# MODEL PROBLEMS FROM NONLINEAR ELASTICITY: PARTIAL REGULARITY RESULTS 

Menita Carozza ${ }^{1}$ and Antonia Passarelli di Napoli ${ }^{2}$


#### Abstract

In this paper we prove that every weak and strong local minimizer $u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right)$ of the functional $$
I(u)=\int_{\Omega}|D u|^{2}+f(\operatorname{Adj} D u)+g(\operatorname{det} D u)
$$ where $u: \Omega \subset \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$, $f$ grows like $|\operatorname{Adj} D u|^{p}, g$ grows like $|\operatorname{det} D u|^{q}$ and $1<q<p<2$, is $C^{1, \alpha}$ on an open subset $\Omega_{0}$ of $\Omega$ such that meas $\left(\Omega \backslash \Omega_{0}\right)=0$. Such functionals naturally arise from nonlinear elasticity problems. The key point in order to obtain the partial regularity result is to establish an energy estimate of Caccioppoli type, which is based on an appropriate choice of the test functions. The limit case $p=q \leq 2$ is also treated for weak local minimizers.


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## 1. Introduction

Let us consider integral functionals of the Calculus of Variations of the type

$$
I(u)=\int_{\Omega} F(D u) \mathrm{d} x
$$

where $\Omega \subset \mathbb{R}^{n}$ and $u: \Omega \rightarrow \mathbb{R}^{N}$.
An interesting class of integral functionals which naturally arise from problems of nonlinear elasticity [3] is the one of polyconvex functionals, i.e. functionals in which the integrand is a convex function of the minors of the matrix $[D u]$. It is well known that polyconvex functionals are also quasiconvex, but they often satisfy anisotropic growth conditions which are not recovered by the results concerning the quasiconvex case $[1,5,6,8,9,13,14]$.

For this reason $[10,11]$ have considered polyconvex integrals with anisotropic growth conditions, which are close to the typical examples arising from nonlinear elasticity theory. A model case included in the results of [10] is, for $n=N=3$

$$
I(u)=\int_{\Omega}|D u|^{2}+|D u|^{p}+|\operatorname{Adj} D u|^{p}+|\operatorname{det} D u|^{p}
$$

[^0]where $p>2$. They proved that absolute minimizers of $I(u)$ are $C^{1, \alpha}$ except for a closed subset of $\Omega$ of zero Lebesgue measure. Recall that a function $u$ is an absolute minimizer of $I(u)$ if $I(u) \leq I(u+\varphi)$ for every $\varphi \in C_{o}^{\infty}(\Omega)$.

Motivated by a recent paper of Ball [4], in [7,15] partial regularity has been proved for a new class of minimizers of $I(u)$, when $F$ is a quasiconvex integrand. In particular, they consider $W^{1, \bar{p}}$ local minimizers, defined as follows

Definition 1.1. Let $1 \leq p \leq \bar{p} \leq+\infty$. A map $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ is a $W^{1, \bar{p}}\left(\Omega ; \mathbb{R}^{N}\right)$ local minimizer of $I(v)$, if there exists $\delta>0$ such that $I(u) \leq I(v)$ whenever $v \in u+W_{o}^{1, p}\left(\Omega ; \mathbb{R}^{N}\right)$ and $\|D v-D u\|_{\bar{p}} \leq \delta$.

We will refer to a $W^{1, \bar{p}}\left(\Omega ; \mathbb{R}^{N}\right)$ local minimizer of $I(v)$ with $1 \leq \bar{p}<\infty$ as a strong local minimizer and to a $W^{1, \infty}\left(\Omega ; \mathbb{R}^{N}\right)$ local minimizer as a weak local minimizer. It has been noted in [15] that, if $p=\bar{p}$, the study of partial regularity of strong local minimizers can be reduced to the study of absolute minimizers. For this reason we confine ourselves to the case $\bar{p}>p$. Since in [4] the study of this class of minimizers is proposed for polyconvex integral functionals, as a natural continuation of the results in [15], in this paper we prove $C^{1, \alpha}$ partial regularity for weak and strong local minimizers of polyconvex functionals of the type

$$
\int_{\Omega}|D u|^{2}+f(\operatorname{Adj} D u)+g(\operatorname{det} D u)
$$

where $u: \Omega \subset \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}, f$ grows like $|\operatorname{Adj} D u|^{p}, g$ grows like $|\operatorname{det} D u|^{q}$ and $1<q<p<2$. We mention that, under the same assumptions on $f$ and $g$ of Theorem A below, the regularity result for absolute minimizers has been obtained in [16].

Theorem A. Let us consider the functional

$$
\begin{equation*}
I(v)=\int_{\Omega}|D v|^{2}+f(\operatorname{Adj} D v)+g(\operatorname{det} D v) \tag{1.1}
\end{equation*}
$$

and suppose that $f: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}, g: \mathbb{R} \rightarrow \mathbb{R}$ are $C^{2}$ convex functions satisfying the following assumptions

$$
\begin{gather*}
c_{1}\left(\mu^{2}+|z|^{2}\right)^{\frac{p-2}{2}}|\xi|^{2} \leq f_{z_{i} z_{j}}(z) \xi_{i} \xi_{j} \leq c_{2}\left(\mu^{2}+|z|^{2}\right)^{\frac{p-2}{2}}|\xi|^{2}  \tag{H1}\\
c_{3}\left(\mu^{2}+t^{2}\right)^{\frac{q-2}{2}} \leq g^{\prime \prime}(t) \leq c_{4}\left(\mu^{2}+t^{2}\right)^{\frac{q-2}{2}} \tag{H2}
\end{gather*}
$$

where $1<q<p<2$ and $\mu \geq 0$. Let $2 \leq \bar{p} \leq \infty$ and assume that $u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right) \cap W_{\text {loc }}^{1, \bar{p}}\left(\Omega, \mathbb{R}^{3}\right)$ be a $W^{1, \bar{p}}$ local minimizer of $I(v)$.

If $\bar{p}=\infty$, we assume in addition that

$$
\begin{equation*}
\limsup _{R \rightarrow 0^{+}}\left\|D u-(D u)_{x, R}\right\|_{L^{\infty}(B(x, R))}<\delta \tag{1.2}
\end{equation*}
$$

holds locally uniformly in $x \in \Omega$, with $\delta$ as in Definition 1. Then, there exists $\alpha \in(0,1)$ such that $u \in C^{1, \alpha}\left(\Omega_{0}\right)$ for some open subset $\Omega_{0}$ of $\Omega$ with meas $\left(\Omega \backslash \Omega_{0}\right)=0$.

As far as we know, no results are available, even for absolute minimizers, if $p=q \leq 2$. However, in the special case of weak local minimizers, we are able to deal also with this assumption. Namely we have
Theorem B. Let $f, g$ satisfy the same assumptions as in Theorem $A$ with $1<p=q \leq 2$. Let $u \in$ $W_{\text {loc }}^{1, \infty}\left(\Omega, \mathbb{R}^{3}\right)$ be a weak local minimizer for the functional (1.1) such that

$$
\begin{equation*}
\limsup _{R \rightarrow 0^{+}}\left\|D u-(D u)_{x, R}\right\|_{L^{\infty}\left(B_{R}(x)\right)}<\delta \tag{1.3}
\end{equation*}
$$

holds locally uniformly in $x \in \Omega$, with $\delta$ as in Definition 1. Then, there exists $\alpha \in(0,1)$ such that $u \in C^{1, \alpha}\left(\Omega_{0}\right)$ for some open subset $\Omega_{0}$ of $\Omega$ with $\operatorname{meas}\left(\Omega \backslash \Omega_{0}\right)=0$.

