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Model pulsar magnetospheres: the perpendicular rotator

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ABSTRACT

The simplest model illustrating the effect of the magnetospheric charge-current field on the structure of a pulsar magnetic field has the region within the light-cylinder filled with the Goldreich–Julian charge density which corotates with the neutron star, but has no electric currents along the magnetic field lines. This model has previously been studied for the axisymmetric case, with the rotation and magnetic dipolar axes aligned. The analogous problem is now solved with the two axes mutually perpendicular, so that not only the material current arising from the rotating charges but also the displacement current contributes. Again, the constructed magnetic field $\boldsymbol{B}_0$ crosses the light-cylinder normally, and there is no energy flux to infinity. However, in a more realistic model there is a flow of current along $\boldsymbol{B}_0$, generating a field $\boldsymbol{B}_1$ which has a non-vanishing toroidal component at the light-cylinder, so yielding a finite integrated Poynting flux.

Key words: magnetic fields – stars: neutron – pulsars: general.

1 QUASI-STEADY, NON-AXISYMMETRIC SYSTEMS

The simplest oblique rotator model for a pulsar – foreshadowed by Pacini (1967), before the actual discovery of pulsars – has a dominantly dipolar magnetic field, frozen into the solid crust of a neutron star which rotates with angular velocity α, and with the dipole axis $\boldsymbol{p}$ inclined at an angle χ to the rotation axis $\boldsymbol{k}$. In a 'quasi-steady state', the variation of any quantity viewed from the non-rotating inertial frame is due just to the rigid rotation of the essentially non-axisymmetric structure. This condition can be written as the equivalence in the inertial frame of the two operators

$$\frac{\partial}{\partial t} = -\alpha \frac{\partial}{\partial \Phi}, \tag{1}$$

where Φ is the azimuthal angle defined by $\boldsymbol{k}$, and the operations are applied to scalars or to cylindrical or spherical polar components. With the constraint (1), the Faraday–Neumann equation of induction becomes

$$-c\nabla \times E = -\alpha \partial \boldsymbol{B}/\partial \Phi = \nabla \times (\alpha \varpi \boldsymbol{t} \times \boldsymbol{B}), \tag{2}$$

where $\boldsymbol{t}$ is the unit azimuthal vector defined by $\boldsymbol{k}$, and the last step – most simply proved in cylindrical polar coordinates (ϖ, Φ, z) based on $\boldsymbol{k}$ – depends on the other homogeneous Maxwell equation $\nabla \cdot B = 0$. Equation (2) then yields the general quasi-static relation

$$\boldsymbol{E} = -(\alpha \varpi / c)\boldsymbol{t} \times \boldsymbol{B} - \nabla \psi, \tag{3}$$

the sum of the 'corotational' and 'non-corotational' electric fields.

Inside the neutron star crust, the perfect conductivity approximation

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}/c = -(\alpha \varpi / c)\boldsymbol{t} \times \boldsymbol{B} \tag{4}$$

forces the non-corotational potential ψ to be a constant ψ_s. To determine ψ outside the star one needs more physical input. A model that avoids unbalanced Maxwell stresses at the stellar surface postulates that the simple plasma condition $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ holds for at least a certain distance away from the star (Goldreich & Julian 1969, hereafter GJ; Mestel 1971; Cohen & Toton 1971). The constant value ψ_s is then propagated outwards along magnetic field lines, defining a 'GJ' magnetospheric domain in which $\boldsymbol{E}$ is again given by (4), and with the associated charge density ρ_e given by the Poisson–Maxwell equation

$$\rho_e = \frac{\nabla \cdot E}{4\pi} = -\frac{\alpha}{2\pi c}\boldsymbol{k} \cdot \left[\boldsymbol{B} - \frac{1}{2}\boldsymbol{r} \times (\nabla \times B)\right]$$
$$= -\frac{\alpha}{2\pi c}\left(\boldsymbol{k} \cdot \boldsymbol{B} - \frac{1}{2}\varpi \boldsymbol{t} \cdot \nabla \times B\right). \tag{5}$$

The GJ assumption converts part of the magnetosphere into a simple perfect conductor.

A realistic pulsar magnetosphere model will contain domains with field lines closing within the light-cylinder, containing charge that simply rotates, and domains with 'open' field lines, along

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which charges can flow out to and in from the light-cylinder. Any charge q moving in the GJ electric field (4) with the velocity

$$\boldsymbol{v} = \alpha \varpi \boldsymbol{t} + \kappa \boldsymbol{B} \tag{6}$$

(corotation plus motion along $\boldsymbol{B}$) is subject to a vanishing Lorentz force $q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$. In a 'charge-separated' domain, with only one sign of charge at each macroscopic point, multiplication by the number density then yields

$$\rho_{\mathrm{e}} \boldsymbol{E} + \frac{\boldsymbol{j} \times \boldsymbol{B}}{c} = 0, \tag{7}$$

where $\boldsymbol{j}$ is a pure *convection* current density $\rho_{\mathrm{e}} \boldsymbol{v}$. Thus, in a charge-separated domain, the simple plasma condition

$$\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c} = 0 \tag{8}$$

implies the force-free condition (7). If, however, the plasma is mixed, then, as in non-relativistic problems, the 'perfect conductivity' and 'force-free' approximations are distinct: charges may adjust themselves so as to 'short out' the component of $\boldsymbol{E}$ along $\boldsymbol{B}$, without *necessarily* requiring that e.g. the inertial forces are so small that the components normal to $\boldsymbol{B}$ of the Lorentz force density must nearly mutually cancel. (However, experience shows that the force-free approximation often remains good until the particles attain very high γ-values.)

