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# Model Reduction for Nonlinear Systems by Incremental Balanced Truncation

Bart Besselink, Nathan van de Wouw, *Member, IEEE*, Jacquelin M. A. Scherpen, *Senior Member, IEEE*, and Henk Nijmeijer, *Fellow, IEEE*

**Abstract**—In this paper, the method of incremental balanced truncation is introduced as a tool for model reduction of nonlinear systems. Incremental balanced truncation provides an extension of balanced truncation for linear systems towards the nonlinear case and differs from existing nonlinear balancing techniques in the definition of two novel energy functions. These incremental observability and incremental controllability functions form the basis for a model reduction procedure in which the preservation of stability properties is guaranteed. In particular, the property of incremental stability, which provides a notion of stability for systems with nonzero inputs, is preserved. Moreover, a computable error bound is given. Next, an extension towards so-called generalized incremental balanced truncation is proposed, which provides a reduction technique with increased computational feasibility at the cost of a (potentially) larger error bound. The proposed reduction technique is illustrated by means of application to an example of an electronic circuit with nonlinear elements.

**Index Terms**— Incremental balanced truncation.

## I. INTRODUCTION

**B**ALANCED truncation, as introduced in [21], is a well-known method for model reduction of (asymptotically stable) linear systems and relies on the definition of two energy functions: the observability function and controllability function (see, e.g., [2], [14]). A reduction procedure on the basis of these energy functions features the desirable properties of the preservation of stability (see [24]) and the availability of a computable bound on the reduction error [10], [14].

An extension of balanced truncation towards nonlinear systems, hereby using the same energy functions, is introduced in [29] and further developed in [12], [13]. As in the linear case, a balanced realization defined according to these energy functions is directly related to the Hankel operator of the system. However, for a reduced-order system obtained by nonlinear balanced truncation, only *local* stability properties of the

equilibrium point for *zero input* are guaranteed. Results on the stability of trajectories for *nonzero* inputs are not available and, as a result, no error bound exists. Other approaches aiming at the extension of balanced truncation towards nonlinear systems such as [31], [32], which is based on linearization around trajectories, or [15], [20], based on the numerical approximation of the observability and controllability function, also do not provide a guarantee on stability properties of the reduced-order system and do not guarantee a computable error bound. Here, it is remarked that other methods for the reduction of nonlinear systems (i.e., methods not based on balanced truncation such as moment matching for nonlinear systems [3] or proper orthogonal decomposition [30]) do generally not satisfy the properties of stability preservation and an error bound either. However, some exceptions are given by [6], [7], [25], in which systems with low-order nonlinearities are considered and reduction is applied to the linear part of the system only.

In this paper, a novel approach is taken in the extension of balanced truncation towards nonlinear systems that are affine in the input. In particular, the method of *incremental* balanced truncation is introduced, hereby addressing the properties of stability preservation (including the nonzero input case) and the computation of an error bound.

Incremental balanced truncation is based on the introduction of two novel energy functions, which replace the observability and controllability function as a basis for model reduction. These so-called *incremental* observability and *incremental* controllability functions measure input and output energy on the basis of the comparison of two system trajectories, rather than on the energy associated to a single trajectory as in their non-incremental counterparts. For linear systems, the incremental energy functions will be shown to be characterized by the observability and controllability Gramian. Thus, for linear systems, the model reduction procedure based on the incremental energy functions is equivalent to conventional balanced truncation.

For nonlinear systems, the introduction of the incremental observability and incremental controllability functions has several advantages. First, the incremental observability function is directly related to the full notion of observability for nonlinear systems, rather than zero-state observability as in [29]. In addition, it provides a link to incremental stability properties. Here, it is noted that incremental stability is a stability property for systems with nonzero input which, in addition, implies internal stability (i.e., stability of the equilibrium point for zero input). As a result, a second advantage is that incremental stability properties are preserved in a balanced truncation procedure

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B. Besselink is with the ACCESS Linnaeus Centre and the Department of Automatic Control, School of Electrical Engineering, KTH Royal Institute of Technology, 10044 Stockholm, Sweden (e-mail: bart.besselink@ee.kth.se).

N. van de Wouw and H. Nijmeijer are with the Dynamics and Control group, Department of Mechanical Engineering, Eindhoven University of Technology, 5600 MB Eindhoven, the Netherlands (e-mail: n.v.d.wouw@tue.nl; h.nijmeijer@tue.nl).

J. M. A. Scherpen is with the Faculty of Mathematics and Natural Sciences, ITM, University of Groningen, 9747 AG Groningen, the Netherlands (e-mail: j.m.a.scherpen@rug.nl).

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based on these incremental energy functions. Third, balancing the incremental observability and incremental controllability function allows for the computation of an error bound in terms of the  $\mathcal{L}_2$  signal norm. Incremental balanced truncation thus provides a model reduction technique that guarantees the preservation of relevant stability properties and the availability of an error bound.

Next, an extension of incremental balanced truncation towards so-called generalized incremental balanced truncation is presented for a subclass of nonlinear systems. Generalized incremental balanced truncation is based on the computation of bounds on the incremental energy functions rather than the energy functions themselves. As the bounds are chosen as quadratic functions, generalized incremental balanced truncation gives an increased computational feasibility of the reduction method while the preservation of stability properties remains guaranteed and a computable error bound remains available. This extension relies heavily on the development of the theory of (non-generalized) incremental balanced truncation.

The remainder of this paper is organized as follows. First, a detailed problem setting and preliminaries are discussed in Section II, after which the incremental observability and incremental controllability functions are introduced in Section III. In Section IV, the reduction method of incremental balanced truncation is discussed, which includes the preservation of stability properties and the derivation of an error bound. Next, generalized incremental balanced truncation is presented in Section V, whereas this procedure is illustrated by means of an example in Section VI. Finally, conclusions are stated in Section VII.

*Notation:* The field of real numbers is denoted by  $\mathbb{R}$ . For a vector  $x \in \mathbb{R}^n$ , the Euclidian norm is given as  $|x| = \sqrt{x^T x}$ . Next, a function  $x : \mathcal{T} \rightarrow \mathbb{R}^n$ , with  $\mathcal{T} \subseteq \mathbb{R}$ , is said to be in  $\mathcal{L}_2^n(\mathcal{T})$  if  $\int_{\mathcal{T}} |x(t)|^2 dt < \infty$ . Finally, a function  $\alpha : [0, a) \rightarrow [0, \infty)$  is said to be of class  $\mathcal{K}$  if it strictly increasing and satisfies  $\alpha(0) = 0$ .

## II. PROBLEM SETTING AND PRELIMINARIES

Nonlinear systems of the form

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x) \end{cases} \quad (1)$$

are considered, with  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$ . It is assumed that  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is locally Lipschitz continuous and satisfies  $f(0) = 0$ , such that the origin is an equilibrium point of  $\Sigma$  for  $u = 0$ . Additionally, the functions  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are continuous, where  $h(0) = 0$ . Input functions  $u(\cdot)$  are chosen from the class  $\mathcal{L}_2^m$ .

