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*Research Article*

# $\mathscr{H}_\infty$ Model Reduction for Discrete-Time Markovian Jump Systems with Deficient Mode Information

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This paper investigates the problem of $\mathscr{H}_\infty$ model reduction for a class of discrete-time Markovian jump linear systems (MJLSs) with deficient mode information, which simultaneously involves the exactly known, partially unknown, and uncertain transition probabilities. By fully utilizing the properties of the transition probability matrices, together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion for the underlying MJLSs is first derived, and then two approaches, namely, the convex linearisation approach and iterative approach, for the $\mathscr{H}_\infty$ model reduction synthesis are proposed. Finally, a simulation example is provided to illustrate the effectiveness of the proposed design methods.

## 1. Introduction

Markovian jump linear systems (MJLSs), initially proposed by Krasovskiĭ and Lidskiĭ [1], have been attracting increasing attention over the past decades. The appeal for the study of this class of hybrid systems is that MJLSs are powerful to model plants whose structures are subject to random abrupt changes, for example, the fault-prone manufacturing systems [2], power systems [3], economic systems [4], networked control systems (NCSs) [5–10], and so on. As a critical factor, the transition probabilities (TPs) in the Markov chain determine the jump rules of different modes and are assumed to be completely known in most of the existing literature [11–14].

However, it is noted that, in many practical engineering systems, not all the mode transition information can be obtained exactly. Recently, there have appeared some results on the analysis and synthesis of MJLSs with uncertain TPs [15–18] or partially unknown TPs [19–21]. To mention a few, the authors in [15] considered the robust stochastic stability analysis problem for a class of MJLSs with norm-bounded uncertainties in the TPs; the author in [16] dealt with the robust stability analysis and stabilization problems for MJLSs with polytopic-type uncertain TPs; the authors in [17] studied the robust stability analysis for a class of discrete-time Markovian jumping neural networks with defective statistics of mode transitions; the authors in [19] investigated the $\mathscr{H}_\infty$ filtering design for MJLSs with partially unknown TPs. In particular, it is noted that the introduction of deficient mode information, which simultaneously contains the exactly known, partially unknown, and uncertain TPs, is very important and useful in both theoretical advances and practical applications of MJLSs.

On the other hand, in many engineering applications, high- or even infinite-order complex mathematical models are frequently used to describe physical systems and processes [22–24]. These high-order models bring serious difficulties to analysis and synthesis of the underlying systems. Therefore, model simplification/reduction with respect to certain criteria has received considerable attention in recent years. Correspondingly, many important results on model reduction have been obtained, involving various efficient approaches such as the Hankel norm approximation method [25], the $\mathscr{H}_2$ approach [26, 27], the $\mathscr{H}_\infty$ approach [28–30], and the $\mathscr{L}_2$-$\mathscr{L}_\infty$ approach [31]. More recently, the linear matrix inequality (LMI) technique has been exploited to solve the model reduction problem for different classes of systems, such as bilinear systems [27], time-delay systems [28], and T-S fuzzy systems [32–36]. Specifically, the authors in [4, 20] considered the $\mathscr{H}_\infty$ model reduction problem for a class of MJLSs with completely known TPs and partially unknown TPs, respectively. However, to the authors' best knowledge, few results have been reported on $\mathscr{H}_\infty$ model reduction for MJLSs with deficient mode information, which simultaneously consists of the exactly known, partially unknown, and uncertain TPs. This motivates us for the present study.

According to the issues mentioned previously, in this paper, we will make an attempt to tackle the $\mathscr{H}_\infty$ model reduction problem for a class of discrete-time MJLSs with deficient mode information. Such deficient mode information simultaneously contains the exactly known, partially unknown, and uncertain TPs, which is a more practical scenario. By fully utilizing the properties of the transition probability matrices, together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion for MJLSs with deficient mode information will be firstly derived. To solve the model reduction problem, two distinctly different approaches will be then proposed. The first approach is based on a linearisation procedure, which casts the model reduction into a convex optimization problem. The second one, which is based on the cone complementarity linearisation (CCL) idea [37, 38], casts the model reduction into a sequential minimization problem subject to LMI constraints. The effectiveness and superiority of the proposed approaches will be illustrated by a simulation example.

*Notations*. The notations used throughout the paper are standard. $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ denote, respectively, the $n$-dimensional Euclidean space and the set of all $m \times n$ real matrices; $\mathbb{N}^+$ represents the set of positive integers; the notation $P > 0$ means that $P$ is real symmetric and positive definite; $\mathbf{I}$ and $\mathbf{0}$ represent the identity matrix and a zero matrix, respectively; $\|\cdot\|$ refers to the Euclidean vector norm; $l_2[0,\infty)$ is the space of square summable infinite sequence and, for $w = \{w(k)\} \in l_2[0,\infty)$, its norm is given by $\|w\|_2 = \sqrt{\sum_{k=0}^{\infty}\|w(k)\|^2}$; $(\mathcal{S}, \mathcal{F}, \mathcal{P})$ denotes a complete probability space, in which $\mathcal{S}$ is the sample space, $\mathcal{F}$ is the $\sigma$ algebra of subsets of the sample space, and $\mathcal{P}$ is the probability measure on $\mathcal{F}$; $\mathscr{E}[\cdot]$ stands for the mathematical expectation and, for the sequence $z = \{z(k)\} \in l_2((\mathcal{S},\mathcal{F},\mathcal{P}),[0,\infty))$, its norm is given by $\|z\|_{\mathscr{E}_2} = \sqrt{\mathscr{E}[\sum_{k=0}^{\infty}\|z(k)\|^2]}$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Problem Formulation and Preliminaries

A discrete-time Markovian jump linear system (MJLS) on a complete probability space $(\mathcal{S}, \mathcal{F}, \mathcal{P})$ can be described as

$$(\Sigma): \quad \begin{aligned} x(k+1) &= A(r(k))x(k) + B(r(k))u(k), \\ z(k) &= C(r(k))x(k) + D(r(k))u(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbf{R}^{n_x}$ is the state vector; $u(k) \in \mathbf{R}^{n_u}$ is the control input vector which belongs to $l_2[0,\infty)$; $z(k) \in \mathbf{R}^{n_z}$ is the output vector. The process $\{r(k), k \geq 0\}$ is described by a discrete-time homogeneous Markov chain, which takes values in a finite set $\mathcal{I} := \{1,\ldots,N\}$ with mode transition probabilities (TPs) $\Pr(r(k+1) = j \mid r(k) = i) = \pi_{ij}$, where $\pi_{ij} \geq 0$, for all $i, j \in \mathcal{I}$, and $\sum_{j=1}^{N}\pi_{ij} = 1$. For $r(k) = i$, $i \in \mathcal{I}$, the system matrices of the $i$th mode are denoted by $(A_i, B_i, C_i, D_i)$, which are real and known. Throughout the paper, we assume that system $(\Sigma)$ is stochastically stable, which is a prerequisite for model reduction design.