We have restricted our study to the case $n=N=3$, which is the most significant from the point of view of the applications, thus avoiding the heavy technicalities needed for the general case $n \geq 3, N \geq 2$, which in any case can carried on without any new idea. A fundamental tool needed to prove partial regularity is a new Caccioppoli type estimate (see Lem. 2.3).

The difficulties here are twofold. The first one is due to the anisotropic growth of the functional which requires the use of suitable test functions obtained (as in [10]) by interpolating the values of $u$ on the boundary.

The second difficulty comes from the definition of local minimizers which imposes a bound on the $W^{1, \bar{p}}$ norm of the test functions. Kristensen and Taheri [15] discovered that this restriction is responsible of the fact that in the proof of the Caccioppoli estimate it is impossible the iteration on small radii and thus they are lead to an inequality which does not involve any two concentric balls. Here, in a different way, we also obtain a pre-iterated form of the Caccioppoli inequality involving only two balls of radii $R$ and $\frac{R}{2}$.

This weaker form of the usual Caccioppoli estimate is however enough to establish the decay estimate by the use of an extra new iteration argument.

## 2. The energy estimate

Let $\Omega$ be a bounded open subset of $\mathbb{R}^{3}$. If $A \in \operatorname{Hom}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right)$ we set

$$
\wedge_{0} A=1, \quad \wedge_{1} A=A, \quad \wedge_{2} A=\operatorname{adj} A, \quad \wedge_{3} A=\operatorname{det} A
$$

Following [11], for every $A, B \in \operatorname{Hom}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right)$ and for $k=2,3$, we shall write,

$$
\begin{equation*}
\wedge_{k}(A+B)=\sum_{i=0}^{k} \wedge_{k-i} A \odot \wedge_{i} B \tag{2.1}
\end{equation*}
$$

where $\wedge_{k-i} A \odot \wedge_{i} B$ denotes a suitable linear combination of products of a component of $\wedge_{k-i} A$ times a component of $\wedge_{i} B$. The explicit expression of $\wedge_{k-i} A \odot \wedge_{i} B$ will not be needed in the sequel.
Definition 2.1. Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^{3}$ and $p>1$. The Sobolev class $\wedge^{3} W^{1, p}\left(\Omega ; \mathbb{R}^{3}\right)$ consists of all functions $u \in W^{1, p}\left(\Omega ; \mathbb{R}^{3}\right)$ such that $\wedge_{i} D u \in L^{p}(\Omega)$ for every $1 \leq i \leq 3$.

Remark that the Sobolev class $\wedge^{3} W^{1, p}$ is not a linear space. Indeed, if $u, v \in \wedge^{3} W^{1, p}$ it may happen that $u+v \notin \wedge^{3} W^{1, p}$.

In what follows $B_{r}(x)$ will be the ball centered in $x$ of radius $r$. If no confusion arises, $B_{r}$ will stands for a ball centered in 0 of radius $r$. When $r=1$, we may use $B$ instead of $B_{1}$. The letter $C$ will denote a generic constant whose value may change from line to line.

Lemma 2.2. Let $\alpha$ be a constant and let $f_{h}, g_{h}$ be two sequences of functions in $L^{1}(B)$ such that $f_{h} \rightarrow \alpha$ for a.e. $x \in B, g_{h} \rightarrow g$ weakly in $L^{1+\eta}(B)$, for some $\eta>0$. Assume that

$$
\int_{B}\left|f_{h} g_{h}\right|^{1+\delta} \mathrm{d} x<\infty
$$

where $\delta \geq \eta$. Then $f_{h} g_{h} \rightarrow \alpha g$ weakly in $L^{1+\eta}(B)$.

Proof. See [11], Lemma 5.5.
If $\Omega \subset \mathbb{R}^{3}$ is a bounded open set and $\lambda \in(0,1)$ is a real number, we consider, for $v \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right)$, the functional

$$
\begin{equation*}
J_{\lambda}(v)=\int_{\Omega}\left[|D v|^{2}+\lambda^{\alpha}|\operatorname{Adj} D v|^{p}+\lambda^{\beta}|\operatorname{det} D v|^{q}\right] \mathrm{d} x \tag{2.2}
\end{equation*}
$$

where $0<\frac{q}{p}<\frac{\beta}{\alpha}$ and $1<q<p<2$.
If $Q \geq 1$ is a real number we say that $u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right)$ is a $W^{1, \bar{p}}\left(\Omega, \mathbb{R}^{3}\right)$ Q-local minimizer of $J_{\lambda}$ if there exists a $\delta>0$ such that $J_{\lambda}(u) \leq Q J_{\lambda}(\varphi)$ for any $\varphi \in u+W_{0}^{1,2}\left(\Omega, \mathbb{R}^{3}\right)$ with

$$
\begin{equation*}
\|D \varphi-D u\|_{L^{\bar{p}}} \leq \delta \tag{2.3}
\end{equation*}
$$

We prove the following Caccioppoli type estimate
Lemma 2.3. Let $2 \leq \bar{p} \leq \infty$ and let $u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right) \cap W_{l o c}^{1, \bar{p}}\left(\Omega, \mathbb{R}^{3}\right)$ be a $W^{1, \bar{p}} Q$ - local minimizer of $J_{\lambda}$. If $\bar{p}=\infty$, assume also that

$$
\begin{equation*}
\limsup _{R \rightarrow 0^{+}}\|D u\|_{L^{\infty}(B(x, R))}<\delta \tag{2.4}
\end{equation*}
$$

where $\delta$ is the number appearing in (2.3). Then there exist a constant $c$ depending only on $Q$ and a radius $\bar{R}$ depending only on $\delta$ and a $\theta \in(0,1)$ such that for any $R<\bar{R}, B_{R} \subset \subset \Omega$

$$
\begin{aligned}
& J_{\lambda}\left(u ; B_{\frac{R}{2}}\right) \leq \theta J_{\lambda}\left(u ; B_{R}\right)+\frac{c}{R^{2}} \int_{B_{R} \backslash B_{\frac{R}{2}}^{2}}\left|u-u_{R}\right|^{2} \mathrm{~d} x \\
& \quad+c \lambda^{\alpha} R^{5-3 p}\left(\int_{B_{R}}|D u|^{2} \mathrm{~d} x\right)^{p}+c \lambda^{\frac{\beta p-\alpha q}{p-q}} R^{3-\frac{3 p q}{2(p-q)}}\left(\int_{B_{R}}|D u|^{2} \mathrm{~d} x\right)^{\frac{p q}{2(p-q)}} .
\end{aligned}
$$

Proof. Fix $B_{R} \subset \Omega$ and define

$$
\begin{aligned}
& E_{R}=\left\{\rho \in\left(\frac{R}{2}, R\right):\right. \int_{\partial B_{\rho}}|D u|^{\bar{p}} \mathrm{~d} \mathcal{H}^{2} \leq \frac{8}{R} \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{\bar{p}} \mathrm{~d} x \\
& \text { and } \quad \\
&\left.\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2} \leq \frac{8}{R} \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2} \mathrm{~d} x\right\} .
\end{aligned}
$$

We have that $\left(\frac{R}{2}, R\right) \backslash E_{R}=\mathcal{C}_{1} \cup \mathcal{C}_{2}$ where

$$
\begin{aligned}
& \mathcal{C}_{1}=\left\{\rho \in\left(\frac{R}{2}, R\right): \int_{\partial B_{\rho}}|D u|^{\bar{p}} \mathrm{~d} \mathcal{H}^{2} \geq \frac{8}{R} \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{\bar{p}}\right\} \\
& \mathcal{C}_{2}=\left\{\rho \in\left(\frac{R}{2}, R\right): \int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2} \geq \frac{8}{R} \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2}\right\} .
\end{aligned}
$$

