajectory for the PWA MPC closed-loop system* (5.6)-(5.2) *with* $x_0 = [-1.9649 \ \ -1.9649]^\top$.

(ii) $\widehat{V}(\cdot)$ is a Lyapunov function *for the closed-loop dynamics* (5.4a) *showing asymptotic stability in* $\mathbb{X}_f(1)$*;*

(iii) *For any* $\varepsilon > 0$, $\widehat{V}(\cdot)$ is not an ISS Lyapunov function *for the closed-loop dynamics* (5.4b) *and disturbances* $w \in \mathcal{B}_\varepsilon \triangleq \{w \in \mathbb{W} \mid \|w\| \leq \varepsilon\}$.

Proof: (i) We have chosen the state (x_1 in Figure 5.1)

$$x^* = [0 \ \ -2.1830]^\top \in \{\partial\overline{\Omega}_{38} \cap \partial\overline{\Omega}_{58}\} \cap \overline{\Omega}_{38}$$

to show that the MPC closed-loop system (5.4a) and $\widehat{V}(\cdot)$ for the above example are not continuous on $\text{int}(\mathbb{X}_f(1))$. We have obtained the following values:

$$A_2x^* + B\hat{u}_{38}(x^*) = [0.0130 \ \ 0.1070]^\top;$$
$$A_3x^* + B\hat{u}_{58}(x^*) = [-0.7791 \ \ -1.3535]^\top;$$
$$\widehat{V}(x^*) = \widehat{V}_{38}(x^*) = 2.6766; \qquad \lim_{x \to x^*,\ x \in \overline{\Omega}_{58}} \widehat{V}(x) = \widehat{V}_{58}(x^*) = 11.7383.$$

(ii) As Assumption 5.2.2 is satisfied via the procedure presented in chapter three, Section 3.4.2, the statement follows from Theorem 5.2.3.

(iii) The MPC closed-loop system (5.4a) corresponding to system (5.6) is such that the dynamics active in region $\overline{\Omega}_{38}$ is employed for $x_k = x^*$. The nominal state trajectory obtained for the initial state

$$x_0 = [-1.9649 \ \ -1.9649]^\top \in \{\partial\overline{\Omega}_{47} \cap \partial\overline{\Omega}_{53}\} \cap \overline{\Omega}_{47}$$

reaches the state $x_1 = x^*$ in one step (see Figure 5.1 for the trajectory plot). Then, for any $\mathcal{K}$-function $\sigma(\cdot)$ we can take an arbitrarily small disturbance w such that $x^* + w \in \overline{\Omega}_{58}$, for which

$$\begin{aligned}\widehat{V}(x^* + w) - \widehat{V}(x_0) &= \widehat{V}_{58}(x^* + w) - \widehat{V}(x_0)\\ &\approx 0.5970\\ &> \sigma(\|w\|) \geq -\alpha_3(\|x_0\|) + \sigma(\|w\|) \end{aligned} \tag{5.8}$$

for any $\alpha_3 \in \mathcal{K}_\infty$. Hence, the ISS inequality (2.4a) of Theorem 2.3.5 does not hold for arbitrarily small w and thus, $\widehat{V}(\cdot)$ is not an ISS Lyapunov function for the closed-loop dynamics (5.4b). ■

The result of Lemma 5.3.1 part (iii) implies that the most likely and natural candidate (i.e. the MPC value function $\widehat{V}(\cdot)$) for proving ISS for the closed-loop system (5.4b) fails. Hence, one should be careful in drawing conclusions on robustness from nominal stability (established via $\widehat{V}(\cdot)$) when dealing with hybrid MPC. At least, there is no obvious way to infer ISS from nominal stability in hybrid MPC, or to modify nominally stabilizing MPC schemes for hybrid systems such that ISS is ensured a priori.

5.4 A posteriori tests for checking robustness

As shown in the previous section, hybrid MPC may be non-robust to arbitrarily small additive disturbances due to the fact that the MPC value function $\widehat{V}(\cdot)$ is not an ISS Lyapunov function in general, which brings us to the second question posed in the introduction of this chapter.

In this section we present a posteriori sufficient conditions for ISS of the perturbed system (5.4b) that can be used to implement a test for checking inherent robustness of nominally stabilizing hybrid MPC schemes.

An alternative to the results presented in this section is to employ the approach of (Kellett and Teel, 2004) for checking robustness of discontinuous discrete-time nonlinear systems. This method can establish robustness of a discontinuous system by checking nominal stability for an upper semicontinuous set-valued regularization of the dynamics. The application of this method to MPC of PWA systems is part of future research.

Let $\mathcal{B}_\mu = \{w \in \mathbb{W} \mid \|w\| \leq \mu\}$ for some $\mu > 0$, $\mathcal{P} \subseteq \mathbb{X}_f(N)$ and let $\mathcal{R}_1(\mathcal{P}) \triangleq \{g(x, \hat{u}(x)) \mid x \in \mathcal{P}\}$ denote the one-step reachable set for system (5.4a).

Theorem 5.4.1 *Suppose that the PWA MPC closed-loop system* (5.4a) *satisfies Assumption 5.2.2 and let $\mathbb{X}_D \subset \mathbb{X}_f(N)$ denote the set of all states at which $\widehat{V}(\cdot)$ is not continuous. Let $\mathcal{P} \subseteq \mathbb{X}_f(N)$ with $0 \in \text{int}(\mathcal{P})$ and suppose that there exists a $\mu > 0$ such that*[4]

$$\mathcal{R}_1(\mathcal{P}) \subseteq \mathcal{P} \sim \mathcal{B}_\mu, \tag{5.9a}$$

$$d(x, \mathbb{X}_D) > \mu \quad \text{for all} \quad x \in \mathcal{R}_1(\mathcal{P}), \tag{5.9b}$$

where $d(x, \mathbb{X}_D) \triangleq \inf_{y \in \mathbb{X}_D} \|x - y\|$. Then, it holds that:

(i) *If Problem 5.2.1 is feasible for state $\tilde{x}_k \in \mathcal{P}$, Problem 5.2.1 is feasible for state $\tilde{x}_{k+1} = \tilde{g}(\tilde{x}_k, \hat{u}(\tilde{x}_k), w_k) = g(\tilde{x}_k, \hat{u}(\tilde{x}_k)) + w_k$ for all $w_k \in \mathcal{B}_\mu$ and all $k \in \mathbb{Z}_+$;* (ii) *The PWA MPC closed-loop system* (5.4b) *is ISS for initial conditions in $\mathcal{P}$ and disturbances in $\mathcal{B}_\mu$.*

Proof: (i) The condition $\mathcal{R}_1(\mathcal{P}) \subseteq \mathcal{P} \sim \mathcal{B}_\mu$ implies that $g(\tilde{x}_k, \hat{u}(\tilde{x}_k)) + w_k \in \mathcal{P} \subseteq \mathbb{X}_f(N)$ for all $\tilde{x}_k \in \mathcal{P}$ and all $w_k \in \mathcal{B}_\mu$, $k \in \mathbb{Z}_+$. Hence, Problem 5.2.1 is feasible for state $\tilde{x}_{k+1} = g(\tilde{x}_k, \hat{u}(\tilde{x}_k)) + w_k$ and any w_k in $\mathcal{B}_\mu$, $k \in \mathbb{Z}_+$;

(ii) The inequality (5.9b) implies that $\widehat{V}(\cdot)$ is continuous on the set (ball) $\{g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu\}$ for all $x \in \mathcal{P}$. Note that this set is a ball of radius μ centered at $g(x, \hat{u}(x))$. Next, we prove that $\widehat{V}(\cdot)$ is Lipschitz continuous on $g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu$ with the same Lipschitz constant for all $x \in \mathcal{P}$. For any two points $y, \bar{y} \in g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu$ connect y and $\bar{y}$ via a straight line and let the points $z_1, \ldots, z_{M-1}$ be the intersection points of the line segment with the boundaries of the regions $\widehat{\Omega}_j$, $j \in \widehat{\mathcal{S}}$. Note that this line segment will not pass through a point in $\mathbb{X}_D$, as it is contained in the ball $g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu$ and $\{g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu\} \cap \mathbb{X}_D = \emptyset$, $\forall x \in \mathcal{P}$. Let $z_0 \triangleq y$ and $z_M \triangleq \bar{y}$. The number of intersection points that needs to be considered is finite. This is because the number of regions $\widehat{\Omega}_j$ is finite. If a non-trivial part of the line segment lies on $\partial\widehat{\Omega}_j$ for some $j \in \widehat{\mathcal{S}}$, select the end-points of the intersection of the line segment with $\partial\widehat{\Omega}_j$ and include only these points in the collection $\{z_1, \ldots, z_{M-1}\}$. Note that the line segment between z_i and z_{i+1} lies in one of the regions $\widehat{\Omega}_{j_i}$ in the state-space partition corresponding to $\widehat{V}(\cdot)$. Due to continuity of $\widehat{V}(\cdot)$ in the region $g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu$, for $z_i \in \partial\Omega_{j_{i-1}} \cap \partial\Omega_{j_i}$,

[4] Note that the condition (5.9a) implies that $\mathcal{P} \subseteq \mathbb{X}_f(N)$ is a RPI set for system (5.4b) and disturbances in $\mathcal{B}_\mu$.

$i = 1, \ldots, M$, we have that $\widehat{V}(z_i) = E_{j_{i-1}} z_i + e_{j_{i-1}} = E_{j_i} z_i + e_{j_i}$. Then, it follows that

$$\begin{aligned}\left|\widehat{V}(y) - \widehat{V}(\bar{y})\right| &= \left|\sum_{i=1}^{M}(\widehat{V}(z_{i-1}) - \widehat{V}(z_i))\right| \leq \sum_{i=1}^{M}\left|\widehat{V}(z_{i-1}) - \widehat{V}(z_i)\right| \\ &= \sum_{i=1}^{M} |E_{j_i}(z_{i-1} - z_i)| \leq \sum_{i=1}^{M} \|E_{j_i}\| \|z_{i-1} - z_i\| \\ &\leq \max_{j_i \in \widehat{\mathcal{S}}} \|E_{j_i}\| \sum_{i=1}^{M} \|z_{i-1} - z_i\| = \max_{j_i \in \widehat{\mathcal{S}}} \|E_{j_i}\| \|y - \bar{y}\|,\end{aligned}$$

where $\|E_{j_i}\| = \|E_{j_i}\|_\infty \triangleq \sum_{p=1}^{n} |E_{j_i}^{\{1p\}}|$ is the corresponding induced matrix norm. Note that $\sum_{i=1}^{M} \|z_{i-1} - z_i\| = \|y - \bar{y}\|$ because the points z_i lie on the same line segment. Hence, $\widehat{V}(\cdot)$ is Lipschitz continuous on $g(x, \hat{u}(x)) \oplus \mathcal{B}_\mu$ for all $x \in \mathcal{P}$ with Lipschitz constant $L_V \triangleq \max_{j_i \in \widehat{\mathcal{S}}} \|E_{j_i}\|$. Then, it follows that

$$\widehat{V}(\tilde{g}(x, \hat{u}(x), w)) - \widehat{V}(g(x, \hat{u}(x))) \leq \sigma(\|w\|)$$

for all $x \in \mathcal{P}$ and $w \in \mathcal{B}_\mu$, where $\sigma(\|w\|) \triangleq L_V \|w\|$. By Assumption 5.2.2, $\widehat{V}(g(x, \hat{u}(x))) - \widehat{V}(x) \leq -\alpha_3(\|x\|)$, which yields:

$$\widehat{V}(\tilde{g}(x, \hat{u}(x), w)) - \widehat{V}(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|),$$

for all $x \in \mathcal{P}$ and all $w \in \mathcal{B}_\mu$. The statement then follows from the general ISS result of Theorem 2.3.5 presented in Chapter 2. ■

In the case when *quadratic costs* are used in Problem 5.2.1, Theorem 8.1 of (Borrelli, 2003) provides sufficient conditions under which the MPC control law (5.2) is a PWA state-feedback (possibly defined on state-space regions that are not polyhedra) and the corresponding MPC value function is piecewise quadratic. Then, the proof of Theorem 5.4.1 can also be applied for hybrid MPC based on quadratic cost functions. Moreover, Theorem 5.4.1 can be extended to any nominally stable PWA system that admits a discontinuous Lyapunov function.

Note that the sufficient condition of Theorem 5.4.1 part (ii) can be further relaxed, as illustrated by the following result.

Proposition 5.4.2 *Suppose that the PWA MPC closed-loop system* (5.4) *satisfies Assumption 5.2.2 and that there exists a* $\mu > 0$ *such that* $\mathbb{X}_f(N)$ *is a RPI set for system* (5.4b) *and* $w \in \mathcal{B}_\mu$. *Let* $\mathbb{X}_D \subset \mathbb{X}_f(N)$ *denote the set of all states at which* $\widehat{V}(\cdot)$ *is not continuous and let*

$$\mathcal{Z} \triangleq \{g(x, \hat{u}(x)) \mid x \in \mathbb{X}_f(N),\ g(x, \hat{u}(x)) \in \mathbb{X}_D\}.$$

For any $z \in \mathcal{Z}$ *let*

$$\mathcal{I}(z) \triangleq \{i \in \widehat{\mathcal{S}} \mid \{z \oplus \mathcal{B}_\mu\} \cap \widehat{\Omega}_i \neq \emptyset\},$$

and consider the following inequalities:

$$\widehat{V}_i(z) \leq \widehat{V}_j(z), \quad \forall z \in \widehat{\Omega}_j \cap \mathcal{Z}, \quad \forall i \in \mathcal{I}(z), \tag{5.10a}$$

$$\widehat{V}_i(z) \leq \widehat{V}(x), \quad \forall z \in \mathcal{Z}, \quad \forall i \in \mathcal{I}(z), \tag{5.10b}$$

where $j \in \widehat{\mathcal{S}}$, $x \in \mathbb{X}_f(N)$ *and* $\widehat{V}_i(z) \triangleq E_i z + e_i$ *for any* $z \in \mathcal{Z}$ *and any* $i \in \widehat{\mathcal{S}}$. *Suppose that either inequality* (5.10a) *or, inequality* (5.10b) *holds. Then, the PWA MPC closed-loop system* (5.4) *is ISS for initial conditions in* $\mathbb{X}_f(N)$ *and disturbances in* $\mathcal{B}_\mu$.

The conditions (5.10) ensure that the discontinuity of $\widehat{V}(\cdot)$ actually works in favor of nominal stability in the sense that $\widehat{V}(\tilde{g}(x, \hat{u}(x), w)) \leq \widehat{V}(g(x, \hat{u}(x)))$ (condition (5.10a)) or, at least, the value of the MPC cost corresponding to a perturbed state is still decreasing, despite the jump caused by the discontinuity (condition (5.10b)).

Suppose now that a PWA MPC closed-loop system of the form (5.4a) and the corresponding set $\mathbb{X}_D$ are known. Note that the set of points at which $\widehat{V}(\cdot)$ is discontinuous is given by:

$$\mathbb{X}_D = \bigcup_{\substack{i,j \in \widehat{\mathcal{S}}, \\ i \neq j}} \left\{x \in \partial\widehat{\Omega}_i \cap \partial\widehat{\Omega}_j \mid (E_i - E_j)x \neq (e_i - e_j)\right\}.$$

The constant μ can be calculated as follows:

$$\mu = \min_{\substack{j \in \mathcal{S}, \\ i \in \overline{\mathcal{S}}}} \{ \min_{\substack{y \in \Omega_j \cap \overline{\Omega}_i \cap \mathcal{P}, \\ \bar{y} \in \mathbb{X}_D}} \|(A_j + B_j L_i)y + (B_j l_i + f_j) - \bar{y}\| \}. \tag{5.11}$$

A solution to the optimization problem (5.11) can be obtained by solving a finite number of LP problems. If $\mu > 0$, then μ is a measure of the (worst case) inherent robustness of system (5.4a).

In order to verify the results of Theorem 5.4.1 we solved the LP problems corresponding to (5.11) for the example presented in Section 5.3 and we obtained $\mu = 0$, as expected. So, the sufficient conditions for ISS of Theorem 5.4.1 fail for this example. Moreover, inequality (5.8) shows that both condition (5.10a) and condition (5.10b) do not hold and therefore, the tests of Proposition 5.4.2 also fail for this example. Then, instead of using a trial-and-error approach to modify the terminal cost and the terminal set in order to satisfy conditions (5.9), it would be desirable to have a systematic way for modifying Problem 5.2.1 so that ISS is guaranteed *a priori*, which brings us to the third question posed in the introduction.

5.5 Robust predictive controllers for discontinuous PWA systems

As concluded in the previous section there is a need for MPC schemes for (discontinuous) PWA systems with an *a priori robustness guarantee* with respect to (small) additive disturbance inputs. In this section we present new design methods based on tightened constraints for setting up ISS MPC schemes for hybrid systems. The novelty of the proposed approach consists in allowing for discontinuous PWA predictions models and discontinuous MPC value functions. The key idea employed in this section for obtaining ISS predictive controllers for PWA systems is to constrain the nominal predicted state sequence such that the *mode sequence* corresponding to a perturbed initial state remains the same as the *nominal predicted mode sequence*.

5.5.1 Input-to-state stabilizing MPC using tightened constraints

Consider first a subclass of the PWA system (5.1), i.e. assume that $0 \in \text{int}(\Omega_{j^*})$ for some $j^* \in \mathcal{S}$ (this implies $\mathcal{S}_0 = \{j^*\}$). In this case we use a common terminal cost, i.e. $F(x) = \|Px\|$ for all $x \in \mathbb{X}_T$. Let $\eta \triangleq \max_{j \in \mathcal{S}} \|A_j\|$, $\xi \triangleq \|P\|$ and define, for any $\mu > 0$ and $i \in \mathbb{Z}_{\geq 1}$,

$$\mathcal{L}_\mu^i \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq \mu \sum_{p=0}^{i-1} \eta^p\}.$$

Consider now the following (tightened) set of admissible input sequences:

$$\widetilde{\mathcal{U}}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid x_{i|k} \in \mathbb{X}_i,\ i = 1, \ldots, N-1,\ x_{N|k} \in \mathbb{X}_T\},\ k \in \mathbb{Z}_+, \tag{5.12}$$

where

$$\mathbb{X}_i \triangleq \cup_{j \in \mathcal{S}} \{\Omega_j \sim \mathcal{L}_\mu^i\} \subseteq \mathbb{X} \quad \text{for all} \quad i = 1, \ldots, N-1$$

and $(x_{1|k}, \ldots, x_{N|k})$ is the state sequence generated from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k$ to the PWA model (5.1a). Let $\widetilde{\mathbb{X}}_f(N)$ denote the set of feasible states for Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$, and let $\widehat{V}(\cdot)$ and $\hat{u}(\cdot)$ denote the corresponding MPC value function and MPC control law, respectively. For any $\mu > 0$, $\mathcal{B}_\mu = \{w \in \mathbb{W} \mid \|w\| \leq \mu\}$ and recall that $\mathbb{X}_\mathbb{U} = \{x \in \mathbb{X} \mid h_{\text{aux}}(x) \in \mathbb{U}\}$. Note that, since $\mathbb{W} \subset \mathbb{R}^n$, it holds that $\mathcal{B}_\mu \subseteq \mathcal{L}_\mu^1$.

Theorem 5.5.1 *Assume that* $0 \in \operatorname{int}(\Omega_{j^*})$ *for some* $j^* \in \mathcal{S}$*. Take* $N \in \mathbb{Z}_{\geq 1}$*,* $\theta > \theta_1 > 0$ *and* $\mu > 0$ *such that* $\mu \leq \frac{\theta-\theta_1}{\xi\eta^{N-1}}$*,*

$$\mathbb{F}_\theta \triangleq \{x \in \mathbb{R}^n \mid F(x) \leq \theta\} \subseteq (\Omega_{j^*} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_{\mathbb{U}}$$

and $g(x, h_{\text{aux}}(x)) \in \mathbb{F}_{\theta_1}$ *for all* $x \in \mathbb{F}_\theta$*. Set* $\mathbb{X}_T = \mathbb{F}_{\theta_1}$*. Furthermore, suppose that Assumption 5.2.2 holds and inequality* (5.3) *with* $P_j = P$ *for all* $j \in \mathcal{S}$ *is satisfied for all* $x \in \mathbb{F}_\theta$*. Then:*

(i) *If* $\tilde{x}_k \in \widetilde{\mathbb{X}}_f(N)$*, then* $\tilde{x}_{k+1} \in \widetilde{\mathbb{X}}_f(N)$ *for all* $w_k \in \mathcal{B}_\mu$ *and all* $k \in \mathbb{Z}_+$*, where* $\tilde{x}_{k+1} = A_j\tilde{x}_k + B_j\hat{u}(\tilde{x}_k) + f_j + w_k$*. Moreover,* $\mathbb{X}_T \subseteq \widetilde{\mathbb{X}}_f(N)$*.*

(ii) *The perturbed PWA system* (5.1b) *in closed-loop with the MPC control* (5.2) *obtained by solving Problem 5.2.1 (with* $\widetilde{\mathcal{U}}_N(x_k)$ *instead of* $\mathcal{U}_N(x_k)$ *and* (5.1a) *as prediction model) at each sampling instant is ISS for initial conditions in* $\widetilde{\mathbb{X}}_f(N)$ *and disturbances in* $\mathcal{B}_\mu$*.*

Proof: Let $(x^*_{1|k}, \ldots, x^*_{N|k})$ denote the state sequence obtained from initial state $x_{0|k} \triangleq \tilde{x}_k$ and by applying the input sequence $\mathbf{u}^*_k$ to (5.1a). Let $(x_{1|k+1}, \ldots, x_{N|k+1})$ denote the state sequence obtained from the initial state $x_{0|k+1} \triangleq \tilde{x}_{k+1} = x_{k+1} + w_k = x^*_{1|k} + w_k$ and by applying the input sequence $\mathbf{u}_{k+1} \triangleq (u^*_{1|k}, \ldots, u^*_{N-1|k}, h_{\text{aux}}(x_{N-1|k+1}))$ to (5.1a).

(i) The constraints imposed in (5.12) ensure that:

$$\text{(P1)} \quad (x_{i|k+1}, x^*_{i+1|k}) \in \Omega_{j_{i+1}} \times \Omega_{j_{i+1}}, \quad j_{i+1} \in \mathcal{S},$$

for all $i = 0, \ldots, N-2$ and, $\|x_{i|k+1} - x^*_{i+1|k}\| \leq \eta^i\mu$ for $i = 0, \ldots, N-1$. This is due to the fact that $x_{0|k+1} = x^*_{1|k} + w_k$, $x_{i|k+1} = x^*_{i+1|k} + \prod_{p=1}^{i} A_{j_p} w_k$ for $i = 1, \ldots, N-1$ and $\|\prod_{p=1}^{i} A_{j_p} w_k\| \leq \eta^i\mu$, which implies that $\prod_{p=1}^{i} A_{j_p} w_k \in \mathcal{L}_\mu^{i+1}$.

Pick the indexes $j_{i+1} \in \mathcal{S}$ such that $x^*_{i+1|k} \in \Omega_{j_{i+1}}$ for all $i = 1, \ldots, N-2$. Then, due to $x^*_{i+1|k} \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^{i+1}$, it follows by Lemma 2 of (Limon et al., 2002a) that $x_{i|k+1} \in \Omega_{j_{i+1}} \sim \mathcal{L}_\mu^{i} \subset \mathbb{X}_i$ for $i = 1, \ldots, N-2$. From

$$x_{N-1|k+1} = x^*_{N|k} + \prod_{p=1}^{N-1} A_{j_p} w_k$$

it follows that

$$F(x_{N-1|k+1}) - F(x^*_{N|k}) \leq \xi\eta^{N-1}\mu,$$

which implies that

$$F(x_{N-1|k+1}) \leq \theta_1 + \xi\eta^{N-1}\mu \leq \theta,$$

due to $x^*_{N|k} \in \mathbb{X}_T = \mathbb{F}_{\theta_1}$ and $\mu \leq \frac{\theta-\theta_1}{\xi\eta^{N-1}}$. Hence,

$$x_{N-1|k+1} \in \mathbb{F}_\theta \subset \mathbb{X}_\mathbb{U} \cap (\Omega_{j^*} \sim \mathcal{L}_\mu^{N-1}) \subset \mathbb{X}_\mathbb{U} \cap \mathbb{X}_{N-1}$$

so that $h_{\text{aux}}(x_{N-1|k+1}) \in \mathbb{U}$ and $x_{N|k+1} \in \mathbb{F}_{\theta_1} = \mathbb{X}_T$. Thus, the sequence of inputs $\mathbf{u}_{k+1}$ is feasible at time $k+1$ and Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ remains feasible. Moreover, from $g(x, h_{\text{aux}}(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$ and $\mathbb{F}_{\theta_1} \subset \mathbb{F}_\theta$ it follows that $\mathbb{F}_{\theta_1}$ is a positively invariant set for system (5.1a) in closed-loop with $h_{\text{aux}}(\cdot)$. Then, since

$$\mathbb{F}_{\theta_1} \subset \mathbb{F}_\theta \subseteq (\Omega_{j^*} \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_\mathbb{U} \subset \mathbb{X}_i \cap \mathbb{X}_\mathbb{U} \quad \text{for all} \quad i = 1, \ldots, N-1$$

and $\mathbb{X}_T = \mathbb{F}_{\theta_1}$, the sequence of control inputs $(h_{\text{aux}}(x_{0|k}), \ldots, h_{\text{aux}}(x_{N-1|k}))$ is feasible with respect to Problem 5.2.1 (with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$) for all $x_{0|k} \triangleq \tilde{x}_k \in \mathbb{F}_{\theta_1}$. Therefore, $\mathbb{X}_T = \mathbb{F}_{\theta_1} \subseteq \widetilde{\mathbb{X}}_f(N)$.

(ii) The result of part (i) implies that $\widetilde{\mathbb{X}}_f(N)$ is a RPI set for system (5.1b) in closed-loop with the MPC control (5.2) and disturbances in $\mathcal{B}_\mu$. Moreover, since $0 \in \text{int}(\mathbb{X}_T)$, we have that $0 \in \text{int}(\widetilde{\mathbb{X}}_f(N))$. The choice of the terminal stage costs already ensures that there exist $\alpha_1(s) \triangleq as$ and $\alpha_2(s) \triangleq bs$, $a, b > 0$, such that $\alpha_1(\|x\|) \leq \widehat{V}(x) \leq \alpha_2(\|x\|)$ for all $x \in \widetilde{\mathbb{X}}_f(N)$. Let $\tilde{x}_{k+1}$ denote the solution of (5.1b) in closed-loop with $\hat{u}(\cdot)$ obtained as indicated in part (i) of the proof and let $x^*_{0|k} \triangleq \tilde{x}_k$. Due to full-column rank of Q there exists $\gamma > 0$ such that $\|Qx\| \geq \gamma\|x\|$ for all x. Then, by optimality, property (P1), $x_{N-1|k+1} \in \mathbb{F}_\theta$ and from inequality (5.3) it follows that:

$$\begin{aligned}
\widehat{V}(\tilde{x}_{k+1}) - \widehat{V}(\tilde{x}_k) &\leq J(\tilde{x}_{k+1}, \mathbf{u}_{k+1}) - J(\tilde{x}_k, \mathbf{u}^*_k) \\
&= -L(x^*_{0|k}, u^*_{0|k}) + F(x_{N|k+1}) + (-F(x_{N-1|k+1}) + F(x_{N-1|k+1})) \\
&\quad - F(x^*_{N|k}) + L(x_{N-1|k+1}, h_{\text{aux}}(x_{N-1|k+1})) \\
&\quad + \sum_{i=0}^{N-2} (L(x_{i|k+1}, \mathbf{u}_{k+1}(i+1)) - L(x^*_{i+1|k}, u^*_{i+1|k})) \\
&\leq -L(x^*_{0|k}, u^*_{0|k}) + F(x_{N|k+1}) - F(x_{N-1|k+1}) \\
&\quad + L(x_{N-1|k+1}, h_{\text{aux}}(x_{N-1|k+1})) + \left(\xi\eta^{N-1} + \|Q\| \sum_{p=0}^{N-2} \eta^p\right) \|w_k\| \\
&\overset{(5.3)}{\leq} -\|Qx^*_{0|k}\| + \sigma(\|w_k\|) \leq -\alpha_3(\|\tilde{x}_k\|) + \sigma(\|w_k\|),
\end{aligned}$$

with $\sigma(s) \triangleq (\xi\eta^{N-1} + \|Q\| \sum_{p=0}^{N-2} \eta^p)s$ and $\alpha_3(s) \triangleq \gamma s$.

Thus, it follows that $\widehat{V}(\cdot)$ satisfies the hypothesis of Theorem 2.3.5, thereby proving ISS of the closed-loop system (5.1b)-(5.2) for initial conditions in $\widetilde{\mathbb{X}}_f(N)$ and disturbances in $\mathcal{B}_\mu$. ∎

The tightened set of admissible input sequences (5.12) may become very conservative as the prediction horizon increases, since it requires that the state trajectory must be kept farther and farther away from the boundaries. This drawback can be reduced by introducing a pre-compensating state-feedback, which is a common solution in robust MPC.

5.5.2 Dual-mode input-to-state stabilizing MPC

Next, we present a new dual-mode technique for setting up ISS MPC schemes for hybrid systems, which generalizes the results of the previous subsection to PWA systems with the origin lying on the boundaries of multiple regions in the state-space partition. In this case it is no longer necessary to consider a common terminal cost, i.e. we allow for $F(x) = \|P_j x\|$ when $x \in \Omega_j \cap \mathbb{X}_T$, $j \in \mathcal{S}$. Let $\mathbb{X}_{\mathrm{RPI}} \subseteq \mathbb{X}_{\mathbb{U}}$ with $0 \in \operatorname{int}(\mathbb{X}_{\mathrm{RPI}})$ be a RPI set for system (5.1b) in closed-loop with $h_{\mathrm{aux}}(\cdot)$. Let $\xi \triangleq \max_{j\in\mathcal{S}} \|P_j\|$, let $\eta = \max_{j\in\mathcal{S}} \|A_j\|$ and, for any $i \in \mathbb{Z}_{\geq 1}$, let $\mathcal{L}_\mu^i = \{x \in \mathbb{R}^n \mid \|x\| \leq \mu \sum_{p=0}^{i-1} \eta^p\}$.

Next, choose (compute) a terminal set $\mathbb{X}_T \subseteq \mathbb{X}_{\mathrm{RPI}} \cap \mathbb{X}_N$ such that $\mathbb{X}_T \oplus \mathcal{L}_\mu^N \subseteq \mathcal{Q}_1(\mathbb{X}_T)$, where $\mathbb{X}_N \triangleq \cup_{j\in\mathcal{S}}\{\Omega_j \sim \mathcal{L}_\mu^N\} \subseteq \mathbb{X}$ and $\mathcal{Q}_1(\mathbb{X}_T) \triangleq \{x \in \mathbb{X} \mid \exists u \in \mathbb{U} \text{ s.t. } g(x,u) \in \mathbb{X}_T\}$ is the one-step controllable set for the dynamics $g(\cdot,\cdot)$ with respect to the target set $\mathbb{X}_T$. Note that for each fixed $\mathbb{X}_T \subseteq \mathbb{X}_{\mathrm{RPI}} \cap \mathbb{X}_N$ there is a trade-off in choosing μ such that $\mathbb{X}_T \oplus \mathcal{L}_\mu^N \subseteq \mathcal{Q}_1(\mathbb{X}_T)$.