In this paper we continue studying the effect of magnetospheric currents on the external magnetic field structure. Clearly, this cannot ultimately be separated from the formidable complete magnetosphere problem, which must involve demarcation of single-sign from mixed-sign domains, and domains of pure rotation from those with outflow, and estimation of the validity of both the simple plasma and force-free approximations. It is nevertheless helpful to begin by constructing simple idealized models, for both mathematical and physical reasons. Thus, in the axisymmetric, 'non-pulsar' problem, the structure out to the light-cylinder of the force-free electromagnetic field, owing to GJ charges which simply corotate (Michel 1973, 1991; Mestel & Wang 1979, hereafter MW; Mestel & Pryce 1992, hereafter MP; Mestel 1999), can be satisfactorily solved by Fourier transformation. The techniques developed are in fact applicable to the different, more realistic models in the literature (e.g. Beskin, Gurevich & Istomin 1993; Mestel & Shibata 1994), which include charge flow along 'open' field lines, and allow for relativistic acceleration near the star and consequent electron–positron pair production.

2 THE RELATIVISTIC FORCE-FREE EQUATION

Equations (4) and (7) combine to yield

$$\boldsymbol{j} - \rho_{\mathrm{e}} \alpha \varpi \boldsymbol{t} = \frac{c}{4\pi} \lambda \boldsymbol{B}; \tag{9}$$

the particle current density is the sum of the current arising from corotation of the charge density ρ_{e} plus the current $(c/4\pi)\lambda\boldsymbol{B}$ along $\boldsymbol{B}$. By charge conservation,

$$\nabla \cdot j = -\frac{\partial \rho_{\mathrm{e}}}{\partial t} = \alpha \frac{\partial \rho_{\mathrm{e}}}{\partial \Phi} \tag{10}$$

in quasi-static states, and $\nabla \cdot (\rho_{\mathrm{e}} \alpha \varpi \boldsymbol{t}) = \boldsymbol{t} \cdot \nabla(\rho_{\mathrm{e}} \alpha \varpi) = \alpha \partial \rho_{\mathrm{e}}/\partial \Phi$; hence the divergence of (9) yields

$$\boldsymbol{B} \cdot \nabla \lambda = 0 \tag{11}$$

on use of $\nabla \cdot B = 0$.

After substitution for ρ_{e} and $\boldsymbol{j}$ from the Poisson–Maxwell and Ampère–Maxwell equations, and for $\partial \boldsymbol{E}/\partial t$ from (4) and (1), (9) reduces to (Mestel 1973; Endean 1974; Mestel, Wright & Westfold 1976)

$$\nabla \times \tilde{\boldsymbol{B}} = \lambda \boldsymbol{B}, \tag{12}$$

where the vector $\tilde{\boldsymbol{B}}$ is defined by

$$\tilde{\boldsymbol{B}} = [B_{\varpi}(1 - x^2), B_{\Phi}, B_z(1 - x^2)] \tag{13}$$

with $x = (\alpha \varpi / c)$, and z is similarly dimensionless. Taking the divergence of (12) confirms the constancy of λ along field lines. The factor $(1 - x^2)$ in (13) is due to the join effect of the corotating GJ charge density and the displacement current associated with the rotating, non-axisymmetric electric field (4).

Equations (12) and (13) may alternatively be derived by making a local Lorentz transformation from the frame S with Cartesian axes (x, y, z) parallel to the unit vectors $(\hat{\boldsymbol{\varpi}}, \hat{\boldsymbol{\Phi}}, \hat{\boldsymbol{z}})$ to the frame S' with axes (x', y', z') instantaneously coinciding with (x, y, z) but moving with the local rotatory velocity $(0, \alpha\varpi, 0)$. In the frame S', the electric field $\boldsymbol{E}' = 0$, and the force-free condition reduces to $\boldsymbol{j}' = \lambda' \boldsymbol{B}'$. Use of the standard relations between electromagnetic quantities in S and S' then yields (12), with the scalars λ, λ' related by $\lambda = \lambda'(1 - x^2)$ (Mestel 1973). The non-relativistic force-free equation $\nabla \times B = \lambda \boldsymbol{B}$ is recovered when terms of order $(v/c)^2$ can be ignored.

If axisymmetry is imposed, $\partial/\partial\Phi = 0$, and 'quasi-steady' becomes 'steady'. The poloidal field $\boldsymbol{B}_{\mathrm{p}} = (B_x, B_z)$ is written in terms of the flux-function P, while currents $(c/4\pi)\lambda\boldsymbol{B}_{\mathrm{p}}$ along $\boldsymbol{B}_{\mathrm{p}}$ generate a toroidal field component B_Φ:

$$\boldsymbol{B} = \left[\frac{\partial P}{x \partial z}, -\frac{S(P)}{x}, -\frac{\partial P}{x \partial x}\right], \tag{14}$$

where $\lambda = \mathrm{d}S/\mathrm{d}P$. If $\lambda = 0$ (no currents along field lines) then P satisfies the Pryce–Michel equation

$$(1 - x^2)\nabla^2 P - \frac{2}{x}\frac{\partial P}{\partial x} = 0, \tag{15}$$

yielding the 'MMPW' field $(\boldsymbol{B}_{\mathrm{p}})_0$ of MW and MP. If the poloidal currents are non-zero but weak, then the associated toroidal field $\boldsymbol{B}_{\mathrm{t}} = B_\Phi \boldsymbol{t} = -[S(P)/x]\boldsymbol{t}$ may be constructed using the MMPW form for P.