The problem discussed in this paper can be stated as follows.

*Problem 1:* Develop a model reduction procedure for nonlinear systems  $\Sigma$  as in (1), hereby guaranteeing the preservation of relevant stability properties and the availability of an *a priori* error bound.

This problem will be addressed by the development of the method of *incremental balanced truncation*, which can

be considered as an extension of balanced truncation for linear systems towards the nonlinear case. However, even though incremental balanced truncation provides a solution to Problem 1, the associated computational complexity is high. Thereto, an extension of incremental balanced truncation will be developed for systems of the form

$$\Sigma_{\text{BC}} : \begin{cases} \dot{x} = f(x) + Bu, \\ y = Cx \end{cases} \quad (2)$$

which provides a subclass of (1). Here, the choice  $g(x) = B$  and  $h(x) = Cx$  can lead to more computationally feasible algorithms. Thus, this extension targets the following problem.

*Problem 2:* Develop an extension of the approach towards Problem 1, herein considering systems  $\Sigma_{\text{BC}}$  as in (2) and aiming at improved computational feasibility while maintaining the properties of stability preservation and the availability of an error bound.

The approach towards Problem 2 will be called generalized incremental balanced truncation, as it will be based on generalizing the tools for reduction as developed towards Problem 1 (albeit for a subclass of systems). It is noted that the approach towards Problem 2 relies heavily on the theoretical foundations developed in the incremental balanced truncation approach towards Problem 1, whereas incremental balanced truncation also provides a clear interpretation of its extension towards generalized incremental balanced truncation.

The model reduction procedure for nonlinear systems as developed in this paper is based on a characterization of observability and controllability properties of nonlinear systems. Thereto, the notions of indistinguishability and observability are recalled (see [16], [22]).

*Definition 1:* Let  $y(\cdot)$  and  $\bar{y}(\cdot)$  be the outputs of (1) for some input function  $u(\cdot)$  and initial conditions  $x(0) = x_0$  and  $x(0) = \bar{x}_0$ , respectively. Then, the states  $x_0$  and  $\bar{x}_0$  are said to be indistinguishable (with respect to  $\Sigma$ ) if, for all  $u(\cdot)$ ,  $y(\cdot) = \bar{y}(\cdot)$  on their common domain of definition. Indistinguishability of  $x_0$  and  $\bar{x}_0$  is denoted by  $x_0 \mathcal{I} \bar{x}_0$ .

*Definition 2:* A system  $\Sigma$  as in (1) is said to be observable if, for all  $x_0, \bar{x}_0 \in \mathbb{R}^n$ ,  $x_0 \mathcal{I} \bar{x}_0$  implies  $x_0 = \bar{x}_0$ .

Furthermore, the following definition of reachability (see, e.g., [29], [33]) will be exploited in the characterization of controllability properties.

*Definition 3:* A system (1) is said to be reachable from  $x^*$  if for each  $x_0 \in \mathbb{R}^n$  there exist a time  $T \geq 0$  and an input function  $u(\cdot) \in \mathcal{L}_2^m([-T, 0])$  such that  $u(\cdot)$  steers the system from  $x(-T) = x^*$  to  $x(0) = x_0$ .

Finally, the preservation of stability properties is of interest. In particular, the notion of incremental stability is exploited.

*Definition 4:* The system (1) is said to be incrementally stable for the class of inputs  $\mathcal{U}$  if there exist a function  $\alpha$  of class  $\mathcal{K}$  and a constant  $c > 0$  such that, for any two solutions  $x(\cdot)$  and  $\bar{x}(\cdot)$  corresponding to any input function  $u(\cdot) \in \mathcal{U}$  and initial conditions  $x(t_0)$  and  $\bar{x}(t_0)$ , respectively, the inequality

$$|x(t) - \bar{x}(t)| \leq \alpha(|x(t_0) - \bar{x}(t_0)|) \quad (3)$$

holds for all  $t \geq t_0$  and for all  $|x(t_0) - \bar{x}(t_0)| < c$ .

The definition of incremental stability corresponds to that of [1], where it is remarked that incremental *asymptotic* stability can be defined by replacing the class  $\mathcal{K}$  function  $\alpha$  in (3) by a suitable function  $\beta(|x(t_0) - \bar{x}(t_0)|, t - t_0)$  of class  $\mathcal{KL}$  (see [19] for a definition). Also, it is stressed that incremental stability provides a stability property for *all* solutions of (1). Consequently, both the stability properties of trajectories for nonzero input and the stability of the equilibrium point for zero input are captured with this stability notion.

### III. INCREMENTAL OBSERVABILITY AND CONTROLLABILITY

The model reduction procedure introduced in this paper is based on the introduction of two novel energy functions, namely the *incremental observability function* and the *incremental controllability function*. The first is defined as follows.

*Definition 5:* Consider the function  $E'_o : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined as

$$E'_o(x_0, \bar{x}_0) = \sup_{u(\cdot) \in \mathcal{L}_2^m([0, \infty))} \int_0^\infty |y(t) - \bar{y}(t)|^2 dt \quad (4)$$

where  $y(\cdot)$  and  $\bar{y}(\cdot)$  denote the outputs of (1) for input function  $u(\cdot)$  and initial conditions  $x(0) = x_0$  and  $\bar{x}(0) = \bar{x}_0$ , respectively. If  $E'_o$  is bounded for all bounded  $(x_0, \bar{x}_0)$ , then  $E_o := E'_o$  is said to be the incremental observability function of  $\Sigma$ .

The incremental observability function thus characterizes the maximum energy associated with the difference in two output trajectories. The incremental controllability function is defined as follows.

*Definition 6:* Consider the function  $E'_c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined as

$$E'_c(x_0, \bar{x}_0) = \inf_{u(\cdot), \bar{u}(\cdot) \in \mathcal{L}_2^m((-\infty, 0])} \int_{-\infty}^0 |u(t) + \bar{u}(t)|^2 dt \quad (5)$$

whenever there exists input functions  $u(\cdot)$  and  $\bar{u}(\cdot)$  such that  $u(\cdot)$  steers the system (1) from  $x(-\infty) = 0$  to  $x(0) = x_0$  and  $\bar{u}(\cdot)$  steers (1) from  $\bar{x}(-\infty) = 0$  to  $\bar{x}(0) = \bar{x}_0$ . If such input functions do not exist, then  $E'_c(x_0, \bar{x}_0) := \infty$ . If  $E'_c$  is bounded for all bounded  $(x_0, \bar{x}_0)$ , then  $E_c := E'_c$  is said to be the incremental controllability function of  $\Sigma$ .