In addition, the TPs of the jumping process are considered to be polytopic-type uncertain and partially accessed; that is, the transition probability matrix (TPM) $\Pi = [\pi_{ij}]_{N\times N}$ is assumed to belong to a given polytope $\mathrm{P}_\Pi := \{\Pi \mid \Pi = \sum_{s=1}^{M}\alpha_s\Pi_s; \alpha_s \geq 0, \sum_{s=1}^{M}\alpha_s = 1\}$, where vertices $\Pi_s = [\pi_{ij}]_{N\times N}$, $i, j \in \mathcal{I}$, $s = 1,2,\ldots,M$ are given TPMs containing unknown elements still. For instance, for system $(\Sigma)$ with four operation modes, the TPM may be as

$$\begin{bmatrix} \pi_{11} & \tilde{\pi}_{12} & \hat{\pi}_{13} & \pi_{14} \\ \hat{\pi}_{21} & \pi_{22} & \tilde{\pi}_{23} & \pi_{24} \\ \pi_{31} & \hat{\pi}_{32} & \pi_{33} & \hat{\pi}_{34} \\ \pi_{41} & \tilde{\pi}_{42} & \hat{\pi}_{43} & \tilde{\pi}_{44} \end{bmatrix}, \tag{2}$$

where the elements labeled with "$\hat{\ }$" and "$\tilde{\ }$" represent the unknown information and polytopic uncertainties on TPs, respectively, and the others are known TPs. For notational clarity, for all $i \in \mathcal{I}$, we denote $\mathcal{I} = \mathcal{I}_{\mathcal{K}}^{(i)} \cup \mathcal{I}_{\mathcal{UC}}^{(i)} \cup \mathcal{I}_{\mathcal{UK}}^{(i)}$ as follows:

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(i)} &:= \left\{j : \pi_{ij} \text{ is known}\right\}, \\ \mathcal{I}_{\mathcal{UC}}^{(i)} &:= \left\{j : \tilde{\pi}_{ij} \text{ is uncertain}\right\}, \\ \mathcal{I}_{\mathcal{UK}}^{(i)} &:= \left\{j : \hat{\pi}_{ij} \text{ is unknown}\right\}. \end{aligned} \tag{3}$$

Moreover, if $\mathcal{I}_{\mathcal{K}}^{(i)} \neq \emptyset$ and $\mathcal{I}_{\mathcal{UC}}^{(i)} \neq \emptyset$, it is further described as

$$\begin{aligned} \mathcal{I}_{\mathcal{K}}^{(i)} &:= \left\{\mathcal{K}_1^{(i)},\ldots,\mathcal{K}_{t_i}^{(i)}\right\}, \quad \forall 1 \leq t_i \leq N-2, \\ \mathcal{I}_{\mathcal{UC}}^{(i)} &:= \left\{\mathcal{U}_1^{(i)},\ldots,\mathcal{U}_{v_i}^{(i)}\right\}, \quad \forall 1 \leq v_i \leq N, \end{aligned} \tag{4}$$

where $\mathcal{K}_{t_i}^{(i)} \in \mathbb{N}^+$ represents the $t_i$th known element with the index $\mathcal{K}_{t_i}^{(i)}$ in the $i$th row of TPM and $\mathcal{U}_{v_i}^{(i)} \in \mathbb{N}^+$ represents the $v_i$th uncertain element with the index $\mathcal{U}_{v_i}^{(i)}$ in the $i$th row of TPM, respectively. Obviously, $1 \leq t_i + v_i \leq N$. Also, we denote

$$\pi_{\mathcal{UK}}^{(is)} := \sum_{j\in\mathcal{I}_{\mathcal{UK}}^{(i)}} \hat{\pi}_{ij} = 1 - \sum_{j\in\mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij} - \sum_{j\in\mathcal{I}_{\mathcal{UC}}^{(i)}} \tilde{\pi}_{ij}^{(s)}, \tag{5}$$

where $\tilde{\pi}_{ij}^{(s)}$ represents an uncertain TP in the $s$th polytope, for all $s = 1,\ldots,M$.

To approximate the original MJLS (1), in this paper, we are interested in designing the following mode-dependent reduced-order model:

$$(\bar{\Sigma}): \quad \begin{aligned} \hat{x}(k+1) &= A_{ri}\hat{x}(k) + B_{ri}u(k), \\ \hat{z}(k) &= C_{ri}\hat{x}(k) + D_{ri}u(k), \end{aligned} \tag{6}$$

where $\widehat{x}(k) \in \mathbf{R}^{n_r}$ $(n_r < n_x)$, $\widehat{z}(k) \in \mathbf{R}^{n_z}$, and $A_{ri} \in \mathbf{R}^{n_r \times n_r}$, $B_{ri} \in \mathbf{R}^{n_r \times n_u}$, $C_{ri} \in \mathbf{R}^{n_z \times n_r}$, and $D_{ri} \in \mathbf{R}^{n_z \times n_u}$ are the gains of the reduced-order models to be determined.

Define $\overline{x}(k) := \left[x^T(k) \;\; \widehat{x}^T(k)\right]^T$ and $\overline{z}(k) := z(k) - \widehat{z}(k)$. Then, by augmenting (1) and (6), the model error dynamics can be described as

$$\left(\overline{\Sigma}\right): \quad \begin{aligned} \overline{x}(k+1) &= \overline{A}_i \overline{x}(k) + \overline{B}_i u(k), \\ \overline{z}(k) &= \overline{C}_i \overline{x}(k) + \overline{D}_i u(k), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \overline{A}_i &:= \begin{bmatrix} A_i & \mathbf{0} \\ \mathbf{0} & A_{ri} \end{bmatrix}, \qquad \overline{B}_i := \begin{bmatrix} B_i \\ B_{ri} \end{bmatrix}, \\ \overline{C}_i &:= \begin{bmatrix} C_i & -C_{ri} \end{bmatrix}, \qquad \overline{D}_i := D_i - D_{ri}. \end{aligned} \tag{8}$$

Therefore, the purpose of this paper is to design a mode-dependent reduced-order model in the form of (6), such that the model error system $(\overline{\Sigma})$ in (7) with deficient mode information is stochastically stable and the induced $l_2$-norm of the operator from $u(k)$ to the model error $\overline{z}(k)$ is less than $\gamma$; that is, $\|\overline{z}(k)\|_{\mathscr{E}_2} < \gamma \|u(k)\|_2$, under zero initial conditions for any nonzero $u(k) \in l_2[0, \infty)$.

Before ending the section, we give the following lemma on the $\mathscr{H}_\infty$ performance analysis of system (7) with *completely known* TPs, which will be used in the proof of our main results.

**Lemma 1** (see [4]). *For the MJLS* (7) *with completely known TPs and a given scalar* $\gamma > 0$, *if the coupled inequalities*

$$\begin{bmatrix} \overline{A}_i^T \mathscr{P}_i \overline{A}_i - P_i + \overline{C}_i^T \overline{C}_i & \overline{A}_i^T \mathscr{P}_i \overline{B}_i + \overline{C}_i^T \overline{D}_i \\ * & -\left(\gamma^2 \mathbf{I} - \overline{B}_i^T \mathscr{P}_i \overline{B}_i - \overline{D}_i^T \overline{D}_i\right) \end{bmatrix} < 0, \quad \forall i \in \mathscr{I}, \tag{9}$$

*where* $\mathscr{P}_i := \sum_{j=1}^N \pi_{ij} P_j$, *have a feasible solution* $P = \{P_1, P_2, \ldots, P_N\}$ *with* $P_i > 0$, *then the MJLS* (7) *with completely known TPs is stochastically stable with an* $\mathscr{H}_\infty$ *performance* $\gamma$.

## 3. Main Results

In this section, we will first derive the $\mathscr{H}_\infty$ performance analysis criterion for the model error system $(\overline{\Sigma})$ in (7) with deficient mode information. Then, two distinctly different approaches will be proposed to solve the $\mathscr{H}_\infty$ model reduction problem formulated in the previous section.