One easily gets that
and then

$$
\operatorname{meas}\left(\mathcal{C}_{i}\right)<\frac{R}{8}
$$

$$
\begin{equation*}
\operatorname{meas}\left(E_{R}\right) \geq \frac{R}{4} \tag{2.5}
\end{equation*}
$$

Now let $\omega=\frac{x}{|x|}$ and for a.e. $\rho \in\left(\frac{R}{2}, R\right)$ consider the function $\omega \rightarrow u(\rho \omega)$. For every $1<m<2$ we get

$$
\begin{equation*}
\left(\int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{m^{*}} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{1}{m^{*}}} \leq c\left(\int_{\partial B_{\rho}}|D u|^{m} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{1}{m}} \tag{2.6}
\end{equation*}
$$

where as usual $m^{*}=\frac{2 m}{2-m}$ denote the Sobolev exponent of $m$.
For each $\rho \in E_{R}$, following [11], we define the function

$$
\varphi(r \omega)= \begin{cases}u_{\partial B_{\rho}} & r \leq \frac{R}{4}  \tag{2.7}\\ \frac{\rho-r}{\rho-\frac{R}{4}} u_{\partial B_{\rho}}+\frac{r-\frac{R}{4}}{\rho-\frac{4}{4}} u(\rho \omega) & \frac{R}{4} \leq r \leq \rho \\ u(r \omega) & \rho \leq r \leq R\end{cases}
$$

and observe that, for $\frac{R}{4}<r<\rho$

$$
\begin{gather*}
|D \varphi(r \omega)| \leq c\left(\frac{\left|u(\rho \omega)-u_{\partial B_{\rho}}\right|}{\rho-\frac{R}{4}}+|D u(\rho \omega)|\right)  \tag{2.8}\\
|\operatorname{Adj} D \varphi(r \omega)| \leq c \frac{\left|u(\rho \omega)-u_{\partial B_{\rho}}\right|}{\rho-\frac{R}{4}}|D u(\rho \omega)|+c|\operatorname{Adj} D u(\rho \omega)|  \tag{2.9}\\
|\operatorname{det} D \varphi(r \omega)| \leq c \frac{\left|u(\rho \omega)-u_{\partial B_{\rho}}\right|}{\rho-\frac{R}{4}}|\operatorname{Adj} D u(\rho \omega)| . \tag{2.10}
\end{gather*}
$$

If $\bar{p}<\infty$, by (2.8) and the assumption $u \in W_{l o c}^{1, \bar{p}}$, one easily gets that there exists a $\bar{R}=\bar{R}(\delta)$ such that

$$
\begin{aligned}
\int_{B_{R}}|D u-D \varphi|^{\bar{p}} \mathrm{~d} x & =\int_{B_{\rho}}|D u-D \varphi|^{\bar{p}} \mathrm{~d} x \leq c \int_{B_{\rho} \backslash B_{\frac{R}{4}}^{4}} \frac{\left|u(\rho \omega)-u_{\partial B_{\rho}}\right|^{\bar{p}}}{\left(\rho-\frac{R}{4}\right)^{\bar{p}}} \mathrm{~d} x+c \int_{B_{R}}|D u|^{\bar{p}} \mathrm{~d} x \\
& \leq \frac{c}{\left(\rho-\frac{R}{4}\right)^{\bar{p}-1}} \int_{\partial B_{\rho}}\left|u(\rho \omega)-u_{\partial B_{\rho}}\right|^{\bar{p}} \mathrm{~d} \mathcal{H}^{2}+c \int_{B_{R}}|D u|^{\bar{p}} \mathrm{~d} x \\
& \leq \frac{c \rho^{\bar{p}}}{\left(\rho-\frac{R}{4}\right)^{\bar{p}-1}} \int_{\partial B_{\rho}}|D u|^{\bar{p}} \mathrm{~d} \mathcal{H}^{2}+c \int_{B_{R}}|D u|^{\bar{p}} \mathrm{~d} x \\
& \leq \frac{c R^{\bar{p}-1}}{\left(\rho-\frac{R}{4}\right)^{\bar{p}-1}} \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{\bar{p}} \mathrm{~d} x+c \int_{B_{R}}|D u|^{\bar{p}} \mathrm{~d} x \leq c \int_{B_{R}}|D u|^{\bar{p}}
\end{aligned}
$$

where we used also Poincaré inequality and the fact that $\frac{R}{2}<\rho<R$. Then, previous inequality ensures that $\varphi$ is an admissible test function, by the absolute continuity of the integral for $\bar{p}<\infty$ and by (2.4) if $\bar{p}=\infty$.

Using (2.8), (2.9), (2.10) and the fact that $u$ is a $W^{1, \bar{p}} \mathrm{Q}$ - local minimizer we get

$$
\begin{align*}
\int_{B_{\rho}}\left[|D u|^{2}\right. & \left.+\lambda^{\alpha}|\operatorname{Adj} D u|^{p}+\lambda^{\beta}|\operatorname{det} D u|^{q}\right] \mathrm{d} x \leq Q \int_{B_{\rho}}\left[|D \varphi|^{2}+\lambda^{\alpha}|\operatorname{Adj} D \varphi|^{p}+\lambda^{\beta}|\operatorname{det} D \varphi|^{q}\right] \mathrm{d} x \\
& \leq \frac{c}{R} \int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{2} \mathrm{~d} \mathcal{H}^{2}+c R \int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2} \\
& +\frac{c \lambda^{\alpha}}{R^{p-1}} \int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{p}|D u|^{p} \mathrm{~d} \mathcal{H}^{2}+c \lambda^{\alpha} R \int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2} \\
& +\frac{c \lambda^{\beta}}{R^{q-1}} \int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{q}|\operatorname{Adj} D u|^{q} \mathrm{~d} \mathcal{H}^{2} \tag{2.11}
\end{align*}
$$

where we used again that $\frac{R}{2}<\rho<R$. Now, we observe that

$$
\begin{align*}
\int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{p}|D u|^{p} \mathrm{~d} \mathcal{H}^{2} & \leq\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{p}{2}}\left(\int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{\frac{2 p}{2-p}} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{2-p}{2}} \\
& \leq c\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{p}{2}}\left(\int_{\partial B_{\rho}}|D u|^{p} \mathrm{~d} \mathcal{H}^{2}\right) \leq c R^{2-p}\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{p} \tag{2.12}
\end{align*}
$$

where we used Hölder inequality and (2.6). Moreover using Young's inequality and (2.6) again we get

$$
\begin{align*}
& \frac{c \lambda^{\beta}}{R^{q-1}} \int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{q}|\operatorname{Adj} D u|^{q} \mathrm{~d} \mathcal{H}^{2} \\
& \leq \frac{c \lambda^{\frac{\beta p-\alpha q}{p-q}}}{R^{\frac{p q}{p-q}-1}}\left(\int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{\frac{p q}{p-q}} \mathrm{~d} \mathcal{H}^{2}\right)+c \lambda^{\alpha} R\left(\int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2}\right) \\
& \leq \frac{c \lambda^{\frac{\beta p-\alpha q}{p-q}}}{R^{\frac{p q}{p-q}-1}}\left(\int_{\partial B_{\rho}}|D u|^{\frac{2 p q}{2 p-2 q+p q}} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{2 p-2 q+p q}{2(p-q)}}+c \lambda^{\alpha} R\left(\int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2}\right) \\
& \leq c \lambda^{\frac{\beta p-\alpha q}{p-q}} R^{3-\frac{p q}{p-q}}\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{p q}{2(p-q)}}+c \lambda^{\alpha} R\left(\int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2}\right) . \tag{2.13}
\end{align*}
$$