The incremental controllability function thus gives the minimum energy in the sum of two input functions that steer  $\Sigma$  to the states  $x_0$  and  $\bar{x}_0$ , respectively. Even though this definition might not seem natural at first sight, it will be shown that the incremental controllability function as in (5) is useful in the development of a model reduction procedure for which a bound on the reduction error can be guaranteed.

It is clear that neither the function  $E'_o$  in (4) nor the function  $E'_c$  in (5) is guaranteed to be bounded for all bounded  $(x_0, \bar{x}_0)$ . In particular, when the nonlinear system  $\Sigma$  exhibits multiple steady-state solutions, the integral in (4) is likely to be unbounded. Moreover, the existence of an input function steering to an arbitrary state is not guaranteed *a priori*, in which case  $E'_c$  is, by definition, infinite. However, it is noted that

these functions only define the incremental observability and incremental controllability function when they are bounded for all bounded  $(x_0, \bar{x}_0)$ , in which case the incremental observability and incremental controllability functions are said to exist. Moreover, the following assumption is made.

*Aassumption 1:* The incremental observability function and incremental controllability function, when they exist (i.e., when the functions in (4) and (5) are bounded for all bounded  $(x_0, \bar{x}_0)$ ), are differentiable functions.

*Remark 1:* The incremental energy functions in Definitions 5 and 6 are based on the energy associated with the comparison of *two* trajectories and thus differ from the energy functions as exploited in nonlinear balanced truncation (see [29]), where the energy associated to a single trajectory is exploited. A further comparison between these energy functions can be found in [5].  $\triangleleft$

The incremental observability function can be characterized in a similar way as the observability function in nonlinear balanced truncation in [29]. Thereto, the theory of dissipative systems (see [8], [33]) is exploited and the auxiliary system

$$\Sigma^o : \begin{cases} \dot{x} = f(x) + g(x)u, \\ \dot{\bar{x}} = f(\bar{x}) + g(\bar{x})u, \\ \Delta y = h(x) - h(\bar{x}) \end{cases} \quad (6)$$

is introduced. Then, it can be observed that the incremental observability function  $E_o$  as in (4) represents the *available storage* for  $\Sigma^o$  with supply rate

$$s(u, \Delta y) = -|\Delta y|^2 = -|y - \bar{y}|^2 \quad (7)$$

such that the following result can be obtained by exploiting the nonlinear Kalman-Yakubovich-Popov lemma.

*Theorem 1:* Let Assumption 1 hold. Then, the incremental observability function  $E_o$  as in Definition 5 exists if and only if there exist functions  $\tilde{E}_o : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $l_o : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$  (for some integer  $q$ ) with  $\tilde{E}_o$  differentiable,  $\tilde{E}_o(0, 0) = 0$  and  $\tilde{E}_o(x, \bar{x}) \geq 0$  such that

$$\frac{\partial \tilde{E}_o}{\partial x}(x, \bar{x})f(x) + \frac{\partial \tilde{E}_o}{\partial \bar{x}}(x, \bar{x})f(\bar{x}) = -|h(x) - h(\bar{x})|^2 - |l_o(x, \bar{x})|^2, \quad (8)$$

$$\frac{\partial \tilde{E}_o}{\partial x}(x, \bar{x})g(x) + \frac{\partial \tilde{E}_o}{\partial \bar{x}}(x, \bar{x})g(\bar{x}) = 0 \quad (9)$$

for all  $x, \bar{x} \in \mathbb{R}^n$ . If such functions  $\tilde{E}_o$  and  $l_o$  exist, then the incremental observability function  $E_o$  is the smallest solution satisfying (8), (9), i.e.,  $E_o$  satisfies  $E_o(x, \bar{x}) \leq \tilde{E}_o(x, \bar{x})$  for all  $x, \bar{x} \in \mathbb{R}^n$  and all  $\tilde{E}_o$  satisfying (8), (9).

*Proof:* The first statement of Theorem 1 results from [18, Theorem 8] (see also [17, Theorem 1] and [8, Lemma 4.87]), i.e., dissipativity is equivalent to the existence of functions satisfying (8), (9). Moreover, it is well known that the available storage  $E_o$  is the smallest solution (in the sense that  $E_o(x, \bar{x}) \leq \tilde{E}_o(x, \bar{x})$  for all  $x, \bar{x} \in \mathbb{R}^n$  and all storage functions  $\tilde{E}_o$ ) satisfying the Kalman-Yakubovich-Popov conditions, see, e.g., [17], [33], which proves the theorem.  $\square$

A characterization of the incremental controllability function can be obtained by using a similar perspective of dissipative systems. Namely, the incremental controllability function in (5) can be considered as the *required supply* for the auxiliary system

$$\Sigma^c : \begin{cases} \dot{x} = f(x) + g(x)u, \\ \dot{\bar{x}} = f(\bar{x}) + g(\bar{x})\bar{u} \end{cases} \quad (10)$$

with supply rate

$$s(u, \bar{u}) = |u + \bar{u}|^2 = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}^T \begin{bmatrix} I & I \\ I & I \end{bmatrix} \begin{bmatrix} u \\ \bar{u} \end{bmatrix} \quad (11)$$

such that the following result can be obtained.

**Theorem 2:** Let the system  $\Sigma$  as in (1) be reachable from 0. Then, the incremental controllability function  $E_c$  exists. If, in addition, Assumption 1 holds, then there exist functions  $\tilde{E}_c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $l_c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^q$  and  $W : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{q \times 2m}$  (for some integer  $q$ ) with  $\tilde{E}_c$  differentiable,  $\tilde{E}_c(0, 0) = 0$  and  $\tilde{E}_c(x, \bar{x}) \geq 0$  such that

$$\frac{\partial \tilde{E}_c}{\partial x}(x, \bar{x})f(x) + \frac{\partial \tilde{E}_c}{\partial \bar{x}}(x, \bar{x})f(\bar{x}) = -l_c^T(x, \bar{x})l_c(x, \bar{x}), \quad (12)$$

$$\begin{bmatrix} \frac{\partial \tilde{E}_c}{\partial x}(x, \bar{x})g(x) & \frac{\partial \tilde{E}_c}{\partial \bar{x}}(x, \bar{x})g(\bar{x}) \end{bmatrix} = -2l_c^T(x, \bar{x})W(x, \bar{x}), \quad (13)$$

$$\begin{bmatrix} I & I \\ I & I \end{bmatrix} = W^T(x, \bar{x})W(x, \bar{x}) \quad (14)$$

for all  $x, \bar{x} \in \mathbb{R}^n$ . Moreover, the incremental controllability function  $E_c$  is the largest solution satisfying (12)–(14), i.e.,  $E_c$  satisfies  $E_c(x, \bar{x}) \geq \tilde{E}_c(x, \bar{x})$  for all  $x, \bar{x} \in \mathbb{R}^n$  and for all  $\tilde{E}_c$  satisfying (12)–(14).