*3.1. $\mathscr{H}_\infty$ Performance Analysis.* In the following, by exploiting the properties of the transition probability matrices (TPMs), together with the convexification of uncertain domains, an $\mathscr{H}_\infty$ performance analysis criterion for the model error system $(\overline{\Sigma})$ in (7) with deficient mode information is presented, which will play an instrumental role in solving the $\mathscr{H}_\infty$ model reduction problem.

**Proposition 2.** *The MJLS* (7) *with deficient mode information is stochastically stable with a guaranteed* $\mathscr{H}_\infty$ *performance* $\gamma$ *if there exist positive-definite symmetric matrices* $P_i \in \mathbf{R}^{(n_x+n_r)\times(n_x+n_r)}$, $i \in \mathscr{I}$, *such that the following matrix inequalities hold:*

$$\begin{aligned} \Phi_{ij}^{(s)} &:= \begin{bmatrix} \overline{A}_i^T \mathscr{P}_j^{(is)} \overline{A}_i - P_i + \overline{C}_i^T \overline{C}_i & \overline{A}_i^T \mathscr{P}_j^{(is)} \overline{B}_i + \overline{C}_i^T \overline{D}_i \\ * & -\left(\gamma^2 \mathbf{I} - \overline{B}_i^T \mathscr{P}_j^{(is)} \overline{B}_i - \overline{D}_i^T \overline{D}_i\right) \end{bmatrix} \\ &< 0, \quad \forall i \in \mathscr{I}, \; j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}, \; s = 1, \ldots, M, \end{aligned} \tag{10}$$

*where*

$$\begin{aligned} \mathscr{P}_j^{(is)} &:= \sum_{j \in \mathscr{I}_{\mathscr{K}}^{(i)}} \pi_{ij} P_j + \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{C}}^{(i)}} \widetilde{\pi}_{ij}^{(s)} P_j + \underbrace{\pi_{\mathscr{U}\mathscr{K}}^{(is)} P_j}_{j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}}, \\ \pi_{\mathscr{U}\mathscr{K}}^{(is)} &:= 1 - \sum_{j \in \mathscr{I}_{\mathscr{K}}^{(i)}} \pi_{ij} - \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{C}}^{(i)}} \widetilde{\pi}_{ij}^{(s)}. \end{aligned} \tag{11}$$

*Proof.* Based on Lemma 1, it is known that system (7) with completely known transition probabilities (TPs) is stochastically stable with an $\mathscr{H}_\infty$ performance $\gamma$ if (9) holds. Now due to $\sum_{j=1}^N \pi_{ij} = 1$ and with deficient mode information, we rewrite the term $\sum_{j=1}^N \pi_{ij} P_j$ as

$$\sum_{j=1}^N \pi_{ij} P_j = \sum_{j \in \mathscr{I}_{\mathscr{K}}^{(i)}} \pi_{ij} P_j + \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{C}}^{(i)}} \left(\sum_{s=1}^M \alpha_s \widetilde{\pi}_{ij}^{(s)}\right) P_j + \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}} \widehat{\pi}_{ij} P_j. \tag{12}$$

Considering the fact that $0 \le \alpha_s \le 1$, $\sum_{s=1}^M \alpha_s = 1$, and $0 \le \widehat{\pi}_{ij}/\pi_{\mathscr{U}\mathscr{K}}^{(is)} \le 1$, $\sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}} (\widehat{\pi}_{ij}/\pi_{\mathscr{U}\mathscr{K}}^{(is)}) = 1$, (12) can be rewritten as

$$\begin{aligned} \sum_{j=1}^N \pi_{ij} P_j &= \sum_{s=1}^M \alpha_s \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}} \frac{\widehat{\pi}_{ij}}{\pi_{\mathscr{U}\mathscr{K}}^{(is)}} \left( \sum_{j \in \mathscr{I}_{\mathscr{K}}^{(i)}} \pi_{ij} P_j + \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{C}}^{(i)}} \widetilde{\pi}_{ij}^{(s)} P_j + \pi_{\mathscr{U}\mathscr{K}}^{(is)} P_j \right). \end{aligned} \tag{13}$$

Thus, with deficient mode information, the left-hand side (LHS) of inequality (9) can be rewritten as

$$\text{LHS}\,(9) = \sum_{s=1}^M \alpha_s \sum_{j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}} \frac{\widehat{\pi}_{ij}}{\pi_{\mathscr{U}\mathscr{K}}^{(is)}} \Phi_{ij}^{(s)}, \tag{14}$$

where $\Phi_{ij}^{(s)}$ is defined in (10). Then (9) holds if and only if $\Phi_{ij}^{(s)} < 0$, which implies that, in the presence of unknown and uncertain TPs, inequality (9) is equivalent to (10). This completes the proof. □

*Remark 3.* By fully exploiting the properties of the TPMs, together with the convexification of uncertain domains, a sufficient condition for the $\mathscr{H}_\infty$ performance analysis has been derived for the model error system $(\overline{\Sigma})$ with deficient mode information in Proposition 2. It is noted that there exist product terms between the Lyapunov matrices and system matrices in condition (10), which will bring some difficulties in the solutions of model reduction problem. By applying decoupling techniques, in the following, two distinctly different approaches to solve the $\mathscr{H}_\infty$ model reduction problem will be proposed.

*3.2. Model Reduction via Convex Linearisation Approach.* In this subsection, based on a linearisation procedure, the model reduction problem described in the previous section will be cast into a convex optimization problem. The following theorem presents a sufficient condition for the existence of admissible reduced-order models.

**Theorem 4.** *Consider MJLS in* (1) *with deficient mode information and reduced-order model in the form of* (6). *The model error system in* (7) *is stochastically stable with an* $\mathscr{H}_\infty$ *performance* $\gamma$ *if there exist positive-definite symmetric matrices* $P_i = \begin{bmatrix} P_{i(1)} & P_{i(2)} \\ * & P_{i(3)} \end{bmatrix} \in \mathbf{R}^{(n_x+n_r)\times(n_x+n_r)}$ *and matrices* $G_{i(1)} \in \mathbf{R}^{n_x\times n_x}$, $G_{i(2)} \in \mathbf{R}^{n_r\times n_r}$, $G_{i(3)} \in \mathbf{R}^{n_x\times n_r}$, $\overline{A}_{ri} \in \mathbf{R}^{n_r\times n_r}$, $\overline{B}_{ri} \in \mathbf{R}^{n_r\times n_u}$, $C_{ri} \in \mathbf{R}^{n_z\times n_r}$, *and* $D_{ri} \in \mathbf{R}^{n_z\times n_u}$, $i \in \mathscr{I}$, *such that the following LMIs hold:*

$$\begin{bmatrix} \mathscr{P}^{(is)}_{j(1)} - G_{i(1)} - G^T_{i(1)} & \mathscr{P}^{(is)}_{j(2)} - HG_{i(2)} - G^T_{i(3)} & \mathbf{0} & G_{i(1)}A_i & H\overline{A}_{ri} & G_{i(1)}B_i + H\overline{B}_{ri} \\ * & \mathscr{P}^{(is)}_{j(3)} - G_{i(2)} - G^T_{i(2)} & \mathbf{0} & G_{i(3)}A_i & \overline{A}_{ri} & G_{i(3)}B_i + \overline{B}_{ri} \\ * & * & -\mathbf{I} & C_i & -C_{ri} & D_i - D_{ri} \\ * & * & * & -P_{i(1)} & -P_{i(2)} & \mathbf{0} \\ * & * & * & * & -P_{i(3)} & \mathbf{0} \\ * & * & * & * & * & -\gamma^2\mathbf{I} \end{bmatrix} < 0,$$
$$i \in \mathscr{I},\ j \in \mathscr{I}^{(i)}_{\mathscr{UK}},\ s = 1,\dots,M, \tag{15}$$