Consider now the updated (tightened) set of admissible input sequences:

$$\widetilde{\mathcal{U}}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid x_{i|k} \in \mathbb{X}_i,\ i = 1,\ldots,N-1,\ x_{N|k} \in \mathbb{X}_T\},\ k \in \mathbb{Z}_+, \tag{5.13}$$

where $\mathbb{X}_i = \cup_{j\in\mathcal{S}}\{\Omega_j \sim \mathcal{L}_\mu^i\} \subseteq \mathbb{X}$ for all $i \in \mathbb{Z}_{[1,N-1]}$ and $(x_{1|k},\ldots,x_{N|k})$ is a state sequence generated from initial state $x_{0|k} \triangleq \tilde{x}_k$ and by applying the input sequence $\mathbf{u}_k$ to the nominal PWA model (5.1a). Just as done previously, let $\widetilde{\mathbb{X}}_f(N)$ denote the set of feasible states for Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$, and let $\widehat{V}(\cdot)$ and $\hat{u}(\cdot)$ denote the corresponding MPC value function and MPC control law, respectively.

We define a dual-mode MPC control law as follows:

$$\hat{u}^{\mathrm{DM}}(x_k) \triangleq \begin{cases} \hat{u}(x_k) & \text{if } x_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}} \\ h_{\mathrm{aux}}(x_k) & \text{if } x_k \in \mathbb{X}_{\mathrm{RPI}} \end{cases} ;\ k \in \mathbb{Z}_+. \tag{5.14}$$

Therefore, the set of feasible states corresponding to $\hat{u}^{\mathrm{DM}}(\cdot)$ is $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\mathrm{RPI}}$, which contains the origin in its interior due to $0 \in \operatorname{int}(\mathbb{X}_{\mathrm{RPI}})$.

Usually, e.g. see (Kerrigan and Mayne, 2002), in dual-mode robust MPC the terminal set is taken as $\mathbb{X}_{\text{RPI}}$. The terminal state is restricted here to a disconnected subset of $\mathbb{X}_{\text{RPI}}$, i.e. $\mathbb{X}_T \subseteq \mathbb{X}_{\text{RPI}} \cap \mathbb{X}_N \subset \mathbb{X}_{\text{RPI}}$, with $0 \notin \mathbb{X}_T$, in order to guarantee robust feasibility of Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ and ISS, as it will be shown next. If the state trajectory reaches either $\mathbb{X}_T$ or $\mathbb{X}_{\text{RPI}} \setminus \mathbb{X}_T$, the dual-mode control law switches to the PWL local controller and then the state trajectory remains in $\mathbb{X}_{\text{RPI}}$ (*and not necessarily in* $\mathbb{X}_T$) forever, due to robust positive invariance of $\mathbb{X}_{\text{RPI}}$ under $h_{\text{aux}}(\cdot)$.

Theorem 5.5.2 *Take $\mu > 0$, $N \in \mathbb{Z}_{\geq 1}$ and $\mathbb{X}_T \subseteq \mathbb{X}_{RPI} \cap \mathbb{X}_N$ such that $\mathbb{X}_{RPI} \cap \mathbb{X}_N \neq \emptyset$ and $\mathbb{X}_T \oplus \mathcal{L}_\mu^N \subseteq \mathcal{Q}_1(\mathbb{X}_T)$. Suppose that $h_{aux}(\cdot)$ and the terminal cost $F(\cdot)$ satisfy* (5.3) *for all $x \in \mathbb{X}_{RPI}$.*

(i) *If Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ is feasible at time $k \in \mathbb{Z}_+$ for state $\tilde{x}_k \in \mathbb{X}$, then Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ is feasible at time $k+1$ for state $\tilde{x}_{k+1} = A_j\tilde{x}_k + B_j\hat{u}^{DM}(\tilde{x}_k) + f_j + w_k$ for all $w_k \in \mathcal{B}_\mu$ and all $k \in \mathbb{Z}_+$;*

(ii) *The perturbed PWA system* (5.1b) *in closed-loop with $\hat{u}^{DM}(\cdot)$ is ISS for initial conditions in $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{RPI}$ and disturbances in $\mathcal{B}_\mu$.*

Proof: (i) There are two situations possible: either $\tilde{x}_k \in \mathbb{X}_{\text{RPI}}$ or $\tilde{x}_k \notin \mathbb{X}_{\text{RPI}}$. If $\tilde{x}_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\text{RPI}}$ for some $k \in \mathbb{Z}_+$, let $(x^*_{1|k}, \ldots, x^*_{N|k})$ denote an optimal predicted state sequence obtained at time k from initial state $x_{0|k} \triangleq \tilde{x}_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\text{RPI}}$ and by applying the input sequence $\mathbf{u}^*_k = (u^*_{0|k}, \ldots, u^*_{N-1|k})$ to the PWA model (5.1a). Let $(x_{1|k+1}, \ldots, x_{N-1|k+1})$ denote the state sequence obtained from the perturbed initial state $x_{0|k+1} \triangleq \tilde{x}_{k+1} = x_{k+1} + w_k = x^*_{1|k} + w_k$ and by applying the inputs $(u^*_{1|k}, \ldots, u^*_{N-1|k})$ to the nominal PWA model (5.1a). The state constraints imposed in (5.13) ensure that: (P1) $(x_{i|k+1}, x^*_{i+1|k}) \in \Omega_{j_{i+1}} \times \Omega_{j_{i+1}}$, $j_{i+1} \in \mathcal{S}$ for all $i = 0, \ldots, N-2$ and, $\|x_{i|k+1} - x^*_{i+1|k}\| \leq \eta^i \mu$ for $i = 0, \ldots, N-1$. Then, as shown in the proof of Theorem 5.5.1, we have that $x_{i|k+1} \in \Omega_{j_{i+1}} \sim \mathcal{L}^i_\mu \subset \mathbb{X}_i$ for $i = 1, \ldots, N-2$. Next, $x_{N-1|k+1} = x^*_{N|k} + \prod_{p=1}^{N-1} A_{j_p} w_k$, $x^*_{N|k} \in \mathbb{X}_T$ and $\|\prod_{p=1}^{N-1} A_{j_p} w_k\| \leq \eta^{N-1}\mu$ imply that $x_{N-1|k+1} \in \mathbb{X}_T \oplus \mathcal{L}^N_\mu$. Since $\mathbb{X}_T \oplus \mathcal{L}^N_\mu \subseteq \mathcal{Q}_1(\mathbb{X}_T)$, it follows that there exists a $\bar{u} \in \mathbb{U}$ such that $x_{N|k+1} \triangleq g(x_{N-1|k+1}, \bar{u}) \in \mathbb{X}_T$. Hence, the sequence of inputs $\mathbf{u}_{k+1} \triangleq (u^*_{1|k}, \ldots, u^*_{N-1|k}, \bar{u})$ is feasible at time $k+1$ and the optimization problem as given in Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ remains feasible.

Consider now the other situation, i.e. $\tilde{x}_k \in \mathbb{X}_{\text{RPI}}$. If the state trajectory enters (or starts in) $\mathbb{X}_{\text{RPI}} \subseteq \mathbb{X}_{\mathbb{U}}$ (note that $\mathbb{X}_T \subset \mathbb{X}_{\text{RPI}}$), feasibility of

$\hat{u}^{\mathrm{DM}}(x_k) = h_{\mathrm{aux}}(x_k)$ is ensured due to robust positive invariance of $\mathbb{X}_{\mathrm{RPI}}$ for system (5.1b) in closed-loop with $u_k = h_{\mathrm{aux}}(x_k)$, $k \in \mathbb{Z}_+$.

(ii) The result of part (i) implies that $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\mathrm{RPI}}$ is a RPI set for system (5.1b) in closed-loop with the dual-mode MPC control $\hat{u}^{\mathrm{DM}}(\cdot)$ and disturbances in $\mathcal{B}_\mu$. To prove ISS, we consider three situations: in Case 1 we assume that $\tilde{x}_k \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ for all $k \in \mathbb{Z}_+$, in Case 2 we assume that $\tilde{x}_0 \in \mathbb{X}_{\mathrm{RPI}}$, and in Case 3 we assume that $\tilde{x}_0 \in \widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ and there exists a $p \in \mathbb{Z}_{\geq 1}$ such that $\tilde{x}_k \notin \mathbb{X}_{\mathrm{RPI}}$ for all $k \in \mathbb{Z}_{<p}$ and $\tilde{x}_p \in \mathbb{X}_{\mathrm{RPI}}$.

Note that Case 1 can only occur when the disturbance is non-zero, i.e. if the disturbance asymptotically converges to zero or it vanishes for some $k \in \mathbb{Z}_+$, then the closed-loop system state will converge to a subset of $\mathbb{X}_{\mathrm{RPI}}$, which brings us to Case 3. In other words, Case 1 cannot happen for asymptotically decreasing disturbances.

In Case 1, the hypothesis already ensures that the MPC value function $\widehat{V}(\cdot)$ satisfies the ISS condition (2.4a) for some $\alpha_1(s) \triangleq as$ and $\alpha_2(s) \triangleq bs$, $a, b > 0$, which ensures that $\alpha_1(\|x\|) \leq \widehat{V}(x) \leq \alpha_2(\|x\|)$ for all $x \in \widetilde{\mathbb{X}}_f(N)$. Let $\tilde{x}_{k+1}$ denote the solution of the perturbed system (5.1b) in closed-loop with $\hat{u}^{\mathrm{DM}}(\cdot)$ obtained as indicated in part (i) of the proof and let $x^*_{0|k} \triangleq \tilde{x}_k$. Due to full-column rank of Q there exists $\gamma > 0$ such that $\|Qx\| \geq \gamma\|x\|$ for all x. Then, as shown in the proof of Theorem 5.5.1 it holds that

$$\widehat{V}(\tilde{x}_{k+1}) - \widehat{V}(\tilde{x}_k) \leq J(\tilde{x}_{k+1}, \mathbf{u}_{k+1}) - J(\tilde{x}_k, \mathbf{u}^*_k) \leq -\alpha_3(\|\tilde{x}_k\|) + \sigma(\|w_k\|),$$

with $\sigma(s) \triangleq (\xi\eta^{N-1} + \|Q\| \sum_{p=0}^{N-2} \eta^p)s$ and $\alpha_3(s) \triangleq \gamma s$. Hence, it follows that $\widehat{V}(\cdot)$ satisfies the hypothesis of Theorem 2.3.5, thereby establishing ISS in this particular case for the closed-loop system (5.1b)-(5.14), for initial conditions in $\widetilde{\mathbb{X}}_f(N) \setminus \mathbb{X}_{\mathrm{RPI}}$ and disturbances in $\mathcal{B}_\mu$.

In Case 2, we prove that the closed-loop system is ISS by showing that the candidate (discontinuous) ISS Lyapunov function $F(x) = \|P_j x\|$ when $x \in \Omega_j$ satisfies the hypothesis of Theorem 2.3.5. Since P_j has full-column rank for all $j \in \mathcal{S}$ there exist positive constants a_j and $b_j \triangleq \|P_j\|$ such that $a_j\|x\| \leq \|P_j x\| \leq b_j\|x\|$ for all $j \in \mathcal{S}$. Hence, the $\mathcal{K}_\infty$-functions $\alpha_1(s) \triangleq \min_{j\in\mathcal{S}} a_j s$ and $\alpha_2(s) \triangleq \max_{j\in\mathcal{S}} b_j s$ satisfy $\alpha_1(\|x\|) \leq F(x) \leq \alpha_2(\|x\|)$ for all $x \in \mathbb{R}^n$. Next, from the hypothesis we have that inequality (5.3) holds for all $x \in \mathbb{X}_{\mathrm{RPI}}$ and all $(j, i) \in \mathcal{S} \times \mathcal{S}$, which yields:

$$\begin{aligned} F((A_j + B_j K_j)x + f_j + w) - F(x) & \\ &\leq \|P_i(A_j + B_j K_j)x + P_i f_j\| + \|P_i w\| - \|P_j x\| \\ &\leq -\|Qx\| + \max_{i\in\mathcal{S}} \|P_i\|\|w\| \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \end{aligned}$$

for all $x \in \mathbb{X}_{\text{RPI}}$, $(j,i) \in \mathcal{S} \times \mathcal{S}$ and disturbances in $\mathcal{B}_\mu$, where $\alpha_3(s) \triangleq \gamma s$ (with $\gamma > 0$ such that $\|Qx\| \geq \gamma\|x\|$) and $\sigma(s) \triangleq \max_{i\in\mathcal{S}} \|P_i\|s$. Then, due to robust positive invariance of $\mathbb{X}_{\text{RPI}}$, ISS for initial conditions in $\mathbb{X}_{\text{RPI}}$ and disturbances in $\mathcal{B}_\mu$ follows from Theorem 2.3.5.

In Case 3 there exists a finite $p \in \mathbb{Z}_{\geq 1}$ such that $\tilde{x}_k \notin \mathbb{X}_{\text{RPI}}$ for all $k \in \mathbb{Z}_{<p}$ and $\tilde{x}_p \in \mathbb{X}_{\text{RPI}}$. Then, from Theorem 2.3.5, Case 1 and Case 2, it follows that there exist $\mathcal{KL}$-functions β_1, β_2 and $\mathcal{K}$-functions γ_1, γ_2 such that for all $p \in \mathbb{Z}_{\geq 1}$ it holds:

$$\begin{aligned}
&\|\tilde{x}_k\| \leq \beta_1(\|\tilde{x}_0\|, k) + \gamma_1(\|w_{[k-1]}\|), \ \forall k \in \mathbb{Z}_{\leq p},\\
&\|\tilde{x}_k\| \leq \beta_2(\|\tilde{x}_p\|, k-p) + \gamma_2(\|w_{[k-p,k-1]}\|), \ \forall k \in \mathbb{Z}_{>p},
\end{aligned}$$

for all $w_{[k-1]} \in \{\mathcal{B}_\mu\}^k$ and all $w_{[k-p,k-1]} \in \{\mathcal{B}_\mu\}^p$, respectively. The functions $\beta_1 \in \mathcal{KL}$, $\gamma_1 \in \mathcal{K}$ and $\beta_2 \in \mathcal{KL}$, $\gamma_2 \in \mathcal{K}$ are obtained as in (2.5) for some constants $\bar{\rho}, \rho \in [0,1)$ and some $\mathcal{K}_\infty$-functions $\bar{\alpha}_1(s) \triangleq \bar{a}s$, $\bar{\alpha}_2(s) \triangleq \bar{b}s$ and $\alpha_1(s) \triangleq as$, $\alpha_2(s) \triangleq bs$, with $\bar{a}, \bar{b}, a, b > 0$, respectively. Let

$$\gamma_3(s) \triangleq \beta_2(2\gamma_1(s), 1) + \gamma_2(s)$$

and consider the inequality:

$$\beta_2(2\beta_1(s,p), k-p) \overset{(2.5)}{=} \alpha_1^{-1}(2\rho^{k-p}\alpha_2(2\bar{\alpha}_1^{-1}(2\bar{\rho}^p\bar{\alpha}_2(s))) \leq 8\frac{b\bar{b}}{a\bar{a}}\tilde{\rho}^k s \triangleq \beta_3(s,k), \tag{5.15}$$

where $\tilde{\rho} \triangleq \max(\rho, \bar{\rho}) \in [0,1)$. Then, for all $k \in \mathbb{Z}_{>p}$ and all $p \in \mathbb{Z}_{\geq 1}$ it follows that

$$\begin{aligned}
\|\tilde{x}_k\| &\leq \beta_2(\beta_1(\|\tilde{x}_0\|, p) + \gamma_1(\|w_{[p-1]}\|), k-p) + \gamma_2(\|w_{[k-p,k-1]}\|)\\
&\leq \beta_2(2\beta_1(\|\tilde{x}_0\|, p), k-p) + \beta_2(2\gamma_1(\|w_{[p-1]}\|), k-p) + \gamma_2(\|w_{[k-p,k-1]}\|)\\
&\overset{(5.15)}{\leq} \beta_3(\|\tilde{x}_0\|, k) + \beta_2(2\gamma_1(\|w_{[p-1]}\|), 1) + \gamma_2(\|w_{[k-p,k-1]}\|)\\
&\leq \beta_3(\|\tilde{x}_0\|, k) + \beta_2(2\gamma_1(\|w_{[k-1]}\|), 1) + \gamma_2(\|w_{[k-1]}\|)\\
&\leq \beta_3(\|\tilde{x}_0\|, k) + \gamma_3(\|w_{[k-1]}\|).
\end{aligned}$$

Since $\beta_3 \in \mathcal{KL}$, $\beta_2 \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$, we obtain that $\gamma_3 \in \mathcal{K}$. Applying Case 1 and Case 2 and combining with the result obtained above for Case 3 it follows that:

$$\|x_k\| \leq \beta(\|\tilde{x}_0\|, k) + \gamma(\|w_{[k-1]}\|),$$

for all $\tilde{x}_0 \in \widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\text{RPI}}$, $w_{[k-1]} \in \{\mathcal{B}_\mu\}^k$ and all $k \in \mathbb{Z}_{\geq 1}$, where

$$\beta(s,k) \triangleq \max(\beta_1(s,k), \beta_2(s,k), \beta_3(s,k))$$

is a $\mathcal{KL}$-function and $\gamma(s) \triangleq \max(\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a $\mathcal{K}$-function. Hence, ISS is proven for system (5.1b) in closed-loop with $\hat{u}^{\mathrm{DM}}(\cdot)$ for all initial conditions in $\widetilde{\mathbb{X}}_f(N) \cup \mathbb{X}_{\mathrm{RPI}}$ and disturbances in $\mathcal{B}_\mu$. ∎

Note that for the particular case when $N = 1$, the terminal constraint set can be simply defined as $\mathbb{X}_T = \{\mathbb{X}_{\mathrm{RPI}} \sim \mathcal{L}_\mu^1\} \cap \mathbb{X}_1$. The statements of Theorem 5.5.2 still hold, as $g(x, \hat{u}^{\mathrm{DM}}(x)) + w \in \mathbb{X}_{\mathrm{RPI}}$ for all $w \in \mathcal{B}_\mu \subseteq \mathcal{L}_\mu^1$ and the dual-mode control law (5.14) switches to the local controller $h_{\mathrm{aux}}(\cdot)$ after one discrete-time instant for all initial conditions $x \in \widetilde{\mathbb{X}}_f(1) \setminus \mathbb{X}_{\mathrm{RPI}}$.

5.6 Illustrative examples

In this section we illustrate the results of Theorem 5.5.1 and Theorem 5.5.2 by means of simulated examples.

5.6.1 Example 1

First, we illustrate the application of Theorem 5.5.1 and how to construct the parameters θ, θ_1 and μ for a given $N \in \mathbb{Z}_{\geq 1}$. Consider the following discontinuous PWA system:

$$x_{k+1} = \tilde{g}(x_k, u_k, w_k) \triangleq g(x_k, u_k) + w_k \triangleq A_j x_k + B_j u_k + w_k \text{ if } x_k \in \Omega_j,\ j \in \mathcal{S}, \quad (5.16)$$

where $\mathcal{S} = \{1, \ldots, 5\}$, $A_1 = \begin{bmatrix} -0.0400 & -0.4610 \\ -0.1390 & 0.3410 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.6552 & 0.2261 \\ 0.5516 & -0.0343 \end{bmatrix}$, $A_3 = \begin{bmatrix} -0.7713 & 0.7335 \\ 0.4419 & 0.5580 \end{bmatrix}$, $A_4 = \begin{bmatrix} -0.0176 & 0.5152 \\ 0.6064 & 0.2168 \end{bmatrix}$, $A_5 = \begin{bmatrix} -0.0400 & -0.4610 \\ -0.0990 & 0.6910 \end{bmatrix}$, $B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ and $B_5 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$. The state and the input of system (5.16) are constrained at all times in the sets $\mathbb{X} = [-3, 3] \times [-3, 3]$ and $\mathbb{U} = [-0.2, 0.2]$, respectively. The state-space partition is plotted in Figure 5.2. The method presented in chapter three, Section 3.4.2, was employed to compute the terminal weight matrix $P = \begin{bmatrix} 2.3200 & 0.3500 \\ -0.2100 & 2.4400 \end{bmatrix}$ and the feedback $K = \begin{bmatrix} -0.04 & -0.35 \end{bmatrix}$ such that inequality (5.3) of Assumption 5.2.2 holds for all $x \in \mathbb{R}^2$, the ∞-norm MPC cost with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 0.01$ and $h_{\mathrm{aux}}(x) = Kx$. Based on inequality (5.3), it can be shown that the sublevel sets of the terminal cost F, i.e. also $\mathbb{F}_\theta$, are λ-contractive sets with $\lambda = 0.6292$ for the dynamics $g(x, h_{\mathrm{aux}}(x))$. Then, for any θ_1 with $\theta > \theta_1 \geq \lambda\theta$ it holds that $g(x, h_{\mathrm{aux}}(x)) \in \mathbb{F}_{\theta_1}$ for all $x \in \mathbb{F}_\theta$. This yields $\mu \leq \frac{(1-\lambda)\theta}{\xi\eta^{N-1}}$. However, μ and θ must also be such that $\mathbb{F}_\theta \subseteq (\Omega_5 \sim \mathcal{L}_\mu^{N-1}) \cap \mathbb{X}_{\mathbb{U}}$. Hence, a trade-off must be made in choosing θ and μ. A large θ implies a large μ, which is desirable since μ is an upper bound on $\|w\|$, but θ must also be small enough to ensure the above inclusion. We chose $\theta = 0.96$ and $\theta_1 = \lambda\theta = 0.6040$.

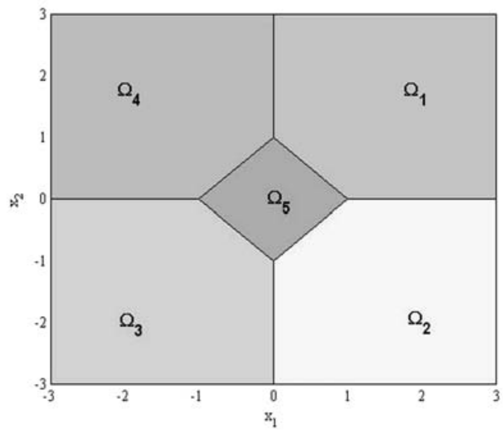

Figure 5.2: State-space partition for system (5.16).

Then, with $\eta = 1.5048$, $\xi = 2.67$ and a prediction horizon $N = 2$ one obtains that any μ with $0 \leq \mu \leq 0.0886$ is an admissible upper bound on $\|w\|$. For $\mu = 0.0886$ it holds that $\mathbb{F}_\theta \subseteq (\Omega_5 \sim \mathcal{L}_\mu^1) \cap \mathbb{X}_\mathbb{U}$ (see Figure 5.3 for an illustrative plot). Hence, the hypothesis of Theorem 5.5.1 is satisfied for any $w \in \mathcal{B}_\mu = \{w \in \mathbb{R}^2 \mid \|w\| \leq 0.0886\}$.

Then, we used the Multi Parametric Toolbox (MPT) (Kvasnica et al., 2004) to calculate the MPC control law (5.2) as an explicit PWA state-feedback, and to simulate the resulting MPC closed-loop system (5.16)-(5.2) for randomly generated disturbances in $\mathcal{B}_\mu$. The explicit MPC controller is defined over 132 state-space regions. The set of feasible states $\widetilde{\mathbb{X}}_f(2)$ is plotted in Figure 5.3 together with the partition corresponding to the explicit MPC control law. Note that, by Theorem 5.5.1, ISS is ensured for the closed-loop system for initial conditions in $\widetilde{\mathbb{X}}_f(2)$ and disturbances in $\mathcal{B}_\mu$, without employing a *continuous* MPC value function Indeed, for example, $\widehat{V}(\cdot)$ and the closed-loop dynamics (5.16)-(5.2) are discontinuous at $x = [0\ 1]^\top \in \text{int}(\widetilde{\mathbb{X}}_f(2))$.

5.6.2 Example 2

Next, we illustrate the result of Theorem 5.5.2 for the PWL system (5.6) introduced in Section 5.3. The terminal weight matrices $P_j = P$ for all $j = 1, \ldots, 4$ and the feedbacks $\{K_j \mid j = 1, \ldots, 4\}$ given in (5.7) are such that inequality (5.3) holds for all $x \in \mathbb{R}^n$. To implement the dual-mode MPC control law one has to compute the terminal set $\mathbb{X}_T$. The MPT (Kvasnica

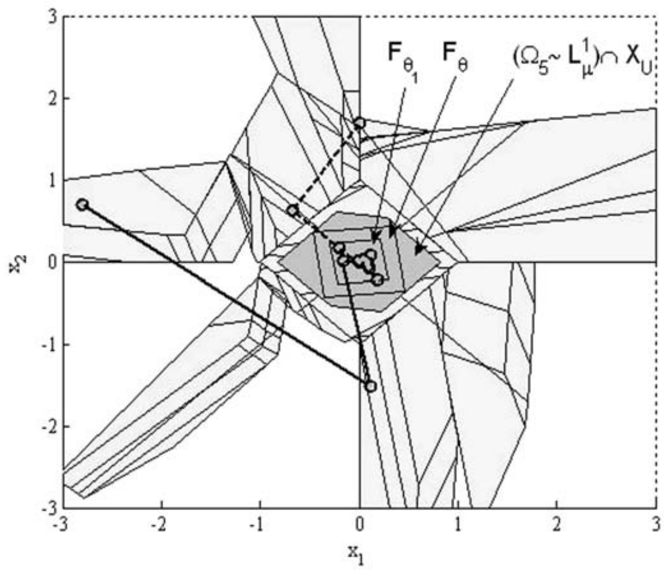

Figure 5.3: State trajectories for the MPC closed-loop system (5.16)-(5.2) *with* $x_0 = [0.003\ 1.7]^\top$ *- dashed line and* $x_0 = [-2.8\ 0.7]^\top$ *- solid line.*

et al., 2004) was employed in order to calculate the maximal RPI set $\mathbb{X}_{\text{RPI}}$ contained in $\mathbb{X}_{\mathbb{U}}$. We choose $\mu = 0.1$ and $N = 1$, for which the terminal constraint set can be chosen as $\mathbb{X}_T = \{\mathbb{X}_{\text{RPI}} \sim \mathcal{L}_\mu^1\} \cap \mathbb{X}_1 \neq \emptyset$, where $\mathbb{X}_1 = \cup_{j=1,\ldots,4}\{\Omega_j \sim \mathcal{L}_\mu^1\}$.

An explicit solution of Problem 5.2.1 with $\widetilde{\mathcal{U}}_N(x_k)$ instead of $\mathcal{U}_N(x_k)$ was calculated with the MPT. The feasible set $\widetilde{\mathbb{X}}_f(1) \cup \mathbb{X}_{\text{RPI}}$ of the dual-mode MPC control law and the state-space partition (138 regions) corresponding to the explicit MPC control law $\hat{u}(\cdot)$ are plotted in Figure 5.4.

Note that, by Theorem 5.5.2, ISS is ensured for the closed-loop system, without employing a *continuous MPC value function*. Indeed, the dual-mode MPC value function $\widehat{V}$ is discontinuous at $x \in \partial\overline{\Omega}_{32} \cap \partial\overline{\Omega}_{80}$, e.g. $\widehat{V}_{32}(x^*) = 2.9038$ and $\widehat{V}_{80}(x^*) = 11.7383$ for $x^* = [0\ -2.1830]^\top$.

To illustrate the ISS property of the dual-mode MPC control law we simulated system (5.6) in closed-loop with $\hat{u}^{\text{DM}}(\cdot)$ for initial states $x_{01} = [-1.9649\ -1.9649]^\top$ (solid line) and $x_{02} = [5\ -5]^\top$ (dashed line) and the disturbance values depicted in Figure 5.5 - (a), (b) for both x_{01} and x_{02}. The control inputs are also plotted in Figure 5.5 - (c), (d) for initial states x_{01} and x_{02}, respectively. Once the disturbance converges to zero, the state

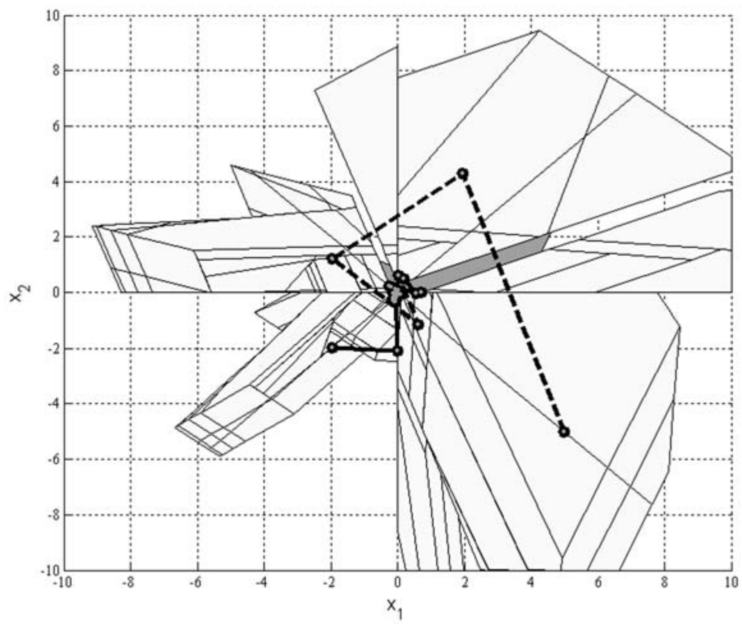

Figure 5.4: The feasible set $\widetilde{\mathbb{X}}_f(1) \cup \mathbb{X}_{RPI}$ *(*$\widetilde{\mathbb{X}}_f(1)$ *- light grey; a part of* $\mathbb{X}_{RPI}$ *- dark grey) and closed-loop state trajectories obtained for initial states* $x_{01} = [-1.9649 \ \ -1.9649]^\top$ *(solid line),* $x_{02} = [5 \ \ -5]^\top$ *(dashed line), for system* (5.6) *in closed-loop with* (5.14)*.*

trajectories also converge to the origin for both initial states, due to the ISS property. It is also worth to point out that the initial state x_{01}, which was a problematic initial condition, as discussed in Section 5.3, is contained in the feasible set of the ISS dual-mode MPC controller. This illustrates the effectiveness of the proposed methodology.

5.7 Conclusions

This chapter considered robust asymptotic stability in terms of ISS for (discontinuous) PWA systems controlled by MPC strategies, as this is an important property from a practical point of view. We focused mainly on the case of MPC cost based on ∞-norms, which gives rise to a PWA closed-loop system, although many results are easily extendable to hybrid MPC based on quadratic costs. We presented an example of a PWA system taken

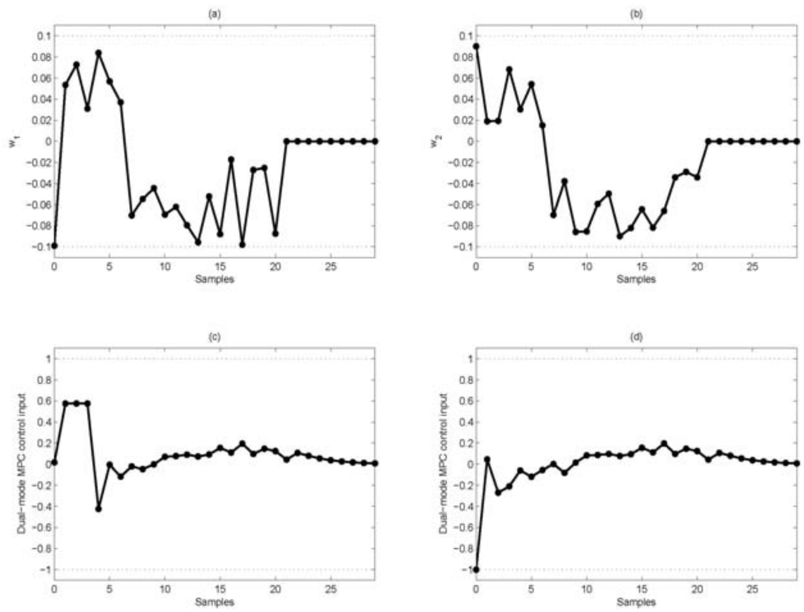

Figure 5.5: History on the interval $[0, 29]$ *of: disturbance inputs* $w = [w_1\ w_2]^\top$ *- (a) and (b);* $\hat{u}^{DM}(\cdot)$ *for* x_{01} *- (c);* $\hat{u}^{DM}(\cdot)$ *for* x_{02} *- (d).*

---

from literature for which a nominally stabilizing MPC scheme generates a (discontinuous) MPC value function that is not an ISS Lyapunov function.