*Proof:* By [33, Theorem 2], a necessary and sufficient condition for dissipativity of  $\Sigma^c$  as in (10) is that the required supply is bounded from below. As the supply rate (11) is nonnegative, the required supply is nonnegative and the auxiliary system  $\Sigma^c$  is dissipative with respect to the supply rate in (11). By [18, Theorem 8] (see also [17] and [8]), the dissipativity of  $\Sigma^c$  with respect to the supply rate (11) is equivalent to the existence of functions  $\tilde{E}_c$  satisfying (12)–(14). Since the auxiliary system  $\Sigma^c$  as in (10) is reachable from the origin, which is implied by reachability from the origin for the original system  $\Sigma$ , boundedness of  $E_c$  is guaranteed. Then, by dissipativity theory, the required supply is the largest storage function for a given supply rate, which proves Theorem 2.  $\square$

**Remark 2:** In the proof of Theorem 2, differentiability of the incremental controllability function is not required to show its existence, Assumption 1 is only needed to guarantee that  $E_c$  is a solution of (12)–(14).  $\triangleleft$

**Remark 3:** For asymptotically stable linear systems (i.e., where  $f(x) = Ax$  in (2)), it can be shown (see [5]) that the incremental observability function  $E_o$  and incremental controllability function  $E_c$  are given as

$$E_o(x_0, \bar{x}_0) = (x_0 - \bar{x}_0)^T Q (x_0 - \bar{x}_0), \quad (15)$$

$$E_c(x_0, \bar{x}_0) = (x_0 + \bar{x}_0)^T P^{-1} (x_0 + \bar{x}_0) \quad (16)$$

with  $Q = Q^T$  and  $P = P^T$  the observability and controllability Gramian, respectively.  $\triangleleft$

From Definitions 5 and 6 it is clear that both the incremental observability function and incremental controllability function are nonnegative. For the incremental observability function, the relation with observability as in Definition 2 leads to a stronger result on its positivity, as stated in the following proposition.

**Proposition 3:** Assume that the incremental observability function  $E_o$  as in Definition 5 exists. Then, the system  $\Sigma$  as in (1) is observable if and only if  $E_o(x_0, \bar{x}_0) > 0$  for all  $x_0, \bar{x}_0 \in \mathbb{R}^n$ ,  $x_0 \neq \bar{x}_0$ .

*Proof:* This follows directly from Definitions 2 and 5. For details, see [5].  $\square$

Moreover, for nonlinear systems with an odd vector field, positivity of the incremental controllability function is related to asymptotic stability. This is formalized in the following result.

**Proposition 4:** Let the system  $\Sigma$  as in (1) with  $f(x) = -f(-x)$  and  $g(x) = g(-x)$  be reachable from 0 and let 0 be an asymptotically stable equilibrium point of  $\Sigma$  for  $u = 0$ . Then, the incremental controllability function  $E_c$  satisfies

$$E_c(x_0, x_0) > 0 \quad (17)$$

for all  $x_0 \in \mathbb{R}^n$ ,  $x_0 \neq 0$ .

*Proof:* It is recalled that, by definition,  $E_c(x_0, x_0) \geq 0$ . Now, let  $E_c(x_0, x_0) = 0$ . By nonnegativity of  $|u + \bar{u}|^2$  in Definition 6, this implies  $u(t) = -\bar{u}(t)$  for all  $t \in (-\infty, 0]$ . However, if  $u$  steers  $\Sigma$  from the origin to  $x(0) = x_0$ , then  $\bar{u}(t) = -u(t)$ ,  $t \in (-\infty, 0]$  steers the system to  $x(0) = -x_0$ , as follows from the properties  $f(x) = -f(-x)$  and  $g(x) = g(-x)$ . Thus,  $E_c(x_0, x_0)$  can only be zero for  $u(t) = \bar{u}(t) = 0$ , for all  $t \in (-\infty, 0]$ . However, by asymptotic stability of the origin, it is impossible to have  $x(-\infty) = 0$  and  $x(0) \neq 0$  for zero input. Thus,  $E_c(x_0, x_0) > 0$  for all  $x_0 \in \mathbb{R}^n$ ,  $x_0 \neq 0$ .  $\square$

#### IV. INCREMENTAL BALANCED TRUNCATION

In Section III, the incremental observability and incremental controllability functions have been defined. Loosely speaking, the incremental observability and incremental controllability function describe the energy transfer from state to output and input to state, respectively, and can thus be used as the basis for a model reduction procedure that aims at the accurate approximation of the input-output behavior. Besides this intuition, the following aspects further motivate this idea. Firstly, for linear systems, the incremental energy functions are directly related to the observability and controllability Gramians (see Remark 3), where the latter are known to give good results when used as the basis for model reduction (in the method of balanced truncation introduced in [21]). Secondly, it will be shown that a reduction procedure based on the *incremental* energy functions ensures certain stability properties of the reduced-order system and allows for the computation of an error bound.

A one-step reduction (in which the state-space dimension is reduced from  $n$  to  $n - 1$ ) is considered, herein exploiting the following definition.

*Definition 7:* A realization (1) of the nonlinear system  $\Sigma$  is said to be an incrementally balanced realization if the incremental observability function  $E_o$  and incremental controllability function  $E_c$  exist and satisfy Assumption 1 as well as the following conditions. First, they can be partitioned as

$$E_o(x, \bar{x}) = E_o^1(x_1, \bar{x}_1) + E_o^2(x_2, \bar{x}_2), \quad (18)$$

$$E_c(x, \bar{x}) = E_c^1(x_1, \bar{x}_1) + E_c^2(x_2, \bar{x}_2) \quad (19)$$

where  $x^T = [x_1^T \ x_2]$  and  $\bar{x}^T = [\bar{x}_1^T \ \bar{x}_2]$  with  $x_1, \bar{x}_1 \in \mathbb{R}^{n-1}$  and  $x_2, \bar{x}_2 \in \mathbb{R}$ . In addition, there exists a constant  $\rho > 0$  such that  $E_o^2$  and  $E_c^2$  satisfy

$$\frac{\partial E_o^2}{\partial \bar{x}_2}(x_2, 0) = -\rho^2 \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, 0) \quad (20)$$

for all  $x_2 \in \mathbb{R}$ .

It will be shown that the properties (18), (19) enable the preservation of stability properties during model reduction, whereas (20) is crucial in the derivation of an error bound. In particular, (20) gives a relation between the parts of the incremental energy functions that are affected by model reduction. The reduced-order model will be obtained by truncation, explaining the substitution  $\bar{x}_2 = 0$  in (20). In addition, it is noted that the properties in Definition 7 are automatically satisfied for linear systems in a balanced realization (i.e., a realization in which the Gramians  $P$  and  $Q$  as in (15) and (16) are equal and diagonal), providing a motivation for this definition.