3 THE PERPENDICULAR COROTATING MAGNETOSPHERE

The analogue of the MMPW problem is again given by putting $\lambda = 0$. Equation (12) then yields the non-dimensional form

$$[B_x(1 - x^2), B_\Phi, B_z(1 - x^2)] = -\nabla \chi. \tag{16}$$

We specialize further to the perpendicular rotator (Fig. 1), with B_x, B_Φ symmetric and B_z antisymmetric in the rotational equator.

Near the origin, $\boldsymbol{B} = -\nabla\chi$, so χ reduces to the scalar potential generated by the stellar field. As before, the star is approximated as a point dipole, but now aligned along the Y-axis; the dipole moment $\boldsymbol{p} = (B_{\mathrm{s}} R^3/2)\hat{\boldsymbol{Y}}$, generating the dimensional potential

$$\chi = p \frac{\varpi \sin \Phi}{(\varpi^2 + z^2)^{3/2}}, \tag{17}$$

with $\Phi = \pi/2$, $z = 0$ defining the magnetic axis, and B_{s} the polar field strength on the star of radius R. Normalized quantities are

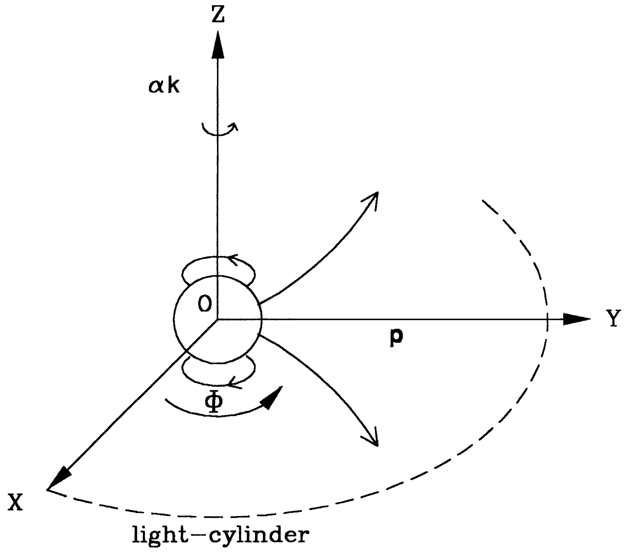

Figure 1. The perpendicular rotator: schematic diagram.

defined by

$$\bar{x} = \alpha\varpi/c, \quad \bar{z} = \alpha z/c, \quad \bar{\chi} = [(c/\alpha)^2/p]\chi. \tag{18}$$

(For convenience, the bars are immediately dropped.) The dimensionless form of (17) is the imaginary part of

$$\frac{x}{(x^2+z^2)^{3/2}}\exp(\mathrm{i}\Phi). \tag{19}$$

We thus look for the function $\chi(x,z)\exp(\mathrm{i}\Phi)$, reducing to (19) near the origin, and satisfying

$$x^2(1-x^2)\frac{\partial^2\chi}{\partial x^2} + x(1+x^2)\frac{\partial\chi}{\partial x} - (1-x^2)^2\chi + x^2(1-x^2)\frac{\partial^2\chi}{\partial z^2} = 0, \tag{20}$$

derived by imposing $\nabla\cdot\boldsymbol{B} = 0$.

The method of solution of (20) is closely parallel to that used by MW and MP (see Mestel 1999). The Fourier transform in z,

$$g(x,k) = \frac{2}{\pi}\int_0^\infty \chi(x,z)\cos(kz)\,\mathrm{d}z, \tag{21}$$

satisfies

$$x^2(1-x^2)g'' + x(1+x^2)g' - [(1-x^2)^2 + k^2x^2(1-x^2)]g = 0, \tag{22}$$

where the prime signifies $\partial/\partial x$.The solution of (22) near the origin is conveniently written

$$g(x) = \frac{2}{\pi}k\mathrm{K}_1(kx) + \frac{2}{\pi}f(x) \tag{23}$$

where K_1 is the modified Bessel function of the second kind (Erdelyi et al. 1954). Then $f(x)$ satisfies (22) with the zero on the right-hand side replaced by $-kx^2[2x\mathrm{K}_1' + (1-x^2)\mathrm{K}_1]$. The first term in (23) is the Fourier transform of (19) [without the $\exp(\mathrm{i}\Phi)$ factor], so the boundary condition at the origin is satisfied if $f(0)$ is finite. Near enough to the origin one can use series expansions for both K_1 and f, arriving at

$$g = \frac{2}{\pi}\left[\frac{1}{x} + \frac{1}{2}(k^2+1)x\log x + \mathrm{O}(x) + \ldots\right] + \frac{2}{\pi}a_k\left[x + (k^2-3)\frac{x^3}{8} + \ldots\right]. \tag{24}$$

Near $x = 1$, there is the solution $g = 1 - k^2(1-x)^2 + \ldots$, which yields B_z singular and so is dropped, and the acceptable solution

$$g = \frac{2}{\pi}b_k\left[(1-x)^2 + \frac{1}{3}(1-x)^3 + \frac{1}{4}\left(1+\frac{k^2}{2}\right)(1-x)^4 + \ldots\right]. \tag{25}$$

Again, the numerical continuation inwards of the solution (25) must link up smoothly with (24) or its numerical continuation, so fixing b_k and a_k/b_k. For moderate values of k, this procedure works satisfactorily with the analytic form (24) adopted for the outgoing solution. For large k the link-up point is beyond the domain of validity of the series approximations. One then extends the outgoing solution numerically, but using values for K_1 and K_1' from the computer library. As in MP, section 2, for very large k the JWKB method yields $a_k \approx 0$, $b_k \approx (\pi/\sqrt{2})k^2\,\mathrm{e}^{-k}$. Away from the singular points $x = 0$ and $x = 1$, $g \approx (2k/\pi)^{1/2}[(1-x^2)/x]^{1/2}\exp(-kx)$.