Then, we presented an a posteriori test for verifying whether or not a nominally stabilizing MPC controller, possibly with a discontinuous value function, has also some inherent robustness properties. In case this test fails, as it happened for the example shown in Section 5.3, there are no systematic ways available for modifying hybrid MPC schemes such that ISS (and thus, robustness) is a priori ensured. Therefore, a new design method for setting up hybrid MPC schemes for general discontinuous PWA systems, with an a priori ISS guarantee, was developed. The novel hybrid MPC algorithms use tightened constraints and do not require continuity of the system, the MPC control law nor of the MPC value function.

The application of the results presented in this chapter for designing computationally friendly feedback min-max hybrid MPC schemes is the object of future research.

6

Input-to-state stabilizing min-max predictive controllers

In this chapter we consider discrete-time nonlinear systems that are affected, possibly simultaneously, by time-varying parametric uncertainties and disturbance inputs. The min-max model predictive control (MPC) methodology is employed to obtain a controller that robustly steers the state of the system towards a desired equilibrium. The goal is to provide a priori sufficient conditions for robust stability of the resulting closed-loop system via the input-to-state stability framework. Moreover, new techniques for computing a terminal cost and a local state-feedback control law that satisfy the developed min-max MPC robust stabilization conditions are presented. These techniques apply to min-max MPC algorithms based on both quadratic and $1,\infty$-norms cost functions.

6.1 Introduction

The problem of robust regulation of discrete-time linear and nonlinear systems affected, possibly simultaneously, by parametric uncertainties and disturbance inputs towards a desired equilibrium has attracted the interest of many researchers. The reason is that this situation is often encountered in practical control applications. In case hard constraints are imposed on state and input variables, the robust MPC methodology provides a reliable solution for addressing this control problem. The research related to robust MPC is focused on solving efficiently the corresponding MPC optimization problems on one hand, e.g. see (Bemporad et al., 2003b; Kerrigan and Maciejowski, 2004; Mayne et al., 2004; Pluymers et al., 2005; Alamo et al., 2005), and guaranteeing stability of the controlled system, on the other hand. In this chapter we are interested in stability issues and therefore, we focus our attention on results developed in this direction. See (Scokaert and Mayne, 1998; Mayne et al., 2000; Bemporad et al., 2003b; Kerrigan and Maciejowski, 2004; Pluymers et al., 2005; Mayne et al., 2004; Alamo et al., 2005; Muñoz de la Peña et al., 2005) and the references therein for a complete overview, also regarding computational aspects.

There are several ways for designing robust MPC controllers for perturbed nonlinear systems. One way is to rely on the inherent robustness properties of nominally stabilizing nonlinear MPC algorithms, e.g. as it was done in (Scokaert et al., 1997; Magni et al., 1998; Limon et al., 2002b; Grimm et al., 2003b). Another approach is to incorporate knowledge about the disturbances in the MPC problem formulation via open-loop worst case scenarios. This includes MPC algorithms based on tightened constraints, e.g. as the one of (Limon et al., 2002a), and MPC algorithms based on open-loop min-max optimization problems, e.g. see the survey (Mayne et al., 2000). To incorporate feedback to the disturbance inputs, the closed-loop or feedback min-max MPC problem set-up was introduced in (Lee and Yu, 1997) and further developed in (Mayne, 2001; Magni et al., 2003; Limon et al., 2006). The open-loop approach is computationally somewhat easier than the feedback approach, e.g. as it is also shown in the example of Section 6.6.3, but the set of feasible states corresponding to the feedback min-max MPC optimization problem is usually much larger. Sufficient conditions for asymptotic stability of nonlinear systems in closed-loop with feedback min-max MPC controllers were presented in (Mayne, 2001) under the a priori assumption that the (additive) disturbance input converges to zero as time tends to infinity. These results were extended in (Limon et al., 2006) to the case when persistent (additive) disturbance inputs affect the system.

In this chapter we employ the input-to-state stability (ISS) framework (Sontag, 1989, 1990; Jiang and Wang, 2001), which was introduced in Chapter 2, to derive new sufficient conditions for robust asymptotic stability of nonlinear min-max MPC. Firstly, we show that in general, only input-to-state practical stability (ISpS) (Jiang, 1993; Jiang et al., 1994, 1996) can be a priori ensured for min-max nonlinear MPC. This is because the min-max MPC controller takes into account the effect of a non-zero disturbance input, even if the disturbance input vanishes in reality. In other words, ISpS does not imply asymptotic stability for zero disturbance inputs, as it is the case for ISS. However, we derive explicit bounds on the evolution of the min-max

MPC closed-loop system state and we prove that the developed ISpS sufficient conditions actually imply that the state trajectory of the closed-loop system is ultimately bounded in a Robust Positively Invariant (RPI) set.

Still, in the case when the disturbance input converges to zero, it is desirable that *asymptotic stability* is recovered for the controlled system, i.e. that the state also converges to zero, which is guaranteed by the ISS property, but not by the ISpS property, as explained above. One of the main results of this chapter is to provide novel a priori sufficient conditions for ISS of min-max nonlinear MPC. ISS is achieved via a dual-mode approach and using a new technique based on $\mathcal{KL}$-estimates of stability, e.g. see (Khalil, 2002). This result is important because it unifies the properties of (Limon et al., 2006) and (Mayne, 2001), i.e. it guarantees both ISpS in the presence of persistent disturbances and robust asymptotic stability in the presence of asymptotically decaying disturbances (without a priori assuming that this property holds for the disturbances).

Another main result of this chapter consists in new techniques for computing a terminal cost and a local state-feedback controller such that the developed robust stabilization conditions for min-max MPC are satisfied. These techniques employ linear matrix inequalities (LMI) for quadratic MPC costs and norm inequalities for MPC costs based on $1, \infty$-norms. The proposed stabilization method works for systems affected by both parametric uncertainties and additive disturbances. One of its advantages is that the resulting MPC cost function is *continuous*, convex and bounded. Also, the techniques for computing the terminal cost presented in this chapter do not depend on the type of min-max MPC numerical set-up. They can be employed to ensure robust stability or input-to-state stability for both *open-loop* and *feedback* min-max MPC schemes, such as the ones in (Kothare et al., 1996; Scokaert and Mayne, 1998; Bemporad et al., 2003b; Kerrigan and Maciejowski, 2004; Mayne et al., 2004; Alamo et al., 2005; Muñoz de la Peña et al., 2005; Pluymers et al., 2005).

6.2 Min-max MPC: Problem set-up

The results presented in this chapter can be applied to both open-loop and feedback min-max MPC strategies. However, it is the common feeling that open-loop min-max formulations are conservative and underestimate the set of feasible input trajectories. For this reason, although we present both problem formulations, the stability results are proven only for feedback min-max nonlinear MPC set-ups. Note that it is possible to prove, via a similar reasoning and using the *same* hypotheses, that all the results developed in this chapter also hold for open-loop min-max MPC schemes.

Consider the discrete-time non-autonomous perturbed nonlinear system:

$$x_{k+1} = g(x_k, u_k, w_k, v_k), \quad k \in \mathbb{Z}_+, \tag{6.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ are the state, the control action, unknown *time-varying parametric uncertainties* and *disturbance inputs* at discrete-time k and, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{d_w} \times \mathbb{R}^{d_v} \to \mathbb{R}^n$ is an arbitrary nonlinear, possibly discontinuous, function. Let $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$ be C-sets that represent state and input constraints for system (6.1).

Open-loop min-max MPC evaluates a single control sequence, i.e. $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k}) \in \mathbb{U}^N$. For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k, \mathbf{w}_k, \mathbf{v}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote a state sequence generated by system (6.1) from initial state $x_{0|k} \triangleq x_k$ and by applying the input sequence $\mathbf{u}_k$, where $\mathbf{w}_k \triangleq (w_{0|k}, \ldots, w_{N-1|k}) \in \mathbb{W}^N$ and $\mathbf{v}_k \triangleq (v_{0|k}, \ldots, v_{N-1|k}) \in \mathbb{V}^N$ are the corresponding disturbance sequences and $x_{i|k} \triangleq g(x_{i-1|k}, u_{i-1|k}, w_{i-1|k}, v_{i-1|k})$ for all $i = 1, \ldots, N$.

Furthermore, let $\mathbb{X}_T \subseteq \mathbb{X}$ with $0 \in \text{int}(\mathbb{X}_T)$ denote a desired target set. The class of *admissible input sequences* defined with respect to $\mathbb{X}_T$ and state $x_k \in \mathbb{X}$ is

$$\begin{aligned}\mathcal{U}_N(x_k) \triangleq \{\mathbf{u}_k \in \mathbb{U}^N \mid & \mathbf{x}_k(x_k, \mathbf{u}_k, \mathbf{w}_k, \mathbf{v}_k) \in \mathbb{X}^N, x_{N|k} \in \mathbb{X}_T, \\ & \forall \mathbf{w}_k \in \mathbb{W}^N, \forall \mathbf{v}_k \in \mathbb{V}^N\}.\end{aligned}$$

Let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0, 0) = 0$ be continuous, convex and bounded functions on bounded sets.

The *open-loop* min-max MPC problem is formulated as follows: Let the target set $\mathbb{X}_T \subseteq \mathbb{X}$ and $N \in \mathbb{Z}_{\geq 1}$ be given. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be given and minimize the cost

$$J(x_k, \mathbf{u}_k) \triangleq \max_{\mathbf{w}_k \in \mathbb{W}^N, \mathbf{v}_k \in \mathbb{V}^N} \left\{ F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}) \right\},$$

with prediction model (6.1), over all sequences $\mathbf{u}_k$ in $\mathcal{U}_N(x_k)$.

An initial state $x \in \mathbb{X}$ is called *feasible* for the above problem if $\mathcal{U}_N(x) \neq \emptyset$. Similarly, the optimization problem is said to be *feasible* for $x \in \mathbb{X}$ if $\mathcal{U}_N(x) \neq \emptyset$. The set of all feasible states for the open-loop min-max MPC optimization problem is denoted by $\mathbb{X}_f(N)$.

Feedback min-max MPC obtains a sequence of feedback control laws that minimizes a worst case cost function, while assuring robust constraint handling. In this chapter we employ the *dynamic programming approach*[1] to feedback min-max nonlinear MPC proposed in (Lee and Yu, 1997) for linear systems and in (Mayne, 2001) for nonlinear systems. In this approach, the feedback min-max optimal control input is obtained as follows:

$$\begin{aligned} V_i(x) \;\triangleq\; & \min_{u\in\mathbb{U}}\{\max_{w\in\mathbb{W},v\in\mathbb{V}}[L(x,u)+V_{i-1}(g(x,u,w,v))] \\ & \text{s.t. } g(x,u,w,v)\in\mathbb{X}_f(i-1), \forall w\in\mathbb{W}, \forall v\in\mathbb{V}\}, \end{aligned} \tag{6.2}$$

where the set $\mathbb{X}_f(i)$ contains all the states $x_i \in \mathbb{X}$ which are such that (6.2) is feasible, $i = 1,\ldots,N$, and "s.t." is a short term for "such that". The optimization problem is defined for $i = 1,\ldots,N$ where N is the prediction horizon.

The boundary conditions are:

$$\begin{aligned} V_0(x_0) &\triangleq F(x_0), \\ \mathbb{X}_f(0) &\triangleq \mathbb{X}_T, \end{aligned} \tag{6.3}$$

where $\mathbb{X}_T \subseteq \mathbb{X}$ is a desired target set that contains the origin in its interior. Taking into account the definition of the min-max problem, $\mathbb{X}_f(i)$ is now the set of all states that can be robustly controlled into the target set $\mathbb{X}_T$ in $i \in \mathbb{Z}_{\geq 1}$ steps.

The control law is applied to system (6.1) in a receding horizon manner. At each sampling time the problem is solved for the current state x and the value function $V_N(x)$ is obtained. The *feedback* min-max MPC control law is defined as

$$\bar{u}(x) \triangleq u_N^*, \tag{6.4}$$

where u_N^* is the optimizer that yields the min-max MPC value function $V(x) = V_N(x)$, which will be used in the next section as a candidate ISpS Lyapunov function to establish ISpS of the nonlinear system (6.1) in closed-loop with the feedback min-max MPC control (6.4).

6.3 ISpS results for min-max nonlinear MPC

In this section we present sufficient conditions for ISpS of system (6.1) in closed-loop with the feedback min-max MPC control (6.4) and we derive

[1]An alternative, equivalent formulation of feedback min-max MPC is the so-called "large scale scenario" approach presented in (Scokaert and Mayne, 1998; Mayne et al., 2000).

explicit bounds on the evolution of the closed-loop system state. Let $h : \mathbb{R}^n \to \mathbb{R}^m$ denote an arbitrary nonlinear function with $h(0) = 0$ and let $\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid h(x) \in \mathbb{U}\}$.

Assumption 6.3.1 There exist $a, b, a_1, \lambda > 0$ with $a \leq b$, non-negative numbers e_1, e_2, a function $h : \mathbb{R}^n \to \mathbb{R}^m$ with $h(0) = 0$ and a $\mathcal{K}$-function σ_1 such that:

1. $\mathbb{X}_T \subseteq \mathbb{X}_{\mathbb{U}}$ and $0 \in \text{int}(\mathbb{X}_T)$;
2. $\mathbb{X}_T$ is a RPI set for system (6.1) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$;
3. $L(x, u) \geq a\|x\|^\lambda$ for all $x \in \mathbb{X}$ and all $u \in \mathbb{U}$;
4. $a_1\|x\|^\lambda \leq F(x) \leq b\|x\|^\lambda + e_1$ for all $x \in \mathbb{X}_T$;
5. $F(g(x, h(x), w, v)) - F(x) \leq -L(x, h(x)) + \sigma_1(\|v\|) + e_2$ for all $x \in \mathbb{X}_T$, $w \in \mathbb{W}$, and all $v \in \mathbb{V}$.

Note that Assumption 6.3.1, which implies that $F(\cdot)$ is a local ISpS (ISS) Lyapunov function, can be regarded as a generalization of the usual sufficient conditions for nominal stability of MPC, which imply that $F(\cdot)$ is a local Lyapunov function, see, for example, the survey (Mayne et al., 2000). Techniques for computing a terminal cost and a function $h(\cdot)$ such that Assumption 6.3.1 is satisfied for a industrially relevant subclass of system (6.1) will be presented in Section 6.5. See also the illustrative example presented in Section 6.6.3.

The next result is directly obtained via Theorem 2.4.4 by showing that the terminal cost $F(\cdot)$ is a local (i.e. for all $x \in \mathbb{X}_T$) ISpS (ISS) Lyapunov function for system (6.1) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$.

Proposition 6.3.2 *Suppose that Assumption 6.3.1 holds. Then, system* (6.1) *in closed-loop with* $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, *is ISpS for initial conditions in* $\mathbb{X}_T$. *Moreover, if Assumption 6.3.1 holds with* $e_1 = e_2 = 0$, *system* (6.1) *in closed-loop with* $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, *is ISS for initial conditions in* $\mathbb{X}_T$.

Now we state the main result of this section.

Theorem 6.3.3 *Suppose that* $F(\cdot)$, $L(\cdot,\cdot)$, $\mathbb{X}_T$ *and* $h(\cdot)$ *are such that Assumption 6.3.1 holds for system* (6.1). *Then, the perturbed nonlinear system* (6.1) *in closed-loop with the feedback min-max MPC control* (6.4) *is ISpS* *for initial conditions in $\mathbb{X}_f(N)$. Moreover, the property (2.8) holds with the following functions:*

$$\beta(s,k) \triangleq \left(\frac{4\theta}{a}\right)^{\frac{1}{\lambda}} \tilde{\rho}^k s, \quad \gamma(s) \triangleq \left(\frac{2\delta}{a(1-\rho)}\right)^{\frac{1}{\lambda}} s, \quad d \triangleq \left(\frac{4\xi}{a}\right)^{\frac{1}{\lambda}}, \qquad (6.5)$$

where $\theta \triangleq \max(b, \frac{\Gamma}{r^\lambda})$ for some constants $\Gamma, r > 0$, $\tilde{\rho} \triangleq \rho^{\frac{1}{\lambda}} \in (0,1)$, $\rho \triangleq 1 - \frac{a}{\theta} \in (0,1)$, $\delta > 0$ can be taken arbitrarily small, and $\xi \triangleq d_1 + \frac{d_2}{1-\rho}$, with $d_1 \triangleq e_1 + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2)$ and $d_2 \triangleq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2$.

Proof: The proof consists in showing that the min-max MPC value function $V(\cdot)$ defined in the previous section is an ISpS Lyapunov function, i.e. it satisfies the hypothesis of Theorem 2.4.4. First, it is known (see (Mayne, 2001; Kerrigan and Maciejowski, 2001)) that under Assumption 6.3.1-1,2, the set $\mathbb{X}_f(N)$ is a RPI set for system (6.1) in closed-loop with the feedback min-max MPC control (6.4).

Now we obtain lower and upper bounding functions on the min-max MPC value function that satisfy the conditions (2.9a), introduced in Chapter 2. From Assumption 6.3.1-3 it follows that $V(x) = V_N(x) \geq L(x, \bar{u}(x)) \geq a\|x\|^\lambda$, for all $x \in \mathbb{X}_f(N)$, where $\bar{u}(x)$ is the feedback min-max MPC control law defined in (6.4).

Next, letting $x_0 \triangleq x \in \mathbb{X}_T$, from Assumption 6.3.1-2,5 it follows that for any $w_{[N-1]} \in \mathbb{W}^N$ and any $v_{[N-1]} \in \mathbb{V}^N$

$$F(x_N) + \sum_{i=0}^{N-1} L(x_i, h(x_i)) \leq F(x_0) + \sum_{i=0}^{N-1} \sigma_1(\|v_i\|) + Ne_2,$$

where $x_i \triangleq g(x_{i-1}, h(x_{i-1}), w_{i-1}, v_{i-1})$ for $i = 1, \ldots, N$. Then, by optimality and Assumption 6.3.1-4 we have that for all $x \in \mathbb{X}_T$,

$$V(x) = V_N(x) \leq F(x) + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2) \leq b\|x\|^\lambda + d_1,$$

where $d_1 \triangleq e_1 + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2) > 0$.

To establish a global upper bound on $V(\cdot)$ in $\mathbb{X}_f(N)$, let $r > 0$ be such that $\mathcal{B}_r \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subseteq \mathbb{X}_T$. Due to compactness of $\mathbb{X}$, $\mathbb{U}$, $\mathbb{W}$, $\mathbb{V}$ and convexity and boundedness of $F(\cdot)$, $L(\cdot,\cdot)$ there exists a number $\Gamma > 0$ such that $V(x) \leq \Gamma$ for all $x \in \mathbb{X}_f(N)$. Letting $\theta \triangleq \max(b, \frac{\Gamma}{r^\lambda})$ we obtain $V(x) \leq \theta\|x\|^\lambda \leq \theta\|x\|^\lambda + d_1$ for all $x \in \mathbb{X}_f(N) \setminus \mathbb{X}_T$. Then, due to $\theta \geq b$ it also follows that $V(x) = V_N(x) \leq b\|x\|^\lambda + d_1 \leq \theta\|x\|^\lambda + d_1$ for all $x \in \mathbb{X}_T$.

Hence, $V(\cdot)$ satisfies condition (2.9a) for all $x \in \mathbb{X}_f(N)$ with $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq \theta s^\lambda$ and $d_1 = e_1 + N(\max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2) > 0$.

Next, we show that $V(\cdot)$ satisfies condition (2.9b) of Chapter 2. By Assumption 6.3.1-5 and optimality, for all $x \in \mathbb{X}_T = \mathbb{X}_f(0)$ we have that:

$$\begin{aligned} V_1(x) - V_0(x) &\leq \max_{w\in\mathbb{W}, v\in\mathbb{V}} [L(x, h(x)) + F(g(x, h(x), w, v))] - F(x) \\ &\leq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2. \end{aligned}$$

Then, we obtain via induction that:

$$V_{i+1}(x) - V_i(x) \leq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2, \ \forall x \in \mathbb{X}_f(i), \ \forall i \in 0, \ldots, N-1. \quad (6.6)$$

At time $k \in \mathbb{Z}_+$, for a given state $x_k \in \mathbb{X}$ and a fixed prediction horizon N the min-max MPC control law $\bar{u}(x_k)$ is calculated and then applied to system (6.1). The state evolves to $x_{k+1} = g(x_k, \bar{u}(x_k), w_k, v_k) \in \mathbb{X}_f(N)$. Then, by Assumption 6.3.1-5 and applying recursively (6.6) it follows that

$$\begin{aligned} &V_N(x_{k+1}) - V_N(x_k) \\ &\quad\leq V_N(x_{k+1}) - \max_{w\in\mathbb{W}, v\in\mathbb{V}} [L(x_k, \bar{u}(x_k)) + V_{N-1}(g(x_k, \bar{u}(x_k), w, v))] \\ &\quad\leq V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) - V_{N-1}(g(x_k, \bar{u}(x_k), w_k, v_k)) \\ &\quad= V_N(x_{k+1}) - L(x_k, \bar{u}(x_k)) - V_{N-1}(x_{k+1}) \\ &\quad\leq -L(x_k, \bar{u}(x_k)) + \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 \\ &\quad\leq -a\|x_k\|^\lambda + \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 \\ &\quad\leq -a\|x_k\|^\lambda + d_2, \end{aligned} \quad (6.7)$$

for all $x_k \in \mathbb{X}_f(N)$, $w_k \in \mathbb{W}$, $v_k \in \mathbb{V}$, $k \in \mathbb{Z}_+$, where $d_2 \triangleq \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 > 0$. Hence, the feedback min-max nonlinear MPC value function $V(\cdot)$ satisfies (2.9b) with $\alpha_3(s) \triangleq as^\lambda$, any $\sigma \in \mathcal{K}$ and $d_2 = \max_{v\in\mathbb{V}} \sigma_1(\|v\|) + e_2 > 0$. The statements then follow from the fundamental ISpS result of chapter two, i.e. Theorem 2.4.4.

The functions $\beta(\cdot,\cdot), \gamma(\cdot)$ and the constant d defined in (6.5) are obtained by letting $\sigma(s) \triangleq \delta s^\lambda$ for some (any) $\delta > 0$ and substituting the functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot), \sigma(\cdot)$ and the constants d_1, d_2 obtained above in relation (2.10). ■

6.4 ISS results for min-max nonlinear MPC

As shown in the proof of Theorem 6.3.3, ISS cannot be proven for system (6.1) in closed-loop with $\bar{u}(\cdot)$, *if the min-max MPC value function* $V(\cdot)$ *is used* *as the candidate ISS Lyapunov function*, which is the most common method of proving stability in MPC, e.g. see the survey (Mayne et al., 2000). This is due to the fact that by construction, $V(\cdot)$ satisfies the conditions (2.9) with $d_1, d_2 > 0$, even if Assumption 6.3.1 holds for $e_1 = e_2 = 0$. In the case of persistent disturbances this is not necessarily a drawback, since ultimate boundedness in a RPI subset of $\mathbb{X}_f(N)$ is guaranteed, as it will be shown in the sequel. However, in the case when the disturbance input vanishes after a certain time it is desirable to have an ISS closed-loop system, since then ISS implies (robust) asymptotic stability.

In this section we present sufficient conditions for ISS of system (6.1) in closed-loop with a dual-mode min-max MPC strategy. Therefore, a standing assumption throughout this section is that the origin is an equilibrium in (6.1) for zero inputs u and v. By this we mean that $g(0, 0, w, 0) = 0$ for all $w \in \mathbb{W}$. The following technical result will be employed to prove the main result for dual-mode min-max nonlinear MPC.

For any τ with $0 < \tau < a$ define

$$\mathbb{M}_\tau \triangleq \left\{ x \in \mathbb{X}_f(N) \mid \|x\|^\lambda \leq \frac{d_2}{a - \tau} \right\} \quad \text{and} \quad \overline{\mathbb{M}}_\tau \triangleq \mathbb{X}_f(N) \setminus \mathbb{M}_\tau, \qquad (6.8)$$

where a is the constant of Assumption 6.3.1-3 and $d_2 = \max_{v \in \mathbb{V}} \sigma_1(\|v\|) + e_2 > 0$. Note that $0 \in \text{int}(\mathbb{M}_\tau)$, as for d_2 and a defined above it holds that $\frac{d_2}{a-\tau} > 0$ and $0 \in \text{int}(\mathbb{X}_T) \subseteq \text{int}(\mathbb{X}_f(N))$.

Theorem 6.4.1 *Suppose that $F(\cdot)$, $L(\cdot, \cdot)$, $\mathbb{X}_T$ and $h(\cdot)$ are such that Assumption 6.3.1 holds for system* (6.1) *and there exists a $\tau \in (0, a)$ such that $\overline{\mathbb{M}}_\tau \neq \emptyset$. Then, for each $x_0 \in \overline{\mathbb{M}}_\tau$ and any disturbances realizations $\{w_j\}_{j \in \mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and $\{v_j\}_{j \in \mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau$. Let $i(x_0)$ denote the minimal one, i.e. $i(x_0) \triangleq \arg\min\{i \in \mathbb{Z}_{\geq 1} \mid x_0 \in \overline{\mathbb{M}}_\tau,\ x_i \in \mathbb{M}_\tau\}$.*

Moreover, there exists a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ such that for all $x_0 \in \overline{\mathbb{M}}_\tau$ and any disturbances realizations $\{w_j\}_{j \in \mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and $\{v_j\}_{j \in \mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, the trajectory of the closed-loop system (6.1)-(6.4) *satisfies $\|x_k\| \leq \beta(\|x_0\|, k)$ for all $k \in \mathbb{Z}_{\leq i(x_0)}$, where $x_k \in \overline{\mathbb{M}}_\tau$ for all $k \in \mathbb{Z}_{<i(x_0)}$.*

Proof: We prove the second statement of the theorem first. As shown in the proof of Theorem 6.3.3, the hypothesis implies that

$$a\|x\|^\lambda \leq V(x) \leq \theta\|x\|^\lambda + d_1, \quad \forall x \in \mathbb{X}_f(N).$$

Let $\tilde{r} > 0$ be such that $\mathcal{B}_{\tilde{r}} \subseteq \mathbb{M}_\tau$. For all the state trajectories that satisfy $x_k \in \overline{\mathbb{M}}_\tau$ (and thus $x_k \notin \mathbb{M}_\tau$) for all $k \in \mathbb{Z}_{<i(x_0)}$ we have that $\|x_k\| \geq \tilde{r}$ for all $k \in \mathbb{Z}_{<i(x_0)}$. This yields:

$$V(x_k) \leq \theta\|x_k\|^\lambda + d_1 \left(\frac{\|x_k\|}{\tilde{r}}\right)^\lambda \leq \left(\theta + \frac{d_1}{\tilde{r}^\lambda}\right)\|x_k\|^\lambda,\ \forall x_k \in \overline{\mathbb{M}}_\tau,\ k \in \mathbb{Z}_{<i(x_0)}.$$

The hypothesis also implies (see (6.7)) that

$$V(x_{k+1}) - V(x_k) \leq -a\|x_k\|^\lambda + d_2,\ \forall x_k \in \mathbb{X}_f(N),\ w_k \in \mathbb{W},\ v_k \in \mathbb{V},\ k \in \mathbb{Z}_+,$$

and by the definition (6.8) it follows that

$$V(x_{k+1}) - V(x_k) \leq -\tau\|x_k\|^\lambda, \quad \forall x_k \in \overline{\mathbb{M}}_\tau,\ w_k \in \mathbb{W},\ v_k \in \mathbb{V},\ k \in \mathbb{Z}_{<i(x_0)}. \tag{6.9}$$

Then, following the steps of the proof of Theorem 2.4.4, it is straightforward to show that the state trajectory satisfies for all $k \in \mathbb{Z}_{\leq i(x_0)}$,

$$\|x_k\| \leq \beta(\|x_0\|, k); \quad \beta(s,k) \triangleq \alpha_1^{-1}(\rho^k\alpha_2(s)) = \left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}} \left(\rho^{\frac{1}{\lambda}}\right)^k s, \tag{6.10}$$

where $\alpha_2(s) \triangleq \bar{b}s^\lambda$, $\bar{b} \triangleq \theta b + \frac{d_1}{\tilde{r}^\lambda}$, $\alpha_1(s) \triangleq as^\lambda$ and $\rho \triangleq 1 - \frac{\tau}{\bar{b}}$. Note that $\rho \in (0,1)$ due to $0 < \tau < a \leq b$.

Next, we prove that there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau$. Let $\bar{r} > \tilde{r} > 0$ be such that $\mathbb{X}_f(N) \subseteq \mathcal{B}_{\bar{r}}$. Such an $\bar{r}$ exists due to the fact that the compactness of $\mathbb{X}$ implies that $\mathbb{X}_f(N)$ is bounded. Assume that there does not exist an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau$. Then, for all $i \in \mathbb{Z}_+$ we have that

$$\|x_i\| \leq \beta(\|x_0\|, i) = \left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}} \|x_0\| \left(\rho^{\frac{1}{\lambda}}\right)^i \leq \left(\rho^{\frac{1}{\lambda}}\right)^i \left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}} \bar{r}.$$

Since $\rho^{\frac{1}{\lambda}} \in (0,1)$, we have that $\lim_{i\to\infty} \left(\rho^{\frac{1}{\lambda}}\right)^i = 0$. Hence, there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathcal{B}_{\tilde{r}} \subseteq \mathbb{M}_\tau$ and we reached a contradiction. ■

Before stating the main result, we make use of Theorem 6.4.1 to prove that the ISpS sufficient conditions of Assumption 6.3.1 ensure ultimate boundedness of the min-max MPC closed-loop system. This property is achieved with respect to a RPI sublevel set of the min-max MPC value function induced by the set $\mathbb{M}_\tau$.

Lemma 6.4.2 *Let*

$$\Upsilon \triangleq \max_{x\in\mathbb{M}_\tau} \left\{V(x) - a\|x\|^\lambda + d_2\right\} \quad \textit{and} \quad \mathcal{V}_\Upsilon \triangleq \{x \in \mathbb{X}_f(N) \mid V(x) \leq \Upsilon\}.$$

Suppose that the hypothesis of Theorem 6.4.1 holds. Then, the closed-loop system (6.1)-(6.4) *is ultimately bounded in the set* $\mathcal{V}_\Upsilon$ *for initial conditions in* $\mathbb{X}_f(N)$.

Proof: By definition of Υ, it follows that $\mathbb{M}_\tau \subseteq \mathcal{V}_\Upsilon$. Suppose that $x_0 \in \mathbb{X}_f(N) \setminus \mathcal{V}_\Upsilon$ and thus, $x_0 \in \overline{\mathbb{M}}_\tau$. Then, by Theorem 6.4.1 it follows that there exists an $i \in \mathbb{Z}_{\geq 1}$ such that $x_i \in \mathbb{M}_\tau \subseteq \mathcal{V}_\Upsilon$.