*where*

$$\mathscr{P}^{(is)}_{j(m)} := \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{K}}} \pi_{ij}P_{j(m)} + \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{UC}}} \widetilde{\pi}^{(s)}_{ij}P_{j(m)} + \underbrace{\pi^{(is)}_{\mathscr{UK}}P_{j(m)}}_{j\in\mathscr{I}^{(i)}_{\mathscr{UK}}}, \quad m = 1,2,3,$$
$$\pi^{(is)}_{\mathscr{UK}} := 1 - \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{K}}} \pi_{ij} - \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{UC}}} \widetilde{\pi}^{(s)}_{ij}, \tag{16}$$
$$H := \begin{bmatrix} \mathbf{I}_{n_r\times n_r} & \mathbf{0}_{n_r\times(n_x-n_r)} \end{bmatrix}^T.$$

*Moreover, if the above conditions have a set of feasible solutions* $(P_{i(1)}, P_{i(2)}, P_{i(3)}, G_{i(1)}, G_{i(2)}, G_{i(3)}, \overline{A}_{ri}, \overline{B}_{ri}, C_{ri}, D_{ri})$, *then an admissible* $n_r$*-order approximation model in the form of* (6) *can be constructed as*

$$A_{ri} = G^{-1}_{i(2)}\overline{A}_{ri}, \quad B_{ri} = G^{-1}_{i(2)}\overline{B}_{ri},$$
$$C_{ri} = C_{ri}, \quad D_{ri} = D_{ri}. \tag{17}$$

*Proof.* It follows from Proposition 2 that if we can show (10), then the claimed results follow. By Schur complement, inequality (10) is equivalent to

$$\begin{bmatrix} \left(\mathscr{P}^{(is)}_j\right)^{-1} & \mathbf{0} & \overline{A}_i & \overline{B}_i \\ * & -\mathbf{I} & \overline{C}_i & \overline{D}_i \\ * & * & -P_i & \mathbf{0} \\ * & * & * & -\gamma^2\mathbf{I} \end{bmatrix} < 0, \tag{18}$$
$$\forall i \in \mathscr{I},\ j \in \mathscr{I}^{(i)}_{\mathscr{UK}},\ s = 1,\dots,M,$$

where

$$\mathscr{P}^{(is)}_j := \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{K}}} \pi_{ij}P_j + \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{UC}}} \widetilde{\pi}^{(s)}_{ij}P_j + \underbrace{\pi^{(is)}_{\mathscr{UK}}P_j}_{j\in\mathscr{I}^{(i)}_{\mathscr{UK}}},$$
$$\pi^{(is)}_{\mathscr{UK}} := 1 - \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{K}}} \pi_{ij} - \sum_{j\in\mathscr{I}^{(i)}_{\mathscr{UC}}} \widetilde{\pi}^{(s)}_{ij}. \tag{19}$$

Perform a congruent transformation to (18) by $\operatorname{diag}\{G_i, \mathbf{I}, \mathbf{I}, \mathbf{I}\}$, and consider the following bounding inequality:

$$-G_i\left(\mathscr{P}^{(is)}_j\right)^{-1}G^T_i \le \mathscr{P}^{(is)}_j - G_i - G^T_i. \tag{20}$$

It is thus easy to see that the following inequality implies (18),

$$\begin{bmatrix} \mathscr{P}^{(is)}_j - G_i - G^T_i & \mathbf{0} & G_i\overline{A}_i & G_i\overline{B}_i \\ * & -\mathbf{I} & \overline{C}_i & \overline{D}_i \\ * & * & -P_i & \mathbf{0} \\ * & * & * & -\gamma^2\mathbf{I} \end{bmatrix} < 0, \tag{21}$$
$$\forall i \in \mathscr{I},\ j \in \mathscr{I}^{(i)}_{\mathscr{UK}},\ s = 1,\dots,M.$$

Furthermore, for model reduction synthesis purpose, we choose the slack matrix $G_i$ as

$$G_i := \begin{bmatrix} G_{i(1)} & HG_{(2)} \\ G_{i(3)} & G_{(4)} \end{bmatrix}, \tag{22}$$

where $H := [\mathbf{I}_{n_r} \;\; \mathbf{0}_{n_r\times(n_x-n_r)}]^T$, $G_{i(1)} \in \mathbf{R}^{n_x\times n_x}$, $G_{i(3)} \in \mathbf{R}^{n_r\times n_x}$, $G_{(2)} \in \mathbf{R}^{n_r\times n_r}$, and $G_{(4)} \in \mathbf{R}^{n_r\times n_r}$. Then, similar to [22], performing a congruent transformation to

$$\begin{bmatrix} G_{i(1)} + G_{i(1)}^T & HG_{(2)} + G_{i(3)}^T \\ * & G_{(4)} + G_{(4)}^T \end{bmatrix} \tag{23}$$

by $\left[\begin{smallmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & G_{(2)}G_{(4)}^{-1} \end{smallmatrix}\right]$ yields

$$\begin{aligned} &\begin{bmatrix} G_{i(1)} + G_{i(1)}^T & HG_{(2)}G_{(4)}^{-T}G_{(2)}^T + G_{i(3)}^T G_{(4)}^{-T}G_{(2)}^T \\ * & G_{(2)}G_{(4)}^{-T}G_{(2)}^T + G_{(2)}G_{(4)}^{-1}G_{(2)}^T \end{bmatrix} \\ &:= \begin{bmatrix} G_{i(1)} + G_{i(1)}^T & H\overline{G}_{(2)} + \overline{G}_{i(3)}^T \\ * & \overline{G}_{(2)} + \overline{G}_{(2)}^T \end{bmatrix}. \end{aligned} \tag{24}$$

Thus, instead of (22), one can directly specify matrix $G_i$ of the following form *without loss of generality*:

$$G_i = \begin{bmatrix} G_{i(1)} & HG_{i(2)} \\ G_{i(3)} & G_{i(2)} \end{bmatrix}, \quad i \in \mathcal{I}. \tag{25}$$

It is noted that in this way the matrix variable $G_{i(2)}$ can be absorbed by the reduced-order model gain variables $A_{ri}$ and $B_{ri}$ by introducing

$$\overline{A}_{ri} := G_{i(2)}A_{ri}, \qquad \overline{B}_{ri} := G_{i(2)}B_{ri}. \tag{26}$$

This feature enables one to make no congruent transformation to the original matrix inequality, and all the slack variables can be set as Markovian switching.

Then, substituting matrices $G_i$ given in (25) into (10), together with the consideration of the matrices defined in (26), leads to (15) exactly.

On the other hand, the conditions in (15) imply that $-G_{i(2)} - G_{i(2)}^T < 0$, which means that $G_{i(2)}$ is nonsingular. Thus, the reduced-order model gains can be constructed by (17). The proof is completed. □

*Remark 5.* Theorem 4 provides a sufficient condition for the solvability of $\mathcal{H}_\infty$ model reduction synthesis problem for the MJLS (1) with deficient mode information. A desired reduced-order model can be determined by solving the following convex optimization problem.

*Problem MRLA (Model Reduction via Linearisation Approach).* Minimize $\gamma$ subject to (15) for $P_{i(1)}, P_{i(2)}, P_{i(3)}, G_{i(1)}, G_{i(2)}, G_{i(3)}, \overline{A}_{ri}, \overline{B}_{ri}, C_{ri}, D_{ri}, i \in \mathcal{I}$.