The whole solution may be constructed from equations (16) and the Fourier back-transform

$$\chi = \int_0^\infty g(x,k)\cos(kz)\,\mathrm{d}k, \tag{26}$$

with the field lines given by integration of the equations

$$\frac{(1-x^2)\,\mathrm{d}x}{(\partial\chi/\partial x)\sin\Phi} = \frac{x\,\mathrm{d}\Phi}{(\chi/x)\cos\Phi} = \frac{(1-x^2)\,\mathrm{d}z}{(\partial\chi/\partial z)\sin\Phi}. \tag{27}$$

Near the origin χ reduces to (17) and $(1-x^2) \approx 1$, whence equations (27) integrate to

$$\frac{z^2}{(x^2+z^2)^{3/2}} = C, \tag{28}$$

$$\cos\Phi = Dz^{1/3}/[1-(Cz)^{2/3}]^{1/2}, \tag{29}$$

where C and D are constants of integration. Values of $C \approx 1$ pertain to field lines that emerge from near the dipolar equator and so remain in the near-vacuum dipole domain for the whole of their length. Smaller values of C pertain to lines that emerge nearer to the poles. With C and D chosen, (28) and (29) fix z and Φ in terms of x a little way from the origin. Further out the field lines are continued by integration of (27) with the computed function χ inserted:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{(\partial\chi/\partial z)}{(\partial\chi/\partial x)} \tag{30}$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}x}\log(\cos\Phi) = \frac{(1-x^2)}{x^2}\left[\frac{\chi}{(\partial\chi/\partial x)}\right]. \tag{31}$$

In Fig. 2(a) there are shown some typical field lines lying in the plane defined by the rotation and magnetic axes; in Fig. 2(b) are shown field lines lying in the equatorial plane defined by the rotation axis.

From (16) and (25), at the light-cylinder, B_x is finite, while B_z and B_Φ vanish like $(1-x)$ and $(1-x)^2$ respectively. The

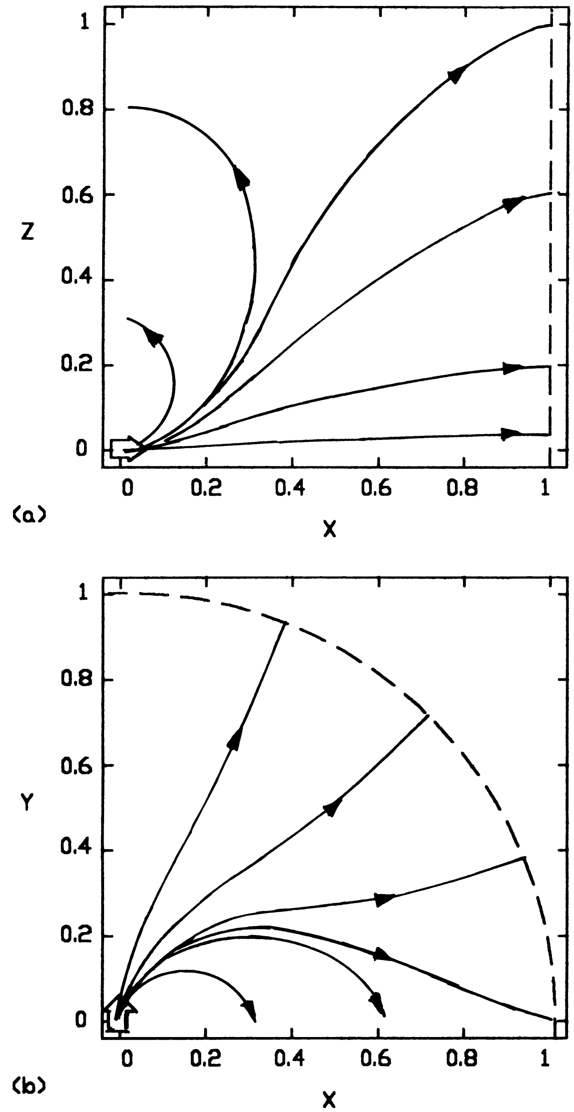

Figure 2. The corotating perpendicular magnetosphere. (a) Field lines in the plane defined by the rotation and magnetic axes. (b) Field lines in the equatorial plane defined by the rotation axis.

x-component of the Poynting vector $\propto -B_x B_\Phi$ vanishes: as in the aligned case, transport of energy and angular momentum by a force-free field requires current flow along the field lines.