Inserting the inequalities (2.12) and (2.13) in (2.11) we obtain

$$
\begin{align*}
& \int_{B_{\rho}}\left[|D u|^{2}+\lambda^{\alpha}|\operatorname{Adj} D u|^{p}+\lambda^{\beta}|\operatorname{det} D u|^{q}\right] \mathrm{d} x \\
& \leq \frac{c}{R} \int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{2} \mathrm{~d} \mathcal{H}^{2}+c R \int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}+c \lambda^{\alpha} R \int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2} \\
& +c \lambda^{\alpha} R^{3-2 p}\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{p}+c \lambda^{\frac{\beta p-\alpha q}{p-q}} R^{3-\frac{p q}{p-q}}\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{\frac{p q}{2(p-q)}} . \tag{2.14}
\end{align*}
$$

Recalling that $\rho$ is in $E_{R}$, we get

$$
\begin{align*}
& \int_{B_{\rho}}\left[|D u|^{2}+\lambda^{\alpha}|\operatorname{Adj} D u|^{p}+\lambda^{\beta}|\operatorname{det} D u|^{q}\right] \mathrm{d} x \\
\leq & \frac{c}{R} \int_{\partial B_{\rho}}\left|u-u_{\partial B_{\rho}}\right|^{2} \mathrm{~d} \mathcal{H}^{2}+c R \int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}+c \lambda^{\alpha} R \int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2} \\
+ & c \lambda^{\alpha} R^{3-3 p}\left(\int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2} \mathrm{~d} x\right)^{p}+c \lambda^{\frac{\beta p-\alpha q}{p-q}} R^{3-\frac{3 p q}{2(p-q)}}\left(\int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2} \mathrm{~d} x\right)^{\frac{p q}{2(p-q)}} . \tag{2.15}
\end{align*}
$$

Integrating (2.15) with respect to $\rho$ in $E_{R}$, using (2.5) we obtain

$$
\begin{align*}
& R \int_{B_{\frac{R}{2}}}\left[|D u|^{2}+\lambda^{\alpha}|\operatorname{Adj} D u|^{p}+\lambda^{\beta}|\operatorname{det} D u|^{q}\right] \mathrm{d} x \\
& \leq \frac{c}{R} \int_{B_{R} \backslash B_{\frac{R}{2}}}\left|u-u_{R}\right|^{2} \mathrm{~d} x+c R \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2} \mathrm{~d} x+c \lambda^{\alpha} R \int_{B_{R} \backslash B_{\frac{R}{2}}}|\operatorname{Adj} D u|^{p} \mathrm{~d} x \\
&  \tag{2.16}\\
& \quad+c \lambda^{\alpha} R^{4-3 p}\left(\int_{B_{R}}|D u|^{2} \mathrm{~d} x\right)^{p}+c \lambda^{\frac{\beta p-\alpha q}{p-q}} R^{4-\frac{33 q}{2(p-q)}}\left(\int_{B_{R}}|D u|^{2} \mathrm{~d} x\right)^{\frac{p q}{2(p-q)}}
\end{align*}
$$

Dividing inequality (2.16) by $R$ and using the standard trick of "hole-filling" we get

$$
\begin{aligned}
J_{\lambda}\left(u ; B_{\frac{R}{2}}\right) \leq \theta J_{\lambda}\left(u ; B_{R}\right)+\frac{c}{R^{2}} \int_{B_{R} \backslash B_{\frac{R}{2}}} & \left|u-u_{R}\right|^{2} \mathrm{~d} x \\
& +c \lambda^{\alpha} R^{3-3 p}\left(\int_{B_{R}}|D u|^{2} \mathrm{~d} x\right)^{p}+c \lambda^{\frac{\beta p-\alpha q}{p-q}} R^{3-\frac{3 p q}{2(p-q)}}\left(\int_{B_{R}}|D u|^{2} \mathrm{~d} x\right)^{\frac{p q}{2(p-q)}}
\end{aligned}
$$

where $\theta=\frac{c}{c+1} \in(0,1)$ i.e. the conclusion follows.

## 3. Proof of Theorem A

Let us consider the excess function

$$
\begin{equation*}
U(x, r)=f_{B_{r}(x)}\left|D u-(D u)_{r}\right|^{2}+\left|\operatorname{Adj}\left(D u-(D u)_{r}\right)\right|^{p}+\left|\operatorname{det}\left(D u-(D u)_{r}\right)\right|^{q} \tag{3.1}
\end{equation*}
$$

we want to establish, as usual, a decay estimate for $U(x, r)$. More precisely we have the following lemma.
Lemma 3.1. Let $u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right) \cap W_{\text {loc }}^{1, \bar{p}}\left(\Omega, \mathbb{R}^{3}\right)$ be a $W^{1, \bar{p}}$ local minimizer of $I$ (satisfying (1.3) if $\bar{p}=+\infty$ ). For any $M>0$ and $\tau \in\left(0, \frac{1}{2}\right)$ there exist two constants $c(M)$ and $\varepsilon(\tau, M)$ such that, if

$$
\begin{equation*}
\left|(D u)_{r}\right| \leq M \quad \text { and } \quad U(x, r)<\varepsilon \tag{3.2}
\end{equation*}
$$

then

$$
\begin{equation*}
U(x, \tau r) \leq c(M) \tau^{\mu} U(x, r) \tag{3.3}
\end{equation*}
$$

for some $\mu$ independent of $M$ and $\tau$.
Proof. Step 1 (Blow up). Fix $M>0$ and $\tau \in\left(0, \frac{1}{2}\right)$. Arguing by contradiction, we assume that there exists a sequence $B_{r_{h}}\left(x_{h}\right) \subset \subset \Omega$ such that

$$
\begin{equation*}
\left|(D u)_{x_{h}, r_{h}}\right| \leq M \quad \text { and } \quad \lambda_{h}^{2}=U\left(x_{h}, r_{h}\right) \rightarrow 0 \tag{3.4}
\end{equation*}
$$

but

$$
\begin{equation*}
\frac{U\left(x_{h}, \tau r_{h}\right)}{\lambda_{h}^{2}}>c(M) \tau^{\mu} \tag{3.5}
\end{equation*}
$$

for some $c(M)$ to be determined later. Setting $A_{h}=(D u)_{x_{h}, r_{h}}$ and

$$
\begin{equation*}
v_{h}=\frac{u\left(x_{h}+r_{h} y\right)-(u)_{x_{h}, r_{h}}-r_{h} A_{h} y}{\lambda_{h} r_{h}} \tag{3.6}
\end{equation*}
$$

for all $y \in B_{1}(0)$, we have

$$
\begin{equation*}
\int_{B_{1}(0)}\left|D v_{h}\right|^{2}+\lambda_{h}^{2 p-2}\left|\operatorname{Adj}\left(D v_{h}\right)\right|^{p}+\lambda_{h}^{3 q-2}\left|\operatorname{det}\left(D v_{h}\right)\right|^{q}=1 \tag{3.7}
\end{equation*}
$$

and $\left(v_{h}\right)_{0,1}=0$. Passing to a subsequence and using the divergence structure of minors, we may assume, without loss of generality, that

$$
\begin{array}{rlrl}
D v_{h} & \rightarrow D v & w-L^{2}\left(B_{1}\right) \\
v_{h} & \rightarrow v & s-L^{2}\left(B_{1}\right) \\
\lambda_{h}^{\frac{2 p-2}{p}} \operatorname{Adj} D v_{h} & \rightarrow 0 & w-L^{p}\left(B_{1}\right) \\
\lambda_{h}^{\frac{3 q-2}{q}} \operatorname{det} D v_{h} & \rightarrow 0 & w-L^{q}\left(B_{1}\right) \\
A_{h} & \rightarrow A & a_{h} \rightarrow a . \tag{3.8}
\end{array}
$$