Next, we prove that $\mathcal{V}_\Upsilon$ is a RPI set for the closed-loop system (6.1)-(6.4). As shown in the proof of Theorem 6.4.1 (see (6.9)), for any $x \in \mathcal{V}_\Upsilon \setminus \mathbb{M}_\tau$ it holds that

$$V(g(x, \bar{u}(x), w, v)) \leq V(x) - \tau \|x\|^\lambda \leq V(x) \leq \Upsilon,$$

for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$. Now let $x \in \mathbb{M}_\tau$. Two situations can occur: either $g(x, \bar{u}(x), w, v) \in \mathbb{M}_\tau$ or $g(x, \bar{u}(x), w, v) \notin \mathbb{M}_\tau$. The first case is trivial due to $\mathbb{M}_\tau \subseteq \mathcal{V}_\Upsilon$. In the latter case, by inequality (6.7) and since $x \in \mathbb{M}_\tau$ it holds that

$$V(g(x, \bar{u}(x), w, v)) \leq V(x) - a\|x\|^\lambda + d_2 \leq \Upsilon.$$

Therefore, for any $x \in \mathcal{V}_\Upsilon$, it holds that $g(x, \bar{u}(x), w, v) \in \mathcal{V}_\Upsilon$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{V}$, which implies that $\mathcal{V}_\Upsilon$ is a RPI set for the closed-loop system (6.1)-(6.4). Hence, the closed-loop system (6.1)-(6.4) is ultimately bounded in $\mathcal{V}_\Upsilon$. ■

Note that in a worst case situation, i.e. when the disturbance input $v \in \mathbb{V}$ is too large and $\mathbb{M}_\tau = \mathbb{X}_f(N)$ for any $\tau \in (0, a)$, the closed-loop system (6.1)-(6.4) is still ultimately bounded in the set $\mathbb{X}_f(N)$ for all $x_0 \in \mathbb{X}_f(N)$, due to robust positive invariance of $\mathbb{X}_f(N)$.

To state the main result, let the dual-mode feedback min-max MPC control law be defined as:

$$\bar{u}^{\mathrm{DM}}(x_k) \triangleq \begin{cases} \bar{u}(x_k) \text{ if } x_k \in \mathbb{X}_f(N) \setminus \mathbb{X}_T \\ h(x_k) \text{ if } x_k \in \mathbb{X}_T \end{cases} ; \quad k \in \mathbb{Z}_+. \tag{6.11}$$

Theorem 6.4.3 *Suppose that Assumption 6.3.1 holds with* $e_1 = e_2 = 0$ *for system* (6.1). *Furthermore, suppose there exists* $\tau \in (0, a)$ *such that* $\mathbb{M}_\tau \subseteq \mathbb{X}_T$. *Then, the perturbed nonlinear system* (6.1) *in closed-loop with the dual-mode feedback min-max MPC control* $\bar{u}^{DM}$ *is ISS for initial conditions in* $\mathbb{X}_f(N)$.

Proof: In order to prove ISS, we consider two situations: in Case 1 we assume that $x_0 \in \mathbb{X}_T$ and in Case 2 we assume that $x_0 \in \mathbb{X}_f(N) \setminus \mathbb{X}_T$. In Case 1, $F(\cdot)$ satisfies the hypothesis of Proposition 6.3.2 with $e_1 = e_2 = 0$ and hence, the closed-loop system (6.1)-(6.4) is ISS. Then, using the reasoning employed in the proof of Theorem 2.4.4, it can be shown that there exist a $\mathcal{KL}$-function $\beta_1(s,k) \triangleq \alpha_1^{-1}(2\rho_1^k\alpha_2(s))$, with $\alpha_1(s) \triangleq a_1 s^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$, $\rho_1 \triangleq 1 - \frac{a}{b}$, and a $\mathcal{K}$-function $\gamma(\cdot)$ such that for all $x_0 \in \mathbb{X}_T$ the state trajectory satisfies

$$\|x_k\| \leq \beta_1(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geq 1}. \tag{6.12}$$

In Case 2, since $\mathbb{M}_\tau \subseteq \mathbb{X}_T$, by Theorem 6.4.1 there exists a $p \in \mathbb{Z}_{\geq 1}$ such that $x_p \in \mathbb{X}_T$. From Theorem 6.4.1 we also have that there exists a $\mathcal{KL}$-function $\beta_2(s,k) \triangleq \bar{\alpha}_1^{-1}(\rho_2^k\bar{\alpha}_2(s))$, with $\bar{\alpha}_1(s) \triangleq as^\lambda$, $\bar{\alpha}_2(s) \triangleq \bar{b}s^\lambda$, $\rho_2 \triangleq 1 - \frac{\tau}{\bar{b}}$ such that for all $x_0 \in \mathbb{X}_f(N) \setminus \mathbb{X}_T$ the state trajectory satisfies

$$\|x_k\| \leq \beta_2(\|x_0\|, k), \quad \forall k \in \mathbb{Z}_{\leq p} \quad \text{and} \quad x_p \in \mathbb{X}_T.$$

Then, for all $p \in \mathbb{Z}_{\geq 1}$ and all $k \in \mathbb{Z}_{\geq p+1}$ it holds that

$$\begin{aligned}\|x_k\| &\leq \beta_1(\|x_p\|, k-p) + \gamma(\|v_{[k-p,k-1]}\|)\\ &\leq \beta_1(\beta_2(\|x_0\|, p), k-p) + \gamma(\|v_{[k-p,k-1]}\|) \leq \beta_3(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|),\end{aligned}$$

where $v_{[k-p,k-1]}$ denotes the truncation between time $k-p$ and $k-1$. In the above inequalities we used

$$\begin{aligned}\beta_1(\beta_2(s,p), k-p) &= \alpha_1^{-1}\left(2\rho_1^{k-p}\alpha_2\left(\left(\frac{\bar{b}}{a}\right)^{\frac{1}{\lambda}} s \left(\rho_2^{\frac{1}{\lambda}}\right)^p\right)\right)\\ &\leq \left(\frac{2b\bar{b}}{a^2}\right)^{\frac{1}{\lambda}} s \left(\rho_3^{\frac{1}{\lambda}}\right)^k \triangleq \beta_3(s,k),\end{aligned}$$

and $\rho_3 \triangleq \max(\rho_1, \rho_2) \in (0,1)$. Hence, $\beta_3 \in \mathcal{KL}$.

Then, we have that

$$\|x_k\| \leq \beta(\|x_0\|, k) + \gamma(\|v_{[k-1]}\|), \quad \forall k \in \mathbb{Z}_{\geq 1},$$

for all $x_0 \in \mathbb{X}_f(N)$, $\{w_j\}_{j\in\mathbb{Z}_+}$ with $w_j \in \mathbb{W}$ for all $j \in \mathbb{Z}_+$ and all $\{v_j\}_{j\in\mathbb{Z}_+}$ with $v_j \in \mathbb{V}$ for all $j \in \mathbb{Z}_+$, where $\beta(s,k) \triangleq \max(\beta_1(s,k), \beta_2(s,k), \beta_3(s,k))$.

Since $\beta_{1,2,3}(\cdot,\cdot) \in \mathcal{KL}$ implies that $\beta(\cdot,\cdot) \in \mathcal{KL}$, and we have $\gamma(\cdot) \in \mathcal{K}$, the statement then follows from the definition of the ISpS property given in Chapter 2, i.e. Definition 2.4.3. ■

The interpretation of the condition $\mathbb{M}_\tau \subseteq \mathbb{X}_T$ is that the min-max MPC controller steers the state of the system inside the terminal set $\mathbb{X}_T$ for all disturbances w in $\mathbb{W}$ and v in $\mathbb{V}$. Then, ISS can be achieved by switching to the local feedback control law when the state enters the terminal set.

Note that in principle it is sufficient that there exist a nonlinear function $\tilde{h}(\cdot)$ that satisfies Assumption 6.3.1, a RPI set $\widetilde{\mathbb{X}} \subseteq \widetilde{\mathbb{X}}_{\mathbb{U}}$ for (6.1) in closed-loop with $u_k = \tilde{h}(x_k)$, $k \in \mathbb{Z}_+$, and a $\tau \in (0, a)$ such that $\mathbb{M}_\tau \subseteq \widetilde{\mathbb{X}}$ for the result of Theorem 6.4.3 to hold, where $\widetilde{\mathbb{X}}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid \tilde{h}(x) \in \mathbb{U}\}$. By this we mean that if one takes the modified dual-mode scheme

$$\bar{u}^{\mathrm{DM}}_{\mathrm{mod}}(x_k) \triangleq \begin{cases} \bar{u}(x_k) \text{ if } x_k \in \mathbb{X}_f(N) \setminus \widetilde{\mathbb{X}} \\ \tilde{h}(x_k) \text{ if } x_k \in \widetilde{\mathbb{X}} \end{cases} ; \quad k \in \mathbb{Z}_+,$$

the result of Theorem 6.4.3 still holds. Hence, it is not necessary to use the terminal set $\mathbb{X}_T$ employed in the min-max MPC optimization problem as the "switch set" (i.e., for example, $\widetilde{\mathbb{X}}$ defined above).

6.5 New methods for computing the terminal cost

As shown in the previous sections, to guarantee ISpS (ISS) stability for the nonlinear system (6.1) in closed-loop with a min-max MPC controller, for a given stage cost $L(\cdot, \cdot)$, one needs to compute the terminal cost $F(\cdot)$, the state-feedback control law $h(\cdot)$ and the terminal set $\mathbb{X}_T$ such that Assumption 6.3.1 holds. This is a highly non-trivial problem which is not solved in its full generality. In this section we consider an industrially relevant subclass of system (6.1) for which the corresponding min-max MPC optimization problems are computationally tractable and for which it is possible to derive systematic ways for solving the above-mentioned problem.

More specifically, we consider the class of constrained discrete-time linear systems affected, possibly simultaneously, by time-varying parametric uncertainties and additive disturbance inputs, i.e.

$$x_{k+1} = g(x_k, u_k, w_k, v_k) \triangleq A(w_k)x_k + B(w_k)u_k + E(w_k)v_k, \quad k \in \mathbb{Z}_+, \tag{6.13}$$

where $x_k \in \mathbb{X} \subset \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subset \mathbb{R}^m$ is the control action, $w_k \in \mathbb{W} \subset \mathbb{R}^{d_w}$ is an unknown parametric uncertainty and $v_k \in \mathbb{V} \subset \mathbb{R}^{d_v}$ is an unknown additive disturbance input. The sets $\mathbb{X}$ and $\mathbb{U}$ are assumed to be compact and to contain the origin in their interior, while $\mathbb{W}$ and $\mathbb{V}$ are assumed to be compact *polyhedral* sets. Here, $A(w) \in \mathbb{R}^{n\times n}$, $B(w) \in \mathbb{R}^{n\times m}$ and $E(w) \in \mathbb{R}^{n\times d_v}$ for all $w \in \mathbb{W}$. The vectors w_k and v_k are unknown exogenous perturbations that take values for all $k \in \mathbb{Z}_+$ in $\mathbb{W}$ and $\mathbb{V}$, respectively. We assume that $A(w)$, $B(w)$ and $E(w)$ are affine functions of $w \in \mathbb{W}$ and there exists a $w \in \mathbb{W}$ such that the matrix $B(w)$ has full-column rank.

Let $\mathbb{W}^{\text{vert}} \subset \mathbb{W}$ and $\mathbb{V}^{\text{vert}} \subset \mathbb{V}$ be the sets of all the vertices of $\mathbb{W}$ and $\mathbb{V}$, arranged in a certain fixed order. Furthermore, let $\mathcal{S} \triangleq \{1, \ldots, S\}$ and $\mathcal{O} \triangleq \{1, \ldots, O\}$ denote the set of indexes s and o such that w^s and v^o take values in $\mathbb{W}^{\text{vert}}$ and $\mathbb{V}^{\text{vert}}$, respectively. Since we assume that $\mathbb{W}$ is a polyhedral set it follows that $[A(w)\;\; B(w)\;\; E(w)] \in \Omega$, where $\Omega \triangleq \text{Co}\{[A(w^s)\;\; B(w^s)\;\; E(w^s)]\}$, $s \in \mathcal{S}$. Also, since we assume that $\mathbb{V}$ is a polyhedral set it follows that $v \in \text{Co}\{v^o\}$, $o \in \mathcal{O}$. In other words, there exist some nonnegative integers $\lambda_1, \ldots, \lambda_S$ and $\mu_1, \ldots, \mu_O$ such that:

$$[A(w)\;\; B(w)\;\; E(w)] = \sum_{s=1}^{S} \lambda_s [A(w^s)\;\; B(w^s)\;\; E(w^s)], \quad \sum_{s=1}^{S} \lambda_s = 1; \tag{6.14a}$$

$$v = \sum_{o=1}^{O} \mu_o v^o, \quad \sum_{o=1}^{O} \mu_o = 1. \tag{6.14b}$$

6.5.1 Specific problem statement

The problem addressed in this section is how to calculate $F(\cdot)$ and $h(\cdot)$ such that Assumption 6.3.1 holds for the perturbed linear system (6.13). This is a non-trivial problem, which depends on the type of MPC cost. For example, in the case of quadratic MPC costs this problem has only been solved partially (Kothare et al., 1996), i.e. for zero additive disturbances. In the case of MPC costs based on $1, \infty$-norms, to the authors' knowledge, there is no systematic method available (which does not resort to a zero terminal cost and a zero stage cost in $\mathbb{X}_T$, as done in (Scokaert and Mayne, 1998; Kerrigan and Maciejowski, 2004)) for solving this problem.

The aim is now to provide solutions, for both quadratic and $1, \infty$-norms MPC costs, to the problem of calculating the terminal cost $F(\cdot)$ and local control law $h(\cdot)$ such that Assumption 6.3.1 is satisfied for linear systems affected by both parametric uncertainties and additive disturbances.

These techniques can also be employed to obtain a terminal cost and a local controller that satisfy Assumption 6.3.1 for particular types of nonlinear systems, see, for example, Section 6.6.3. Moreover, they can be straightforwardly extended to the class of perturbed PWA systems via the results of chapters three and four.

6.5.2 MPC costs based on quadratic forms

Consider the case when quadratic forms are used to define the MPC cost function, i.e. $F(x) \triangleq \|P^{\frac{1}{2}}x\|_2^2 = x^\top Px$ and $L(x,u) \triangleq \|Q^{\frac{1}{2}}x\|_2^2 + \|R^{\frac{1}{2}}u\|_2^2 = x^\top Qx + u^\top Ru$, where $P, Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are assumed to be positive definite and symmetric matrices. Then, it follows that $L(x,u) \geq x^\top Qx \geq \lambda_{\min}(Q)\|x\|_2^2$ for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$ and $\lambda_{\min}(P)\|x\|_2^2 \leq F(x) \leq \lambda_{\max}(P)\|x\|_2^2$ for all $x \in \mathbb{R}^n$. Hence, Assumption 6.3.1-3,4 is satisfied for $a \triangleq \lambda_{\min}(Q) > 0$, $a_1 \triangleq \lambda_{\min}(P) > 0$, $b \triangleq \lambda_{\max}(P) > 0$ and $\lambda = 2$.

Next, we present LMI based sufficient conditions for Assumption 6.3.1-5 to be satisfied. Let Q and R be known positive definite and symmetric matrices and let $\tau > 0$ be a given number. Consider now the following matrix inequality:

$$\Delta(w) \triangleq \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} > 0, \tag{6.15}$$

where

$$\Delta_{11} \triangleq \begin{pmatrix} Z & -(A(w)Z + B(w)Y)^\top \\ -(A(w)Z + B(w)Y) & \tau Z \end{pmatrix},$$

$$\Delta_{22} \triangleq \operatorname{diag}\left(\left[\begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}, \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix}, \begin{pmatrix} R^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix}\right]\right),$$

$$\Delta_{12} = \Delta_{21}^\top \triangleq \begin{pmatrix} (A(w)Z + B(w)Y)^\top & 0 & Z & 0 & Y^\top & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In (6.15), the operator $\operatorname{diag}([Z_1, \ldots, Z_n])$ denotes a diagonal matrix of appropriate dimensions with the matrices $Z_1, \ldots, Z_n$ on the main diagonal, and 0 denotes a zero matrix of appropriate dimensions. The unknown variables in the matrix inequality (6.15) (an LMI for fixed $w \in \mathbb{W}$) are the matrix $Z \in \mathbb{R}^{n\times n}$, which is required to be positive definite and symmetric, and $Y \in \mathbb{R}^{m\times n}$.

Theorem 6.5.1 *Suppose that the LMI*

$$\operatorname{diag}([\Delta(w^1), \Delta(w^2), \ldots, \Delta(w^S)]) > 0, \tag{6.16}$$

in the unknowns Z, Y is feasible, where $\{w^s \mid s \in \mathcal{S} = \{1, \ldots, S\}\}$ are the vertices of $\mathbb{W}$. Let $Z > 0$ and Y be a solution of (6.16). Then, the terminal cost $F(x) = x^\top Px$ with $P \triangleq Z^{-1}$ and the control law $h(x) = Kx$ with the feedback gain $K \triangleq YZ$ satisfy Assumption 6.3.1-5 for all $x \in \mathbb{R}^n$, $w \in \mathbb{W}$ and all $v \in \mathbb{R}^{d_v}$, and with $\sigma(\|v\|) \triangleq (1+\tau)\|P^{\frac{1}{2}}\|_2^2 \max_{w\in\mathbb{W}} \|E(w)\|_2^2\|v\|_2^2$.

Proof: Using property (6.14a) and the hypothesis (6.16) it follows that

$$\Delta(w) = \Delta\left(\sum_{s=1}^{S} \lambda_s w^s\right) = \sum_{s=1}^{S} \lambda_s \Delta(w^s) > 0, \quad \forall w \in \mathbb{W}.$$

Then, applying the Schur complement to (6.15) we obtain

$$\Delta_{11} - \Delta_{12}\Delta_{22}^{-1}\Delta_{21} > 0 \Leftrightarrow \Delta_{11} - \Delta_{21}^{\top}\Delta_{22}^{-1}\Delta_{21} > 0,$$

which yields the equivalent matrix inequality:

$$\Theta_Z(w) \triangleq \begin{pmatrix} \Xi_Z(w) & -(A(w)Z + B(w)Y)^{\top} \\ -(A(w)Z + B(w)Y) & \tau Z \end{pmatrix} > 0$$

for all $w \in \mathbb{W}$, where

$$\begin{aligned}\Xi_Z(w) \triangleq Z &- (A(w)Z + B(w)Y)^{\top} Z^{-1}(A(w)Z + B(w)Y) \\ &- ZQZ - Y^{\top} RY.\end{aligned}$$

Next, by pre- and post-multiplying the matrix inequality $\Theta_Z(w) > 0$ with the positive definite matrix $\begin{pmatrix} Z^{-1} & 0 \\ 0 & Z^{-1} \end{pmatrix}$ and by replacing Z^{-1} with P and YZ^{-1} with K in the resulting matrix inequality we obtain:

$$\Theta_P(w) \triangleq \begin{pmatrix} \Xi_P(w) & -(A(w) + B(w)K)^{\top} P \\ -P(A(w) + B(w)K) & \tau P \end{pmatrix} > 0$$

for all $w \in \mathbb{W}$, where

$$\begin{aligned}\Xi_P(w) \triangleq P &- (A(w) + B(w)K)^{\top} P(A(w) + B(w)K) \\ &- Q - K^{\top} RK.\end{aligned}$$

This implies that

$$\begin{bmatrix} x^{\top} & v^{\top} E(w)^{\top} \end{bmatrix} \Theta_P(w) \begin{bmatrix} x \\ E(w)v \end{bmatrix} \geq 0, \tag{6.17}$$

for all $x \in \mathbb{R}^n$, $w \in \mathbb{W}$ and all $v \in \mathbb{R}^{d_v}$, which yields:

$$\begin{aligned}&x^{\top}\Xi_P(w)x - v^{\top}E(w)^{\top}P(A(w) + B(w)K)x \\ &\quad - x^{\top}(A(w) + B(w)K)^{\top}PE(w)v - v^{\top}E(w)^{\top}PE(w)v \\ &\quad + (1+\tau)v^{\top}E(w)^{\top}PE(w)v \geq 0 \Leftrightarrow x^{\top}Px - x^{\top}Qx - x^{\top}K^{\top}RKx \\ &\quad - ((A(w) + B(w)K)x + E(w)v)^{\top}P((A(w) + B(w)K)x + E(w)v) \\ &\quad + (1+\tau)v^{\top}E(w)^{\top}PE(w)v \geq 0. \qquad (6.18)\end{aligned}$$

Note that in (6.18) we added and subtracted the term $v^\top E(w)^\top P E(w) v$. Next, consider the function

$$\bar{\sigma}(w, v) \triangleq (1+\tau) v^\top E(w)^\top P E(w) v = (1+\tau)\|P^{\frac{1}{2}} E(w) v\|_2^2,$$

and choose $\sigma(\|v\|) \triangleq (1+\tau)\|P^{\frac{1}{2}}\|_2^2 \max_{w\in\mathbb{W}} \|E(w)\|_2^2 \|v\|_2^2$. Using the fact that $\bar{\sigma}(w, v) \leq \sigma(\|v\|)$ for all $w \in \mathbb{W}$ and all $v \in \mathbb{R}^{d_v}$, inequality (6.18), and letting $h(x) \triangleq Kx$ we obtain:

$$F(A(w)x + B(w)h(x) + E(w)v) - F(x) \leq -L(x, h(x)) + \sigma(\|v\|).$$

Hence, Assumption 6.3.1-5) is satisfied for all $x \in \mathbb{R}^n$, $w \in \mathbb{W}$ and all $v \in \mathbb{R}^{d_v}$, and with

$$\sigma(\|v\|) \triangleq (1+\tau)\|P^{\frac{1}{2}}\|_2^2 \max_{w\in\mathbb{W}} \|E(w)\|_2^2 \|v\|_2^2.$$

The only thing left to prove is that $\lambda_{\min}(Q) \triangleq a \leq b \triangleq \lambda_{\max}(P)$. As shown above, for all $w \in \mathbb{W}$ it holds that:

$$\Theta_P(w) > 0 \Rightarrow \Xi_P(w) > 0 \Rightarrow P - Q > 0 \Rightarrow x^\top (P - Q) x \geq 0,$$

for all $x \in \mathbb{R}^n$. Then, it follows that $a\|x\|_2^2 \leq x^\top Q x \leq x^\top P x \leq b\|x\|_2^2$, for all $x \in \mathbb{R}^n$ and therefore, $a \leq b$. ■

Once the terminal cost $F(\cdot)$ and the state-feedback control law $h(\cdot)$ have been calculated by solving the LMI (6.16), Assumption 6.3.1-1,2 is satisfied by taking $\mathbb{X}_T$ as the maximal RPI set for the closed-loop system $x_{k+1} = (A(w_k) + B(w_k)K)x_k + E(w_k)v_k$ that is contained in $\mathbb{X}_{\mathbb{U}}$. The maximal RPI set can be computed efficiently using the algorithms of (Blanchini, 1994; Kolmanovsky and Gilbert, 1998), which are implemented, for example, in the multi parametric toolbox (MPT) (Kvasnica et al., 2004).

The matrix Q is related to the decrease of the state norm, and hence, to the transient behavior. If robust stability is the only goal, Q can be chosen less positive definite to reduce conservativeness of the LMI (6.16). The number $\tau > 0$ and the matrix P yield the gain of the $\mathcal{K}$-function $\sigma(\cdot)$, which is related to the robust performance of the closed-loop system, as shown in Section 6.3 and Section 6.4. Note that τ can also be defined as an unknown variable in (6.16), which results in a bilinear matrix inequality (i.e. due to τP). However, since the unknown τ is a scalar, this problem can be solved efficiently via semi-definite programming solvers by setting lower and upper bounds for τ and doing bisections.

6.5.3 MPC costs based on $1, \infty$-norms

In this section we consider the case when $1, \infty$-norms are used to define the MPC cost function, i.e. $F(x) \triangleq \|Px\|$ and $L(x,u) \triangleq \|Qx\| + \|Ru\|$, where $P \in \mathbb{R}^{n_p \times n}$, $Q \in \mathbb{R}^{n_q \times n}$ and $R \in \mathbb{R}^{m_r \times m}$ are assumed to be matrices that have full-column rank and $\|\cdot\|$ denotes either the 1-norm or the ∞-norm.

For any full-column rank matrix $Z \in \mathbb{R}^{n_z \times n}$ there exist $\zeta_2(Z) \geq \zeta_1(Z) > 0$ such that

$$\zeta_1(Z)\|x\| \leq \|Zx\| \leq \zeta_2(Z)\|x\|, \quad \forall x \in \mathbb{R}^n.$$

Then, we have that $L(x,u) \geq \|Qx\| \geq \zeta_1(Q)\|x\|$ for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$ and $\zeta_1(P)\|x\| \leq F(x) \leq \zeta_2(P)\|x\|$ for all $x \in \mathbb{R}^n$. Hence, Assumption 6.3.1-3,4 is satisfied for $a \triangleq \zeta_1(Q) > 0$, $a_1 \triangleq \zeta_1(P) > 0$, $b \triangleq \zeta_2(P) > 0$ and $\lambda = 1$.

Next, we present norm inequalities based sufficient conditions for Assumption 6.3.1-5 to be satisfied. Let Q and R be known full-column rank matrices and consider the following norm inequalities:

$$1 - \|P(A(w^s) + B(w^s)K)P^{-L}\| - \|QP^{-L}\| - \|RKP^{-L}\| \geq 0 \qquad (6.19a)$$
$$\zeta - \|P(A(w^s) + B(w^s)K)\| - \|Q\| - \|RK\| \geq 0, \qquad (6.19b)$$

where $P^{-L} = (P^\top P)^{-1}P^\top$, $\zeta > 0$ is a fixed number and $\{w^s \mid s \in \mathcal{S} = \{1, \dots, S\}\}$ are the vertices of $\mathbb{W}$. The unknown variables in (6.19) are the matrices P and K.

Theorem 6.5.2 *Consider the following hypotheses:*

(i) *There exist a full-column rank matrix P and a feedback gain K that satisfy inequality* (6.19a) *for all $s \in \mathcal{S}$;*

(ii) *There exist a matrix P and a feedback gain K that satisfy inequality* (6.19b) *for all $s \in \mathcal{S}$ and $\|Px\| \geq \zeta\|x\|$ for all $x \in \mathbb{R}^n$.*

Suppose that either hypothesis (i) *or hypothesis* (ii) *holds. Then, the terminal cost $F(x) \triangleq \|Px\|$ and the function $h(x) \triangleq Kx$ satisfy Assumption 6.3.1-5 for all $x \in \mathbb{R}^n$, $w \in \mathbb{W}$ and all $v \in \mathbb{R}^{d_v}$ and with*

$$\sigma(\|v\|) \triangleq \tau\|P\| \max_{w \in \mathbb{W}} \|E(w)\|\|v\|$$

for any $\tau \geq 1$.

Proof: (i) Using property (6.14a) and the inequality (6.19a) we obtain that for all $w \in \mathbb{W}$:

$$\begin{aligned}
&1 - \|P(A(w) + B(w)K)P^{-L}\| - \|QP^{-L}\| - \|RKP^{-L}\| \\
&\quad = 1 - \|P\sum_{s=1}^{S}\lambda_s(A(w^s) + B(w^s)K)P^{-L}\| - \|QP^{-L}\| - \|RKP^{-L}\| \\
&\quad \geq \sum_{s=1}^{S}\lambda_s(1 - \|P(A(w^s) + B(w^s)K)P^{-L}\| - \|QP^{-L}\| - \|RKP^{-L}\|) \geq 0.
\end{aligned} \tag{6.20}$$

Then, by post-multiplying the above inequality with $\|Px\|$ and using the triangle inequality yields:

$$\begin{aligned}
0 &\leq \|Px\| - \|P(A(w) + B(w)K)P^{-L}\|\|Px\| - \|QP^{-L}\|\|Px\| \\
&- \|RKP^{-L}\|\|Px\| \leq \|Px\| - \|P(A(w) + B(w)K)P^{-L}Px\| \\
&- \|QP^{-L}Px\| - \|RKP^{-L}Px\| \leq \|Px\| - \|Qx\| - \|RKx\| \\
&- \|P(A(w) + B(w)K)x\| + (\tau - 1)\|PE(w)v\| \\
&\leq \|Px\| - \|P((A(w) + B(w)K)x + E(w)v)\| - \|Qx\| \\
&- \|RKx\| + \tau\|PE(w)v\|, \quad \forall \tau \geq 1.
\end{aligned}$$

Letting $h(x) \triangleq Kx$ and $\sigma(\|v\|) \triangleq \tau\|P\| \max_{w\in\mathbb{W}} \|E(w)\|\|v\|$ we obtain

$$F(A(w)x + B(w)h(x) + E(w)v) - F(x) \leq -L(x, h(x)) + \sigma(\|v\|).$$

Hence, Assumption 6.3.1-5 is satisfied for all $x \in \mathbb{R}^n$, $w \in \mathbb{W}$ and all $v \in \mathbb{R}^{d_v}$ and with $\sigma(\|v\|) \triangleq \tau\|P\| \max_{w\in\mathbb{W}} \|E(w)\|\|v\|$ for any $\tau \geq 1$. Finally, from the above inequality we obtain:

$$\zeta_1(Q)\|x\| \leq \|Qx\| \leq \|Px\| \leq \zeta_2(P)\|x\|, \quad \forall x \in \mathbb{R}^n,$$

and therefore, $\zeta_1(Q) \triangleq a \leq b \triangleq \zeta_2(P)$.

(ii) The proof is obtained analogously to the proof of part (i), by post-multiplying the inequality obtained from (6.19b) as in (6.20) with $\|x\|$ and using $\|Px\| \geq \zeta\|x\|$ for all $x \in \mathbb{R}^n$. ■

A solution that satisfies the norm inequalities (6.19) can be obtained by minimizing the costs

$$\begin{aligned}
J_1(P, K) &\triangleq \max_{s\in\mathcal{S}} \left\{\|P(A(w^s) + B(w^s)K)P^{-L}\| + \|QP^{-L}\| + \|RKP^{-L}\|\right\}, \\
J_2(P, K) &\triangleq \max_{s\in\mathcal{S}} \left\{\|P(A(w^s) + B(w^s)K)\| + \|Q\| + \|RK\|\right\},
\end{aligned}$$

if the resulting value functions are less than 1 or ζ, respectively. These are non-convex nonlinear optimization problems, which can be solved using black-box optimization solvers, such as *fmincon* and *fminunc* of Matlab. Alternatively one can construct an optimization problem with a zero cost and impose the nonlinear constraint $J_1(P, K) \leq 1$. Such a problem can be solved with the *fmincon* solver. The nonlinear nature of these optimization problems is not critical, since they are solved off-line.

Moreover, the optimization problems associated to (6.19a) and (6.19b) can be simplified if one of the unknowns is fixed using an educated guess. For example, a locally stabilizing feedback K can be calculated via the LMI approach of Theorem 6.5.1. Then, fixing K in (6.19b) and solving in P amounts to searching for a piecewise linear Lyapunov function for a system which already admits a quadratic Lyapunov function, which is not conservative, e.g. see (Blanchini, 1994). Alternatively, P can be chosen as the matrix that defines a 0-symmetric polyhedral RPI set for system (6.13) in closed-loop with $h(\cdot)$, e.g. obtained using MPT. Then, fixing P in (6.19a) or (6.19b) and solving in K boils down to searching for a different feedback which renders the polyhedral set induced by $F(\cdot)$ RPI and, moreover, is such that (6.19a) or (6.19b), respectively, holds.