This conclusion illustrates strikingly the stringent nature of the GJ postulate $\boldsymbol{E}\cdot\boldsymbol{B}=0$. The solution for a non-aligned dipole *in vacuo* – extended by Deutsch (1955) to the case with the dipolar flux emanating from a perfectly conducting, rigidly rotating star of finite radius – does indeed have a non-vanishing Poynting integral, but has also a component of $\boldsymbol{E}$ along $\boldsymbol{B}$ that remains comparable to that normal to $\boldsymbol{B}$ well into the wave zone. The GJ condition alters radically the mathematical structure of the problem. In the vacuum case the differential equation has no singularities at a finite distance, and the radius of the light-cylinder is a measure of the distance at which the induction-dominated field goes over into a radiation field. By contrast, in the GJ force-free case, the light-cylinder is a singularity of the differential equation to P, and as seen above, in the absence of field-aligned currents, a non-singular solution of (12) (with zero λ) is forced to have B_Φ zero at the light-cylinder, yielding a vanishing Poynting flux.

In fact, if one tries to extend to infinity the above model by attaching a vacuum field, continuous at the light-cylinder in B_x, E_z and E_Φ, and going to zero at infinity, the resulting discontinuities in B_Φ, B_z and E_x again imply a surface charge-current distribution with non-cancelling Maxwell stresses (cf. Mestel & Wang 1982). There must therefore be a charge-current field beyond the light-cylinder, and it is again instructive to make the illicit extrapolation of the corotating solution, imposing continuity of $\boldsymbol{B}$. Analysis analogous to that in section 5 of Mestel & Shibata (1994) yields $g_k \approx (2k/\pi)^{1/2}[(x^2-1)/x]^{1/2}\exp[k(x-2)]$, which in turn yields a χ that blows up beyond $x=2$: again, the absence of a second non-singular solution at the light-cylinder prevents the construction of a smoothly continuous solution extending to infinity. Thus any realistic generalization of the present model must have currents flowing along the field lines that cross the light-cylinder. The ultimate complete global solution should predict both the strength and the structure of the currents, and also the likely departure of $\boldsymbol{E}$ from the GJ condition, both of which contribute to the Poynting flux. In this paper, we are content to illustrate the effect of field-aligned currents but with $\boldsymbol{E}\cdot\boldsymbol{B}=0$ retained, i.e. we consider solutions of (12) with $\lambda \neq 0$.

4 THE PERPENDICULAR ROTATOR: FIELD-ALIGNED CURRENTS

In the present illustrative study, the currents flowing along the field lines are treated as prescribed input. An acceptable model will distinguish between field lines that close within the light-cylinder and those that extend beyond. As in the axisymmetric problem, the closed field lines can carry the corotating GJ charges, a pure electron cloud in domains where $\boldsymbol{B}\cdot\boldsymbol{k}>0$ and a pure ion cloud where $\boldsymbol{B}\cdot\boldsymbol{k}<0$. Field lines emanating from the polar caps will cross the light-cylinder, and as in the axisymmetric problem it is along these that we expect field-aligned currents to flow, carrying energy and angular momentum outwards. However, as pointed out originally by Kahn (1971), and later by da Costa & Kahn (1982), a distinction must be drawn also between field lines that are respectively forward-pointing and backward-pointing about the rotation axis. Outward-flowing, single-sign currents can continue to follow backward-pointing field lines across the light-cylinder, but will inevitably break away from forward-pointing field lines as the light-cylinder is approached. As for the aligned problem, quasi-magnetohydrodynamic behaviour may very well break down for *all* field streamlines; the difference is that the critical surface on which breakdown occurs coincides in part with the light-cylinder.

It is clear that the construction of a strictly self-consistent model even just out to the breakdown surface will require much iteration. However, for all but the shortest period pulsars, the polar funnels along which the currents flow occupy a small fraction of a volume $(c/\alpha)^3$. Also, in this perpendicular case the GJ charge density in the funnel is smaller than for the aligned case by a factor $|\boldsymbol{k}\cdot\boldsymbol{B}|/B$, so that the current density owing to the convection of this density will also be small, suggesting that the field model of the last section remains a good zeroth-order approximation. Even so, the computation of the dissipative flow beyond the light-cylinder of the forward-directed currents, and the subsequent construction of the modified field, is a formidable task.

For the moment we are content just to illustrate the effect of the field-aligned currents. From now on, the suffix 0 refers to the pure corotation field of the last section.We indeed solve the relativistic force-free equation (12) as a perturbation problem, with the new currents taken to flow along $\boldsymbol{B}_0$, and generating a correcting field $\boldsymbol{B}_1$. However, we consider the mathematically simplest example, taking λ as constant in each half-space $|z| > 0$:

$$\lambda = -|\lambda|\mathrm{sign}(z). \tag{32}$$

The equatorial symmetry of the problem ensures that the total outflow of charge is zero. From Maxwell's equations (including the quasi-steady displacement current term), the perturbed $\boldsymbol{B}$-field has B_ϖ and B_Φ symmetric and B_z antisymmetric in z, just like $\boldsymbol{B}_0$.

Note that in the region near the north magnetic pole, and with $z > 0$, $\rho_e \propto -\boldsymbol{B}\cdot\boldsymbol{k} < 0$, so that a GJ convection current consists of outflowing electrons; whereas, in the corresponding region near the south magnetic pole $\rho_e > 0$, so that the GJ charge density consists of positively charged ions, and the negative inward current $-(c/4\pi)|\lambda|\boldsymbol{B}_0$ is in fact an outflow of ions. Likewise, in the analogous regions with $z < 0$ the signs are reversed, with ions flowing out near the north pole and electrons near the south.