We introduce the rescaled functionals

$$
f_{h}(\xi)=\frac{1}{\lambda_{h}^{2}}\left[f\left(\operatorname{Adj}\left(A_{h}+\lambda_{h} \xi\right)\right)-f\left(\operatorname{Adj}\left(A_{h}\right)\right)-D f\left(\operatorname{Adj}\left(A_{h}\right)\right)\left(\operatorname{Adj}\left(A_{h}+\lambda_{h} \xi\right)-\operatorname{Adj}\left(A_{h}\right)\right)\right]
$$

and

$$
g_{h}(s)=\frac{1}{\lambda_{h}^{2}}\left[g\left(\operatorname{det}\left(A_{h}+\lambda_{h} s\right)\right)-g\left(\operatorname{det}\left(A_{h}\right)\right)-g^{\prime}\left(\operatorname{det}\left(A_{h}\right)\right)\left(\operatorname{det}\left(A_{h}+\lambda_{h} s\right)-\operatorname{det}\left(A_{h}\right)\right)\right]
$$

and for any $h$, we set

$$
\begin{equation*}
I_{h}(w)=\int_{B_{1}}|D w|^{2}+f_{h}(D w)+g_{h}(D w) \tag{3.9}
\end{equation*}
$$

It is easy to check that $I_{h}\left(v_{h}\right) \leq I_{h}\left(v_{h}+\varphi\right)$ provided $\varphi \in W_{0}^{1, \bar{p}}\left(B_{1}, R^{N}\right)$ and

$$
\|D \varphi\|_{\bar{p}} \leq \delta_{h}= \begin{cases}\frac{\delta}{\lambda_{h} r_{h}^{\frac{n}{p}}} & \bar{p}<\infty \\ \frac{\delta}{\lambda_{h}} & \bar{p}=\infty\end{cases}
$$

Step $2\left(v\right.$ solves a linear system). By formula (2.1) one easily deduces that, for every $\phi \in C_{0}^{\infty}\left(B_{1}\right)$,

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} t} \wedge_{i}\left(A_{h}+\lambda_{h}\left(D v_{h}+t D \phi\right)\right)\right|_{t=0}=\wedge_{i-1}\left(A_{h}+\lambda_{h} D v_{h}\right) \odot \lambda_{h} D \phi
$$

Then the minimality of $v_{h}$ implies that they solve the Euler Lagrange systems:

$$
\begin{align*}
\int_{B_{1}(0)} D v_{h} D \phi \mathrm{~d} x & +\int_{B_{1}(0)}\left(\int_{0}^{1} D^{2} f\left(\operatorname{Adj}\left(A_{h}\right)+t\left(\operatorname{Adj}\left(A_{h}+\lambda_{h} D v_{h}\right)-\operatorname{Adj}\left(A_{h}\right)\right)\right) \mathrm{d} t\right) \\
& \times\left(\lambda_{h} \operatorname{Adj}\left(D v_{h}\right)+A_{h} \odot D v_{h}\right) \cdot\left[\left(A_{h}+\lambda_{h} D v_{h}\right) \odot D \phi\right] \mathrm{d} x \\
& +\int_{B_{1}(0)}\left(\int_{0}^{1} g^{\prime \prime}\left(\operatorname{det}\left(A_{h}\right)+t\left(\operatorname{det}\left(A_{h}+\lambda_{h} D v_{h}\right)-\operatorname{det}\left(A_{h}\right)\right)\right) \mathrm{d} t\right) \\
& \times\left(\operatorname{Adj}\left(A_{h}\right) \odot D v_{h}+\lambda_{h} D v_{h} \odot \operatorname{adj}\left(D v_{h}\right)\right) \cdot\left[\operatorname{Adj}\left(A_{h}+\lambda_{h} D v_{h}\right) \odot D \phi\right] \mathrm{d} x=0 \tag{3.10}
\end{align*}
$$

for all $\phi \in C_{0}^{\infty}\left(B_{1}\right)$. Letting $h \rightarrow \infty$, using (3.8) and Lemma 2.2 we get

$$
0=\int_{B_{1}(0)} D v D \phi+D^{2} f(\operatorname{Adj} A)(A \odot D v)(A \odot D \phi)+D^{2} g(\operatorname{det} A)(\operatorname{Adj} A \odot D v)(\operatorname{Adj} A \odot D \phi) \mathrm{d} x
$$

then $v$ solves a linear elliptic system, with constant coefficients. By standard regularity result (see [12]) we have that for any $\sigma \in\left(0, \frac{1}{2}\right]$

$$
\begin{equation*}
f_{B_{\sigma}}\left|D v-(D v)_{\sigma}\right|^{2} \leq c \sigma^{2} f_{B_{1}}\left|D v-(D v)_{\sigma}\right|^{2} \leq c \sigma^{2} \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|(D v)_{2 \sigma}-(D v)_{\sigma}\right|^{2} \leq c \sigma^{2} \tag{3.12}
\end{equation*}
$$

where the constant $c$ depends only on $M$. Setting

$$
\begin{equation*}
w_{h}(y)=v_{h}(y)-\left(D v_{h}\right)_{\sigma} y-\left(v_{h}\right)_{2 \sigma} \tag{3.13}
\end{equation*}
$$

and using the fact that $v_{h}$ minimizes the functional (3.9), one easily see that $w_{h}$ minimizes the functional

$$
\begin{equation*}
w \rightarrow \int_{B_{1}}|D w|^{2}+f_{h}\left(D w+\left(D v_{h}\right)_{\sigma}\right)+g_{h}\left(D w+\left(D v_{h}\right)_{\sigma}\right) \tag{3.14}
\end{equation*}
$$

Now, we claim that

$$
\begin{align*}
\left|f_{h}(\xi)\right| & \leq c(M)\left(|\xi|^{2}+\lambda_{h}^{2 p-2}|\operatorname{Adj} \xi|^{p}\right)  \tag{3.15}\\
\left|g_{h}(\xi)\right| & \leq c(M)\left(|\xi|^{2}+\lambda_{h}^{3 q-2}|\operatorname{det} \xi|^{q}\right) .
\end{align*}
$$