From the proof of Theorem 4, it is easy to see that the matrix inequality linearisation procedure is based on the bounding inequality in (20) with slack variables of a structural constraint in (25). This inevitably brings some degree of design conservatism. Another avenue to solve the nonlinear matrix inequality problem in (10) is by utilizing the cone complementarity linearisation (CCL) technique [37]. To this end, we will resort to an iterative approach to solve the model reduction problem.

### 3.3. Model Reduction via Iterative Approach

In this subsection, we will solve the model reduction problem by using the CCL technique. The following theorem gives another sufficient condition for the existence of admissible reduced-order models.

**Theorem 6.** *Consider MJLS in* (1) *with deficient mode information and reduced-order model in the form of* (6). *The model error system in* (7) *is stochastically stable with an* $\mathcal{H}_\infty$ *performance* $\gamma$ *if there exist positive-definite symmetric matrices* $\{P_i, X_i\} \in \mathbf{R}^{(n_x+n_r)\times(n_x+n_r)}$ *and matrices* $A_{ri} \in \mathbf{R}^{n_r\times n_r}$, $B_{ri} \in \mathbf{R}^{n_r\times n_u}$, $C_{ri} \in \mathbf{R}^{n_z\times n_r}$, *and* $D_{ri} \in \mathbf{R}^{n_z\times n_u}$, $i \in \mathcal{I}$, *such that*

$$\begin{bmatrix} \mathcal{X}_j^{(i)} & \mathbf{0} & F_i^{(s)}(\mathcal{A}_i + E\mathcal{G}_{ri}R) & F_i^{(s)}(\mathcal{B}_i + E\mathcal{G}_{ri}S) \\ * & -\mathbf{I} & \mathcal{C}_i + F\mathcal{G}_{ri}R & \mathcal{D}_i + F\mathcal{G}_{ri}S \\ * & * & -P_i & \mathbf{0} \\ * & * & * & -\gamma^2\mathbf{I} \end{bmatrix} < 0,$$
$$i \in +F\mathcal{G}_{ri}R\mathcal{I}, \; j \in \mathcal{I}_{\mathcal{UK}}^{(i)}, \; s = 1,\ldots,M, \tag{27}$$

$$P_iX_i = \mathbf{I}, \quad i \in \mathcal{I}, \tag{28}$$

*where*

$$\begin{aligned} &\mathcal{X}_j^{(i)} := \operatorname{diag}\left\{X_{\mathcal{K}_1^{(i)}}, \ldots, X_{\mathcal{K}_{t_i}^{(i)}}, X_{\mathcal{U}_1^{(i)}}, \ldots, X_{\mathcal{U}_{v_i}^{(i)}}, X_j\right\}, \\ &F_i^{(s)} \\ &:= \left[\sqrt{\pi_{i\mathcal{K}_1^{(i)}}}\mathbf{I}, \ldots, \sqrt{\pi_{i\mathcal{K}_{t_i}^{(i)}}}\mathbf{I}, \sqrt{\tilde{\pi}_{i\mathcal{U}_1^{(i)}}^{(s)}}\mathbf{I}, \ldots, \sqrt{\tilde{\pi}_{i\mathcal{U}_{v_i}^{(i)}}^{(s)}}\mathbf{I}, \sqrt{\pi_{\mathcal{UK}}^{(is)}}\mathbf{I}\right]^T, \\ &\mathcal{G}_{ri} := \begin{bmatrix} A_{ri} & B_{ri} \\ C_{ri} & D_{ri} \end{bmatrix}, \qquad \mathcal{A}_i := \begin{bmatrix} A_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathcal{B}_i := \begin{bmatrix} B_i \\ \mathbf{0} \end{bmatrix}, \\ &\mathcal{C}_i := \begin{bmatrix} C_i & \mathbf{0} \end{bmatrix}, \qquad \mathcal{D}_i := D_i, \qquad E := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \\ &F := \begin{bmatrix} \mathbf{0} & -\mathbf{I} \end{bmatrix}, \qquad R := \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad S := \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}, \\ &\pi_{\mathcal{UK}}^{(is)} := 1 - \sum_{j\in\mathcal{I}_{\mathcal{K}}^{(i)}} \pi_{ij} - \sum_{j\in\mathcal{I}_{\mathcal{UC}}^{(i)}} \tilde{\pi}_{ij}^{(s)}. \end{aligned} \tag{29}$$

*Proof.* From Proposition 2, we know that for $i \in \mathcal{I}$ there exists a reduced-order model in the form of (6) such that the model error system (7) is stochastically stable with a guaranteed $\mathcal{H}_\infty$ error performance $\gamma$ if there exist matrices $P_i$ satisfying (10). First, rewrite the matrices defined in (8) in the following form:

$$\begin{aligned} \overline{A}_i &:= \mathcal{A}_i + E\mathcal{G}_{ri}R, \qquad \overline{B}_i := \mathcal{B}_i + E\mathcal{G}_{ri}S, \\ \overline{C}_i &:= \mathcal{C}_i + F\mathcal{G}_{ri}R, \qquad \overline{D}_i := \mathcal{D}_i + F\mathcal{G}_{ri}S, \end{aligned} \tag{30}$$

where

$$\mathscr{G}_{ri} := \begin{bmatrix} A_{ri} & B_{ri} \\ C_{ri} & D_{ri} \end{bmatrix}, \qquad \mathscr{A}_i := \begin{bmatrix} A_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathscr{B}_i := \begin{bmatrix} B_i \\ \mathbf{0} \end{bmatrix},$$

$$\mathscr{C}_i := \begin{bmatrix} C_i & \mathbf{0} \end{bmatrix}, \qquad \mathscr{D}_i := D_i, \qquad E := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix},$$

$$F := \begin{bmatrix} \mathbf{0} & -\mathbf{I} \end{bmatrix}, \qquad R := \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad S := \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}. \tag{31}$$

Then, by Schur complement to (10), and with the notations $\mathscr{I}_{\mathscr{K}}^{(i)} := \{\mathscr{K}_1^{(i)}, \dots, \mathscr{K}_{t_i}^{(i)}\}, \mathscr{I}_{\mathscr{U}\mathscr{C}}^{(i)} := \{\mathscr{U}_1^{(i)}, \dots, \mathscr{U}_{v_i}^{(i)}\}$, we have that (10) is equivalent to

$$\begin{bmatrix} \overline{\mathscr{P}}_j^{(i)} & \mathbf{0} & F_i^{(s)}\left(\mathscr{A}_i + E\mathscr{G}_{ri}R\right) & F_i^{(s)}\left(\mathscr{B}_i + E\mathscr{G}_{ri}S\right) \\ * & -\mathbf{I} & \mathscr{C}_i + F\mathscr{G}_{ri}R & \mathscr{D}_i + F\mathscr{G}_{ri}S \\ * & * & -P_i & \mathbf{0} \\ * & * & * & -\gamma^2\mathbf{I} \end{bmatrix} < 0,$$
$$i \in \mathscr{I}, \; j \in \mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}, \; s = 1, \dots, M, \tag{32}$$

where

$$\overline{\mathscr{P}}_j^{(i)} := \operatorname{diag}\left\{P_{\mathscr{K}_1^{(i)}}^{-1}, \dots, P_{\mathscr{K}_{t_i}^{(i)}}^{-1}, P_{\mathscr{U}_1^{(i)}}^{-1}, \dots, P_{\mathscr{U}_{v_i}^{(i)}}^{-1}, P_j^{-1}\right\},$$
$$F_i^{(s)} := \left[\sqrt{\pi_{i\mathscr{K}_1^{(i)}}}\mathbf{I}, \dots, \sqrt{\pi_{i\mathscr{K}_{t_i}^{(i)}}}\mathbf{I}, \sqrt{\widetilde{\pi}_{i\mathscr{U}_1^{(i)}}^{(s)}}\mathbf{I}, \dots, \sqrt{\widetilde{\pi}_{i\mathscr{U}_{v_i}^{(i)}}^{(s)}}\mathbf{I}, \sqrt{\pi_{\mathscr{U}\mathscr{K}}^{(is)}}\mathbf{I}\right]^T. \tag{33}$$

Setting $X_i := P_i^{-1}$ and considering (33), it is easy to see that (32) is equivalent to (27) and (28). This completes the proof. □

*Remark 7.* Theorem 6 provides another sufficient condition for testing the solvability of $n_r$-order $\mathscr{H}_\infty$ model reduction synthesis for MJLS (1) with deficient mode information. It is noted that the conditions in Theorem 6 are not strict LMI-based conditions due to the matrix equality in (28). However, with the CCL algorithm in [37, 38], we can resolve this nonconvex feasibility problem by formulating it into a sequential optimization problem subject to LMI constraints.