The limitations of the problem so formulated are manifest. Near and beyond the light-cylinder it takes no account of the distinction between forward-pointing and backward-pointing field lines. Within the light-cylinder it postulates current flow along all field lines, including those (the majority) that close within the light-cylinder; and along the funnel that crosses the light-cylinder, the constancy of λ in each hemisphere implies a discontinuous change at the equator in the sign of the current rather than a smooth transition through zero. The principal argument for this preliminary study is that the resulting mathematical problem is linear and is again easily solvable by Fourier transformation.

The discussion so far has been implicitly in terms of a 'classical' model magnetosphere, e.g. as in Mestel et al. (1985) and Fitzpatrick & Mestel (1988a,b). It is in fact possible that, in the negatively charged domain ($\boldsymbol{B}\cdot\boldsymbol{k} > 0$), the electrons emitted from the polar regions will be locally accelerated to high γ-values near the star, with consequent $\pm$e pair production near the star by the gamma-rays emitted by the primary electrons (Sturrock 1971, and many papers since, e.g. Ruderman & Sutherland 1975; Arons & Scharlemann 1979; Jones 1980; Arons 1981; Daugherty & Harding 1982; Shibata 1991; Mestel & Shibata 1994). A dense $\pm$e plasma will attempt to short out the accelerating $\boldsymbol{E}_\parallel$-field by the usual process of charge polarization, so that, in the post-pair production domain, the GJ condition may be re-established. The generation of a dense $\pm$e plasma allows a *conduction* current to flow – again nearly parallel to $\boldsymbol{B}$ – owing to mutual motion of oppositely signed charges, so allowing an increase in the magnitude of the parameter $|\lambda|$. However, the mathematical technique adopted in this section assumes implicitly that the perturbing field is small compared with the basic field $\boldsymbol{B}_0$.

Recall that the suffix 0 refers to the pure corotation field of Section 3, and the suffix 1 to the perturbation field associated with the field-aligned currents $\lambda\boldsymbol{B}_0$. With λ constant in each half-space, (12) integrates to

$$\tilde{\boldsymbol{B}}_1 = \lambda\boldsymbol{A}_0 - \nabla\chi_1, \tag{33}$$

where $\boldsymbol{A}_0$ is a vector potential describing the zeroth-order field $\boldsymbol{B}_0$:

$$\frac{1}{x}\left(\mathrm{i}A_{0z} - x\frac{\partial A_{0\Phi}}{\partial z}\right) = -\frac{1}{1-x^2}\frac{\partial\chi_0}{\partial x}, \tag{34}$$

$$\frac{\partial}{\partial z}(A_{0x}) - \frac{\partial}{\partial x}(A_{0z}) = -\frac{\mathrm{i}\chi_0}{x}, \tag{35}$$

$$\frac{1}{x}\left[\frac{\partial}{\partial x}(xA_{0\Phi}) - \mathrm{i}A_{0x}\right] = -\frac{1}{(1-x^2)}\frac{\partial\chi_0}{\partial z}. \tag{36}$$

It is convenient to choose the specially simple form for $\boldsymbol{A}_0$:

$$A_{0z} = 0, \qquad \frac{\partial A_{0\Phi}}{\partial z} = \frac{\partial\chi_0/\partial x}{(1-x^2)}, \qquad \frac{\partial A_{0x}}{\partial z} = -\frac{\mathrm{i}\chi_0}{x}. \tag{37}$$

Equations (34) and (35) are immediately satisfied, while substitution in (36) yields just the z-derivative of (20). With use of (37), the condition $\nabla\cdot B_1 = 0$ reduces in the northern hemisphere to

$$x^2(1-x^2)\xi_1'' + x(1+x^2)\xi_1' - (1-x^2)^2\xi_1 + x^2(1-x^2)(\xi_1)_{zz} = -2\mathrm{i}|\lambda|x^2\chi_0, \tag{38}$$

where

$$\xi_1 \equiv \partial\chi_1/\partial z. \tag{39}$$

In the southern hemisphere, the right-hand side of (38) has a +-sign. Recall that χ_0 has even parity (and so is written in Section 3 as a Fourier cosine-integral). Thus $\pm\chi_0$ has odd parity and so is written as a Fourier sine-integral: in the northern hemisphere,

$$\chi_0(x,z) = \int_0^\infty h_0(x,k)\sin(kz)\,\mathrm{d}k, \tag{40}$$

$$h_0(x,k) = \frac{2}{\pi}\int_0^\infty \chi_0(x,z)\sin(kz)\,\mathrm{d}z. \tag{41}$$

Since ξ_1 is of odd parity, vanishing on the rotational equator $z = 0$, it is convenient to define its Fourier *sine* transform

$$g_{1k} = \frac{2}{\pi}\int_0^\infty \left(\frac{\xi_1}{2\mathrm{i}|\lambda|}\right)\sin(kz)\,\mathrm{d}z. \tag{42}$$

The Fourier sine transform of (38) then becomes

$$x^2(1-x^2)g_{1k}'' + x(1+x^2)g_{1k}' - [(1-x^2)^2 + k^2x^2(1-x^2)]g_{1k} = -x^2h_{0k}, \tag{43}$$

where use is made of (33) and (37), and of the boundary condition on $z = 0$:

$$\xi_1 \equiv \frac{\partial\chi_1}{\partial z} = -B_{1z} = 0. \tag{44}$$

In principle, solution for $\boldsymbol{B}_1$ proceeds as follows. The left-hand-side operator in (43) is identical with that occurring in (22). Near $x = 1$, the non-singular complementary function solution is written

$$\frac{2}{\pi}c_k\left[(1-x)^2 + \frac{1}{3}(1-x)^3 + \frac{1}{4}\left(1+\frac{k^2}{2}\right)(1-x)^4 + \ldots\right]. \tag{45}$$