Namely, by definition of $f_{h}$, we have that

$$
\begin{aligned}
f_{h}(\xi) & =\frac{1}{\lambda_{h}^{2}}\left[f\left(\operatorname{Adj}\left(A_{h}+\lambda_{h} \xi\right)\right)-f\left(\operatorname{Adj}\left(A_{h}\right)\right)-D f\left(\operatorname{Adj}\left(A_{h}\right)\right)\left(\operatorname{Adj}\left(A_{h}+\lambda_{h} \xi\right)-\operatorname{Adj}\left(A_{h}\right)\right)\right] \\
& =\frac{1}{\lambda_{h}^{2}}\left[f\left(\operatorname{Adj} A_{h}+\lambda_{h}^{2} \operatorname{Adj} \xi+A_{h} \odot \lambda_{h} \xi\right)-f\left(\operatorname{Adj}\left(A_{h}\right)\right)-D f\left(\operatorname{Adj}\left(A_{h}\right)\right)\left(\lambda_{h}^{2} \operatorname{Adj} \xi+A_{h} \odot \lambda_{h} \xi\right)\right]
\end{aligned}
$$

and setting

$$
\zeta=\lambda_{h}^{2} \operatorname{Adj} \xi+A_{h} \odot \lambda_{h} \xi
$$

we can write

$$
\begin{align*}
\lambda_{h}^{2}\left|f_{h}(\xi)\right| & =\left|\int_{0}^{1}(1-s) D^{2} f\left(\operatorname{Adj} A_{h}+s \zeta\right) \zeta \zeta \mathrm{d} s\right| \\
& \leq|\zeta|^{2} \int_{0}^{1}\left(\mu^{2}+\left|\operatorname{Adj} A_{h}+s \zeta\right|^{2}\right)^{\frac{p-2}{2}} \\
& \leq c|\zeta|^{2}\left(\mu^{2}+\left|\operatorname{Adj} A_{h}\right|^{2}+\left|\zeta+\operatorname{Adj} A_{h}\right|^{2}\right)^{\frac{p-2}{2}} \tag{3.16}
\end{align*}
$$

where we used Lemma 2.1 in [2] and assumption (H2). On the other hand we observe that if

$$
\left|\lambda_{h}^{2} \operatorname{Adj}(\xi)\right| \leq\left|A_{h} \odot \lambda_{h} \xi\right|,
$$

then by (3.16) we have

$$
\lambda_{h}^{2}\left|f_{h}(\xi)\right| \leq c \lambda_{h}^{2}\left|A_{h} \odot \xi\right|^{2} \leq c(M) \lambda_{h}^{2}|\xi|^{2}
$$

If

$$
\left|\lambda_{h}^{2} \operatorname{Adj}(\xi)\right|>\left|A_{h} \odot \lambda_{h} \xi\right|
$$

then by (3.16)

$$
\lambda_{h}^{2}\left|f_{h}(\xi)\right| \leq c \lambda_{h}^{2 p}|\operatorname{Adj}(\xi)|^{p}
$$

The second inequality in (3.15) is analogue. Using Lemma 2.1 in [2] again and assumptions (H1) and (H2) we obtain

$$
\begin{aligned}
&\left|f_{h}(\xi)\right| \geq c(M)\left(|\xi|^{2}+\lambda_{h}^{2 p-2}|\operatorname{Adj} \xi|^{p}\right), \\
&\left|g_{h}(\xi)\right| \geq c(M)\left(|\xi|^{2}+\lambda_{h}^{3 q-2}|\operatorname{det} \xi|^{q}\right) .
\end{aligned}
$$

Hence the functional defined at (3.14) is equivalent to the following

$$
w \rightarrow \int_{B_{1}}|D w|^{2}+\lambda_{h}^{2 p-2}|\operatorname{Adj}(D w)|^{p}+\lambda_{h}^{3 q-2}|\operatorname{det}(D w)|^{q}
$$

and then $w_{h}$ is a $W^{1, \bar{p}} \mathrm{Q}$-local minimizer of $J_{\lambda_{h}}\left(B_{1}\right)$ for some $Q=Q(M)$, with $\alpha=2 p-2$ and $\beta=3 q-2$.
Step 3 (Conclusion). Rescaling the excess function defined by (3.1), we get

$$
\begin{aligned}
U\left(x_{h}, \sigma r_{h}\right) & =f_{B_{\sigma r_{h}}\left(x_{h}\right)}\left|D u-(D u)_{\sigma r_{h}}\right|^{2}+\left|\operatorname{Adj}\left(D u-(D u)_{\sigma r_{h}}\right)\right|^{p}+\left|\operatorname{det}\left(D u-(D u)_{\sigma r_{h}}\right)\right|^{q} \\
& =f_{B_{\sigma}(0)} \lambda_{h}^{2}\left|D v_{h}-\left(D v_{h}\right)_{\sigma}\right|^{2}+\lambda_{h}^{2 p}\left|\operatorname{Adj}\left(D v_{h}-\left(D v_{h}\right)_{\sigma}\right)\right|^{p}+\lambda_{h}^{3 q}\left|\operatorname{det}\left(D v_{h}-\left(D v_{h}\right)_{\sigma}\right)\right|^{p} \\
& =f_{B_{\sigma}(0)} \lambda_{h}^{2}\left|D w_{h}\right|^{2}+\lambda_{h}^{2 p}\left|\operatorname{Adj}\left(D w_{h}\right)\right|^{p}+\lambda_{h}^{3 q}\left|\operatorname{det}\left(D w_{h}\right)\right|^{p}
\end{aligned}
$$

Since $w_{h}$ satisfies the assumptions of Lemma 2.3 (in particular if $\bar{p}=+\infty$ assumption (1.3) implies that $w_{h}$ satisfies (2.4) with $\delta_{h}=\frac{\delta}{\lambda_{h}}$ ), for $h$ sufficiently large we obtain

$$
\begin{align*}
\frac{U\left(x_{h}, \sigma r_{h}\right)}{\lambda_{h}^{2}}= & f_{B_{\sigma}(0)}\left|D w_{h}\right|^{2}+\lambda_{h}^{2 p-2}\left|\operatorname{Adj}\left(D w_{h}\right)\right|^{p}+\lambda_{h}^{3 q-2}\left|\operatorname{det}\left(D w_{h}\right)\right|^{q} \\
& \leq \theta \frac{U\left(x_{h}, 2 \sigma r_{h}\right)}{\lambda_{h}^{2}}+\frac{c}{\sigma^{2}} f_{B_{2 \sigma}}\left|w_{h}-\left(w_{h}\right)_{2 \sigma}\right|^{2} \\
& +c \sigma^{5-3 p} \lambda_{h}^{2 p-2}\left(f_{B_{2 \sigma}}\left|D w_{h}\right|^{2}\right)^{p}+c \sigma^{3-\frac{3 p q}{2(p-q)}} \lambda_{h}^{\frac{p q}{p-q}-2}\left(f_{B_{2 \sigma}}\left|D w_{h}\right|^{2}\right)^{\frac{p q}{2(p-q)}} \tag{3.17}
\end{align*}
$$

Passing to the limit as $h \rightarrow \infty$ in (3.17) and using (3.8), (3.11), (3.12) and the assumption $1<p<q<2$ we get

$$
\begin{align*}
& \limsup _{h \rightarrow \infty} \frac{U\left(x_{h}, \sigma r_{h}\right)}{\lambda_{h}^{2}} \leq \theta \limsup _{h \rightarrow \infty} \frac{U\left(x_{h}, 2 \sigma r_{h}\right)}{\lambda_{h}^{2}}+\frac{c}{\sigma^{2}} f_{B_{2 \sigma}}\left|v-(v)_{2 \sigma}-(D v)_{\sigma} y\right|^{2} \mathrm{~d} y \\
& \leq \theta \limsup _{h \rightarrow \infty} \frac{U\left(x_{h}, 2 \sigma r_{h}\right)}{\lambda_{h}^{2}}+\frac{c}{\sigma^{2}} f_{B_{2 \sigma}}\left[\left|v-(v)_{2 \sigma}-(D v)_{2 \sigma} y\right|^{2}+\left|(D v)_{2 \sigma} y-(D v)_{\sigma} y\right|^{2}\right] \mathrm{d} y \\
& \leq \theta \limsup _{h \rightarrow \infty} \frac{U\left(x_{h}, 2 \sigma r_{h}\right)}{\lambda_{h}^{2}}+c f_{B_{2 \sigma}}\left|D v-(D v)_{2 \sigma}\right|^{2} \mathrm{~d} y+c \sigma^{2} \\
& \leq \theta \limsup _{h \rightarrow \infty} \frac{U\left(x_{h}, 2 \sigma r_{h}\right)}{\lambda_{h}^{2}}+c_{1}(M) \sigma^{2} \tag{3.18}
\end{align*}
$$