Based on the previous discussions and using a CCL technique, the nonconvex feasibility problem given in (27) and (28) is converted into the following nonlinear minimization problem that involves LMI conditions.

*Problem MRIA (Model Reduction via Iterative Approach).* Minimize Trace $(\sum_{i=1}^{N} P_i X_i)$ subject to (27) and

$$\begin{bmatrix} P_i & \mathbf{I} \\ * & X_i \end{bmatrix} \geq 0, \quad \forall i \in \mathscr{I}. \tag{34}$$

Then, the suboptimal performance of $\gamma$ can be found by the following algorithm. The convergence of this algorithm is guaranteed in terms of similar results in [37, 38].

*Algorithm MRIA: Suboptimal Performance of $\gamma$*

*Step 1.* Choose a sufficiently large initial $\gamma > 0$, such that there exists a feasible solution to (27) and (34). Set $\gamma_0 = \gamma$.

*Step 2.* Find a feasible set $(P_i^{(0)}, X_i^{(0)}, A_{ri}^{(0)}, B_{ri}^{(0)}, C_{ri}^{(0)}, D_{ri}^{(0)},$ for all $i \in \mathscr{I})$ that satisfies the conditions in (27) and (34). Set $q = 0$.

*Step 3.* Solve the following LMI problem over the variables $P_i$, $X_i$, $A_{ri}$, $B_{ri}$, $C_{ri}$, and $D_{ri}$:

$$\text{Minimize Trace}\left(\sum_{i=1}^{N}\left(P_i^{(q)}X_i + P_iX_i^{(q)}\right)\right) \tag{35}$$

subject to (27) and (34).

Set $P_i^{(q+1)} = P_i$ and $X_i^{(q+1)} = X_i$.

*Step 4.* Substituting the gains $A_{ri}$, $B_{ri}$, $C_{ri}$, and $D_{ri}$ obtained in Step 3 into (10) and if the LMIs in (10) are feasible with respect to the variables $P_i$, then set $\gamma_0 = \gamma$ and return to Step 2 after decreasing $\gamma$ to some extent. If (10) are infeasible within the maximum number of iterations allowed, then exit. Otherwise, set $q = q + 1$, and go to Step 3.

*Remark 8.* It is noted that the $\mathscr{H}_\infty$ model reduction problem for discrete-time MJLSs has been considered in [20]. However, there are some remarkable differences between our results and those in [20]. Firstly, it is noted that in this paper the deficient mode information, which simultaneously contains the exactly known, partially unknown, and uncertain transition probabilities (TPs), is considered for MJLS (1), which is more general and practical for engineering applications, whereas the TPs considered in [20] are only partially unknown. Secondly, by fully utilizing the properties of the TPM, including the relation $\sum_{j\in\mathscr{I}_{\mathscr{U}\mathscr{K}}^{(i)}} \widehat{\pi}_{ij} = 1 - \sum_{j\in\mathscr{I}_{\mathscr{K}}^{(i)}} \pi_{ij} - \sum_{j\in\mathscr{I}_{\mathscr{U}\mathscr{C}}^{(i)}} \widetilde{\pi}_{ij}^{(s)}$, Theorem 4 gives a unified formulation of $\mathscr{H}_\infty$ model reduction design for MJLS (1), while in [20] the above relation is neglected, which may cause possible conservatism. In addition, two different approaches, namely, the convex linearisation approach and iterative approach, to solve the model reduction problem are proposed, while in [20] only the linearisation approach was presented. It will be shown in the simulation part that the previous points are crucial to reduce the conservatism of model reduction design for MJLS in (1).

*Remark 9.* It is worth mentioning that the conditions given in Theorem 4 are convex and thus can be readily solved with commercially available software. The design conservatism of Theorem 4 mainly comes from the bounding inequality in (20) with the slack variable of a structural constraint in (25). In the iterative approach, the conditions given in (27) and (34) are equivalent to the corresponding performance analysis results. This is the main advantage of Theorem 6 over Theorem 4. However, the numerical computation cost involved in Algorithm MRIA is also much larger than that involved in Theorem 4 (MRLA), especially when the number of iterations increases.

## 4. An Illustrative Example

In this section, we will present an illustrative example to demonstrate the effectiveness and superiority of the proposed approaches.

Consider a Markovian jump linear system (MJLS) described by $(\Sigma)$ with parameters as follows:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{cccc|c} 0.05 & -0.27 & 0.44 & 0.39 & 0.5 \\ 0.55 & 0.33 & 0.38 & 0.55 & 0.2 \\ 0.1 & 0.17 & 0.27 & 0.44 & 0.3 \\ 0.05 & 0.22 & 0.16 & 0.11 & 0.1 \\ \hline 1 & 0.1 & 0.2 & -0.3 & 0.5 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cccc|c} 0.11 & -0.17 & 0.27 & 0.44 & -0.8 \\ 0.55 & 0.06 & 0.22 & 0.55 & -0.2 \\ 0.05 & 0.17 & 0.28 & 0.44 & -0.1 \\ 0.17 & 0.05 & 0.06 & -0.11 & -1 \\ \hline 0.5 & -0.8 & 0.3 & 0.5 & 0.5 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_3 & B_3 \\ \hline C_3 & D_3 \end{array}\right] = \left[\begin{array}{cccc|c} 0.16 & 0.06 & -0.02 & 0.18 & 0.2 \\ 0.04 & -0.37 & 0.53 & -0.04 & -0.2 \\ -0.08 & -0.32 & -0.05 & -0.11 & -0.1 \\ -0.17 & 0.40 & 0.04 & 0.29 & 0.1 \\ \hline 1.4 & 0.7 & 0.2 & -0.8 & 0.5 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_4 & B_4 \\ \hline C_4 & D_4 \end{array}\right] = \left[\begin{array}{cccc|c} 0.23 & 0.01 & -0.55 & -0.38 & 0.9 \\ -0.33 & 0.36 & -0.48 & -0.10 & -1.1 \\ -0.20 & -0.45 & 0.1 & -0.19 & -0.7 \\ 0.23 & 0.16 & 0.5 & -0.30 & -1.2 \\ \hline -0.7 & 1.2 & 1.2 & -0.6 & 0.5 \end{array}\right]. \tag{36}$$

Four different cases for the transition probability matrix (TPM) are given in Table 1, where the transition probabilities (TPs) labeled "^" and "~" represent the unknown and uncertain elements, respectively. Specifically, Case 1, Case 2, Case 3, and Case 4 stand for the completely known TPs, deficient mode information (including known, partially unknown, and uncertain TPs), partially unknown TPs, and completely unknown TPs, respectively.