6.6 Illustrative examples

In this section we present three examples that show how the results developed in this chapter can be applied to solve robust control problems.

6.6.1 An active suspension system

Consider the problem of robustly regulating to the origin the following active suspension system (Bemporad et al., 2003b):

$$x_{k+1} = \begin{bmatrix} 0.809 & 0.009 & 0 & 0 \\ -36.93 & 0.80 & 0 & 0 \\ 0.191 & -0.009 & 1 & 0.01 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0.0005 \\ 0.0935 \\ -0.005 \\ -0.01 \end{bmatrix} u_k + \begin{bmatrix} -0.009 \\ 0.191 \\ -0.0006 \\ 0 \end{bmatrix} v_k, \tag{6.21}$$

$k \in \mathbb{Z}_+$, where the additive disturbance v_k takes values in the set $\mathbb{V} \triangleq \{v \in \mathbb{R} \mid -0.4 \leq v \leq 0.4\}$, for all $k \in \mathbb{Z}_+$. The first and the third state are constrained between -0.02 and 0.02, and -0.05 and 0.05, respectively. The control input is constrained between -5 and 5.

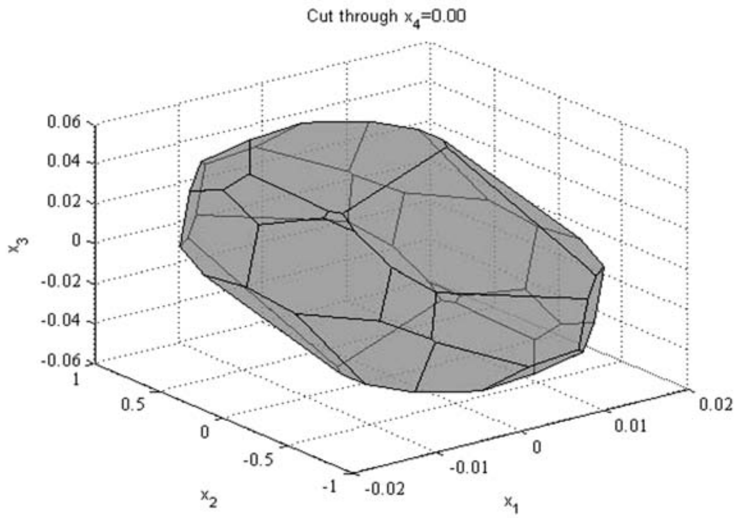

Figure 6.1: 3D section of the terminal constraint set $\mathbb{X}_T$.

To ensure robust stability, we employed the LMI approach of Theorem 6.5.1 in order to compute a quadratic forms based terminal cost and a local state-feedback controller that satisfy Assumption 6.3.1. The following terminal weight P and feedback K were obtained for the same stage cost weights used in (Bemporad et al., 2003b), i.e. $Q = \text{diag}([5000, 0.1, 400, 0.1])$, $R = 1.8$, and for $\tau = 100$:

$$P = 10^4 \begin{bmatrix} 7.5651 & 0.0079 & 0.3702 & -0.0156 \\ 0.0079 & 0.0017 & 0.0001 & 0.0001 \\ 0.3702 & 0.0001 & 0.7241 & 0.0250 \\ -0.0156 & 0.0001 & 0.0250 & 0.0303 \end{bmatrix},$$

$$K = \begin{bmatrix} 20.7682 & -0.8657 & 16.8717 & 2.1688 \end{bmatrix}.$$

The terminal constraint set $\mathbb{X}_T$ (a polyhedron with 56 vertices, see Figure 6.1 for a 3D section plot) was calculated as the maximal RPI set inside $\mathbb{X}_{\mathbb{U}}$ for system (6.21) in closed-loop with $h(x) = Kx$ using the MPT (Kvasnica et al., 2004). The evolution in time of the states, additive disturbance input and min-max MPC control action is plotted in Figure 6.2. The min-max MPC control action was calculated for $N = 6$ using the numerical set-up of (Muñoz de la Peña et al., 2005) to solve the min-max MPC optimization problem. As it can be observed in Figure 6.2, the closed-loop system is robustly stable and the state converges to the origin when the additive disturbance vanishes.

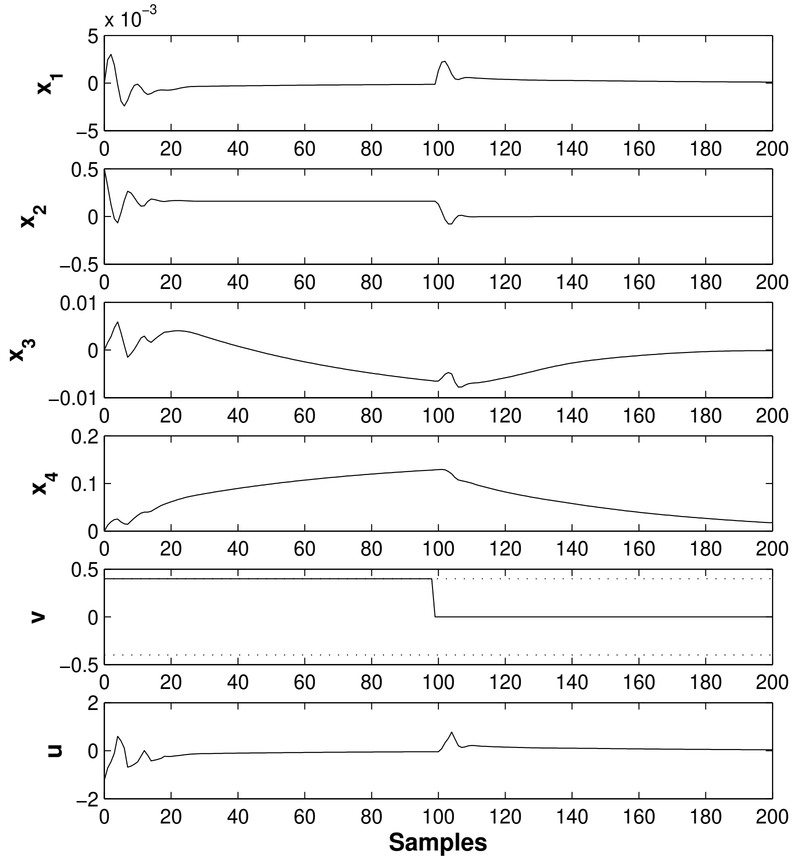

Figure 6.2: States, additive disturbance input and control action histories.

6.6.2 A perturbed double integrator

Consider a discrete-time perturbed double integrator, i.e.

$$x_{k+1} = A(w_k)x_k + B(w_k)u_k + Ev_k, \quad k \in \mathbb{Z}_+, \tag{6.22}$$

which was obtained from a continuous-time double integrator via a sample-and-hold device with a time-varying sampling period T_s as follows: $A(w) = \left[\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right]$, $B(w) = \begin{bmatrix} \frac{w^2}{2} \\ w \end{bmatrix}$, $E = I_2$. The uncertainty parameter $w \in \mathbb{W} \triangleq [0.6, 1]$ represents the sampling period T_s and the additive disturbance takes values in the set $\mathbb{V} \triangleq \{v \in \mathbb{R}^2 \mid \|v\| \leq 0.1\}$. Both states are constrained between

-10 and 10 and the input is constrained between -2 and 2. Note that $B(w)$ is not an affine function of w (i.e. the first element of $B(w)$ is a quadratic function of w). For simplicity, we only used the four pairs of vertices $[A(1)\ B(1)]$, $[A(1)\ B(0.6)]$, $[A(0.6)\ B(1)]$ and $[A(0.6)\ B(0.6)]$ to model the parametric uncertainty.

An optimization problem based on the inequality (6.19a) was solved using the Matlab *fminunc* solver taking into account all the fours pairs of vertices given above. The following terminal weight P and feedback K were obtained for $Q = 0.2I_2$ and $R = 0.01$:

$$P = \begin{bmatrix} 2.5335 & 2.0038 \\ 0.2554 & 3.2492 \end{bmatrix}, \ K = \begin{bmatrix} -0.6304 & -1.5246 \end{bmatrix}. \tag{6.23}$$

Then, by Theorem 6.5.2 the terminal cost $F(x) = \|Px\|$ and local control law $h(x) = Kx$ satisfy Assumption 6.3.1 for system (6.22). The terminal constraint set $\mathbb{X}_T$ (a polyhedron with 10 vertices, see Figure 6.3) was calculated as the maximal RPI set inside $\mathbb{X}_{\mathbb{U}}$ for system (6.22) in closed-loop with $h(x) = Kx$ using the MPT (Kvasnica et al., 2004) for the above four pairs of vertices corresponding to the parametric uncertainty and additive disturbances in $\mathbb{V}$.

Two simulations were performed. First, the additive disturbance was set equal to zero at all times and system (6.22) was affected by parametric uncertainty only. The state trajectory of system (6.22) in closed-loop with a feedback min-max MPC controller based on an ∞-norms cost with $N = 6$ is plotted in Figure 6.3. The evolution in time of the sampling period and of the control input is plotted in Figure 6.4.

In the second simulation $T_s = w_k = 1$ for all $k \in \mathbb{Z}_+$ and system (6.22) was affected by additive disturbances only. The state trajectory of system (6.22) in closed-loop with a feedback min-max MPC controller based on an ∞-norms cost with $N = 6$ is plotted in Figure 6.5. The evolution in time of the additive disturbances and of the control input is plotted in Figure 6.6. In both simulations, a recently developed optimization algorithm(Muñoz de la Peña et al., 2005) based on Bender's decomposition was used to calculate the feedback min-max control input. As guaranteed by the theory presented in this chapter, the closed-loop system is robustly stable in both cases and, when the additive disturbance vanishes in the second simulation, the state converges to the origin.

Note that we have used the same terminal cost matrix P given in (6.23) and terminal constraint set $\mathbb{X}_T$ in both simulations. This demonstrates that the stabilization method developed in this chapter works for linear systems that are affected by both parametric uncertainties and additive disturbances.

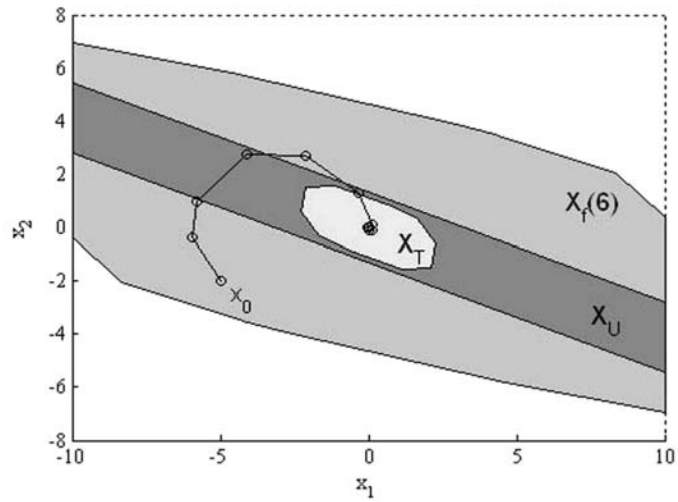

Figure 6.3: The closed-loop state trajectory for the first simulation.

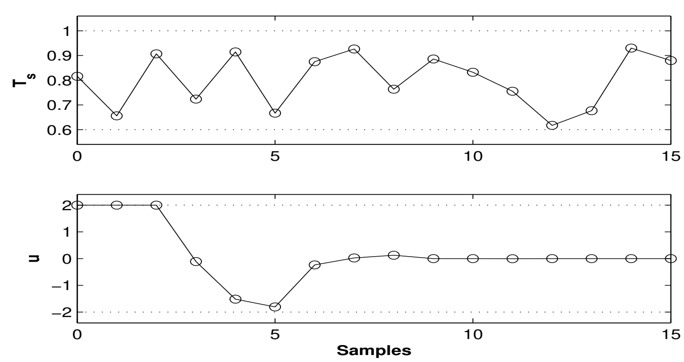

Figure 6.4: Sampling period ($T_s = w$) and control input histories.

6.6.3 A perturbed nonlinear double integrator

Consider a perturbed discrete-time nonlinear double integrator obtained from a continuous-time double integrator via a sample-and-hold device with a sampling period equal to one, as follows:

$$x_{k+1} = Ax_k + Bu_k + f(x_k) + v_k, \quad k \in \mathbb{Z}_+, \tag{6.24}$$

where $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0.5 \\ 1 \end{smallmatrix}\right]$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) \triangleq 0.025 \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] x^\top x$ is a nonlinear additive term and $v_k \in \mathbb{V} \triangleq \{v \in \mathbb{R}^2 \mid \|v\| \leq 0.03\}$ for all $k \in \mathbb{Z}_+$ is

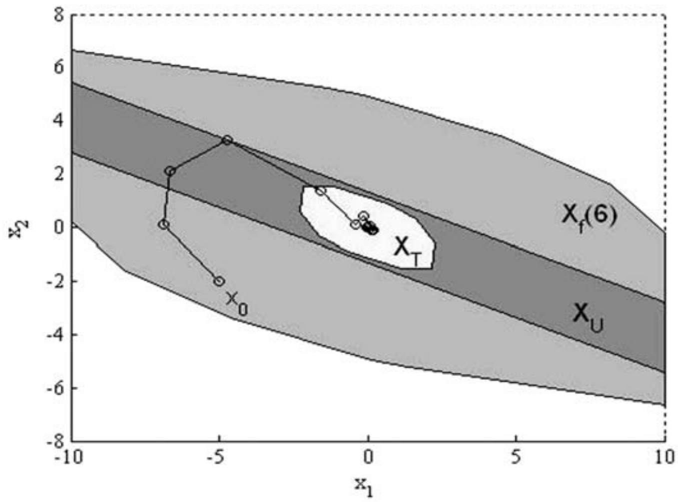

Figure 6.5: The closed-loop state trajectory for the second simulation.

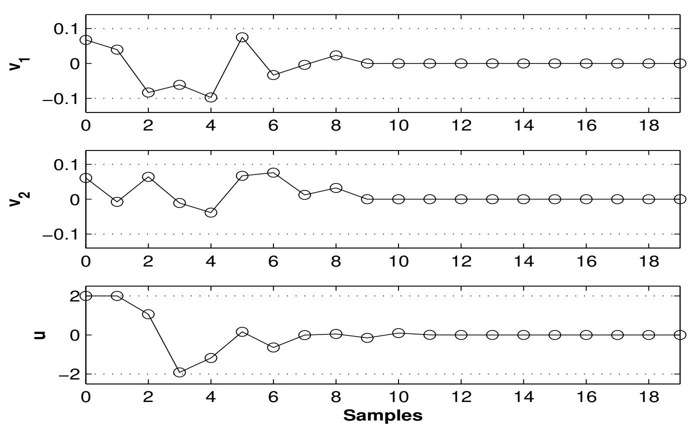

Figure 6.6: Additive disturbances ($v = [v_1 v_2]^\top$) and control input histories.

an additive disturbance input (we use $\|\cdot\|$ to denote the infinity norm for shortness). The state and the input are constrained at all times in the C-sets

$$\mathbb{X} \triangleq \{x \in \mathbb{R}^2 \mid \|x\| \leq 10\} \text{ and } \mathbb{U} \triangleq \{u \in \mathbb{R} \mid |u| \leq 2\}.$$

The MPC cost function is defined using ∞-norms, i.e.

$$F(x) \triangleq \|Px\|, \quad L(x,u) \triangleq \|Qx\| + \|Rx\|,$$

where P is a full-column rank matrix (to be determined), $Q = 0.8I_2$ and $R = 0.1$. Note that the stage cost $L(\cdot,\cdot)$ satisfies Assumption 6.3.1-3 for $\lambda = 1$ and any $a \in (0, 0.8)$.

We take the function $h(\cdot)$ as $h(x) \triangleq Kx$, where $K \in \mathbb{R}^{1\times 2}$ is the gain matrix. To compute the terminal cost matrix P and the gain matrix K such that Assumption 6.3.1-5 holds, we first obtain a solution for the following linearization of system (6.24):

$$x_{k+1} = Ax_k + Bu_k + v_k, \quad k \in \mathbb{Z}_+, \tag{6.25}$$

which is obtained from (6.24) by removing the nonlinear term $f(\cdot)$. To accommodate for the nonlinear term $f(\cdot)$, we employ a "larger" stage cost weight matrix for the state, i.e. $\widetilde{Q} = 2.4I_2$, instead of $Q = 0.8I_2$, for which it holds that $\|\widetilde{Q}x\| \geq \|Qx\|$ for all $x \in \mathbb{R}^2$. The terminal cost $F(x) = \|Px\|$ and local control law $h(x) = Kx$ with the matrices

$$P = \begin{bmatrix} 12.1274 & 7.0267 \\ 0.4769 & 11.6072 \end{bmatrix}, \; K = \begin{bmatrix} -0.5885 & -1.4169 \end{bmatrix}, \tag{6.26}$$

were computed (using a technique based on inequality (6.19a), as described in Section 6.5) such that the following inequality holds for the linear system (6.25), i.e.

$$\|P((A + BK)x + v)\| - \|Px\| \leq -\|\widetilde{Q}x\| - \|RKx\| + \sigma_1(\|v\|), \tag{6.27}$$

for all $x \in \mathbb{R}^2$ and all $v \in \mathbb{R}^2$, where $\sigma_1(s) \triangleq \|P\|s$.

The terminal cost $F(x) \triangleq \|Px\|$ satisfies Assumption 6.3.1-4 for $\lambda = 1$, $b \triangleq \|P\| = 19.1541$, $a_1 = 0.1$ and $e_1 = 0$. The safe set is the following (see Figure 6.7 for a plot of $\mathbb{X}_{\mathbb{U}}$):

$$\mathbb{X}_{\mathbb{U}} \triangleq \{x \in \mathbb{X} \mid \|x\| \leq 1.72, \; |Kx| \leq 2\}.$$

The terminal set $\mathbb{X}_T$, also plotted in Figure 6.7, is taken as the maximal RPI set contained in the set $\mathbb{X}_{\mathbb{U}}$ (and which is non-empty) for the linear system (6.18), in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, and disturbances in the set $\{v \in \mathbb{R}^2 \mid \|v\| \leq 0.18\}$. One can easily check that $\max_{x\in\mathbb{X}_T} \|f(x)\| < 0.15$ and thus, it follows that the terminal set $\mathbb{X}_T$ chosen as specified above is a RPI set for the nonlinear system (6.24) in closed-loop with $u_k = h(x_k)$, $k \in \mathbb{Z}_+$, and all disturbances v in $\mathbb{V} = \{v \in \mathbb{R}^2 \mid \|v\| \leq 0.03\}$.

Using the fact that (in the second equality below, $\|\cdot\|$ denotes the induced infinity matrix norm)

$$\|\widetilde{Q}x\| \geq 2.3515\|x\|, \; \forall x \in \mathbb{R}^2, \qquad \max_{x\in\mathbb{X}_T} \|P0.025 \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] x^\top\| = 1.5515,$$

inequality (6.27) and the triangle inequality, for all $x \in \mathbb{X}_T$ and all $v \in \mathbb{R}^2$ we obtain:

$$\begin{aligned}
&\|P((A+BK)x+v)+Pf(x)\|-\|Px\| \\
&\quad\le \|P((A+BK)x+v)\|-\|Px\|+\|Pf(x)\| \\
&\quad\le -\|\widetilde{Q}x\|-\|RKx\|+\sigma_1(\|v\|)+\|Pf(x)\| \\
&\quad\le -2.3515\|x\|-\|RKx\|+\sigma_1(\|v\|)+\max_{x\in\mathbb{X}_T}\left(\|P0.025\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]x^\top\|\right)\|x\| \\
&\quad\le -2.3515\|x\|-\|RKx\|+\sigma_1(\|v\|)+1.5515\|x\| \\
&\quad= -0.8\|x\|-\|RKx\|+\sigma_1(\|v\|) \\
&\quad= -\|Q\|\|x\|-\|RKx\|+\sigma_1(\|v\|) \\
&\quad\le -\|Qx\|-\|RKx\|+\sigma_1(\|v\|) \\
&\quad= -L(x,Kx)+\sigma_1(\|v\|).
\end{aligned}$$

Hence, the terminal cost $F(x) = \|Px\|$ and the control law $h(x) = Kx$, with the matrices P and K given in (6.26), satisfy Assumption 6.3.1-5) for the nonlinear system (6.24) with $e_2 = 0$ and with $\sigma_1(s) = \|P\|s$.

Consider now the set $\mathbb{M}_\tau$, which needs to be determined to establish ISS of the nonlinear system (6.24) in closed-loop with the dual-mode min-max MPC control law (6.4) that robustly optimizes the calculated MPC cost function. We can choose the constant $a = 0.79 < 0.8$, which ensures that $\|Qx\| \geq a\|x\|$ for all $x \in \mathbb{R}^2$, and since $d_2 = \max_{v\in\mathbb{V}} \sigma_1(\|v\|) = 0.5746$, it follows that a necessary condition to be satisfied is $\tau \in (0, 0.79)$ (with the smallest set $\mathbb{M}_\tau$ obtained for $\lim_{\tau\to 0} \frac{d_2}{a-\tau} = 0.7273$). For $\tau = 0.0718$, which yields $\frac{d_2}{a-\tau} = 0.8001$, it holds that $\mathbb{M}_\tau \subset \mathbb{X}_T$, see Figure 6.7 for an illustrative plot. Therefore, the closed-loop system (6.24)-(6.4) is ISS for $x_0 \in \mathbb{X}_f(N)$ and disturbances in $\mathbb{V}$, as guaranteed by Theorem 6.4.3.

Unfortunately, the *feedback* min-max MPC optimization problem was computationally untractable for the nonlinear model (6.24). Because of this, we have used an open-loop min-max MPC problem set-up, as the one described in Section 6.2, to calculate the control input.

Note that, as remarked in the beginning of Section 6.2, it can be proven that the hypothesis of Theorem 6.4.3 is also sufficient for ISS when an open-loop min-max MPC controller is employed in the dual-mode set-up of Section 6.4 instead of the feedback min-max MPC controller $\bar{u}(\cdot)$.

Although the resulting open-loop min-max optimization problem still has a very high computational burden, we could obtain a solution using the *fmincon* Matlab solver. The closed-loop state trajectory for initial state $x_0 =$

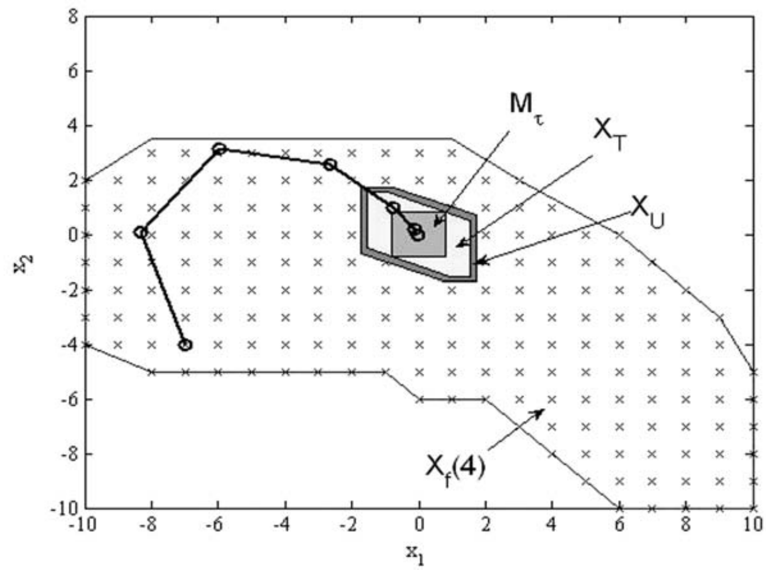

Figure 6.7: State trajectory for the nonlinear system (6.24) *in closed-loop with a dual-mode min-max MPC controller and an estimate of the feasible set* $\mathbb{X}_f(4)$.

$[-7\ \ -4]^\top$ and prediction horizon $N = 4$ is plotted in Figure 6.7. The dual-mode min-max MPC control input and (randomly generated) disturbance input histories are plotted in Figure 6.8.

6.7 Conclusions

In this chapter we have revisited the robust stability problem of min-max nonlinear model predictive control. The input-to-state practical stability framework has been employed to study stability of perturbed nonlinear systems in closed-loop with min-max MPC controllers. A priori sufficient conditions for ISpS were presented and explicit bounds on the evolution of the closed-loop system state were derived. Moreover, it was proven that these conditions also ensure ultimate boundedness. Then, new sufficient conditions under which ISS can be achieved in min-max nonlinear MPC were derived via a dual-mode approach. This result is important because it unifies the properties of (Limon et al., 2006) and (Mayne, 2001), i.e. it guarantees both ISpS in the presence of persistent disturbances and robust asymptotic stability in

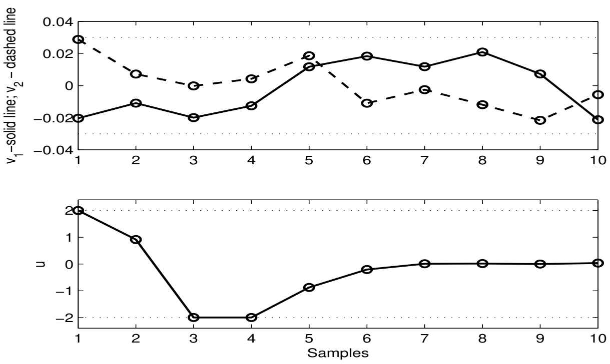

Figure 6.8: Dual-mode min-max nonlinear MPC control input and disturbance input histories.

the presence of decaying uncertainties.

New techniques for computing the terminal cost and a local state-feedback controller such that the developed input-to-state stabilization conditions are satisfied were developed for min-max MPC controllers based on both quadratic and $1,\infty$-norms costs. These techniques employ linear matrix inequalities (which can be solved efficiently) in the case of quadratic MPC cost functions and norm inequalities in the case of MPC cost functions based on $1,\infty$-norms. The effectiveness of the developed methods was illustrated by means of simulated examples.

Future work is concerned with the extension of the techniques developed for computing the terminal cost to the class of perturbed PWA systems, as well as with the design of computationally efficient feedback min-max MPC problem set-ups.

7

Robust sub-optimal model predictive controllers

This chapter focuses on the synthesis of *computationally friendly* robust stabilizing sub-optimal model predictive control (MPC) algorithms for perturbed nonlinear and hybrid systems. This goal is achieved via new, simpler stabilizing constraints, that can be implemented as a finite number of linear inequalities. A case study on the control of DC-DC converters that contains preliminary real-time computational estimates is included to illustrate the potential of the developed theory for practical applications.

7.1 Introduction

One of the most studied properties of model predictive control (MPC) is the stability of the controlled system. Perhaps the most embraced stabilization method is the so-called terminal cost and constraint set approach, e.g. see the survey (Mayne et al., 2000) for an overview. This method, already utilized in Chapter 3 and Chapter 5, uses the value function of the MPC cost as a candidate Lyapunov function for the closed-loop system and achieves stability via a particular terminal cost and an additional constraint on the terminal state, i.e. the predicted state at the end of the prediction horizon. Its advantage consists in the fact that initial feasibility of the MPC optimization problem implies feasibility all the way and the finite horizon MPC cost is proven to be a good approximation of the infinite horizon MPC cost. However, these properties are only guaranteed under the standing assumption that the global optimum of the MPC optimization problem is attained on-line, at each sampling instant. Clearly, when dealing with nonlinear or hybrid prediction models and hard constraints, it is difficult if not impossible to guarantee this assumption in practice, where numerical solvers usually provide (in the limited computational time available) a feasible, sub-optimal input sequence, rather than a globally optimal one. Such a sub-optimal input sequence needs to have certain properties to still guarantee (robust) stability of the MPC closed-loop system. Therefore, in practice, there is a need for sub-optimal MPC algorithms based on simpler optimization problems, which can be solved faster, and that still have an a priori stability guarantee.

An important result regarding sub-optimal MPC of nonlinear systems was presented in (Scokaert et al., 1999), where it is shown that feasibility of the MPC optimization problem rather than optimality is sufficient for stability. In (Scokaert et al., 1999), stability is achieved without optimality, by forcing the MPC value function to decrease at each sampling-instant. This requirement can be expressed in terms of an additional, so called stabilizing constraint. However, if nonlinear prediction models are used, this constraint becomes highly nonlinear and difficult to implement from a computational point of view, as the MPC value function depends on the whole sequence of unknown predicted future inputs. Feasibility is guaranteed for the nominal case in (Scokaert et al., 1999) by adding a terminal equality or inequality constraint.

This chapter investigates the possibility of designing input-to-state stabilizing (ISS) (Jiang and Wang, 2001), but computationally friendly sub-optimal MPC algorithms for nonlinear and hybrid systems. We propose to achieve this goal via new, simpler stabilizing constraints, that can be implemented as a finite number of linear inequalities and whose complexity does not depend on the length of the prediction horizon. Two sub-optimal nonlinear MPC algorithms are presented. The first one is based on a contraction argument, i.e. it is proven that, if the norm of the state of the closed-loop system is sufficiently decreasing at each sampling instant, then input-to-state stability is guaranteed. The second MPC scheme resorts to an ∞-norm based artificial Lyapunov function, which only depends on the measured state and the first element of the sub-optimal sequence of predicted future inputs. Both these schemes have an input-to-state stability guarantee with respect to additive disturbance inputs. For the class of PWA systems, it is shown how the sub-optimal MPC scheme based on an artificial Lyapunov function can be modified to ensure input-to-state stability with respect to measurement noise. This modified scheme is particularly relevant because it can be used in interconnection with an asymptotically stable observer, resulting (if cascade conditions are satisfied) in an asymptotically stable closed-loop system. Methods for computing ∞-norms artificial Lyapunov functions off-line for linear or PWA models are also indicated. A case study on the control of DC-DC converters is included to illustrate the potential of the developed theory for practical applications.

Remark 7.1.1 Compared to (Scokaert et al., 1999), we do not guarantee that initial feasibility implies feasibility all the way for the proposed algorithms. However, note that we consider perturbed systems. In this case, feasibility all the way is also not guaranteed for the algorithms of (Scokaert et al., 1999). From a computational point of view, we obtain faster MPC algorithms, as our stabilizing constraints can be written as a finite number of linear inequalities. Moreover, we also provide an ISS guarantee, which ensures an explicit bound on the norm of the MPC closed-loop system state, via the results presented in Chapter 2.

7.2 Sub-optimal MPC algorithms for fast nonlinear systems

We consider nominal and perturbed discrete-time nonlinear systems of the form:

$$x_{k+1} = f(x_k) + g(x_k)u_k, \quad k \in \mathbb{Z}_+, \tag{7.1a}$$

$$\tilde{x}_{k+1} = f(\tilde{x}_k) + g(\tilde{x}_k)u_k + w_k, \quad k \in \mathbb{Z}_+, \tag{7.1b}$$

where $x_k, \tilde{x}_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $w_k \in \mathbb{W} \subset \mathbb{R}^n$ are the state, the input and the additive disturbance, respectively, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are nonlinear, possibly discontinuous functions with $f(0) = 0$. In the sequel we will consider the case when sub-optimal MPC is used to generate the control input in (7.1). We assume that the state and the input vectors are constrained for both systems (7.1a) and (7.1b), in a compact subset $\mathbb{X}$ of $\mathbb{R}^n$ and a compact subset $\mathbb{U}$ of $\mathbb{R}^m$, respectively, which contain the origin in their interior. For a fixed $N \in \mathbb{Z}_{\geq 1}$, let $\mathbf{x}_k(x_k, \mathbf{u}_k) \triangleq (x_{1|k}, \ldots, x_{N|k})$ denote the state sequence generated by the nominal system (7.1a) from initial state $x_{0|k} \triangleq x_k$ and by applying an input sequence $\mathbf{u}_k \triangleq (u_{0|k}, \ldots, u_{N-1|k})$. Let $F : \mathbb{R}^n \to \mathbb{R}_+$ with $F(0) = 0$ and $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ with $L(0, 0) = 0$ be mappings. At time $k \in \mathbb{Z}_+$ let $x_k \in \mathbb{X}$ be available. The basic MPC scenario consists in minimizing at each sampling instant $k \in \mathbb{Z}_+$ a finite horizon cost function of the form

$$J(x_k, \mathbf{u}_k) \triangleq F(x_{N|k}) + \sum_{i=0}^{N-1} L(x_{i|k}, u_{i|k}), \tag{7.2}$$

with prediction model (7.1a), over all input sequences $\mathbf{u}_k$, subject to the state and input constraints.