Near $x = 0$, we must now select for the complementary function just the non-singular solution (the imposed dipole already forms part of the solution for χ_0):

$$\frac{2}{\pi}d_k\left[x + \frac{1}{8}(k^2-3)x^3 + \ldots\right]. \tag{46}$$

The complete solution will include particular integrals of (43). Near $x = 1$, by (25) and (26),

$$\chi_0 \simeq \left[(2/\pi)\int_0^\infty b_k\cos(kz)\,\mathrm{d}k\right](1-x)^2, \tag{47}$$

with b_k known from the corotating magnetosphere solution already constructed; hence, from (41),

$$h_0 = \frac{2}{\pi}\int_0^\infty \chi_0 \sin(kz)\,\mathrm{d}z \equiv K_k(1-x)^2, \tag{48}$$

where

$$K_k = \frac{4}{\pi^2}\int_0^\infty \sin(kz)\left[\int_0^\infty b'_k \cos(k'z)\,\mathrm{d}k'\right](1-x)^2. \tag{49}$$

A particular integral of (43) near $x=1$ is

$$-\frac{K_k}{6}(1-x)^3. \tag{50}$$

Near $x=0$, by (19), $\chi_0 \simeq x/(x^2+z^2)^{3/2}$, so

$$h_0 = \frac{2}{\pi}x\int_0^\infty \frac{\sin(kz)}{(x^2+z^2)^{3/2}}\,\mathrm{d}z = \frac{2}{\pi}x\left(-\frac{\pi}{2}\frac{k}{x}\right)[\mathrm{I}_1(kx) - \mathrm{L}_{-1}(kx)], \tag{51}$$

where I_1 is as usual a Bessel function of imaginary argument and L_{-1} is a Struve function (Gradshteyn & Ryzhik 1980, pp. 426, 961, 982). When kx is small, $\mathrm{I}_1 \approx kx/2 + \mathrm{O}(kx)^3$, $\mathrm{L}_{-1} \simeq 2/\pi + 2(kx)^2/3\pi$, so that

$$-x^2 h_0 = -\frac{2k}{\pi}x^2 + \frac{k^2}{2}x^3 + \mathrm{O}(k^2x^4). \tag{52}$$

A convenient particular integral of (43) is then

$$-\frac{2k}{3\pi}x^2. \tag{53}$$

Thus, near $x=1$, the solution is the sum of (45) and (50); near $x=0$ it is the sum of (46) and (53). Again, for each value of k, c_k and d_k are to be determined by smooth link-up (continuity of each g_{1k} and its x-derivative) at a conveniently chosen point.

5 THE ENERGY OUTFLOW

Because the problem as formulated has an equatorial discontinuity in the field-aligned currents (shown by the change of sign in λ), the Fourier back-transform will show the 'Gibbs phenomenon' (e.g. Jeffreys & Jeffreys 1966): the inevitable cut-off in the Fourier integral at a finite k-value will always yield a poor approximation near $z=0$. This is the price paid for the mathematical simplification of taking λ constant in each hemisphere. In a more realistic problem, with convection currents of strength fixed by the GJ charge density, λ must be constant on each line of $\boldsymbol{B}_0$, but will change sign smoothly, going to zero like $\boldsymbol{k}\cdot\boldsymbol{B}$ at the equator, so there will be no Gibbs phenomenon. However, this will be more than offset by the complications ensuing from the variation of λ between field lines. For example, one will no longer be able to factor out $\exp(\mathrm{i}\Phi)$: a strictly dipolar field structure imposed on the star will generate harmonics in the magnetosphere.

To complete study of our idealized, illustrative model, we note that – just as in the aligned case – the current flow along the field lines yields a non-zero $B_{1\phi}$-component, and so a flow of energy and angular momentum across the light-cylinder. Further, this energy loss can be found without the tedium of constructing the field $\boldsymbol{B}_1$. By (4) and the GJ approximation, the ϖ-component of the Poynting vector $(c/4\pi)(\boldsymbol{E}\times\boldsymbol{B})$ becomes

$$-(c/4\pi)(B_\varpi B_\Phi)_{\mathrm{lc}}, \tag{54}$$

or

$$-(c/4\pi)[B_{0x}B_{1\Phi}]_{x=1} \tag{55}$$

to first order in the parameter $|\lambda|$. By (16), (25) and (26), the normalized B_{0x} is given by

$$\frac{B_{0x}(1,z)}{(\alpha/c)^3 p} = -\left[\frac{1}{(1-x^2)}\frac{\partial\chi_0}{\partial x}\right]_{x\to 1}\sin\Phi$$
$$= \left[\frac{2}{\pi}\int_0^\infty b_k\cos(kz)\,\mathrm{d}k\right]\sin\Phi = \left[\frac{\chi_0(x,z)}{(1-x)^2}\right]_{x\to 1}\sin\Phi, \tag{56}$$

where the imaginary part has now been taken. By (33) [$\exp(\mathrm{i}\Phi)$ temporarily omitted],