*Remark 4:* An arbitrary realization of a nonlinear system will generally not be an incrementally balanced realization as in Definition 7. Instead, given a realization as in (1), the objective is to find a coordinate transformation  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  (i.e., a diffeomorphism), such that the conditions (18), (19), and (20) hold in the new coordinates  $z = \varphi^{-1}(x)$ . Whether such a transformation exists or whether conditions guaranteeing this existence can be found is unknown in general.  $\triangleleft$

In the remainder of this section, it is assumed that the realization (1) of  $\Sigma$  is an incrementally balanced realization as in Definition 7. After partitioning the state as  $x^T = [x_1^T \ x_2]$  with  $x_1 \in \mathbb{R}^{n-1}$  and  $x_2 \in \mathbb{R}$ , the functions  $f$ ,  $g$  and  $h$  in (1) can be split accordingly as

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}, \quad g(x) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix}, \quad (21)$$

$$h(x) = h(x_1, x_2). \quad (22)$$

Using this partitioning, a reduced-order approximation of  $\Sigma$  is obtained by truncation. Substitution of  $x_2 = 0$  in the partitioned incrementally balanced realization and discarding the dynamics concerning  $x_2$  results in the one-step reduced-order system

$$\hat{\Sigma}_{n-1} : \begin{cases} \dot{\xi} = f_1(\xi, 0) + g_1(\xi, 0)u, \\ \hat{y} = h(\xi, 0) \end{cases} \quad (23)$$

with  $\xi \in \mathbb{R}^{n-1}$  the reduced-order state, which approximates  $x_1$ . This reduction procedure will be referred to as incremental balanced truncation.

In the analysis of properties of  $\hat{\Sigma}_{n-1}$  as in (23), results on the (boundedness and characterization of the) incremental

observability and incremental controllability function for the reduced-order system are exploited. First, the following statement regarding the incremental observability function of the reduced-order nonlinear system  $\hat{\Sigma}_{n-1}$  holds.

*Lemma 5:* Let (1) be an incrementally balanced realization of the system  $\Sigma$  and let  $\hat{\Sigma}_{n-1}$  as in (23) be a one-step reduced-order system obtained by incremental balanced truncation. Then, the incremental observability function  $\hat{E}_o$  of  $\hat{\Sigma}_{n-1}$  exists and is upper bounded as

$$\hat{E}_o(\xi, \bar{\xi}) \leq E_o^1(\xi, \bar{\xi}) \quad (24)$$

for all  $\xi, \bar{\xi} \in \mathbb{R}^{n-1}$ , where  $E_o^1$  is defined in (18).

*Proof:* The proof is stated in Appendix A.  $\square$

Hence, even though  $E_o^1$  satisfies the conditions of Theorem 1 (as shown in the proof of Lemma 5),  $E_o^1$  does not necessarily represent the incremental observability function  $\hat{E}_o$ . Namely,  $E_o^1$  is not necessarily the minimal solution of (57), (58) in the Appendix when the function  $l_o$  can be chosen arbitrarily as in the statement of Theorem 1.

A similar result can be obtained for the incremental controllability function for the reduced-order system, even though its existence cannot be guaranteed *a priori*. This is formalized in the following lemma.

*Lemma 6:* Let (1) be an incrementally balanced realization of the system  $\Sigma$  and let  $\hat{\Sigma}_{n-1}$  as in (23) be a one-step reduced-order system obtained by incremental balanced truncation. Then, when the incremental controllability function  $\hat{E}_c$  of  $\hat{\Sigma}_{n-1}$  exists, it is lower bounded as

$$\hat{E}_c(\xi, \bar{\xi}) \geq E_c^1(\xi, \bar{\xi}) \quad (25)$$

for all  $\xi, \bar{\xi} \in \mathbb{R}^{n-1}$ , where  $E_c^1$  is defined in (19).

*Proof:* The proof can be found in Appendix B.  $\square$

The properties of the incremental observability and incremental controllability functions of the reduced-order system as in Lemmas 5 and 6 can be used to derive stability properties of  $\hat{\Sigma}_{n-1}$ . In particular, boundedness of solutions and incremental stability as in Definition 4 can be guaranteed, as formalized in the following theorem.

*Theorem 7:* Let (1) be an incrementally balanced realization as in Definition 7 of the system  $\Sigma$  and let the incremental observability and incremental controllability function satisfy  $E_o(x, \bar{x}) > 0$  for all  $x, \bar{x} \in \mathbb{R}^n$ ,  $x \neq \bar{x}$  and  $E_c(x, x) > 0$  for all  $x \in \mathbb{R}^n$ ,  $x \neq 0$ , respectively. In addition, let  $\hat{\Sigma}_{n-1}$  as in (23) be the one-step reduced-order system obtained by incremental balanced truncation. Then, the following statements hold:

- 1) There exists a set of initial conditions  $\hat{\mathcal{X}} \subseteq \mathbb{R}^{n-1}$  with  $\hat{\mathcal{X}} \ni 0$  and a class of input functions  $\mathcal{U} \subseteq \mathcal{L}_2^m([0, \infty))$  such that any state trajectory  $\xi(\cdot)$  of the reduced-order system (23) corresponding to  $\xi(0) = \xi_0 \in \hat{\mathcal{X}}$  and  $u(\cdot) \in \mathcal{U}$  is bounded for all  $t \geq 0$ . If  $E_c(x, x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ , then  $\hat{\mathcal{X}} = \mathbb{R}^{n-1}$  and  $\mathcal{U} = \mathcal{L}_2^m([0, \infty))$ ;
- 2) The reduced-order nonlinear system  $\hat{\Sigma}_{n-1}$  is incrementally stable for the class of inputs  $\mathcal{L}_2^m([0, \infty))$ ;
- 3) For any bounded input function  $u(\cdot) \in \mathcal{L}_2^m([0, \infty))$  and initial conditions  $\xi(0) = \xi_0$ ,  $\bar{\xi}(0) = \bar{\xi}_0$  such that the state

trajectories  $\xi(\cdot)$  and  $\bar{\xi}(\cdot)$  of (23) are bounded for all  $t \geq 0$ , the corresponding outputs converge, i.e.

$$\lim_{t \rightarrow \infty} |h(\xi(t), 0) - h(\bar{\xi}(t), 0)| = 0. \quad (26)$$

*Proof:* The proof can be found in Appendix C.  $\square$

It is noted that, in the proof of Theorem 7, the availability of the incremental observability and incremental controllability function of the reduced-order system is not required. Instead, the bounds as derived in Lemmas 5 and 6 are used. As a result, the stability conditions in Theorem 7 can be evaluated *a priori*.