where we used Poincaré inequality. Setting for $\sigma \in\left(0, \frac{1}{2}\right]$

$$
\varphi(\sigma)=\limsup _{h \rightarrow \infty} \frac{U\left(x_{h}, \sigma r_{h}\right)}{\lambda_{h}^{2}}
$$

inequality (3.18) can be rewritten as

$$
\varphi(\sigma) \leq \theta \varphi(2 \sigma)+c_{1}(M) \sigma^{2}
$$

and one can easily check that the function

$$
\psi(\sigma)=\varphi(\sigma)+c_{1}(M) \sigma^{2}
$$

is such that, for all $\sigma \in\left(0, \frac{1}{2}\right]$,

$$
\begin{equation*}
\psi(\sigma) \leq \max \left\{\theta, \frac{1}{2}\right\} \psi(2 \sigma)=\gamma \psi(2 \sigma) \quad 0<\gamma<1 \tag{3.19}
\end{equation*}
$$

An iteration procedure yields that, for any $k=0,1,2, \ldots$ and $\sigma \in\left(0, \frac{1}{2}\right]$

$$
\begin{equation*}
\psi\left(\frac{\sigma}{2^{k}}\right) \leq \gamma^{k} \psi(\sigma) \tag{3.20}
\end{equation*}
$$

Now, take $\tau$ in the interval $\left(\frac{1}{2^{k+1}} ; \frac{1}{2^{k}}\right)$ and observe, by the definition of $\psi$, that

$$
\begin{equation*}
\psi(\tau) \leq C \psi\left(\frac{1}{2^{k}}\right) \tag{3.21}
\end{equation*}
$$

Putting together estimates (3.20) and (3.21), we obtain

$$
\begin{align*}
\varphi(\tau) & \leq \psi(\tau) \leq \gamma^{k-1} \psi\left(\frac{1}{2}\right) \\
& =2^{(k-1) \log _{2} \gamma} \psi\left(\frac{1}{2}\right)=\left(\frac{2^{k+1}}{4}\right)^{\log _{2} \gamma} \psi\left(\frac{1}{2}\right) \\
& \leq\left(\frac{1}{4}\right)^{\log _{2} \gamma}\left(\frac{1}{\tau}\right)^{\log _{2} \gamma} \psi\left(\frac{1}{2}\right) \\
& =\frac{1}{\gamma^{2}} \tau^{-\log _{2} \gamma} \psi\left(\frac{1}{2}\right) \leq C_{2}(M) \tau^{\mu} \tag{3.22}
\end{align*}
$$

Inequality (3.22) contradicts (3.5) if we choose $c(M)$ larger then $C_{2}(M)$.
We now are in position to give the proof of Theorem A, that relies on a standard iteration argument involving the excess function.

Lemma 3.2. Let $0<\alpha<1$ and $M>0$. Then there exist $\tau \in\left(0, \frac{1}{2}\right)$ and $\varepsilon>0$ both depending on $\alpha$ and $M$ such that if

$$
B(x, r) \subset \Omega, \quad\left|(D u)_{r}\right| \leq M, \quad \text { and } \quad U(x, r)<\varepsilon
$$

then

$$
U\left(x, \tau^{l} r\right) \leq\left(\tau^{l}\right)^{\alpha} U(x, r)
$$

for every $l \in N$.

Proof. See Lemma 6.1 [10].

Lemma 3.3. Let $u \in \wedge_{k} W^{1, p}(\Omega)$. Then

$$
\lim _{r \rightarrow 0} f_{B_{r}(x)} \mid \wedge_{i}\left(D u-\left.(D u)_{r}\right|^{p}=0\right.
$$

for almost every $x \in \Omega$ and $1 \leq i \leq k$.

Proof. See Lemma 6.2 [10].
The proof of Theorem A is now consequence of a iteration procedure based on Lemma 4.1. The singular set turns out to be contained in the complement of

$$
\begin{gather*}
\Omega_{o}=\left\{x \in \Omega: \lim _{r \rightarrow 0}(D u)_{r}=0, \lim _{r \rightarrow 0} f_{B_{r}(x)} \mid \operatorname{Adj}\left(D u-\left.(D u)_{r}\right|^{p}=0\right.\right. \\
 \tag{3.23}\\
\lim _{r \rightarrow 0} f_{B_{r}(x)} \mid \operatorname{det}\left(D u-\left.(D u)_{r}\right|^{q}=0\right\}
\end{gather*}
$$

## 4. The limit case

In this section we treat the limit case in which the growth exponents $p=q \leq 2$. Also in this case the partial regularity of weak local minimizers is based on a decay estimate for the excess function

$$
\begin{equation*}
U(x, r)=f_{B_{r}(x)}\left|D u-(D u)_{r}\right|^{2}+\left|\operatorname{Adj}\left(D u-(D u)_{r}\right)\right|^{p}+\left|\operatorname{det}\left(D u-(D u)_{r}\right)\right|^{p} \tag{4.1}
\end{equation*}
$$

The proof of Theorem B differs from the one of Theorem A only in the Caccioppoli type estimate.
Lemma 4.1. Let $p \leq 2$, and $u \in W^{1,2}\left(\Omega, \mathbb{R}^{3}\right) \cap W_{\text {loc }}^{1, \infty}\left(\Omega, \mathbb{R}^{3}\right)$ be a $W^{1, \infty} Q$ - local minimizer of

$$
J_{\lambda}(v)=\int_{\Omega}|D v|^{2}+\lambda^{2 p-2}|\operatorname{Adj} D v|^{p}+\lambda^{3 p-2}|\operatorname{det} D v|^{p}
$$

such that

$$
\begin{equation*}
\limsup _{R \rightarrow 0^{+}}\|D u\|_{L^{\infty}\left(B_{R}(x)\right)}<\frac{\delta}{\lambda} \tag{4.2}
\end{equation*}
$$

where $\delta$ is the number appearing in (1.3). Then there exist a constant $c$ depending only on $Q$, a radius $\bar{R}$ depending only on $\delta$ and a number $\theta \in(0,1)$ such that for any $R<\bar{R}$, we have

$$
J_{\lambda}\left(u ; B_{\frac{R}{2}}\right) \leq \theta J_{\lambda}\left(u ; B_{R}\right)+\frac{c}{R^{2}} \int_{B_{R} \backslash B_{\frac{R}{2}}}\left|u-u_{R}\right|^{2}+c \lambda^{2 p-2} R^{5-3 p}\left(\int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2}\right)^{p}
$$

if $p<2$,

$$
J_{\lambda}\left(u ; B_{\frac{R}{2}}\right) \leq \theta J_{\lambda}\left(u ; B_{R}\right)+\frac{c}{R^{2}} \int_{B_{R} \backslash B_{\frac{R}{2}}}\left|u-u_{R}\right|^{2},
$$

if $p=2$.