Let $\mathbb{X}_f(N) \subseteq \mathbb{X}$ denote the set of *feasible states* with respect to the above optimization problem, i.e. the set of all states for which there exists a sequence of inputs that satisfies the input constraints and results in a predicted state trajectory that satisfies the state constraints. Then,

$$V_{\mathrm{MPC}} : \mathbb{X}_f(N) \to \mathbb{R}_+, \quad V_{\mathrm{MPC}}(x_k) \triangleq \inf_{\mathbf{u}_k \in \mathcal{U}_N(x_k)} J(x_k, \mathbf{u}_k)$$

is the MPC value function corresponding to the cost (7.2). If there exists an optimal sequence of controls $\mathbf{u}_k^* \triangleq (u_{0|k}^*, u_{1|k}^*, \ldots, u_{N-1|k}^*)$ that minimizes (7.2), the infimum above is a minimum and $V_{\mathrm{MPC}}(x_k) = J(x_k, \mathbf{u}_k^*)$. Then, an *optimal* MPC control law is defined as $u^{\mathrm{MPC}}(x_k) \triangleq u_{0|k}^*$, $k \in \mathbb{Z}_+$. Stability of the resulting MPC closed-loop system is usually guaranteed by adding a particular constraint on the terminal state $x_{N|k}$, e.g. see Chapter 3 or the survey (Mayne et al., 2000).

As mentioned in the introduction, in practice, the available solvers provide only a feasible, sub-optimal sequence of inputs, i.e.

$$\bar{\mathbf{u}}_k \triangleq (\bar{u}_{0|k}, \bar{u}_{1|k}, \ldots, \bar{u}_{N-1|k}),$$

and the control applied to the plant, i.e. $\bar{u}_{0|k}$, is a *sub-optimal* MPC control. The resulting value function is then $\overline{V}(x_k) \triangleq J(x_k, \bar{\mathbf{u}}_k)$. The stability of the resulting sub-optimal MPC closed-loop system may be unclear, and can be lost. Next, we present sub-optimal MPC algorithms that still guarantee stability a priori for the controlled system.

7.2.1 A contraction approach

In this section we consider ∞-norm based MPC costs, i.e. $F(x) \triangleq \|Px\|$ and $L(x,u) \triangleq \|Qx\| + \|R_u u\|$, where $\|\cdot\|$ denotes the ∞-norm for shortness and $P \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{q\times n}$ and $R_u \in \mathbb{R}^{r_u\times m}$ are assumed to be known matrices that have full-column rank. To set-up the sub-optimal MPC algorithm we assume[1] that a sector growth condition holds for the dynamics $f(\cdot), g(\cdot)$ in the sense that there exist constants $\mathcal{L}_f, \mathcal{L}_g > 0$ such that

$$\|f(x) + g(x)u\| \leq \mathcal{L}_f\|x\| + \mathcal{L}_g\|u\|, \quad \forall x \in \mathbb{X}, \forall u \in \mathbb{U}. \tag{7.3}$$

[1]Note that the continuity assumption is only required for the contraction based sub-optimal MPC algorithm presented in this section, and it is not necessary for the other sub-optimal MPC algorithms.

We also assume that all the controls in the sequence of predicted future inputs satisfy the regularity condition

$$\|u_{i|k}\| \leq \theta_i \|x_{0|k}\|, \quad i = 0, \ldots, N-1, \tag{7.4}$$

for some constants $\theta_i > 0$. Since the control laws $u_{i|k}$ are not known explicitly, in order to ensure that (7.4) holds we will choose the constants θ_i a priori and impose (7.4) as an additional constraint to the MPC optimization problem. Note that the above assumptions are usually employed in smooth nonlinear MPC to guarantee robust stability, e.g. see (Scokaert et al., 1997). Then, using (7.3) and (7.4) successively, one can easily establish a class $\mathcal{K}$ upper bound on $J(x, \mathbf{u})$ for any $x \in \mathbb{X}$ and feasible $\mathbf{u}$, i.e.

$$J(x, \mathbf{u}) \leq \alpha_2(\|x\|) \quad \text{with} \quad \alpha_2(s) \triangleq \mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)s,$$

where $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N) > 0$ is a constant that depends on $\mathcal{L}_f$, $\mathcal{L}_g$, $\theta \triangleq (\theta_i, \ldots, \theta_n)$ and N (e.g., see (Scokaert et al., 1997) for details). Since Q has full-column rank, there exists a $\xi_Q > 0$ such that $\|Qx\| > \xi_Q \|x\|$ for all $x \in \mathbb{X}$. Then, it holds that $J(x, \mathbf{u}) \geq \alpha_1(\|x\|)$ for all $x \in \mathbb{X}$ and any $\mathbf{u}$, where $\alpha_1(s) \triangleq \xi_Q s$. Let $\tau \in (0, 1)$ be a known constant.

Algorithm 7.2.1

1. At time $k \in \mathbb{Z}_+$ measure the state x_k, let $x_{0|k} \triangleq x_k$ and find a control sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ that satisfies:

$$x_{i+1|k} = f(x_{i|k}) + g(x_{i|k})u_{i|k}, \; i = 0, \ldots, N-1, \tag{7.5a}$$
$$\alpha_2(2\|f(x_{0|k}) + g(x_{0|k})u_{0|k}\|) - (1-\tau)\alpha_1(\|x_{0|k}\|) \leq 0, \tag{7.5b}$$
$$x_{i|k} \in \mathbb{X}, \quad i = 1, \ldots, N, \tag{7.5c}$$
$$u_{i|k} \in \mathbb{U}, \quad i = 0, \ldots, N-1, \tag{7.5d}$$
$$\|u_{i|k}\| \leq \theta_i \|x_{0|k}\|, \quad i = 0, \ldots, N-1. \tag{7.5e}$$

2. Let $\bar{\mathbf{u}}_k$ be a feasible sequence of inputs calculated at Step 1. Apply to the perturbed system (7.1b) the control input $\bar{u}^{\mathrm{MPC}}(x_k) \triangleq \bar{u}_{0|k}$.

Theorem 7.2.2 *Let $\mathbb{X}_f(N)$ be the set of states $x \in \mathbb{X}$ for which the optimization problem in Step 1 of Algorithm 7.2.1 is feasible and let $\widetilde{\mathbb{X}}_f(N) \subseteq \mathbb{X}_f(N)$ be a RPI set for system (7.1b) in closed-loop with the sub-optimal MPC control $\bar{u}^{MPC}(\cdot)$ with $0 \in \operatorname{int}(\widetilde{\mathbb{X}}_f(N))$.*

Then, the perturbed system (7.1b) *in closed-loop with the sub-optimal MPC control* $\bar{u}^{MPC}(\cdot)$ *is ISS for initial conditions in* $\widetilde{\mathbb{X}}_f(N)$ *and disturbances in* $\mathbb{W}$.

Proof: The proof consists in showing that $\overline{V}(x_k) \triangleq J(x_k, \bar{\mathbf{u}}_k)$ is an ISS Lyapunov function. Let $\mathcal{C}$ denote $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)$ for shortness. By construction and from constraint (7.5e) we have that $\overline{V}(\cdot)$ satisfies (2.4a) for all $x \in \widetilde{\mathbb{X}}_f(N)$ with $\alpha_1(\|x\|) = \xi_Q\|x\|$ and $\alpha_2(\|x\|) = \mathcal{C}\|x\|$. Moreover, from constraint (7.5b) we have that for all $x \in \widetilde{\mathbb{X}}_f(N)$ and any feasible $\bar{\mathbf{u}}$ (recall that $\|\cdot\|$ denotes the ∞-norm):

$$\begin{aligned}
&\overline{V}(f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x) + w) - \overline{V}(x)\\
&\qquad\qquad \leq \alpha_2(\|f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x) + w\|) - \alpha_1(\|x\|)\\
&\qquad\qquad \leq \alpha_2(\|f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x)\| + \|w\|) - \alpha_1(\|x\|)\\
&\qquad\qquad \leq \alpha_2(\|f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x)\|) + \alpha_2(\|w\|) - \alpha_1(\|x\|)\\
&\qquad\qquad \leq -\alpha_3(\|x\|) + \sigma(\|w\|),
\end{aligned}$$

where $\alpha_3(s) \triangleq \tau\alpha_1(s) = \tau\xi_Q s$ and $\sigma(s) \triangleq \alpha_2(s) = \mathcal{C}s$. The statement then follows from Theorem 2.3.5. ■

In Step 1 of Algorithm 7.2.1, one has to search for a feasible sequence of inputs, which is sufficient for guaranteeing ISS of the closed-loop system. However, as in practice it is also important to achieve robust performance, one can still minimize the cost (7.2), while searching for a feasible sequence of inputs. ISS of the closed-loop system is still guaranteed in this case, even if the global optimum is not attained.

The drawback of the sub-optimal MPC Algorithm 7.2.1 is that the gain of $\alpha_2(\cdot)$, i.e. $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)$, is a strictly increasing function of N, which implies that for long prediction horizons, the contractive constraint (7.5b) may become very conservative. Moreover, the constant $\mathcal{C}(\mathcal{L}_f, \mathcal{L}_g, \theta, N)$ also depends on the ∞-norms of P, Q and R_u. Hence, one cannot freely choose the MPC cost weights, e.g. following performance motivations, since a large ∞-norm implies a large gain of the $\mathcal{K}$-function $\gamma(\cdot)$, which in turn yields a large ISS gain for the closed-loop system, via $\sigma(\cdot)$, $\alpha_1(\cdot)$ and the relation (2.5) established in chapter two.

7.2.2 An artificial Lyapunov function approach

In practice it would be desirable that the design of the MPC cost, i.e. choosing the parameters $F(\cdot)$, $L(\cdot,\cdot)$ and N, is separated from guaranteeing stability, so that the MPC cost can be tuned for best performance. A possible solution to achieve this goal is to resort to an artificial Lyapunov function, which is designed independently of the MPC cost function.

In this section, an ∞-norms artificial Lyapunov function is employed to derive an ISS sub-optimal MPC algorithm. Consider the candidate ISS Lyapunov function

$$V(x) \triangleq \|P_V x\|,$$

where $P_V \in \mathbb{R}^{p_v \times n}$ is a full-column rank matrix. Let $Q_V \in \mathbb{R}^{q_v \times n}$ denote a known matrix with full-column rank. The sub-optimal MPC algorithm is now formulated as follows.

Algorithm 7.2.3

1. At time $k \in \mathbb{Z}_+$ measure the state x_k, let $x_{0|k} \triangleq x_k$ and find a control sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ (alternatively, also minimize the cost (7.2)) that satisfies:

$$x_{i+1|k} = f(x_{i|k}) + g(x_{i|k})u_{i|k}, \ i = 0, \ldots, N-1, \tag{7.6a}$$
$$\|P_V(f(x_{0|k}) + g(x_{0|k})u_{0|k})\| - \|P_V x_{0|k}\| \leq -\|Q_V x_{0|k}\|, \tag{7.6b}$$
$$x_{i|k} \in \mathbb{X}, \quad i = 1, \ldots, N, \tag{7.6c}$$
$$u_{i|k} \in \mathbb{U}, \quad i = 0, \ldots, N-1. \tag{7.6d}$$

2. Let $\bar{\mathbf{u}}_k$ be a feasible sequence of inputs calculated at Step 1. Apply to the perturbed system (7.1b) the control input $\bar{u}^{\mathrm{MPC}}(x_k) \triangleq \bar{u}_{0|k}$.

Theorem 7.2.4 *Let $\mathbb{X}_f(N)$ be the set of states $x \in \mathbb{X}$ for which the optimization problem in Step 1 of Algorithm 7.2.3 is feasible and let $\widetilde{\mathbb{X}}_f(N) \subseteq \mathbb{X}_f(N)$ be a RPI set for system* (7.1b) *in closed-loop with the sub-optimal MPC control $\bar{u}^{MPC}(\cdot)$ with $0 \in \mathrm{int}(\widetilde{\mathbb{X}}_f(N))$.*

Then, the perturbed system (7.1b) *in closed-loop with the sub-optimal MPC control $\bar{u}^{MPC}(\cdot)$ is ISS for initial conditions in $\widetilde{\mathbb{X}}_f(N)$ and disturbances in $\mathbb{W}$.*

Proof: The proof consists in showing that $V(x_k) = \|P_V x_k\|$ is an ISS Lyapunov function for system (7.1b) in closed-loop with $\bar{u}^{\mathrm{MPC}}(\cdot)$. Since P_V has full-column rank, there exist $c_2 \geq c_1 > 0$ such that $c_1\|x\| \leq \|P_V x\| \leq c_2\|x\|$ for all x. Hence, $V(\cdot)$ satisfies condition (2.4a) for $\alpha_1(\|x\|) \triangleq c_1\|x\|$ and $\alpha_2(\|x\|) \triangleq c_2\|x\|$. Next, we show that $V(\cdot)$ satisfies condition (2.4b). The hypothesis of the theorem implies that $\widetilde{\mathbb{X}}_f(N)$ is a RPI set for system

(7.1b) in closed-loop with $\bar{u}^{\mathrm{MPC}}(\cdot)$ and disturbances in $\mathbb{W}$. Moreover, from constraint (7.6b) and using the triangle inequality, we have that for all $x \in \widetilde{\mathbb{X}}_f(N)$ and any feasible $\bar{\mathbf{u}}$:

$$\begin{aligned} V(f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x) + w) - V(x) \\ &= \|P_V(f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x) + w)\| - \|P_V x\| \\ &\leq \|P_V(f(x) + g(x)\bar{u}^{\mathrm{MPC}}(x))\| + \|P_V w\| - \|P_V x\| \\ &\leq -\|Q_V x\| + \|P_V w\| \leq -\alpha_3(\|x\|) + \sigma(\|w\|), \end{aligned}$$

where $\alpha_3(s) \triangleq \xi_{Q_V} s$ (ξ_{Q_V} is such that $\|Q_V x\| \geq \xi_{Q_V}\|x\|$ for all x) and $\sigma(s) \triangleq c_2 s$. The statement then follows from Theorem 2.3.5. ■

Next, we present a method for computing the ∞-norm based artificial Lyapunov function $V(\cdot)$ off-line. Let

$$x_{k+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \tag{7.7}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, be a linear approximation of (7.1a) about $(0, 0)$. We assume that there exists a neighborhood $\mathcal{N} \subset \mathbb{X}$ of the origin where $Ax + Bu \approx f(x) + g(x)u$ for all $x \in \mathcal{N}$ and all $u \in \mathbb{U}$. For a given full-column rank matrix Q_V, in order to compute the matrix P_V, we consider a linear state-feedback $u_k = Kx_k$, $K \in \mathbb{R}^{m\times n}$, $k \in \mathbb{Z}_+$, and we make use of the following result, which yields an ISS Lyapunov function for the closed-loop system $x_{k+1} = (A + BK)x_k + w_k$, $w_k \in \mathbb{W}$ for all $k \in \mathbb{Z}_+$.

Lemma 7.2.5 *Suppose that a full-column rank matrix P_V and a gain K satisfy*

$$1 - \|P_V(A + BK)P_V^{-L}\| - \|Q_V P_V^{-L}\| \geq 0, \tag{7.8}$$

where $P_V^{-L} \triangleq (P_V^\top P_V)^{-1}P_V^\top$ is a left Moore-Penrose inverse of P_V. Then, it holds that $\|P_V(A + BK)x\| - \|P_V x\| \leq -\|Q_V x\|$ for all x and, the function $V(x) = \|P_V x\|$ is an ISS Lyapunov function for the closed-loop system $x_{k+1} = (A + BK)x_k + w_k$.

The proof of Lemma 7.2.5 is a particular case of the proof of the more general result presented in Section 6.5.3. We also refer the reader to Section 6.5.3 for ways to find a solution to inequality (7.8).

Note that, due to the use of an artificial Lyapunov function, the weights of the MPC cost function and the length of the prediction horizon can now be freely chosen in order to achieve physical performance requirements, in case one also minimizes the cost (7.2) in Step 1 of Algorithm 7.2.3. The value function of the MPC cost is no longer used as an ISS Lyapunov function and the norms of P, Q and R_u no longer influence the gain of the $\mathcal{K}$-function $\gamma(\cdot)$, which represents the ISS gain of the closed-loop system.

Remark 7.2.6 The hypothesis of Theorem 7.2.2 (Theorem 7.2.4) assumes robust feasibility of the optimization problem that has to be solved in Step 1 of Algorithm 7.2.1 (Algorithm 7.2.3), which cannot be guaranteed a priori in general. In practice, the constraint $x_{1|k} \in \mathbb{X} \sim \mathbb{W}$ is usually added to the optimization problem to ensure that the closed-loop system state, i.e. $\tilde{x}_{k+1} = x_{1|k} + w_k$, $k \in \mathbb{Z}_+$, does not violate the state constraints at time $k+1$ for any disturbance in $\mathbb{W}$.

Remark 7.2.7 When the sets $\mathbb{X}$, $\mathbb{U}$ (and $\mathbb{W}$) are polyhedral, which is often the case in practice, the constraints (7.5b)-(7.5e), as well as the constraints (7.6b)-(7.6d), can be written as a finite number of linear inequalities since the measured state $x_{0|k}$ is known and $\|\cdot\|$ denotes the ∞-norm. Moreover, since the nonlinear system (7.1) is affine with respect to the input, for $N = 1$, the optimization problem that has to be solved at Step 1 of Algorithm 7.2.1 or Algorithm 7.2.3 can be formulated as a single linear program.

7.3 Application to PWA systems

In this section we consider discrete-time piecewise affine systems affected by *measurement noise*, i.e.

$$x_{k+1} = A_j x_k + B_j u(\hat{x}_k) + f_j \text{ when } x_k \in \Omega_j,\ j \in \mathcal{S},\ k \in \mathbb{Z}_+, \qquad (7.9)$$

where $x_k \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u_k \in \mathbb{U} \subseteq \mathbb{R}^m$ is the control input, $\hat{x}_k \triangleq x_k + e_k$ is the measured state and $e_k \in \mathbb{E} \subset \mathbb{R}^n$ represents an unknown bounded measurement noise.

The sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{E}$ are assumed to be polyhedral C-sets. Here, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $f_j \in \mathbb{R}^n$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \mathcal{S}$ with $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ a *finite set* of indices and s denotes the number of discrete modes. The collection $\{\Omega_j \mid j \in \mathcal{S}\}$ defines a partition of $\mathbb{X}$, meaning that $\cup_{j\in\mathcal{S}}\Omega_j = \mathbb{X}$ and $\mathrm{int}(\Omega_i) \cap \mathrm{int}(\Omega_j) = \emptyset$ for $i \neq j$. Each Ω_j is assumed to be a polyhedron (not necessarily closed). Let $\mathcal{S}_0 \triangleq \{j \in \mathcal{S} \mid 0 \in \mathrm{cl}(\Omega_j)\}$ and let $\mathcal{S}_1 \triangleq \{j \in \mathcal{S} \mid 0 \notin \mathrm{cl}(\Omega_j)\}$, so that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$. We assume that the origin is an equilibrium state for (7.9) with $u = 0$ and we require that $f_j = 0$ for all $j \in \mathcal{S}_0$. This includes PWA systems that are discontinuous.

Inherent robustness to measurement noise is usually guaranteed in the case of (Lipschitz) continuous system dynamics, or even stronger, continuous control laws and candidate Lyapunov functions. See, for example, (Magni et al., 1998) and the more recent article (Messina et al., 2005), where only continuity of the control system (but not necessarily of the feedback law) is assumed. However, in the case of PWA systems, when the system dynamics may be discontinuous, the problem of guaranteeing robustness to measurement noise is much more difficult. The goal is to develop a sub-optimal MPC scheme for (discontinuous) PWA systems that is robust to measurement noise. This is achieved in the present section via an artificial Lyapunov function approach, similar to the one used in the previous section to deal with additive disturbances.

Note that if the robust MPC algorithms presented in Chapter 5, where additive disturbances were considered, are used to generate the control input $u(\hat{x}_k)$, then the measurement noise input affects the PWA system (7.9) in the same way as an additive disturbance input, due to the fact that the MPC control input is a PWA state feedback in this case. Hence, if the initial state is known, i.e. $e_0 = 0$ and thus $\hat{x}_0 = x_0$. Then, the MPC schemes developed in chapter five are ISS with respect to measurement noise inputs. This is because the tightened set of admissible input sequences ensures that the effect of the measurement noise will not change the mode of the PWA system (i.e. the region Ω_j where the real state x_k lies). More precisely, the state will still be measured wrong, but the mode will be measured right, i.e. $(\hat{x}_k, x_k) \in \Omega_j \times \Omega_j$, $j \in \mathcal{S}$. Then, the effect of measurement noise is just an additive disturbance and the results developed in Chapter 5 hold *mutatis mutandis*.

However, in practice it is often the case that the initial state and/or mode are not known, e.g. when an observer is used to estimate the state. Therefore, we develop in the sequel a sub-optimal MPC algorithm that is ISS with respect to measurement noise, without assuming that the initial state or mode is known.

To do so, we consider an ∞-norm based artificial Lyapunov function of the form:

$$V(x) \triangleq \|P_j^V x\| \quad \text{if} \quad x \in \Omega_j, \quad j \in \mathcal{S}.$$

Such a function can be calculated for the PWA system (7.9) in closed-loop with a state-feedback $u_k \triangleq K_j x_k$ when $x_k \in \Omega_j$ using the techniques presented in Section 3.4.2.

Next, let the terminal cost and the stage cost be defined using ∞-norms, i.e. $F(x) \triangleq \|P_j x\|$ when $x \in \Omega_j$ and $L(x, u) = \|Qx\| + \|Ru\|$, where the full-column rank matrices $\{P_j \mid j \in \mathcal{S}\}$, Q and R can now be freely chosen to ensure performance.

Algorithm 7.3.1

1. At time $k \in \mathbb{Z}_+$ obtain the measured state $\hat{x}_k$, let $x_{0|k} \triangleq \hat{x}_k$ and compute the set $\mathcal{B}_e(\hat{x}_k) \triangleq \hat{x}_k \oplus \mathbb{E}$. If $\mathcal{B}_e(\hat{x}_k) \cap \Omega_j \neq \emptyset$ for $j \in \mathcal{S}$, then add the index j to a set of indices $\mathcal{S}_{1_k}$.

2. Solve the LP problems

$$\min_{j \in \mathcal{S}_{1_k}, x \in \mathcal{B}_e(\hat{x}_k) \cap \Omega_j} \|P_j^V x\| \quad \text{and} \quad \max_{x \in \mathcal{B}_e(\hat{x}_k) \cap \Omega_j} \|Qx\|,$$

   and let

$$(P_{j_k^*}^V, x_{1_k}^*) \triangleq \arg \min_{j \in \mathcal{S}_{1_k}, x \in \mathcal{B}_e(\hat{x}_k) \cap \Omega_j} \|P_j^V x\|, \quad x_{2_k}^* \triangleq \arg \max_{x \in \mathcal{B}_e(\hat{x}_k)} \|Qx\|.$$

3. For all $j \in \mathcal{S}_{1_k}$ perform the following operations: (i) Compute the one-step reachable set

$$\mathcal{X}_j \triangleq \{y \in \mathbb{X} \mid y = A_j x + B_j u + f_j; x \in \mathcal{B}_e(\hat{x}) \cap \Omega_j, u \in \mathbb{U}\};$$

   (ii) If $\mathcal{X}_j \cap \Omega_i \neq \emptyset$ for $i \in \mathcal{S}$, then add the index i to a set of indices $\mathcal{S}_{2_k}^j$.

4. Find a control sequence $\mathbf{u}_k = (u_{0|k}, \ldots, u_{N-1|k})$ (alternatively, also minimize the cost (7.2)) that satisfies:

$$\begin{aligned}
x_{i+1|k} &\triangleq A_j x_{i|k} + B_j u_{i|k} + f_j \quad \text{when} \quad x_{i|k} \in \Omega_j, \\
&\qquad i = 0, \ldots, N-1, && \text{(7.10a)} \\
\max_{i \in \mathcal{S}_{2_k}^j} &\left(\|P_i^V\|\right) \|A_j x_{0|k} + B_j u_{0|k} + f_j\| - \|P_{j_k^*}^V x_{1_k}^*\| \leq -\|Q x_{2_k}^*\|, \\
&\qquad \text{for all} \quad j \in \mathcal{S}_{1_k}, && \text{(7.10b)} \\
x_{i|k} &\in \mathbb{X}, \quad i = 1, \ldots, N, && \text{(7.10c)} \\
u_{i|k} &\in \mathbb{U}, \quad i = 0, \ldots, N-1. && \text{(7.10d)}
\end{aligned}$$

5. Let $\bar{\mathbf{u}}_k$ be a feasible sequence of inputs calculated at Step 4. Apply to the perturbed PWA system (7.9) the sub-optimal MPC control input $\bar{u}^{\text{MPC}}(\hat{x}_k) \triangleq \bar{u}_{0|k}$.

Theorem 7.3.2 *Let $\mathbb{X}_f(N)$ be the set of states $x \in \mathbb{X}$ for which the optimization problem in Step 4 of Algorithm 7.3.1 is feasible and let $\widetilde{\mathbb{X}}_f(N) \subseteq \mathbb{X}_f(N)$ be a RPI set for the PWA system (7.9) in closed-loop with the sub-optimal MPC control $\bar{u}^{MPC}(\cdot)$ for all $e \in \mathbb{E}$, with $0 \in \operatorname{int}(\widetilde{\mathbb{X}}_f(N))$.*

Then, the PWA system (7.9) in closed-loop with $\bar{u}^{MPC}(\cdot)$ and affected by measurement noise is ISS for states in $\widetilde{\mathbb{X}}_f(N)$ and measurement noise in $\mathbb{E}$.

Proof: The proof consists in showing that the "artificial" local Lyapunov function $V(x) \triangleq \|P_j^V x\|$ when $x \in \Omega_j$, $j \in \mathcal{S}$, is an ISS Lyapunov function for the MPC closed-loop system. It is straightforward to show that $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ for all $x \in \widetilde{\mathbb{X}}_f(N)$, where $\alpha_1(\|x\|) \triangleq \tau\|x\|$ for some $\tau > 0$ and $\alpha_2(\|x\|) \triangleq \max_{j\in\mathcal{S}} \|P_j^V\|\|x\|$. Next, for all $k \in \mathbb{Z}_+$, $\hat{x}_k \in \widetilde{\mathbb{X}}_f(N)$ and $x_k \in \mathcal{B}_e(\hat{x}_k)$ we have that:

$$
\begin{aligned}
V(x_{k+1}) - V(x_k) &= V(A_j x_k + B_j \bar{u}^{\mathrm{MPC}}(\hat{x}_k) + f_j) - V(x_k) \\
&= \|P_i^V(A_j x_k + B_j \bar{u}^{\mathrm{MPC}}(\hat{x}_k) + f_j)\| - \|P_j^V x_k\| \\
&\quad \text{when } (x_k, x_{k+1}) \in \Omega_j \times \Omega_i \\
&\leq \max_{i\in\mathcal{S}_{2_k}^j} \left(\|P_i^V\|\right) \|A_j(\hat{x}_k - e_k) + B_j \bar{u}^{\mathrm{MPC}}(\hat{x}_k) + f_j\| - \|P_{j_k^*}^V x_{1_k}^*\| \\
&\leq \max_{i\in\mathcal{S}_{2_k}^j} \left(\|P_i^V\|\right) \|A_j \hat{x}_k + B_j \bar{u}^{\mathrm{MPC}}(\hat{x}_k) + f_j\| - \|P_{j_k^*}^V x_{1_k}^*\| \\
&\quad + \max_{i\in\mathcal{S}_{2_k}^j} \left(\|P_i^V\|\right) \max_{j\in\mathcal{S}} \left(\|A_j\|\right) \|e_k\| \\
&\leq -\|Q x_{2_k}^*\| + \max_{i\in\mathcal{S}_{2_k}^j} \left(\|P_i^V\|\right) \max_{j\in\mathcal{S}} \left(\|A_j\|\right) \|e_k\| \\
&\leq -\|Q x_k\| + \max_{i\in\mathcal{S}_{2_k}^j} \left(\|P_i^V\|\right) \max_{j\in\mathcal{S}} \left(\|A_j\|\right) \|e_k\| \\
&\leq -\alpha_3(\|x_k\|) + \sigma(\|e_k\|),
\end{aligned}
$$

with

$$
\sigma(s) \triangleq \max_{i\in\mathcal{S}_{2_k}^j} \left(\|P_i^V\|\right) \max_{j\in\mathcal{S}} \left(\|A_j\|\right) s, \quad \alpha_3(s) \triangleq \gamma s.
$$

Hence, $V(\cdot)$ is an ISS Lyapunov function for the closed-loop MPC system. Then the statement follows from Theorem 2.3.5. ■

As done in the previous subsection on sub-optimal MPC we rely on the fact that (robust) feasibility implies (input-to-state) stability. Also, the prediction horizon and the MPC cost no longer play a role in proving stability and their only purpose is to improve performance.

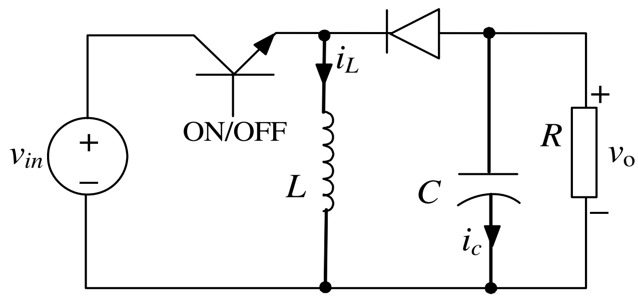

Figure 7.1: A schematic view of a Buck-Boost converter.

Note that the ISS constraint (7.10b) can be written as a finite number of linear inequality constraints in $u_{0|k}$. Hence, the optimization problem that has to be solved at Step 4 of the above algorithm is a mixed integer linear programming (MILP) problem for $N \in \mathbb{Z}_{>1}$ and a LP problem for $N = 1$. In addition to solving this MILP problem, at most $(2\,\text{card}(\mathcal{S}) + 1)$ LP problems must be solved in Step 2 and $\text{card}(\mathcal{S})(\text{card}(\mathcal{S})+1)$ intersections of polyhedra must be computed in Step 1 and Step 3. However, the worst case is equivalent to using a common artificial Lyapunov function weight matrix, i.e. $P_j^V = P^V$ for all $j \in \mathcal{S}$. If this is the case, one can search for a common Lyapunov function based on ∞-norms, i.e. $V(x) = \|P^V x\|$ for all $x \in \mathbb{X}$. Then, only two LP problems must be solved (in Step 2), without computing any intersection. Algorithm 7.3.1 can be directly implemented using the MPT (Kvasnica et al., 2004), as it contains all the necessary numerical tools.