Moreover, it is remarked that the stability results of Theorem 7 also hold for the original high-order system  $\Sigma$  as in (1). This can be shown by exploiting the incremental energy functions  $E_o$  and  $E_c$  rather than the partitioned counterparts  $E_o^1$  and  $E_c^1$  in the proof of Theorem 7 (see also [5]). As a result, boundedness of solutions, incremental stability and converging outputs are necessary conditions on the high-order system for the incremental observability and incremental controllability function to exist and to have the desirable properties  $E_o(x, \bar{x}) > 0$  for all  $x, \bar{x} \in \mathbb{R}^n$ ,  $x \neq \bar{x}$  and  $E_c(x, x) > 0$  for all  $x \in \mathbb{R}^n$ ,  $x \neq 0$ . Consequently, the results of Theorem 7 can be interpreted as a guarantee on stability *preservation*.

*Remark 5:* The proof of the (incremental) stability properties in Theorem 7 relies on the incrementally balanced realization as in Definition 7. However, only the partitioning of the incremental observability and incremental controllability as in (18) and (19) is used, whereas the condition (20) is not required to guarantee incremental stability of the reduced-order system. In fact, the results in Lemmas 5 and 6 and Theorem 7 hold for reduced-order systems of arbitrary order  $k$  ( $k < n$ ) as long as the partitioning (18), (19) holds for  $x_1, \bar{x}_1 \in \mathbb{R}^k$  and  $x_2, \bar{x}_2 \in \mathbb{R}^{n-k}$ .  $\triangleleft$

Besides the results on stability of the reduced-order system  $\hat{\Sigma}_{n-1}$  as stated in Theorem 7, it can be shown that an error bound in terms of the  $\mathcal{L}_2$  signal norm holds. This is formalized in the following theorem.

*Theorem 8:* Let (1) be an incrementally balanced realization as in Definition 7 of the system  $\Sigma$  and let  $\hat{\Sigma}_{n-1}$  as in (23) be the one-step reduced-order system obtained by incremental balanced truncation. For trajectories  $x(\cdot)$  of  $\Sigma$  and  $\xi(\cdot)$  of  $\hat{\Sigma}_{n-1}$  corresponding to a common input function  $u(\cdot) \in \mathcal{L}_2^n([0, \infty))$  and initial conditions  $x(0) = 0$  and  $\xi(0) = 0$ , respectively, the outputs  $y(\cdot)$  and  $\hat{y}(\cdot)$  satisfy the error bound

$$\|y - \hat{y}\|_2 \leq 2\rho \|u\|_2 \quad (27)$$

with  $\rho$  as in (20).

*Proof:* The proof can be found in Appendix D.  $\square$

The error bound in Theorem 8 is only dependent on the properties of the incrementally balanced realization of the high-order system and can thus be evaluated *a priori*.

*Remark 6:* The error bound in Theorem 8 is only valid for a one-step reduction. However, repeated application of Theorem 8 and the triangle inequality leads to an error bound for reduced-order systems of arbitrary state dimension, as long as each reduced-order subsystem obtained after a one-step reduction satisfies a property similar to (20).  $\triangleleft$

The results in Theorems 7 and 8 show that the desired properties of the preservation of relevant stability properties and

the availability of a computable error bound are a direct consequence of the introduction of the incremental energy functions in Definitions 5 and 6. These fundamental properties are thus inherent to the approach of incremental balanced truncation.

## V. GENERALIZED INCREMENTAL BALANCED TRUNCATION

In Section IV, the model reduction technique of incremental balanced truncation has been introduced. This reduction procedure for nonlinear systems guarantees certain stability properties of the reduced-order system as well as an error bound. This method is based on the incremental observability and incremental controllability functions and heavily relies on properties of the so-called incrementally balanced realization (see Definition 7), whose existence is difficult to check. An extension to incremental balanced truncation that is computationally more attractive and easier to apply is given in the current section, hereby circumventing this problem. The developments on incremental balanced truncation of Section IV are instrumental for this extension.

The extension will be referred to as *generalized* incremental balanced truncation and largely follows the ideas of incremental balanced truncation as in Section IV. However, rather than using the incremental observability and incremental controllability functions, generalized incremental balancing is based on energy functions that provide *bounds* on these incremental observability and incremental controllability functions. This is beneficial from two perspectives. Firstly, these bounds are generally easier to obtain than the incremental energy functions themselves, which increases the computational feasibility of generalized incremental balanced truncation over incremental balanced truncation. Secondly, the usage of bounds on the energy functions gives the added flexibility of freely choosing the structure of these energy functions. Here, the choice of *quadratic* functions allows for the computation of a realization that satisfies the properties of the incrementally balanced realization and thus satisfies the potentially limiting assumption of its existence.

### A. Generalized Incremental Energy Functions

In order to replace the incremental observability function in an alternative model reduction procedure, the following definition is used.

*Definition 8:* The function

$$\tilde{E}_o(x, \bar{x}) = (x - \bar{x})^T \tilde{Q} (x - \bar{x}) \quad (28)$$

with  $\tilde{Q} = \tilde{Q}^T \succ 0$  is said to be a generalized incremental observability function of system (1) if it is a storage function of the auxiliary system  $\Sigma^o$  as in (6) with the supply rate (7).

Since  $\tilde{E}_o$  is defined as a storage function for  $\Sigma^o$  with supply rate (7), the proof of Theorem 1 directly shows that  $\tilde{E}_o$  is equivalently defined as a solution (of the form (28)) satisfying the conditions (8), (9). Moreover, it is recalled that the incremental observability function  $E_o$  is given as the *available storage*, which, by dissipativity theory (see [33]), is the smallest

storage function (see Theorem 1). Consequently, the following corollary holds.

*Corollary 9:* Let the generalized incremental observability function  $\tilde{E}_o$  as in (28) exist. Then, the incremental controllability function  $E_o$  as in (4) exists and is bounded as  $E_o(x, \bar{x}) \leq \tilde{E}_o(x, \bar{x})$  for all  $x, \bar{x} \in \mathbb{R}^n$ .

Similarly, the following definition will be used to generalize the incremental controllability function.

*Definition 9:* The function

$$\tilde{E}_c(x, \bar{x}) = (x + \bar{x})^T \tilde{R}(x + \bar{x}) \quad (29)$$

with  $\tilde{R} = \tilde{R}^T \succ 0$  is said to be a generalized incremental controllability function of system (1) if it is a storage function of the auxiliary system  $\Sigma^c$  as in (10) with the supply rate (11).

The following corollary is immediate from Theorem 2.

*Corollary 10:* Let the generalized incremental controllability function  $\tilde{E}_c$  as in (29) exist. In addition, assume that the incremental controllability function  $E_c$  as in (5) exists. Then,  $E_c(x, \bar{x}) \geq \tilde{E}_c(x, \bar{x})$  for all  $x, \bar{x} \in \mathbb{R}^n$ .