Table 1: Four different TPMs.

| Case 1: completely known TPM | Case 2: deficient TPM1 |
|---|---|
| $\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ 0.3 & 0.2 & 0.3 & 0.2 \\ 0.1 & 0.5 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{bmatrix}$ | $\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ \widehat{\pi}_{21} & \widehat{\pi}_{22} & 0.3 & 0.2 \\ \widehat{\pi}_{31} & \widetilde{\pi}_{32} & \widehat{\pi}_{33} & \widetilde{\pi}_{34} \\ 0.2 & \widehat{\pi}_{42} & \widehat{\pi}_{43} & \widehat{\pi}_{44} \end{bmatrix}$ |
| **Case 3: deficient TPM2** | **Case 4: completely unknown TPM** |
| $\begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ \widehat{\pi}_{21} & \widehat{\pi}_{22} & 0.3 & 0.2 \\ \widehat{\pi}_{31} & \widehat{\pi}_{32} & \widehat{\pi}_{33} & \widehat{\pi}_{34} \\ 0.2 & \widehat{\pi}_{42} & \widehat{\pi}_{43} & \widehat{\pi}_{44} \end{bmatrix}$ | $\begin{bmatrix} \widehat{\pi}_{11} & \widehat{\pi}_{12} & \widehat{\pi}_{13} & \widehat{\pi}_{14} \\ \widehat{\pi}_{21} & \widehat{\pi}_{22} & \widehat{\pi}_{23} & \widehat{\pi}_{24} \\ \widehat{\pi}_{31} & \widehat{\pi}_{32} & \widehat{\pi}_{33} & \widehat{\pi}_{34} \\ \widehat{\pi}_{41} & \widehat{\pi}_{42} & \widehat{\pi}_{43} & \widehat{\pi}_{44} \end{bmatrix}$ |

Table 2: Comparison of minimum $\mathscr{H}_\infty$ performance with deficient TPM2 in Case 3.

| Methods | Three-order | Two-order | One-order |
|---|---|---|---|
| Theorem 4 | 1.2423 | 1.5231 | 2.5952 |
| [20] | 1.3815 | 1.7320 | 2.9094 |

For Case 2 shown in Table 1, the uncertain TPs comprise two vertices $\Pi_s$, $s = 1, 2$, where the third rows $\Pi_{s(3)}$, $s = 1, 2$, are given by

$$\begin{aligned} \Pi_{1(3)} &= \left[\widehat{\pi}_{31} \;\; 0.1 \;\; \widehat{\pi}_{33} \;\; 0.3\right], \\ \Pi_{2(3)} &= \left[\widehat{\pi}_{31} \;\; 0.3 \;\; \widehat{\pi}_{33} \;\; 0.4\right], \end{aligned} \tag{37}$$

and the other rows in the two vertices are defined with the same elements; that is,

$$\begin{aligned} \Pi_{s(1)} &= \left[0.3 \;\; 0.2 \;\; 0.1 \;\; 0.4\right], \\ \Pi_{s(2)} &= \left[\widehat{\pi}_{21} \;\; \widehat{\pi}_{22} \;\; 0.3 \;\; 0.2\right], \\ \Pi_{s(4)} &= \left[0.2 \;\; \widehat{\pi}_{42} \;\; \widehat{\pi}_{43} \;\; \widehat{\pi}_{44}\right], \quad s = 1, 2. \end{aligned} \tag{38}$$

The objective is to design an $n_r$-order model of the form (6) to approximate the above system such that the model error system (7) is stochastically stable with an $\mathscr{H}_\infty$ performance index $\gamma$. Here, we first assume that the MJLS is subject to the partially unknown TPs in Case 3, and then a detailed comparison between the minimum $\mathscr{H}_\infty$ performance indices $\gamma_{\min}$ obtained based on Theorem 4 in this paper and the results in [20] is summarized in Table 2. It is observed from Table 2 that the design approach proposed in Theorem 4 is less conservative than [20].

In the following, we will further demonstrate the advantage and less conservatism of Theorem 6 over Theorem 4 proposed in this paper. By solving the problems MRLA and MRIA, another detailed comparison of the minimum $\mathscr{H}_\infty$ performance indices $\gamma_{\min}$ is listed in Table 3, where

TABLE 3: Comparison of minimum $\mathscr{H}_\infty$ performance for different TPMs.

| TPMs | Three-order | | Two-order | | One-order | |
|---|---|---|---|---|---|---|
| | Theorem 4 (MRLA) | Theorem 6 (MRIA) | Theorem 4 (MRLA) | Theorem 6 (MRIA) | Theorem 4 (MRLA) | Theorem 6 (MRIA) |
| Case 1 | 0.8798 | 0.4 (**35**) | 0.9595 | 0.8 (**139**) | 2.0057 | 1.5 (**38**) |
| Case 2 | 1.1687 | 0.6 (**39**) | 1.4829 | 1.2 (**145**) | 2.4748 | 1.9 (**40**) |
| Case 3 | 1.2423 | 0.7 (**87**) | 1.5231 | 1.3 (**157**) | 2.5952 | 2 (**43**) |
| Case 4 | 1.7332 | 1 (**147**) | 2.1914 | 1.6 (**165**) | 3.3299 | 2.4 (**137**) |

the bold numbers in the brackets represent the number of iterations. By inspection of Tables 1 and 2, it is easy to see that the results in Theorem 6 are less conservative than those in Theorem 4. It is also shown that the more information on TPs is available, the better $\mathscr{H}_\infty$ performance can be obtained, which is effective to reduce the conservatism of model reduction design. Therefore, the introduction of the polytopic-type uncertain TPs is significant.

Specifically, with $n_r = 3$, we obtain $\gamma_{\min} = 1.1687$ by Theorem 4 with deficient TPM1 shown in Table 1, and the three-order model parameters are given by

$$\left[\begin{array}{c|c} A_{r1} & B_{r1} \\ \hline C_{r1} & D_{r1} \end{array}\right] = \left[\begin{array}{ccc|c} -0.0540 & 0.2529 & 0.2176 & -0.8490 \\ 0.3274 & 0.6285 & 0.4862 & 0.1883 \\ -0.1543 & 0.3250 & 0.2913 & -0.8224 \\ \hline -0.9425 & 0.1661 & -0.0302 & 0.6282 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{r2} & B_{r2} \\ \hline C_{r2} & D_{r2} \end{array}\right] = \left[\begin{array}{ccc|c} 0.1724 & 0.1095 & 0.3894 & 0.8742 \\ 0.3654 & 0.4674 & 0.2785 & 0.3566 \\ -0.0484 & 0.3457 & 0.2416 & 0.1962 \\ \hline -0.6496 & 0.6678 & -0.7150 & 0.6001 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{r3} & B_{r3} \\ \hline C_{r3} & D_{r3} \end{array}\right] = \left[\begin{array}{ccc|c} 0.2655 & 0.2048 & -0.1861 & -0.2537 \\ 0.1033 & -0.1618 & 0.2405 & 0.1205 \\ -0.1747 & -0.3936 & 0.1018 & 0.3127 \\ \hline -1.2638 & -0.0046 & -0.0662 & 0.4385 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{r4} & B_{r4} \\ \hline C_{r4} & D_{r4} \end{array}\right] = \left[\begin{array}{ccc|c} 0.1162 & -0.5423 & -0.5004 & -0.7796 \\ -0.3767 & 0.0416 & -0.3589 & 1.1859 \\ 0.0724 & -0.3765 & 0.0533 & 0.6505 \\ \hline 0.5008 & -0.9345 & -1.1617 & 0.3712 \end{array}\right]. \quad (39)$$