$$B_{1\Phi}(1,z) = -|\lambda|A_{0\Phi} - \mathrm{i}\chi_1, \tag{57}$$

and so, by (37),

$$\frac{\partial}{\partial z}B_{1\Phi} = -\mathrm{i}\frac{\partial\chi_1}{\partial z} - |\lambda|\frac{\partial A_{0\Phi}}{\partial z} = -|\lambda|\left[\frac{1}{(1-x^2)}\frac{\partial\chi_0}{\partial x}\right]_{x\to 1}, \tag{58}$$

since, by (45) and (50), $\xi_1(1) = (\partial\chi_1/\partial z)(1) = 0$. Substitution from (25) yields further

$$\frac{\partial}{\partial z}B_{1\Phi} = \frac{2}{\pi}|\lambda|\int_0^\infty b_k\cos(kz)\,\mathrm{d}k = |\lambda|\left[\frac{\chi_0}{(1-x)^2}\right]_{x\to 1}, \tag{59}$$

whence

$$B_{1\Phi}(1,z) = -|\lambda|\sin\Phi\int_z^\infty\left[\frac{\chi_0}{(1-x)^2}\right]_{x\to 1}\mathrm{d}z, \tag{60}$$

where the condition $\boldsymbol{B}\to 0$ as $z\to\infty$ has been used, and the imaginary part again taken. Thus, from (56),

$$B_{1\Phi}(1,z) = -|\lambda|\int_z^\infty B_{0x}(1,z)\,\mathrm{d}z. \tag{61}$$

Whereas $B_{0\Phi}$ is $\pi/2$ out of phase with B_{0x}, $B_{1\Phi}$ and B_{0x} are in phase, yielding for the ϖ-component (55) of the Poynting vector at the light-cylinder

$$\left(\frac{\alpha}{c}\right)^6 p^2\frac{c}{4\pi}|\lambda|\sin^2\Phi B_{0x}(1,z)\left[\int_z^\infty B_{0x}(1,z)\,\mathrm{d}z\right]. \tag{62}$$

The Poynting integral over the light-cylinder from $z=-\infty$ to $+\infty$ and from $\Phi=0$ to 2π then becomes

$$\frac{\alpha^4p^2}{4c^3}|\lambda|\left[\int_0^\infty B_{0x}(1,\bar{z})\mathrm{d}\bar{z}\right]^2, \tag{63}$$

where $\bar{z}$ is again the normalized length defined in (18). It is seen that, to first order in the perturbing parameter $|\lambda|$, the energy and angular momentum loss rate can be computed from knowledge of the *zeroth-order* field (cf. Mestel & Selley 1970). The basic reason is that the available voltage is fixed by the magnetic flux crossing the light-cylinder, and the current density is given by $\lambda\boldsymbol{B}$. Thus, to determine the energy flow to first order in λ, one needs just the zeroth-order field $\boldsymbol{B}_0$ in both the current and the voltage.

REFERENCES

Arons J., 1981, ApJ, 248, 1099
Arons J., Scharlemann E. T., 1979, ApJ, 231, 854
Beskin V. S., Gurevich A. V., Istomin Ya.N., 1993, Physics of the Pulsar Magnetosphere, Cambridge Univ. Press, Cambridge
Cohen J. M., Toton E. T., 1971, Astrophys. Lett., 7, 21
da Costa A. A., Kahn F. D., 1982, MNRAS, 199, 211
Daugherty J. K., Harding A. K., 1982, ApJ, 252, 337
Deutsch A. J., 1955, Ann. Astrophys., 18, 1

Endean V. G., 1974, ApJ, 187, 359
Erdelyi A., Magnus W., Oberhettinger F., Tricomi F. G., 1954, Tables of integral transforms, McGraw-Hill, New York
Fitzpatrick R., Mestel L., 1988a, MNRAS, 232, 277
Fitzpatrick R., Mestel L., 1988b, MNRAS, 232, 303
Goldreich P., Julian W. H., 1969, ApJ, 157, 869 (GJ)
Gradshteyn I. S., Ryzhik I. M., 1980, Tables of Integrals, Series and Products, corrected and enlarged edition, Academic Press Inc, London
Jeffreys H., Jeffreys B. S., 1966, Methods of Mathematical Physics, 3rd edn. Cambridge Univ. Press, Cambridge
Jones P. B., 1980, MNRAS, 192, 847
Kahn F. D., 1971, paper read at Royal Astronomical Society, unpublished
Mestel L., 1971, Nat. Phys. Sci., 233, 149
Mestel L., 1973, Ap&SS, 24, 289
Mestel L., 1999, Stellar magnetism, Oxford Univ. Press, Oxford
Mestel L., Pryce M. H. L., 1992, MNRAS, 254, 355 (MP)
Mestel L., Selley C. S., 1970, MNRAS, 149, 197
Mestel L., Shibata S., 1994, MNRAS, 271, 621
Mestel L., Wang Y.-M., 1979, MNRAS, 188, 799 (MW)
Mestel L., Wang Y.-M., 1982, MNRAS, 198, 405
Mestel L., Wright G. A. E., Westfold K. C., 1976, MNRAS, 175, 257
Mestel L., Robertson J. A., Wang Y. M., Westfold K. C., 1985, MNRAS, 217, 443
Michel F. C., 1973, ApJ, 180, 207
Michel F. C., 1991, Theory of neutron star magnetospheres, Univ. Chicago Press, Chicago
Pacini F., 1967, Nat, 216, 567
Ruderman M. A., Sutherland P. G., 1975, ApJ, 196, 51
Shibata S., 1991, ApJ, 378, 239
Sturrock P. A., 1971, ApJ, 164, 529

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