Even though the generalized incremental energy functions as in Definitions 8 and 9 can be defined for nonlinear systems of the form (1), they can be more conveniently characterized when nonlinear systems  $\Sigma_{BC}$  as in (2) are considered. For such systems, the generalized incremental observability function can be characterized as follows.

*Theorem 11:* Consider the system  $\Sigma_{BC}$  as in (2). Then, the function  $\tilde{E}_o$  as in (28) is a generalized incremental observability function if and only if the matrix  $\tilde{Q}$  satisfies

$$(x - \bar{x})^T \tilde{Q} (f(x) - f(\bar{x})) \leq -\frac{1}{2} (x - \bar{x})^T C^T C (x - \bar{x}) \quad (30)$$

for all  $x, \bar{x} \in \mathbb{R}^n$ .

*Proof:* First, assume that  $\tilde{E}_o$  as in (28) is a generalized incremental observability function. Then, it is a solution of the conditions (8), (9) in Theorem 1 for some function  $l_o$ . Here, the structure of  $\tilde{E}_o$  and  $g(x) = B$  ensure that (9) is automatically satisfied. Hence, the condition (9) can be discarded. Moreover, substitution of (28) in (8) directly leads to (30). Herein, the term  $l_o(x, \bar{x})$  in (8) is removed, such that the equality in (8) is replaced by the inequality in (30).

The converse follows directly by reversing the reasoning above.  $\square$

It is noted that, even though Theorem 11 provides a necessary and sufficient condition of the existence of  $\tilde{E}_o$  as in Definition 8, (30) only provides a *sufficient* condition for dissipativity of the auxiliary system corresponding to  $\Sigma_{BC}$ . The converse does not hold since it is in general not guaranteed that a *quadratic* storage function exists.

Similar to the characterization of the generalized incremental observability function in Theorem 11, the following result is obtained for the generalized incremental controllability function.

*Theorem 12:* Consider the system  $\Sigma_{BC}$  as in (2). Then, the function  $\tilde{E}_c$  as in (29) is a generalized incremental controllability function if the matrix  $\tilde{R}$  satisfies

$$(x + \bar{x})^T \tilde{R} (f(x) + f(\bar{x})) \leq -\frac{1}{2} (x + \bar{x})^T \tilde{R} B B^T \tilde{R} (x + \bar{x}) \quad (31)$$

for all  $x, \bar{x} \in \mathbb{R}^n$ .

*Proof:* To prove the theorem, it will be shown that (31) implies that the conditions in the characterization of the incremental controllability function in Theorem 2 are satisfied. Namely, when the functions  $l_c$  and  $W$  in (12)–(14) are chosen as

$$l_c(x, \bar{x}) = - \begin{bmatrix} B^T \tilde{R}(x + \bar{x}) \\ \tilde{l}_c(x, \bar{x}) \end{bmatrix}, \quad W(x, \bar{x}) = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad (32)$$

with  $\tilde{l}_c$  an arbitrary function, it is easily seen that the function  $\tilde{E}_c$  as in (29) satisfies (13) and (14) for the dynamics (2). Substitution of  $l_c$  as in (32) in (12) yields

$$(x + \bar{x})^T \tilde{R} (f(x) + f(\bar{x})) = -\frac{1}{2} (x + \bar{x})^T \tilde{R} B B^T \tilde{R} (x + \bar{x}) - \frac{1}{2} \tilde{l}_c^T(x, \bar{x}) \tilde{l}_c(x, \bar{x}) \quad (33)$$

thus leading to the condition (31). Hence, (31) implies that the conditions of Theorem 2 are satisfied, such that  $\tilde{E}_c$  as in (29) is a storage function for the supply rate  $s(u, \bar{u}) = |u + \bar{u}|^2$  as in the proof of Theorem 2. This completes the proof.  $\square$

For linear systems, the inequalities (30) and (31) boil down to Lyapunov inequalities describing the *generalized* observability and controllability Gramian (see [9]), which find application in, e.g., model reduction for uncertain systems [4] or linear time-varying systems [27]. This further motivates the names for the functions in Definitions 8 and 9.

*Remark 7:* From the inequalities (30) and (31) it is clear that the generalized incremental observability function and generalized incremental controllability function are not unique. For example, when  $\tilde{Q}$  and  $\tilde{R}$  satisfy (30) and (31), then  $\alpha \tilde{Q}$  and  $\alpha^{-1} \tilde{R}$ , with  $\alpha > 1$ , are solutions as well. It is expected that the best reduced-order approximation is obtained when the bounds on the incremental observability and incremental controllability function as provided by  $\tilde{E}_o$  and  $\tilde{E}_c$  are as tight as possible. Thus, since the quadratic form of the bounds is used, the smallest solution  $\tilde{Q}$  to (30) and the largest solution  $\tilde{R}$  to (31) should be sought. Later, it will be shown that the bound on the reduction error can indeed be reduced by seeking tight bounds.  $\triangleleft$

*Remark 8:* The choice of the *quadratic* generalized incremental controllability function in (29) implies that a solution to (31) can only be found when the vector field satisfies  $f(x) = -f(-x)$  for all  $x \in \mathbb{R}^n$ . More details on the relation between an odd vector field  $f$  and the (generalized) incremental controllability function (see also Proposition 4) can be found in [5].  $\triangleleft$

As the reduction procedure developed in this section is based on generalized incremental energy functions in Definitions 8 and 9, it can be applied to all nonlinear systems of the form (2) for which solutions to (30) and (31) exist. This class of systems can be further characterized by the following proposition.

*Proposition 13:* Consider the system  $\Sigma_{BC}$  as in (2) and assume that  $\Sigma_{BC}$  is quadratically incrementally stable, i.e., there exist matrices  $M = M^T \succ 0$  and  $N = N^T \succ 0$  such that

$$(x - \bar{x})^T M (f(x) - f(\bar{x})) \leq -\frac{1}{2} (x - \bar{x})^T N (x - \bar{x}) \quad (34)$$

for all  $x, \bar{x} \in \mathbb{R}^n$ . Then, a solution  $\tilde{Q}$  to (30) exists. If, in addition,  $f$  as in (2) satisfies  $f(x) = -f(-x)$ , then a solution  $\tilde{R}$  to (31) exists as well.



*Proof:* Since  $N$  is positive definite, there exists a parameter  $\alpha > 0$  such that  $\alpha C^T C \preceq N$ . Then, it follows immediately that (30) is satisfied for  $\tilde{Q} = \alpha^{-1}M$ . The existence of a solution  $\tilde{R}$  to (31) can be proven similarly when the property  $f(x) = -f(-x)$  is exploited.  $\square$

Thus, the generalized incremental observability and generalized incremental controllability function can be computed for systems that satisfy the quadratic incremental stability property. Results for such systems are available in, e.g., [11]. Results on so-called quadratic convergence, which implies quadratic incremental stability, are discussed in [23], which also includes computational approaches, e.g., on the basis of linear matrix inequalities or the circle criterion.