Proof. The proof goes as the one of Lemma 2.3 with some minor changes. Assumption (4.2) implies that there exists a radius $\bar{R}$ depending only on $\delta$ such that for any $R<\bar{R}$,

$$
\|D u\|_{L^{\infty}\left(B_{R}(x)\right)}<\frac{\delta}{\lambda}
$$

Fix $R<\bar{R}$ and consider $\rho$ in the set

$$
E_{R}=\left\{\rho \in\left(\frac{R}{2}, R\right): \int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2} \leq \frac{4}{R} \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2} \mathrm{~d} x\right\}
$$

Since $u$ is Lipschitz continuous

$$
\begin{equation*}
\left|u(\rho \omega)-u_{\partial B_{\rho}}\right| \leq 2 R\|D u\|_{L^{\infty}\left(B_{\rho}\right)} \leq 2 \frac{\delta}{\lambda} R \tag{4.3}
\end{equation*}
$$

holds.
By the assumption $u \in W_{l o c}^{1, \infty}(\Omega)$ and (4.2), one easily gets that the function $\varphi$, defined at (2.6), is an admissible test function. Arguing as in Lemma 2.1, using that $u$ is a $W^{1, \infty} \mathrm{Q}$ - local minimizer, we get
$\int_{B_{\rho}}\left[|D u|^{2}+\lambda^{2 p-2}|\operatorname{Adj} D u|^{p}+\lambda^{3 p-2}|\operatorname{det} D u|^{p}\right] \mathrm{d} x$

$$
\begin{align*}
& \leq Q \int_{B_{\rho} \backslash B_{\frac{R}{2}}}\left[|D \varphi|^{2}+\lambda^{2 p-2}|\operatorname{Adj} D \varphi|^{p}+\lambda^{3 p-2}|\operatorname{det} D \varphi|^{p}\right] \mathrm{d} x \\
\leq & c \int_{\partial B_{\rho}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{2}}{R}+c R \int_{\partial B_{\rho}}|D u|^{2}+c R \lambda^{2 p-2} \int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \\
& \quad+c \lambda^{2 p-2} \int_{\partial B_{\rho}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{p}}{R^{p-1}}|D u|^{p}+c \lambda^{3 p-2} \int_{\partial B_{\rho}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{p}}{R^{p-1}}|\operatorname{Adj} D u|^{p} . \tag{4.4}
\end{align*}
$$

Let us treat separately the two cases $p<2$ and $p=2$. Assume that $p<2$. Using (4.3) we get

$$
\begin{equation*}
\int_{\partial B_{\rho}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{p}}{R^{p-1}}|\operatorname{Adj} D u|^{p} \leq 2 R \frac{\delta^{p}}{\lambda^{p}} \int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \tag{4.5}
\end{equation*}
$$

Inserting the inequality (4.5) in (4.4) and using (2.12), we obtain

$$
\begin{align*}
& \int_{B_{\rho}}\left[|D u|^{2}+\lambda^{2 p-2}|\operatorname{Adj} D u|^{2}+\lambda^{3 p-2}|\operatorname{det} D u|^{p}\right] \mathrm{d} x \\
& \qquad c \int_{\partial B_{\rho}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{2}}{R} \mathrm{~d} \mathcal{H}^{2}+c R \int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}+c\left(1+\delta^{p}\right) \lambda^{2 p-2} R \int_{\partial B_{\rho}}|\operatorname{Adj} D u|^{p} \mathrm{~d} \mathcal{H}^{2} \\
&  \tag{4.6}\\
& \quad+c \lambda^{2 p-2} R^{3-2 p}\left(\int_{\partial B_{\rho}}|D u|^{2} \mathrm{~d} \mathcal{H}^{2}\right)^{p}
\end{align*}
$$

Recalling that $\rho$ is in $E_{R}$, integrating with respect to $\rho$ in $E_{R}$ and using the fact that $\left|E_{R}\right| \geq \frac{R}{4}$, we obtain
$R \int_{B_{\frac{R}{2}}}|D u|^{2}+\lambda^{2 p-2}|\operatorname{Adj} D u|^{p}+\lambda^{3 p-2}|\operatorname{det} D u|^{p}$

$$
\begin{align*}
& \leq c \int_{B_{R} \backslash B_{\frac{R}{2}}} \frac{\left|u-u_{R}\right|^{2}}{R}+c R \int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2}+c \lambda^{2 p-2} R \int_{B_{R} \backslash B_{\frac{R}{2}}}|\operatorname{Adj} D u|^{p} \\
& \quad+c \lambda^{2 p-2} R^{4-3 p}\left(\int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2}\right)^{p} . \tag{4.7}
\end{align*}
$$

Dividing inequality (4.7) by $R$ and using the standard trick of "hole-filling" we get

$$
J_{\lambda}\left(u ; B_{\frac{R}{2}}\right) \leq \theta J_{\lambda}\left(u ; B_{R}\right)+\frac{c}{R^{2}} \int_{B_{R} \backslash B_{\frac{R}{2}}}\left|u-u_{R}\right|^{2}+c \lambda^{2 p-2} R^{3-3 p}\left(\int_{B_{R} \backslash B_{\frac{R}{2}}}|D u|^{2}\right)^{p}
$$

where $\theta \in(0,1)$. This is conclusion in the case $p<2$.
Suppose $p=2$. Formula (4.4) becomes

$$
\begin{align*}
\int_{B_{\rho}}\left[|D u|^{2}+\lambda^{2}|\operatorname{Adj} D u|^{2}+\lambda^{4}|\operatorname{det} D u|^{2}\right] \mathrm{d} x
\end{aligned} \quad \begin{aligned}
& \quad \leq Q \int_{B_{\rho} \backslash B_{\frac{R}{2}}}\left[|D \varphi|^{2}+\lambda^{2}|\operatorname{Adj} D \varphi|^{2}+\lambda^{4}|\operatorname{det} D \varphi|^{2}\right] \mathrm{d} x \\
& \left.\leq c \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{2}}{R^{2}}+c \int_{B_{\rho} \backslash B_{\frac{R}{2}}}|D u|^{2}+c \lambda^{2} \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \right\rvert\, \operatorname{AdjDu|^{2}} \\
& \\
& \left.\quad+c \lambda^{2} \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{2}}{R^{2}}|D u|^{2}+c \lambda^{4} \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{2}}{R^{2}} \right\rvert\, \operatorname{AdjDu|^{2}.} \tag{4.8}
\end{align*}
$$

Using (4.3) in (4.8) we find

$$
\begin{aligned}
& \int_{B_{\rho}}\left[|D u|^{2}+\lambda^{2}|\operatorname{Adj} D u|^{2}+\lambda^{4}|\operatorname{det} D u|^{2}\right] \mathrm{d} x \\
& \qquad \left.c \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \frac{\left|u-u_{\partial B_{\rho}}\right|^{2}}{R^{2}}+c \int_{B_{\rho} \backslash B_{\frac{R}{2}}}|D u|^{2}+c \lambda^{2} \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \right\rvert\, \operatorname{AdjDu|^{2}} \\
& \left.\quad+c \lambda^{2} \frac{\delta^{2}}{\lambda^{2}} \int_{B_{\rho} \backslash B_{\frac{R}{2}}}|D u|^{2}+c \lambda^{4} \frac{\delta^{2}}{\lambda^{2}} \int_{B_{\rho} \backslash B_{\frac{R}{2}}} \right\rvert\, \operatorname{AdjDu|^{2}}
\end{aligned}
$$

Arguing as before we conclude the proof.

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    ${ }^{1}$ Dipartimento Pe.Me.Is., Piazza Arechi II, 82100 Benevento, Italy; carozza@unisannio.it
    2 Dipartimento di Matematica e Appl. "R.Caccioppoli" Università di Napoli "Federico II" Via Cintia, 80126 Napoli, Italy; antonia.passarelli@unina.it