7.4 Application to the control of DC-DC converters

In this section we implement Algorithm 7.2.3 to control a Buck-Boost DC-DC converter power circuit. DC-DC converters are some of the most important circuits within the family of power circuits. They are extensively used in power supplies for electronic equipment to control the energy flow between two DC systems. Control of a DC-DC converter power circuit is based, explicitly or implicitly, on a model that describes how control actions and disturbances are expected to affect the behavior of the plant. Usually, the control problem consists in defining the desired nominal operating condition, and then regulating the circuit so that it stays close to the nominal, when the plant is subject to disturbances and modeling errors that cause its operation to deviate from the nominal. The following nonlinear averaged model of a

Buck-Boost DC-DC converter (see Figure 7.1 for a schematic representation of an ideal circuit (i.e. neglecting the parasite components), which was developed in (Lazar and De Keyser, 2004) by applying the theory of (Kassakian et al., 1992), is used to obtain a prediction model:

$$x_{k+1}^m = \begin{bmatrix} x_{1,k}^m + \frac{T}{L}x_{2,k}^m - \frac{T}{L}(x_{2,k}^m - V_{in})u_k^m \\ -\frac{T}{C}x_{1,k}^m + \frac{T}{C}x_{1,k}^m u_k^m + (1-\frac{T}{C})x_{2,k}^m \end{bmatrix}, \quad k \in \mathbb{Z}_+, \tag{7.11}$$

where $x_k^m = [x_{1,k}^m \; x_{2,k}^m]^\top \in \mathbb{R}^2$ and $u_k^m \in \mathbb{R}$ are the state and the input, respectively. x_1^m (i_L) represents the current flowing through the inductor, x_2^m (v_o) the output voltage and u^m represents the duty cycle (i.e. the fraction of the sampling period during which the transistor is kept ON). The sampling period is $T = 0.65$ milliseconds. The parameters of the circuit are the inductance $L = 4.2$mH, the capacitance $C = 2200\mu$F, the load resistance $R = 165\Omega$ and the source input voltage v_{in}, with nominal value $V_{in} = 15$V.

The control objective is to reach a desired steady state value of the output voltage, i.e. x_2^{ss}, as fast as possible and with minimum overshoot. From x_2^{ss} one can obtain the steady state duty cycle and inductor current as follows:

$$u^{ss} = \frac{x_2^{ss}}{x_2^{ss} - V_{in}}, \quad x_1^{ss} = \frac{x_2^{ss}}{R(u^{ss}-1)}. \tag{7.12}$$

Furthermore, the following physical constraints must be fulfilled at all times $k \in \mathbb{Z}_+$:

$$x_{1,k}^m \in [0.001, 5], \quad x_{2,k}^m \in [-20, 0], \quad u_k^m \in [0.1, 0.9]. \tag{7.13}$$

To implement Algorithm 7.2.3, we first perform the following coordinate transformation on (7.11):

$$x_{1,k} = x_{1,k}^m - x_1^{ss}, \quad x_{2,k} = x_{2,k}^m - x_2^{ss}, \quad u_k = u_k^m - u^{ss}. \tag{7.14}$$

We obtain the following nonlinear system description

$$x_{k+1} = \begin{bmatrix} x_{1,k} + \alpha x_{2,k} + (\beta - \frac{T}{L}x_{2,k})u_k \\ (\frac{T}{C}x_{1,k} + \gamma)u_k + (1 - \frac{T}{RC})x_{2,k} + \delta x_{1,k} \end{bmatrix}, \tag{7.15}$$

where the constants α, β, γ and δ depend on the fixed steady state value x_2^{ss} as follows

$$\alpha = \frac{T}{L}(1 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}}), \quad \beta = \frac{T}{L}(V_{in} - x_2^{ss}),$$
$$\gamma = \frac{T}{RCV_{in}}x_2^{ss}(x_2^{ss} - V_{in}), \quad \delta = \frac{T}{C}\left(\frac{x_2^{ss}}{x_2^{ss} - V_{in}} - 1\right).$$

Using (7.14) and (7.12), the constraints given in (7.13) can be converted to:

$$x_{1,k} \in [\underline{b}^{x_1}, \overline{b}^{x_1}], \quad x_{2,k} \in [\underline{b}^{x_2}, \overline{b}^{x_2}], \quad u_k \in [\underline{b}^{u}, \overline{b}^{u}], \tag{7.16}$$

where

$$\underline{b}^{x_1} = 0.001 - \frac{1}{RV_{in}} x_2^{ss}(x_2^{ss} - V_{in}), \quad \overline{b}^{x_1} = 5 - \frac{1}{RV_{in}} x_2^{ss}(x_2^{ss} - V_{in}),$$
$$\underline{b}^{x_2} = -20 - x_2^{ss},\ \overline{b}^{x_2} = -x_2^{ss}, \quad \underline{b}^{u} = 0.1 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}},\ \overline{b}^{u} = 0.9 - \frac{x_2^{ss}}{x_2^{ss} - V_{in}}.$$

The control objective can now be formulated as to stabilize (7.15) around the equilibrium $(0,0)$ while fulfilling the constraints given in (7.16).

Next, to compute an ∞-norm Lyapunov function via Lemma 7.2.5, we linearize system (7.15) around the equilibrium $(0,0)$ (for zero input $u_k = 0 \in [\underline{b}^{u}, \overline{b}^{u}]$). The linearized equations are given by

$$\Delta x_{k+1} = A\Delta x_k + B\Delta u_k, \tag{7.17}$$

where Δx_k and Δu_k represent "small" deviations from the equilibrium $(0,0)$ and zero input $u_k = 0$, respectively. The matrices A and B are given by

$$A \triangleq \frac{\partial f}{\partial x}\Big|_{\substack{x=0,\\ u=0}} = \begin{bmatrix} 1 & \alpha \\ \delta & 1 - \frac{T}{RC} \end{bmatrix}, \quad B \triangleq \frac{\partial f}{\partial u}\Big|_{\substack{x=0,\\ u=0}} = \begin{bmatrix} \beta \\ \gamma \end{bmatrix}.$$

For the linear model corresponding to a steady state output voltage $x_2^{ss} = -4$V (which yields $u^{ss} = 0.2105$ and $x_1^{ss} = 0.0307$A), by applying one of the methods presented in Section 6.5.3 to find the matrix P_V and the feedback gain K satisfying (7.8) for $Q_V = \left[\begin{smallmatrix} 0.001 & 0 \\ 0 & 0.001 \end{smallmatrix}\right]$, we have obtained the solution

$$P_V = \begin{bmatrix} 0.9197 & -0.6895 \\ -0.5815 & 1.8109 \end{bmatrix}, \quad K = \begin{bmatrix} -0.4648 & 0.4125 \end{bmatrix}.$$

The MPC cost matrices have been chosen as follows, to ensure a good performance:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad R_u = 0.1.$$

To assess the real-time applicability of the developed theory for this type of very fast system with a sampling period well below one milisecond, we chose $N = 1$ and, for the nonlinear model (7.15), we formulated the optimization problem in Step 1 of Algorithm 7.2.3 as a Linear Programming (LP) problem. The resulting LP problem has 3 optimization variables and 18 constraints.

In one simulation, we tested first the start-up behavior (see Figure 7.2 column one) and then, after reaching the desired operating point, we tested the disturbance rejection (see Figure 7.2 column two). The dynamics were simultaneously affected by an asymptotically decreasing additive disturbance of the form $w = [\frac{1}{k} \quad 0]^\top$ and a 50% drop of the load (i.e. R=82.5Ω) for $k = 80, 81, \ldots, 180$ (or from time instant 0.052 until time instant 0.117 - in seconds). For $k > 180$ the disturbance was set equal to zero and the load was set to its nominal value (i.e. R=165Ω) to show that the closed-loop system is ISS, i.e. that asymptotic stability is recovered when the disturbance input vanishes. The trajectories over the time interval [0 0.1495] (in seconds, or 230 sampling periods T) of the state and sub-optimal NMPC control input are plotted in Figure 7.2. Moreover, in Figure 7.2 (first plot in the second column) one can observe that during the disturbance rejection phase of the simulation, the output voltage is well within the operating margin required in industry for DC-DC converters, i.e ±3% of the desired operating value.

Note that, although the simulations were performed for the transformed system (7.15), we chose to plot all variables in the original coordinates corresponding to system (7.11), which have more physical meaning.

The LP problem equivalent to the sub-optimal NMPC optimization problem in Step 1 of Algorithm 7.2.3 was always solved[2] within the allowed sampling interval, with an worst case CPU time of 0.6314 milliseconds over 20 runs. The very good closed-loop performance obtained for $N = 1$ collaborated with the computational time estimate is encouraging for further development of the real-time application of the presented theory to control DC-DC power converters, especially using faster platforms, such as Digital Signal Processors (DSP).

7.5 Concluding remarks

Two new computationally friendly sub-optimal MPC algorithms with an a priori input-to-state stability guarantee were presented. The first one employs a contraction constraint on the norm of the closed-loop system state, while the second algorithm uses an ∞-norm based artificial Lyapunov function. For both sub-optimal MPC schemes, the input-to-state stabilization constraints can be written as a finite number of linear inequalities. For the class of PWA systems, it is shown how the sub-optimal MPC scheme based

[2]The simulation platform was Matlab 7.0.4 (R14) (CDD Dual Simplex LP solver) running on a Linux Fedora Core 5 operating system powered by an Intel Pentium 4 with a 3.2 GHz CPU.

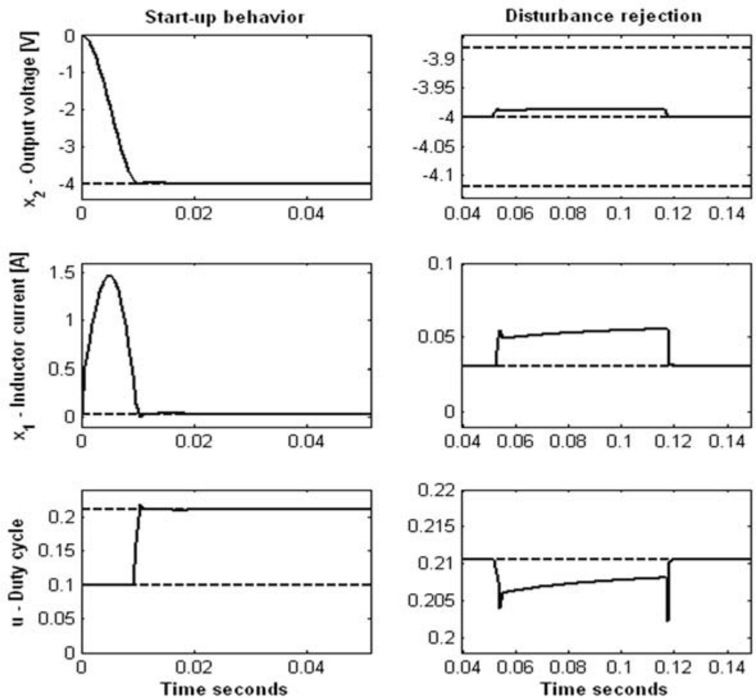

Figure 7.2: State trajectories and sub-optimal NMPC input histories for $N = 1$ - solid lines, desired steady state values, input constraint (in first column, bottom plot) and industrial operating margins for DC-DC converters ($\pm 3\%$ of the desired output voltage, in second column, first plot) - dashed lines.

---

on an artificial Lyapunov function can be modified to ensure input-to-state stability with respect to measurement noise. This modified scheme is particularly relevant because it can be used in interconnection with an input-to-state stable observer, resulting in an asymptotically stable closed-loop system. Methods for computing such artificial Lyapunov functions off-line for linear or PWA models were also indicated. A case study on the control of a Buck-Boost DC-DC power converter includes preliminary real-time numerical data was also presented to illustrate the applicability of the developed theory.

8

Conclusions

A summary of the main contributions and a collection of several possible directions for future research conclude this thesis.

8.1 Contributions

The major contributions are in the domains of

- Stability Theory of Hybrid Systems;
- Stabilizing Model Predictive Control;
- Robust Model Predictive Control;
- Set Invariance Theory;
- Linear Matrix Inequalities in Control.

We discuss the obtained results in more detail below.

8.1.1 Stability Theory of Hybrid Systems

Classical stability results on Lyapunov asymptotic stability, input-to-state stability (ISS) and input-to-state practical stability (ISpS) are revisited in the hybrid context, where one has to deal with discontinuous dynamics and discontinuous Lyapunov functions. It is shown that, for discrete-time, possibly discontinuous nonlinear systems, continuity at the equilibrium point rather than continuity on a neighborhood of the equilibrium point of the system and the candidate (ISS) Lyapunov function is sufficient for Lyapunov stability and input-to-state stability. For system dynamics that are also discontinuous at the equilibrium point, the input-to-state practical stability framework can be employed to establish robustness. Several relaxations with respect to the existing results are presented. The usefulness of these relaxations for stability theory of hybrid systems has been demonstrated via various simple examples of hybrid systems, for which the classical stability (robust stability) arguments that rely on continuous system dynamics and continuous Lyapunov functions do not apply. Stability and input-to-state stability can be established for these examples via the new results derived in this thesis.

The concept of *norms as Lyapunov functions*, mostly employed for linear systems (Polanski, 1995), has been extended to hybrid systems. Techniques for computing Lyapunov functions defined using ∞-norms for piecewise affine (PWA) systems are presented in Chapter 3, as a new way to compute a suitable terminal cost for MPC algorithms based on PWA prediction models. Lyapunov functions based on ∞-norms are particularly attractive, as opposed to the popular quadratic Lyapunov functions, since their level sets are *polyhedral* positively invariant sets.

A warning has been issued via the construction of a globally exponentially stable piecewise affine system that has zero robustness to *arbitrarily small* additive disturbances. The absence of a continuous Lyapunov function is the main reason for this phenomenon. Although such examples are known for nonlinear systems (Grimm et al., 2004; Kellett and Teel, 2004), this is the first example of zero robustness in PWA systems. This result, presented in Chapter 4, is very important as stability is often established for piecewise affine systems via discontinuous Lyapunov functions. In particular, this is also the case of model predictive control of hybrid systems, which typically produces a discontinuous piecewise affine closed-loop system and a discontinuous MPC value function (which is usually used as the candidate Lyapunov function to establish stability in MPC). The practical relevance of this warning cannot be overstated, since controllers that are designed to be nominally stabilizing, can result in closed-loop systems that have zero robustness when applied in practice, where disturbances are always present.

To prevent this undesired situation in real-life applications, the concept of input-to-state stability has been employed in the fourth chapter for piecewise affine and hybrid systems to develop sufficient conditions for robust stability. Techniques based on linear matrix inequalities for solving the global input-to-state stability and stabilization problem for piecewise affine systems have also been developed in Chapter 4. This methodology, that is applicable to discontinuous systems, can be used to synthesize piecewise linear state-feedback controllers for PWA systems that a priori guarantee input-to-state stability for the closed-loop system. *Therefore, these controllers can be safely implemented in practice, as they will not result in undesired phenomena, such* *as zero robustness.*

The LMI methods for computing piecewise quadratic ISS Lyapunov function as well as input-to-state stabilizing state-feedbacks for PWA systems presented in Chapter 4 contribute to the domain of **Linear matrix inequalities in control**. The contribution consists in a new technique for applying the Schur complement block wise, without changing any of the matrix elements that are not part of a desired block.

8.1.2 Stabilizing Model Predictive Control

A general theorem on stability of non-smooth nonlinear model predictive control, which unifies and extends many of the previous stability results in MPC is presented in Chapter 3. This theorem explicitly allows for discontinuous system dynamics and MPC value functions and therefore, can be directly applied to establish Lyapunov stability for model predictive control of hybrid systems. Until now, a proof of Lyapunov stability for hybrid MPC in its full generality was missing in the literature.

In the third chapter, new methods for computing a terminal cost that satisfies the developed stabilization conditions are presented for the class of piecewise affine systems and MPC costs based on both quadratic forms and $1, \infty$-norms. These methods also contribute to the field of **Linear matrix inequalities in control**, as for quadratic MPC costs, the stabilization conditions are expressed as a linear matrix inequality. This approach can be employed for both simple stabilizing state-feedback synthesis and terminal cost and local controller computation in stabilizing MPC. The existing LMI methods for computing *piecewise quadratic Lyapunov functions for PWA systems* cannot offer a solution for the terminal cost and local controller MPC stabilization problem.

New methods for computing a terminal constraint set that satisfies the developed stabilization conditions are also presented in Chapter 3. An essential condition that the terminal constraint set must satisfy is the condition of positive invariance. This brings us to the contributions of this thesis in the domain of **Set invariance theory**.

Computing invariant sets for piecewise affine systems is a notoriously difficult problem, but of great interest. In general, invariant sets for PWA systems are not convex, and they are usually obtained, via standard controllability or reachability based recursive algorithms, as the union of a number (possibly very large or even infinite) of polyhedra. Chapter 3 presents new techniques for computing positively invariant sets for piecewise affine systems that have two very attractive properties:

- they are *piecewise polyhedral* positively invariant sets - i.e. they consist of a union of a finite number of polyhedra;
- they have a low complexity - i.e. the number of polyhedra that form the positively invariant set is at most equal to the number of affine sub-systems of the piecewise affine system.

A method for computing *convex* and *polyhedral* positively invariant sets for PWA systems is also presented. Such sets are extremely appealing for use in optimization problems, as the hybrid MPC problems, and also for estimation of domains of stability, due to their low complexity.

An optimization technique for computing Lyapunov functions based on $1, \infty$-norms is presented in Chapter 3. The sublevel sets of such functions are implicitly polyhedral (or piecewise polyhedral) invariant sets. Therefore, this technique can be employed to calculate $1, \infty$-norm induced positively invariant sets for PWA systems.

Apart from these results keyed to the particular class of PWA systems, a new approach to the computation of invariant and contractive sets for general, possibly discontinuous nonlinear systems is also presented. The only requirement for the implementation of this method is that the system admits a quadratic or piecewise quadratic Lyapunov function. This method, due to its unique geometrical approach that is independent of the system dynamics, can be applied to a wide class of relevant systems including linear systems subject to input saturation, linear systems affected by parametric uncertainties and/or additive disturbances, switched linear systems under arbitrary switching and conewise piecewise linear systems. We prove that a solution to the problem of computing a polyhedral or piecewise polyhedral invariant (contractive) set can be obtained by solving a finite number of quadratic programs (QP). An explicit upper bound on the number of QP problems that have to be solved is also derived.

8.1.3 Robust Model Predictive Control

The input-to-state stability framework is employed in Chapter 5 to investigate the robustness of discrete-time (discontinuous) PWA systems in closed-loop with MPC.

Inherent robustness issues

A strong warning is issued by showing via an example that hybrid MPC can generate (discontinuous) MPC values functions *that are not input-to-state Lyapunov functions*. Therefore, one should be careful in drawing conclusions on robustness from nominal stability in hybrid MPC, as earlier examples presented in Chapter 4 showed that the absence of a continuous Lyapunov function can lead to the zero robustness phenomenon.

The fifth chapter provides an a posteriori test for checking robustness of nominally stable hybrid MPC schemes (and, in principle, of any PWA system), given a discontinuous MPC value function. This test comes down to solving a finite number of linear programs.

Robust MPC via tightened constraints

However, if the a posteriori robustness test fails for a specific hybrid MPC closed-loop system, there are no systematic ways available that can be used to modify the nominally stabilizing MPC schemes so that robustness is ensured. A new design method for setting up hybrid MPC schemes with an a priori ISS guarantee with respect to bounded additive disturbance inputs is presented in Chapter 5. This method restricts the predicted future states to a tightened, possibly disconnected subset of the state-space, and does not require continuity of the MPC value function, nor of the PWA system dynamics.

Robust MPC via min-max formulations

Chapter 6 employs the ISS framework to derive new sufficient conditions for robust asymptotic stability of min-max nonlinear MPC. It is shown that in general, only input-to-state practical stability can be a priori ensured for min-max nonlinear MPC. This is because the min-max MPC controller takes into account the effect of a non-zero disturbance input, even if the disturbance input vanishes in reality. However, explicit bounds on the evolution of the min-max MPC closed-loop system state are derived and ultimate boundedness is proven for the closed-loop system.

ISS is achieved via a dual-mode approach and using a new technique based on $\mathcal{KL}$-estimates of stability. This result is important because it unifies in one MPC scheme the properties of the MPC controllers of (Limon et al., 2006) and (Mayne, 2001), i.e. it guarantees both input-to-state practical stability (ultimate boundedness) in the presence of persistent disturbances and robust asymptotic stability in the presence of asymptotically decaying disturbances (without a priori assuming that this property holds for the disturbances).

New techniques for calculating a terminal cost and local state-feedback controller such that the developed robust stabilization conditions for min-max MPC are satisfied are presented in Chapter 6. These techniques employ linear matrix inequalities for quadratic MPC costs and norm inequalities for MPC costs based on $1, \infty$-norms. The proposed stabilization method works for systems affected by both time-varying parametric uncertainties and additive disturbances. One of its advantages is that *the resulting MPC cost function is continuous, convex and bounded*, which is desirable from an optimization point of view. Also, the techniques for computing a terminal cost presented in this chapter do not depend on the type of min-max MPC numerical set-up. They can be employed to ensure robust stability or input-to-state stability for a wide variety of both *open-loop* and *feedback* min-max MPC schemes existing in the literature.

Robust sub-optimal MPC

Most of the stabilization methods developed within the MPC framework rely on the assumption of optimality. Clearly, when dealing with nonlinear or hybrid prediction models and hard constraints, it is difficult if not impossible to guarantee this assumption in practice, where numerical solvers usually provide, in the limited computational time available, a feasible, sub-optimal input sequence, rather than a globally optimal one.

Chapter 7 proposes methods for designing input-to-state stabilizing, but computationally friendly sub-optimal MPC algorithms for nonlinear and hybrid systems. This is achieved via new, simpler stabilizing constraints, that can be implemented as a finite number of linear inequalities. Two sub-optimal nonlinear MPC algorithms that are ISS with respect to *additive disturbances* are presented. The first one is based on a contraction argument, while the second MPC scheme resorts to an ∞-norm based artificial Lyapunov function.

For the class of PWA systems, in the seventh chapter it is shown how the sub-optimal MPC scheme based on an artificial Lyapunov function can be modified to ensure input-to-state stability with respect to *measurement noise*. This modified scheme is particularly relevant because it can be used in interconnection with an asymptotically stable observer, resulting, if cascade conditions are satisfied, in an asymptotically stable closed-loop system.

Application to the control of DC-DC converters

Power circuits are widely used in the computer industry and in consumer electronics including cameras and they are critical for mission success on space platforms. Some of the most important circuits within the family of power circuits are the DC-DC converters. They are employed in power supplies for electronic equipment to control the energy flow between two DC systems. The control problem for DC-DC converters is usually tackled using PID (PI to be more specific) controllers. However, this approach has its limitations, due to the highly nonlinear characteristic of DC-DC converters. That is why there is an increasing interest for advanced control of DC-DC converters (Lazar and De Keyser, 2004). An efficient predictive controller for DC-DC converters is designed in Chapter 7, where successful preliminary real-time computational data is also provided for this "very fast" system (with a sampling period well below one millisecond). This opens up a complete new application domain, next to the traditional process control for typically slow systems.

8.2 Ideas for future research

There are several interesting research directions possible on the basis of the results presented in this thesis. In what follows we will briefly present some future lines of research that can be pursued.

8.2.1 Stability Theory of Hybrid Systems

In this thesis we have considered PWA systems with the switching triggered by the state only. In practice however, it is also the case that the switching between the affine sub-systems is triggered by both the state and the input. The extension of the stability results developed in Chapter 3 and Chapter 4 is not immediate. It would be of interest to investigate how the developed stabilization results for PWA systems can be extended to this case.

Another practically relevant, more general class of PWA systems is the one where the affine sub-models can have different state dimensions. This situation arises, for example, in optimal control of energy management of electrical power networks that have limited transporting capacities, as multi-dimensional PWA systems provide a good way to approximate the associated nonlinear costs (Jokic et al., 2006). Therefore, it is worth exploring how the stability results presented in this thesis can be transferred to such multi-dimensional PWA systems.

For simplicity and clarity of exposition, in Chapter 4 we employed piecewise quadratic candidate ISpS (ISS) Lyapunov functions. However, the results can be extended to piecewise polynomial or piecewise affine candidate ISpS (ISS) Lyapunov functions. Extensions to PWA systems affected by time-varying parametric uncertainties and the use of norm based candidate ISS Lyapunov functions are also of interest.

8.2.2 Set Invariance Theory

Recently, an increased interest has been shown in the development of efficient numerical algorithms for computing polyhedral robust invariant sets for perturbed linear and PWA systems. In this thesis we have presented a number of approaches to address this problem. One of them, that uses norm Lyapunov functions to induce polyhedral invariant sets, is of particular interest. In Chapter 6 we have presented solutions for the computation of norm based ISS Lyapunov functions. It would be interesting to investigate how norm based ISS Lyapunov functions can be employed to induce robust invariant sets for perturbed linear and PWA systems.

8.2.3 Robust Model Predictive Control

A topic of great interest to robust model predictive control is how to design computationally efficient feedback min-max MPC schemes. Several solutions exist for linear and continuous piecewise affine systems that either rely on numerical solutions for solving the standard feedback min-max MPC optimization problem presented in Chapter 6, or on new ways for introducing feedback to the disturbance. The challenge is to design computationally efficient feedback min-max MPC algorithms for *discontinuous* PWA systems. The corresponding MPC optimization problem is numerically too complex in the hybrid case, so the route of searching for new methods to introduce feedback to the disturbance seems more appropriate.

The inherent drawback of the robust sub-optimal MPC algorithms presented in this thesis is that recursive feasibility of the optimization problem is difficult to be guaranteed a priori. Therefore, it is of great interest to develop methods for estimating positively invariant regions of feasibility for the sub-optimal MPC schemes presented in Chapter 7. This seems to be possible for the sub-optimal MPC algorithm based on an ∞-norm artificial Lyapunov function. The sublevel sets of this function can be used in combination with the computation of robust one-step controllable sets to obtain a priori a polyhedral robust invariant set of feasible states for the closed-loop system.

The real-life implementation of the predictive controller for DC-DC converters will be pursued in the near future by employing the LP based nonlinear MPC controller on a DSP.

Bibliography

Alamo, T., Muñoz de la Peña, D., Limon, D., Camacho, E. F., 2005. Constrained min-max predictive control: Modifications of the objective function leading to polynomial complexity. IEEE Transactions on Automatic Control 50 (5), 710–714.

Alessio, A., Bemporad, A., Addis, B., Pasini, A., 2005. An algorithm for PWL approximations of nonlinear functions. Tech. rep., Universita di Siena, Italy, web: http://control.dii.unisi.it/research/ABAP05.pdf.

Alessio, A., Lazar, M., Bemporad, A., Heemels, W. P. M. H., 2006. Squaring the circle: An algorithm for obtaining polyhedral invariant sets from ellipsoidal ones, provisionally accepted for publication in Automatica.

Baotic, M., Christophersen, F. J., Morari, M., 2003. Infinite time optimal control of hybrid systems with a linear performance index. In: 42nd IEEE Conference on Decision and Control. Maui, Hawaii, pp. 3191–3196.

Bemporad, A., 2003. Hybrid Toolbox-User's Guide. Toolbox available on-line at http://www.dii.unisi.it/hybrid/toolbox.

Bemporad, A., 2004. Efficient conversion of mixed logical dynamical systems into an equivalent piecewise affine form. IEEE Transactions on Automatic Control 49 (5), 832–838.

Bemporad, A., Borodani, P., Mannelli, M., 2003a. Hybrid control of an automotive robotized gearbox for reduction of consumptions and emissions. In: Hybrid Systems: Computation and Control. Vol. 2623 of Lecture Notes in Computer Science. Springer Verlag, pp. 81–96.

Bemporad, A., Borrelli, F., Morari, M., 2000. Optimal controllers for hybrid systems: Stability and piecewise linear explicit form. In: 39th IEEE Conference on Decision and Control. Sydney, Australia, pp. 1810–1815.

Bemporad, A., Borrelli, F., Morari, M., 2003b. Min-max control of constrained uncertain discrete-time linear systems. IEEE Transactions on Automatic Control 48 (9), 1600–1606.

Bemporad, A., Giorgetti, N., 2003. A logic-based hybrid solver for optimal control of hybrid systems. In: 42nd IEEE Conference on Decision and Control. Vol. 1. pp. 640–645.

Bemporad, A., Morari, M., 1999. Control of systems integrating logic, dynamics, and constraints. Automatica 35 (3), 407–427.

Blanchini, F., 1994. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. IEEE Transactions on Automatic Control 39 (2), 428–433.

Blanchini, F., 1995. Nonquadratic Lyapunov functions for robust control. Automatica 31 (3), 451–461.

Blanchini, F., Mesquine, F., Miani, S., 1995. Constrained stabilization with an assigned initial condition set. International Journal of Control 62 (3), 601–617.

Borrelli, F., 2003. Constrained optimal control of linear and hybrid systems. Vol. 290 of Lecture Notes in Control and Information Sciences. Springer.

Boyd, S., El Ghaoui, L., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequalities in Control Theory. Vol. 15 of Studies in Applied Mathematics. SIAM.

Branicky, M. S., Borkar, V. S., Mitter, S. K., 1998. A unified framework for hybrid control: model and optimal control theory. IEEE Transactions on Automatic Control 43 (1), 31–45.

Brogliato, B., Niculescu, S. I., Orthant, P., 1997. On the control of finite-dimensional mechanical systems with unilateral constraints. IEEE Transactions on Automatic Control 42, 200–215.

Brogliato, B., Rio, A. Z., 2000. On the control of complementary-slackness mechanical juggling systems. IEEE Transactions on Automatic Control 45, 235–246.

Cai, C., Teel, A., 2005. Results on input-to-state stability for hybrid systems. In: Proceedings of the 44th IEEE Conference on Decision and Control. Seville, Spain, pp. 5403–5408.

Camacho, E. F., Bordons, C., 2004. Model Predictive Control. Springer-Verlag, London.

Cassandras, C. G., Pepyne, D. L., Wardi, Y., 2001. Optimal control of a class of hybrid systems. IEEE Transactions on Automatic Control 46 (3), 398–415.

Chen, H., Allgöwer, F., 1996. A quasi-infinite horizon predictive control scheme for constrained nonlinear systems. In: 16th Chinese Control Conference. Quindao, China, pp. 309–316.

Christophersen, F. J., Baotic, M., Morari, M., 2004. Stability analysis of hybrid systems with a linear performance index. In: 43rd IEEE Conference on Decision and Control. Paradise Island, Bahamas, pp. 4589–4594.

Clarke, D. W., Mohtadi, C., Tuffs, P. S., 1987. Generalized predictive control: I - the basic algorithm and II - Extensions and interpretations. Automatica 23, 137–160.

Corona, D., Lazar, M., De Schutter, B., Heemels, W. P. M. H., 2006. A hybrid MPC approach to the design of a Smart adaptive cruise controller. In: IEEE Conference on Control Applications. Munich, Germany.

Cutler, C. R., Ramaker, B. L., 1980. Dynamic matrix control - A computer control algorithm. In: Joint Automatic Control Conference. San Francisco, U.S.A.

Daafouz, J., Riedinger, P., Iung, C., 2002. Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach. IEEE Transactions on Automatic Control 47 (11), 1883–1887.

De Keyser, R. M. C., van Cauwenberghe, A. R., 1985. Extended prediction self-adaptive control. In: IFAC Symposium on Identification and System Parameter Estimation. York, U.K., pp. 1255–1260.