With $n_r = 3$, we obtain $\gamma = 0.6$ by solving the Algorithm MRIA after 39 iterations with deficient TPM1 shown in Table 1, and the three-order model parameters are given by

$$\left[\begin{array}{c|c} A_{r1} & B_{r1} \\ \hline C_{r1} & D_{r1} \end{array}\right] = \left[\begin{array}{ccc|c} 0.2800 & 0.0772 & 0.6328 & -0.4352 \\ 0.5261 & 0.2307 & 0.7350 & -0.1521 \\ 0.0462 & 0.2468 & 0.3092 & -0.0833 \\ \hline -1.5239 & 0.2064 & 0.4439 & 0.4937 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{r2} & B_{r2} \\ \hline C_{r2} & D_{r2} \end{array}\right] = \left[\begin{array}{ccc|c} 0.3132 & -0.0048 & 0.7263 & 0.5510 \\ 0.4900 & 0.0264 & 0.7604 & -0.2455 \\ 0.0826 & 0.1022 & -0.0038 & 0.5862 \\ \hline -0.7215 & 0.8753 & -0.3862 & 0.4679 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{r3} & B_{r3} \\ \hline C_{r3} & D_{r3} \end{array}\right] = \left[\begin{array}{ccc|c} 0.1946 & -0.0921 & 0.0870 & -0.0826 \\ 0.1133 & -0.2445 & -0.5027 & 0.2327 \\ -0.1452 & 0.1980 & 0.3471 & -0.0378 \\ \hline -1.9205 & -0.3685 & 0.8553 & 0.4941 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{r4} & B_{r4} \\ \hline C_{r4} & D_{r4} \end{array}\right] = \left[\begin{array}{ccc|c} -0.0592 & -0.2981 & -0.6252 & -0.3605 \\ -0.5600 & -0.0061 & 0.0120 & 0.5109 \\ 0.1490 & -0.0012 & -0.3114 & 0.8336 \\ \hline 0.5634 & -2.4550 & -0.7422 & 0.4789 \end{array}\right]. \quad (40)$$

The feasible solutions for the other cases are omitted for brevity.

In order to further illustrate the effectiveness of the designed approximation models, simulations have been carried out. Specifically, choose the zero initial condition and input $u(k)$ as

$$u(k) = \begin{cases} e^{(-0.2k+1)} \sin(0.2k), & 10 \le k \le 30, \\ 0.3 \cos(0.25k), & 50 \le k \le 90, \\ 0, & \text{otherwise}. \end{cases} \quad (41)$$

With the above obtained approximation models under Case 2 given in Table 1, and one of the possible realizations of

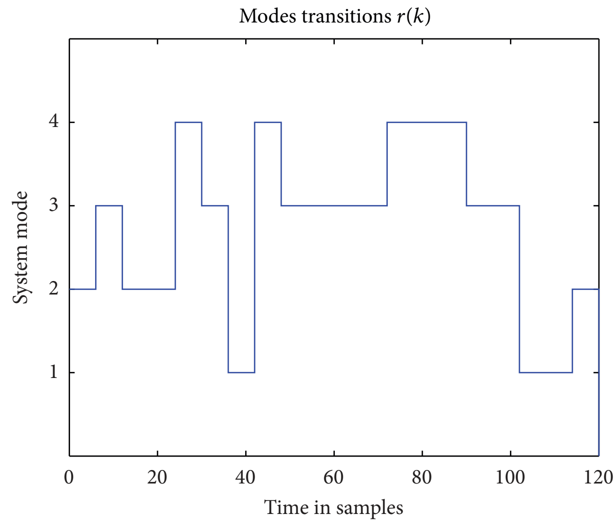


Figure 1: One possible system mode evolution.

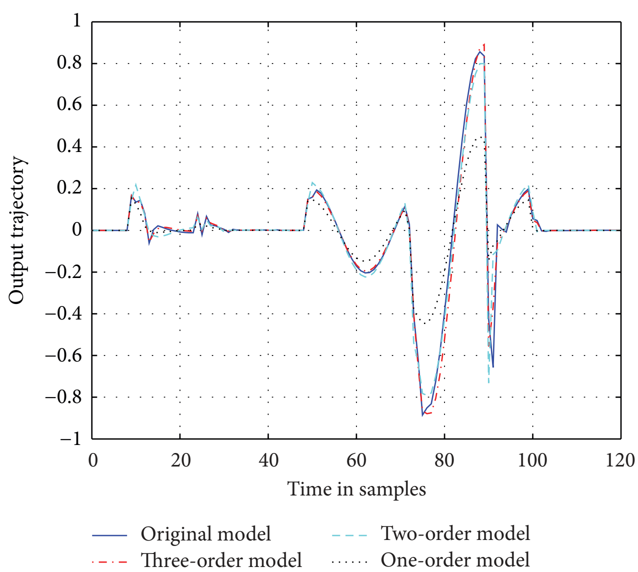


Figure 2: Output trajectories of the original system and reduced-order models with deficient TPM1 based on Theorem 4 (MRLA).

the Markovian jumping mode shown in Figure 1, the output trajectories of the original system and approximation models obtained based on Theorem 4 (MRLA) and Theorem 6 (MRIA) are shown in Figures 2 and 3, respectively. It can be clearly observed from the simulation curves that, despite the deficient TPs, the designed reduced-order models can approximate the original system very well.

## 5. Conclusions

This paper has studied the problem of robust $\mathscr{H}_\infty$ model reduction for a class of discrete-time Markovian jump linear systems (MJLSs) with deficient mode information, which simultaneously consists of the exactly known, partially

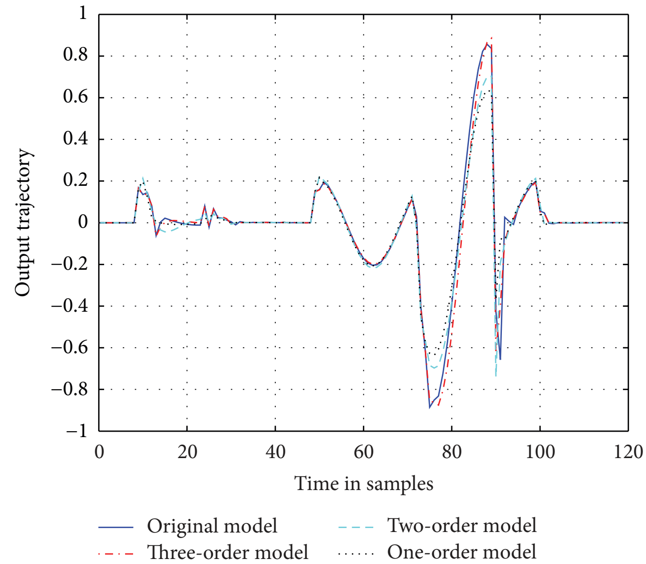


Figure 3: Output trajectories of the original system and reduced-order models with deficient TPM1 based on Theorem 6 (MRIA).

unknown, and polytopic-type uncertain transition probabilities and is thus more general and practical. By fully utilizing the properties of the transition probability matrices, together with the convexification of uncertain domains, a new $\mathscr{H}_\infty$ performance analysis criterion has been developed. Two distinctly different approaches, namely, the convex linearisation approach and iterative approach, have been proposed to solve the model reduction problem. A simulation example has been given to illustrate the less conservatism and effectiveness of the proposed approaches.

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