De Schutter, B., van den Boom, T. J. J., 2001. Model predictive control for max-plus-linear discrete event systems. Automatica 37 (7), 1049–1056.

Drulhe, S., Ferrari-Trecate, G., de Jong, H., Viari, A., 2006. Reconstruction of switching thresholds in piecewise-affine models of genetic regulatory networks. In: Hybrid Systems: Computation and Control. Vol. 3927 of Lecture Notes in Computer Science. Springer Verlag, pp. 184–199.

Feng, G., 2002. Stability analysis of piecewise discrete-time linear systems. IEEE Transactions on Automatic Control 47 (7), 1108–1112.

Ferrari-Trecate, G., Cuzzola, F. A., Mignone, D., Morari, M., 2002. Analysis of discrete-time piecewise affine and hybrid systems. Automatica 38 (12), 2139–2146.

Findeisen, R., Imsland, L., Allgöwer, F., Foss, B. A., 2003. State and output feedback nonlinear model predictive control: An overview. European Journal of Control 9 (2–3), 190–206.

Freeman, H., 1965. Discrete-time systems. John Wiley & Sons, Inc.

Garcia, C. E., Prett, D. M., Morari, M., 1989. Model predictive control: theory and practice - a survey. Automatica 25 (3), 335–348.

Geyer, T., Papafotiou, G., Morari, M., 2005. Model Predictive Control in Power Electronics: A Hybrid Systems Approach. In: IEEE Conference on Decision and Control. Seville, Spain.

Goebel, R., Teel, A. R., 2006. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. Automatica 42 (4), 573–587.

Golub, G. H., Van Loan, C. F., 1989. Matrix computations. The Johns Hopkins University Press.

Goodwin, G. C., Seron, M. M., De Dona, J. A., 2005. Constrained control and estimation An optimization approach. Communications and control engineering. Springer.

Grieder, P., 2004. Efficient computation of feedback controllers for constrained systems. Ph.D. thesis, Swiss Federal Institute of Technology (ETH) Zürich, Switzerland.

Grieder, P., Kvasnica, M., Baotic, M., Morari, M., 2005. Stabilizing low complexity feedback control of constrained piecewise affine systems. Automatica 41 (10), 1683–1694.

Grimm, G., Messina, M. J., Tuna, S., Teel, A. R., 2005. Model predictive control: for want of a local control Lyapunov function, all is not lost. IEEE Transactions on Automatic Control 50 (5), 546–558.

Grimm, G., Messina, M. J., Tuna, S. E., Teel, A. R., 2003a. Examples of zero robustness in constrained model predictive control. In: 42nd IEEE Conference on Decision and Control. Maui, Hawaii, pp. 3724–3729.

Grimm, G., Messina, M. J., Tuna, S. E., Teel, A. R., 2003b. Nominally robust model predictive control with state contraints. In: 42nd IEEE Conference on Decision and Control. Maui, Hawaii, pp. 1413–1418.

Grimm, G., Messina, M. J., Tuna, S. E., Teel, A. R., 2004. Examples when nonlinear model predictive control is nonrobust. Automatica 40 (10), 1729–1738.

Gutman, P. O., Cwikel, M., 1987. An algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states. IEEE Transactions on Automatic Control AC-32 (3), 251–254.

Hahn, W., 1958. On the application of the method of Lyapunov to difference equations (in german). Mathematische Annalen 136 (5), 403–441.

Hahn, W., 1967. Stability of motion. Springer-Verlag.

Heemels, W. P. M. H., De Schutter, B., Bemporad, A., 2001. Equivalence of hybrid dynamical models. Automatica 37 (7), 1085–1091.

Heemels, W. P. M. H., Schumacher, J. M., Weiland, S., 2000. Linear complementarity systems. SIAM journal on applied mathematics 60 (4), 1234–1269.

Hespanha, J. P., Liberzon, D., Teel, A. R., 2005. On input-to-state stability of impulsive systems. In: Proceedings of the 44th IEEE Conference on Decision and Control. Seville, Spain, pp. 3992–3997.

Hu, T., Lin, Z., Chen, M., 2002. Analysis and design for discrete-time linear systems subject to actuator saturation. Systems and Control Letters 45, 97–112.

Jiang, Z.-P., 1993. Quelques résultats de stabilisation robuste. Application á la commande (in french). Ph.D. thesis, École des Mines de Paris, France.

Jiang, Z.-P., Lin, Y., Wang, Y., 2004. Nonlinear small-gain theorems for discrete-time feedback systems and applications. Automatica 40, 2129–2136.

Jiang, Z.-P., Mareels, I., Wang, Y., 1996. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. Automatica 32 (8), 1211–1215.

Jiang, Z.-P., Teel, A. R., Praly, L., 1994. Small gain theorem for ISS systems and applications. Journal of Mathematics of Control, Signals and Systems 7, 95–120.

Jiang, Z.-P., Wang, Y., 2001. Input-to-state stability for discrete-time nonlinear systems. Automatica 37, 857–869.

Jiang, Z.-P., Wang, Y., 2002. A converse Lyapunov theorem for discrete-time systems with disturbances. Systems & Control Letters 45, 49–58.

Johansson, M., Rantzer, A., 1998. Computation of piecewise quadratic Lyapunov functions for hybrid systems. IEEE Transactions on Automatic Control 43 (4), 555–559.

Jokic, A., Wittebol, E. H. M., van den Bosch, P. P. J., 2006. Dynamic market behavior of autonomous network-based power systems. European transactions on electrical power 16, 1–12.

Kalman, R. E., Bertram, J. E., 1960a. Control system analysis and design via the second method of Lyapunov, I: Continuous-time systems. Transactions of the ASME, Journal of Basic Engineering 82, 371–393.

Kalman, R. E., Bertram, J. E., 1960b. Control system analysis and design via the second method of Lyapunov, II: Discrete-time systems. Transactions of the ASME, Journal of Basic Engineering 82, 394–400.

Kassakian, J. G., Schlecht, M. F., Verghese, G. C., 1992. Principles of Power Electronics. Adisson-Wesley, Publishing Company, Inc.

Keerthi, S. S., Gilbert, E. G., 1988. Optimal, infinite horizon feedback laws for a general class of constrained discrete time systems: Stability and moving-horizon approximations. Journal of Optimization Theory and Applications 57 (2), 265–293.

Kellett, C. M., Teel, A. R., 2004. Smooth Lyapunov functions and robustness of stability for difference inclusions. Systems & Control Letters 52, 395–405.

Kerrigan, E. C., Maciejowski, J. M., 2001. Robust feasibility in model predictive control: Necessary and sufficient conditions. In: 40th IEEE Conference on Decision and Control. Orlando, Florida, pp. 728–733.

Kerrigan, E. C., Maciejowski, J. M., 2004. Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. International Journal of Robust and Nonlinear Control 14, 395–413.

Kerrigan, E. C., Mayne, D. Q., 2002. Optimal control of constrained, piecewise affine systems with bounded disturbances. In: 41st IEEE Conference on Decision and Control. Las Vegas, Nevada, pp. 1552–1557.

Khalil, H., 2002. Nonlinear Systems, Third Edition. Prentice Hall.

Kolmanovsky, I., Gilbert, E. G., 1998. Theory and computation of disturbance invariant sets for discrete-time linear systems. Mathematical Problems in Engineering 4, 317–367.

Kothare, M. V., Balakrishnan, V., Morari, M., 1996. Robust constrained model predictive control using linear matrix inequalities. Automatica 32 (10), 1361–1379.

Kvasnica, M., Grieder, P., Baotic, M., Morari, M., 2004. Multi Parametric Toolbox (MPT). In: Hybrid Systems: Computation and Control. Lecture Notes in Computer Science, Volume 2993. Springer Verlag, Pennsylvania, Philadelphia, USA, pp. 448–462, toolbox available for download at http://control.ee.ethz.ch/∼mpt.

LaSalle, J. P., 1976. The stability of dynamical systems. In: SIAM (Ed.), Regional Conference Series in Applied Mathematics. No. 25. Philadelphia.

Lazar, M., Alessio, A., Bemporad, A., Heemels, W. P. M. H., 2006a. Squaring the circle: An algorithm for obtaining polyhedral invariant sets from ellipsoidal ones. In: 25th American Control Conference. Minneapolis, Minnessota, pp. 3007–3012.

Lazar, M., De Keyser, R., 2004. Nonlinear predictive control of a DC-to-DC converter. In: Symposium on Power Electronics, Electrical Drives, Automation & Motion - SPEEDAM. Capri, Italy.

Lazar, M., Heemels, W. P. M. H., 2003. A semi-explicit MPC set-up for constrained piecewise affine systems. In: European Control Conference. Cambridge, UK.

Lazar, M., Heemels, W. P. M. H., 2006a. Global input-to-state stability and stabilization of discrete-time piece-wise affine systems. In: 2nd IFAC Conference on Analysis and Design of Hybrid Systems. Alghero, Italy, pp. 296–301.

Lazar, M., Heemels, W. P. M. H., 2006b. Global input-to-state stability and stabilization of discrete-time piecewise affine systems, submitted to a journal.

Lazar, M., Heemels, W. P. M. H., 2006c. A new dual-mode hybrid MPC scheme with a robust stability guarantee. In: 2nd IFAC Conference on Analysis and Design of Hybrid Systems. Alghero, Italy, pp. 321–328.

Lazar, M., Heemels, W. P. M. H., 2006d. Robust stabilization of discontinuous piecewise affine systems using model predictive control, submitted to a journal.

Lazar, M., Heemels, W. P. M. H., Bemporad, A., Weiland, S., 2005a. On the stability and robustness of non-smooth nonlinear model predictive control. In: Workshop on Assessment and Future Directions of NMPC. Freudenstadt-Lauterbad, Germany, pp. 327–334.

Lazar, M., Heemels, W. P. M. H., Weiland, S., Bemporad, A., 2004a. Stabilization conditions for model predictive control of constrained PWA systems. In: 43rd IEEE Conference on Decision and Control. Paradise Island, Bahamas, pp. 4595–4600.

Lazar, M., Heemels, W. P. M. H., Weiland, S., Bemporad, A., 2004b. Stabilizing receding horizon control of PWL systems: An LMI approach. In: 16th Symposium on Mathematical Theory of Networks and Systems. Leuven, Belgium.

Lazar, M., Heemels, W. P. M. H., Weiland, S., Bemporad, A., 2005b. On the stability of quadratic forms based Model Predictive Control of constrained PWA systems. In: 24th American Control Conference. Portland, Oregon, pp. 575–580.

Lazar, M., Heemels, W. P. M. H., Weiland, S., Bemporad, A., 2006b. Stabilizing model predictive control of hybrid systems, to appear in IEEE Transactions on Automatic Control.

Lazar, M., Heemels, W. P. M. H., Weiland, S., Bemporad, A., Pastravanu, O., 2005c. Infinity norms as Lyapunov functions for model predictive control of constrained PWA systems. In: Hybrid Systems: Computation and Control. Vol. 3414 of Lecture Notes in Computer Science. Springer Verlag, Zürich, Switzerland, pp. 417–432.

Lazar, M., Muñoz de la Peña, D., Heemels, W. P. M. H., Alamo, T., 2006c. Min-max nonlinear model predictive control with guaranteed input-to-state stability. In: 17th Symposium on Mathematical Theory for Networks and Systems. Kyoto, Japan.

Lazar, M., Muñoz de la Peña, D., Heemels, W. P. M. H., Alamo, T., 2006d. On input-to-state stability of min-max nonlinear model predictive control, submitted to a journal.

Lazar, M., Muñoz de la Peña, D., Heemels, W. P. M. H., Alamo, T., 2006e. Stabilizing feedback min-max linear model predictive control: New methods for computing the terminal cost, submitted to a conference.

Lazar, M., Roset, B. J. P., Heemels, W. P. M. H., Nijmeijer, H., van den Bosch, P. P. J., 2006f. Input-to-state stabilizing sub-optimal nonlinear MPC algorithms with an application to DC-DC converters. In: IFAC Workshop on Nonlinear Model Predictive Control for Fast Systems. Grenoble, France.

Lee, J. H., Yu, Z., 1997. Worst-case formulations of model predictive control for systems with bounded parameters. Automatica 33 (5), 763–781.

Leenaerts, D. M. W., 1996. Further extensions to Chua's explicit piecewise linear function descriptions. International Journal of Circuit Theory and Applications 24, 621–633.

Li, T., 1934. The stability problem in difference equations (in german). Acta Mathematica 63, 99–141.

Liberzon, D., 2003. Switching in systems and control. Birkhauser, Boston.

Limon, D., Alamo, T., Camacho, E. F., 2002a. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties. In: 41st IEEE Conference on Decision and Control. Las Vegas, Nevada, pp. 4619–4624.

Limon, D., Alamo, T., Camacho, E. F., 2002b. Stability analysis of systems with bounded additive uncertainties based on invariant sets: Stability and feasibility of MPC. In: American Control Conference. Anchorage, pp. 364–369.

Limon, D., Alamo, T., Salas, F., Camacho, E. F., 2006. Input to state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. Automatica 42, 797–803.

Löfberg, J., 2002. YALMIP: Matlab toolbox for rapid prototyping of optimization problems, web: http://control.ee.ethz.ch/ joloef/yalmip.msql.

Lyapunov, A. M., 1907. Problème général de la stabilité du mouvement (in french). Ann. Fac. Sci. Toulouse 9, 203–474, reprinted in Ann. Math. Study No. 17, 1949, Princeton University Press.

Lygeros, J., Johansson, K. H., Simic, S. N., Zhang, J., Sastry, S., 2003. Dynamical properties of hybrid automata. IEEE Transactions on Automatic Control 48, 2–17.

Magni, L., De Nicolao, G., Magnani, L., Scattolini, R., 2001. A stabilizing model-based predictive control algorithm for nonlinear systems. Automatica 37 (9), 1351–1362.

Magni, L., De Nicolao, G., Scattolini, R., 1998. Output feedback receding-horizon control of discrete-time nonlinear systems. In: 4th IFAC NOLCOS. Vol. 2. Oxford, UK, pp. 422–427.

Magni, L., De Nicolao, G., Scattolini, R., Allgöwer, F., 2003. Robust model predictive control for nonlinear discrete-time systems. International Journal of Robust and Nonlinear Control 13, 229–246.

Mayne, D. Q., 2001. Control of constrained dynamic systems. European Journal of Control 7, 87–99.

Mayne, D. Q., Rakovic, S. V., 2003. Model predictive control of constrained piecewise affine discrete-time systems. International Journal of Robust and Nonlinear Control 13, 261–279.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., Scokaert, P. O. M., 2000. Constrained model predictive control: Stability and optimality. Automatica 36 (6), 789–814.

Mayne, D. Q., Seron, M. M., Rakovic, S. V., 2004. Robust model predictive control of constrained linear systems with bounded disturbances. Automatica 41, 219–224.

Messina, M. J., Tuna, S. E., Teel, A. R., 2005. Discrete-time certainty equivalence output feedback: allowing discontinuous control laws including those from model predictive control. Automatica 41, 617–628.

Michalska, H., Mayne, D. Q., 1993. Robust receding horizon control of constrained nonlinear systems. IEEE Transactions on Automatic Control 38 (11), 1623–1633.

Mignone, D., Ferrari-Trecate, G., Morari, M., 2000a. Stability and stabilization of piecewise affine and hybrid systems: An LMI approach. In: 39th IEEE Conference on Decision and Control. pp. 504–509.

Mignone, D., Ferrari-Trecate, G., Morari, M., 2000b. Stability and stabilization of piecewise affine systems: An LMI approach. Tech. Rep. AUT00-12, Automatic Control Laboratory, ETH Zürich, Switzerland.

Mosca, E., Zappa, G., Manfredi, C., 1984. Multistep horizon self-tuning controllers. In: 9th IFAC World Congress. Budapest, Hungary.

Muñoz de la Peña, D., Alamo, T., Bemporad, A., December 2005. A decomposition algorithm for feedback min-max model predictive control. In: Proceedings of the 44rd IEEE Conference on Decision and Control. Seville, Spain.

Muñoz de la Peña, D., Alamo, T., Ramirez, D. R., Camacho, E. F., July 2005. Min-max model predictive control as a quadratic program. In: Proceedings of the 16th IFAC World Congress. Prague, Czech Republic.

Musters, M. W. J. M., van Riel, N. A. W., 2004. Analysis of the transformating growth factor-beta1 pathway and extracellular matrix formation as a hybrid system. In: 26th Conference of the IEEE Engineering in Medicine and Biology Society. San Francisco, California, pp. 2901–2904.

Pfeiffer, F., Glocker, C., 1996. Multibody dynamics with unilateral contacts. Wiley, Chichester.

Pluymers, B., Rossiter, J., Suykens, J., De Moor, B., 2005. Interpolation based MPC for LPV systems using polyhedral invariant sets. In: American Control Conference. Portland, Oregon, pp. 810–815.

Polanski, A., 1995. On infinity norms as Lyapunov functions for linear systems. IEEE Transactions on Automatic Control 40 (7), 1270–1273.

Prajna, S., Papachristodoulou, A., 2003. Analysis of switched and hybrid systems - beyond piecewise qudratic methods. In: 22nd American Control Conference. Denver, Colorado, pp. 2779–2784.

Qin, S. J., Badgwell, T. A., 2003. A survey of industrial model predictive control technology. Control Engineering Practice 11, 733–764.

Rakovic, S. V., Grieder, P., Kvasnica, M., Mayne, D. Q., Morari, M., 2004. Computation of invariant sets for piecewise affine discrete time systems subject to bounded disturbances. In: 43nd IEEE Conference on Decision and Control. Paradise Island, Bahamas, pp. 1418–1423.

Rakovic, S. V., Mayne, D. Q., 2004. Robust model predictive control of contrained piecewise affine discrete time systems. In: 6th IFAC NOLCOS. Stuttgart, Germany.

Richalet, J. A., Rault, A., Testud, J. L., Papon, J., 1978. Model predictive heuristic control: applications to an industrial process. Automatica 14, 413–428.

Roset, B. J. P., Lazar, M., Nijmeijer, H., Heemels, W. P. M. H., 2006a. A stabilizing output based nonlinear model predictive control scheme, submitted to a conference.

Roset, B. J. P., Lazar, M., Nijmeijer, H., Heemels, W. P. M. H., 2006b. Stabilizing output feedback nonlinear model predictive control: An extended observer approach. In: 17th Symposium on Mathematical Theory for Networks and Systems. Kyoto, Japan.

Scokaert, P. O. M., Mayne, D. Q., 1998. Min-max feedback model predictive control for constrained linear systems. IEEE Transactions on Automatic Control 43 (8), 1136–1142.

Scokaert, P. O. M., Mayne, D. Q., Rawlings, J. B., 1999. Suboptimal model predictive control (feasibility implies stability). IEEE Transactions on Automatic Control 44 (3), 648–654.

Scokaert, P. O. M., Rawlings, J. B., Meadows, E. B., 1997. Discrete-time stability with perturbations: Application to model predictive control. Automatica 33 (3), 463–470.

Soeterboek, R., 1992. Predictive Control - A unified approach. Englewood Cliffs, NJ: Prentice-Hall.

Sontag, E. D., 1981. Nonlinear regulation: the piecewise linear approach. IEEE Transactions on Automatic Control 26 (2), 346–357.

Sontag, E. D., 1989. Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control AC–34, 435–443.

Sontag, E. D., 1990. Further facts about input to state stabilization. IEEE Transactions on Automatic Control AC–35, 473–476.

Sontag, E. D., 1996. Interconnected automata and linear systems: A theoretical framework in discrete-time. In: Alur, R., Henzinger, T. A., Sontag, E. D. (Eds.), Hybrid Systems III: Verification and Control. Springer, New York, pp. 436–448.

Sontag, E. D., Wang, Y., 1995. On characterizations of the input-to-state stability property. Systems and Control Letters 24, 351–359.

Sturm, J. F., 2001. SeDuMi: Matlab toolbox for solving optimization problems over symmetric cones, toolbox available online at: http://sedumi.mcmaster.ca/.

van der Schaft, A. J., Schumacher, J. M., 1998. Complementarity modeling of hybrid systems. IEEE Transactions on Automatic Control 43 (3), 483–490.

van der Schaft, A. J., Schumacher, J. M., 2000. An introduction to hybrid dynamical systems. Vol. 251 of Lecture Notes in Control and Information Sciences. Springer.

Vasak, M., Baotic, M., Morari, M., Petrovic, I., Peric, N., 2006. Constrained optimal control of an electronic throttle. International Journal of Control 79 (5), 465–478.

Veres, S. M., 1995. Geometric Bounding Toolbox (GBT) for Matlab. Official website: http://www.sysbrain.com.

Vidyasagar, M., 1993. Nonlinear Systems Analysis (Second Edition). Prentice-Hall, Englewood Cliffs, NJ.

Vu, L., Chatterjee, D., Liberzon, D., 2005. ISS of switched systems and applications to switching adaptive control. In: 44th IEEE Conference on Decision and Control. Seville, Spain, pp. 120–125.

Willems, J. L., 1970. Stability theory of dynamical systems. Thomas Nelson and Sons, London, England.

Witsenhausen, H. S., 1966. A class of hybrid-state continuous-time dynamic systems. IEEE Transactions on Automatic Control 11 (2), 161–167.

Yepremyan, L., Falk, J. E., 2005. Delaunay partitions in $\mathbb{R}^n$ applied to non-convex programs and vertex/facet enumeration problems. Computers & Operations Research 32, 793–812.

Samenvatting

Model gebaseerd voorspellend regelen voor hybride systemen - stabiliteit en robuustheid -

Dit proefschrift behandelt de stabilisatie en de robuuste stabilisatie van zogenaamde hybride systemen door middel van model gebaseerd voorspellend regelen oftewel "Model Predictive Control" (MPC). Hybride systemen zijn in hun algemeenheid dynamische systemen, die zowel een discreet (vaak beschreven middels een eindige automaat) als een continu (beschreven door differentie- of differentiaalvergelijkingen) karakter hebben. Deze systemen ontstaan onder meer door tijd-continue processen te koppelen aan digitale tijd-asynchrone regelsystemen (embedded systemen). Andere voorbeelden van hybride dynamica worden gevonden in veel toepassingsgebieden en onderzoekspecialismen, zoals schakelende elektrische netwerken, mechanische en biologische processen, economie, etc. Deze systemen zijn inherent niet-lineair en discontinu. Bovendien vertonen ze verschillende werkgebieden (ook wel "modes" of "discrete toestanden" genoemd), die elk hun eigen karakteristieke bewegingsvergelijkingen hebben. Dientengevolge zijn de bestaande technieken voor stabiliteitsanalyse en synthese van (robuuste) stabiliserende regelaars voor lineaire en niet-lineaire (gladde) systemen niet toepasbaar in de hybride context. Dit vormt een belangrijke motivatie voor het ontwikkelen van een nieuwe systematische methode van regelaarontwerp, die om kan gaan met discontinu systeem gedrag.

Model gebaseerd voorspellend regelen (kortweg MPC) is een regelstrategie, die zeer aantrekkelijk is voor het regelen van tijd-discrete lineaire of niet-lineaire systemen en wordt al succesvol toegepast in de industrie, met name in de proces industrie. MPC voorspelt op basis van een model van het te regelen proces wat het toekomstig effect van een keuze van ingangssignalen over een bepaalde (vaak eindige) tijdshorizon is en selecteert vervolgens een optimaal ingangssignaal op basis van een kostenfunctie, die geminimaliseerd dient te worden. De eerste ingangswaarde in het optimale ingangssignaal wordt vervolgens geïmplementeerd op het proces en op basis van nieuwe metingen van uitgangssignalen van het proces wordt het optimalisatieprobleem opnieuw opgelost. Een van de redenen van het succes van MPC is de mogelijkheid om te gaan met harde beperkingen op de regelingangen, toestanden en uitgangssignalen van het systeem. Onmisbaar voor praktische

implementatie zijn de stabiliteit en robuustheid van de regellus. Deze twee eigenschappen vormen dan ook de eigenschappen van MPC regelaars, die het meest bestudeerd zijn en een complete theorie is dan ook voorhanden in de context van lineaire en (gladde) niet-lineaire systemen. Echter, deze resultaten zijn niet overdraagbaar naar de situatie van hybride systemen op een eenvoudige wijze. Dit vormt dan ook een van de uitdagingen, die onderwerp van studie zijn in dit proefschrift.

Als startpunt wordt in hoofdstuk 2 een theoretische kader ontwikkeld voor stabiliteit en ingang-naar-toestand stabiliteit ("input-to-state stability" afgekort tot ISS), dat om kan gaan met discontinue en niet-lineaire dynamica. Deze resultaten vormen de theoretische basis van het proefschrift, die het mogelijk maakt om voor hybride systemen en verschillende MPC strategieën (robuuste) stabiliteit te onderzoeken.

Het (nominale) stabiliteitsprobleem voor hybride systemen geregeld door MPC wordt in zijn algemeenheid opgelost in hoofdstuk 3. De focus is vervolgens op een belangrijke klasse van hybride systemen, die stuksgewijs lineaire systemen ("piecewise affine" systemen, afgekort tot PWA systemen) genoemd worden. Deze klasse van hybride systemen is zeer aantrekkelijk, daar ze van de ene kant een eenvoudige mathematische structuur heeft en van de andere kant een verscheidenheid aan praktisch relevante processen kan modelleren. Voor PWA systemen en bepaalde keuzes van MPC kostenfuncties worden nieuwe algoritmes gepresenteerd voor het berekenen van de kosten aan het eind van de horizon ("terminal costs"), die aan de ontwikkelde stabilisatie condities voldoen. Voor de bepaling van de toestandsbeperkingen ("terminal set"), die opgelegd dienen te worden aan de toestand aan het eind van de horizon, worden ook nieuwe methodes gepresenteerd. Hierbij wordt er nadrukkelijk opgelet dat de complexiteit van deze toegevoegde toestandsbeperkingen eenvoudig zijn. Zeker ten aanzien van bestaande algoritmes wordt een significante reductie in complexiteit bewerkstelligd. Dit betekent dat naast de studie van de fundamentele eigenschappen van stabiliteit en robuustheid, ook het doel is om regelalgoritmen te ontwikkelen met een lage complexiteit zodanig dat praktische implementatie haalbaar is.

Voordat we robuuste stabilisatie bekijken in hoofdstuk 5, worden twee voorbeelden beschreven in hoofdstuk 4. Deze voorbeelden laten twee PWA systemen zien die beiden een discontinue stuksgewijs kwadratische Lyapunov functie toelaten en beiden exponentieel stabiel zijn. Echter, een van de twee is globaal ISS, terwijl de andere geen enkele robuustheid heeft. Deze voorbeelden laten zien dat men behoedzaam moet zijn om nominaal stabiliserende regelingen in de praktijk toe te passen, aangezien het gebrek aan robuuste stabiliteit ongewenste effecten kan opleveren. Voor het robuuste voorbeeld is het mogelijk om ISS aan te tonen middels een discontinue stuksgewijs kwadratische Lyapunov functie, maar niet via een continue. Dit impliceert dat het gebruik van continue Lyapunov functies conservatief is in de studie van hybride systemen en dat er dus een noodzaak is voor een framework waarin discontinue Lyapunov robuuste stabiliteit kunnen aantonen. Dit is het onderwerp van hoofdstuk 4. Voor PWA systemen worden er dan ook condities afgeleid voor globale ISS in termen van lineaire matrix ongelijkheden, die efficient opgelost kunnen worden. Deze kunnen zowel gebruikt worden voor de analyse van de robuuste stabiliteit voor geregelde PWA systemen als het synthetiseren van stuksgewijs lineaire toestandsterugkoppeling, die ISS garanderen.

In hoofdstuk 5 wordt het probleem van robuuste stabilisatie van PWA systemen middels MPC bestudeerd. In de bestaande MPC literatuur is dit probleem alleen opgelost voor het geval waarbij de dynamica van het PWA systeem continu verondersteld is. Dit is een zeer beperkende aanname in het geval van hybride systemen. In hoofdstuk 5 wordt dan ook een ISS MPC strategie voorgesteld voor PWA systemen gebaseerd op aanscherping van toestandsbeperkingen. Het voordeel van deze nieuwe aanpak - naast het feit dat het de eerste robuust stabiliserende MPC strategie voor discontinue PWA systemen is - is dat de resulterende MPC optimalisatieproblemen nog steeds via standaard numerieke algoritmen als toegepast binnen de hybride MPC opgelost kunnen worden.

Een zogenaamde "min-max" aanpak van robuuste stabilisatie van algemene niet-lineaire systemen middels MPC is het onderwerp van hoofdstuk 6. Alhoewel min-max MPC rekentechnisch intensiever is, levert deze terugkoppeling aan de verstoring, hetgeen kan leiden tot beter prestaties van het geregelde systemen, wanneer deze aan verstoringen onderhevig is. In het algemeen kan slechts ingang-naar-toestand praktische stabiliteit ("input-to-state practical stability" of kortweg ISpS) realiseren voor verstoorde niet-lineaire systemen geregeld door min-max MPC strategieën. Echter, door gebruik te maken van een geschakelde regelstructuur (zogenaamde "dual-mode") kan toch ISS behaald worden. Ook hier worden nieuwe methodes gepresenteerd waarmee de toe te voegen kosten en de toestandsbeperkingen aan de eind van de horizon efficient berekend kunnen worden.

Het laatste deel van het proefschrift bekijkt het ontwerp van robuust stabiliserende sub-optimale MPC regelingen met een lage complexiteit ten aanzien van on-line rekencapaciteit. Dit doel wordt bereikt voor hybride systemen via eenvoudigere stabiliteitscondities, die uitgedrukt kunnen worden als lineaire vergelijkingen. Het potentieel voor praktische applicaties wordt aangetoond via een case study op het gebied van het regelen van Buck-

Boost DC-DC converters, een voorbeeld van een vaak gebruikt schakelend elektrisch circuit. Initiële real-time resultaten zijn bemoedigend, aangezien de MPC optimalisatieproblemen altijd binnen de gewenste sample tijd, die onder de 1 milliseconde ligt, opgelost worden.

Concluderend, dit proefschrift geeft een volledig kader voor het ontwerp van model gebaseerd voorspellende (MPC) regelaars voor hybride systemen, die leiden tot stabiele en robuuste geregelde systemen. Deze eigenschappen zijn onmisbaar voor elke toepassing van deze regelalgoritmen in de praktijk. Een duidelijke nadruk bij het ontwikkelen van de MPC strategieën lag op lage complexiteit van de optimalisatieproblemen, zodanig dat real-time implementatie mogelijk wordt. Het voorbeeld van de DC-DC converter toont aan dit inderdaad mogelijk is voor (zeer) snelle systemen. De ontwikkelde analyse en synthese methodieken voor MPC van hybride systemen bieden dan ook veel perspectief voor toekomstige praktische applicaties.

Curriculum Vitae

Mircea Lazar was born on March 4, 1978 in Iaşi, Romania.

He received the Dipl.Ing. degree in Control Engineering and Industrial Informatics from the Faculty of Automatic Control and Computer Science, Technical University "Gh. Asachi" of Iaşi, Romania in 2001.

He received the Master of Science degree in Control Engineering from the same faculty in 2002 for the thesis "Nonlinear model predictive control".

From September 2002 to September 2006 he pursued his Ph.D. degree with the Control Systems Group at the Faculty of Electrical Engineering, Eindhoven University of Technology, The Netherlands, under the supervision of Prof. Paul van den Bosch. The research conducted in this period is presented in this dissertation.

He received the *Outstanding reviewer* distinction from the *IEEE Transactions on Automatic Control* journal for three years consecutively, starting with 2003.