### B. Model Reduction

Based on the generalized incremental energy functions introduced in Section V-A, a generalized incrementally balanced realization can be defined as follows.

*Definition 10:* A realization (2) of the system  $\Sigma_{BC}$  is said to be a generalized incrementally balanced realization if there exists a diagonal matrix  $\Sigma$  as

$$\Sigma = \begin{bmatrix} \sigma_1 I_{m_1} & 0 & \cdots & 0 \\ 0 & \sigma_2 I_{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_q I_{m_q} \end{bmatrix} \quad (35)$$

where the parameters  $\sigma_i$  satisfy  $\sigma_1 > \sigma_2 > \cdots > \sigma_q > 0$  and have multiplicity  $m_i$  with  $\sum_{i=1}^q m_i = n$ , such that (30) and (31) hold with  $\tilde{Q} = \Sigma$  and  $\tilde{R} = \Sigma^{-1}$ .

Then, the following result is immediate.

*Theorem 14:* Let the system  $\Sigma_{BC}$  as in (2) be such that there exist positive definite symmetric matrices  $\tilde{Q}$  and  $\tilde{R}$  satisfying (30) and (31), respectively, thus characterizing the generalized incremental observability function as in (28) and generalized incremental controllability function as in (29). Then, there exists a coordinate transformation  $x = Tz$  such that the system  $\Sigma_{BC}$  is a generalized incrementally balanced realization in the new coordinates  $z$ . Moreover, the parameters  $\sigma_i^2$  in (35) equal the eigenvalues of the product  $\tilde{Q}\tilde{R}^{-1}$ .

*Proof:* This is essentially the same coordinate transformation as used in balanced truncation for linear systems, such that the results from [21] (see also [2]) hold.  $\square$

In the remainder of this section, it is assumed that the realization (2) of  $\Sigma_{BC}$  is a generalized incrementally balanced realization. For this realization, the generalized incremental observability and generalized incremental controllability function can be partitioned as

$$\tilde{E}_o(x, \bar{x}) = \tilde{E}_o^1(x_1, \bar{x}_1) + \tilde{E}_o^2(x_2, \bar{x}_2), \quad (36)$$

$$\tilde{E}_c(x, \bar{x}) = \tilde{E}_c^1(x_1, \bar{x}_1) + \tilde{E}_c^2(x_2, \bar{x}_2) \quad (37)$$

with

$$\tilde{E}_o^i(x_i, \bar{x}_i) := (x_i - \bar{x}_i)^T \Sigma_i (x_i - \bar{x}_i), \quad (38)$$

$$\tilde{E}_c^i(x_i, \bar{x}_i) := (x_i + \bar{x}_i)^T \Sigma_i^{-1} (x_i + \bar{x}_i) \quad (39)$$

and  $i \in \{1, 2\}$ , which is a direct consequence of (35). Here,  $x^T = [x_1^T \ x_2^T]$  and  $\bar{x}^T = [\bar{x}_1^T \ \bar{x}_2^T]$  with  $x_1, \bar{x}_1 \in \mathbb{R}^k$  and  $x_2, \bar{x}_2 \in \mathbb{R}^{n-k}$ . The partitioning (36), (37) holds for any  $k < n$ , but it is assumed that  $k$  is chosen such that (35) is split according to the multiplicities of the parameters  $\sigma_i$ , i.e., there exists  $r$  such that  $k = \sum_{i=1}^r m_i$ . Then,  $\Sigma_1$  and  $\Sigma_2$  in (36), (37) are given as

$$\Sigma_1 = \text{blkdiag} \{ \sigma_1 I_{m_1}, \sigma_2 I_{m_2}, \dots, \sigma_r I_{m_r} \}, \quad (40)$$

$$\Sigma_2 = \text{blkdiag} \{ \sigma_{r+1} I_{m_{r+1}}, \sigma_{r+2} I_{m_{r+2}}, \dots, \sigma_q I_{m_q} \}. \quad (41)$$

It is noted that the generalized incrementally balanced realization as given by Definition 10 and resulting in (36), (37) satisfies the conditions of Definition 7. As discussed in Section IV, it is unknown whether an incrementally balanced realization as in Definition 7 can be found in general. Thus, since a *generalized* incrementally balanced realization is easily obtained (see Theorem 14), it can serve as an alternative realization on which model reduction can be based. Moreover, the diagonal structure in  $\Sigma$  as in (35) will turn out to be instrumental in the computation of an error bound for reduced-order models of arbitrary order.

In order to find a reduced-order approximation by truncation, the function  $f$  and matrices  $B$  and  $C$  as in (2) are partitioned according to (36) and (37), such that

$$f(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2]. \quad (42)$$

Then, the reduced-order system  $\hat{\Sigma}_{BC,k}$  is given as

$$\hat{\Sigma}_{BC,k} : \begin{cases} \dot{\xi} = f_1(\xi, 0) + B_1 u, \\ \hat{y} = C_1 \xi \end{cases} \quad (43)$$

with  $\xi \in \mathbb{R}^k$  an approximation for  $x_1$ . Motivated by earlier definitions, this reduction procedure will be referred to as generalized incremental balanced truncation.

### C. Stability Preservation and Error Bound

Properties of the reduced-order model  $\hat{\Sigma}_{BC,k}$  as in (43) obtained by generalized incremental balanced truncation are analyzed in the current section. The following statement holds for the generalized incremental observability function.

*Lemma 15:* Let (2) be a generalized incrementally balanced realization of the system  $\Sigma_{BC}$  and let  $\hat{\Sigma}_{BC,k}$  as in (43) be a reduced-order system obtained by generalized incremental balanced truncation. Then, the function  $\tilde{E}_o^1$  as in (36) is a generalized incremental observability function for  $\hat{\Sigma}_{BC,k}$ . Moreover, the incremental observability function  $\tilde{E}_o$  of  $\hat{\Sigma}_{BC,k}$  exists.

*Proof:* Evaluation of (30) in generalized incrementally balanced coordinates and substitution of  $x_1 = \xi$ ,  $\bar{x}_1 = \bar{\xi}$  and  $x_2 = \bar{x}_2 = 0$  gives

$$\begin{aligned} & (\xi - \bar{\xi})^T \Sigma_1 (f_1(\xi, 0) - f_1(\bar{\xi}, 0)) \\ & \leq -\frac{1}{2} (\xi - \bar{\xi})^T C_1^T C_1 (\xi - \bar{\xi}) \end{aligned} \quad (44)$$

for all  $\xi, \bar{\xi} \in \mathbb{R}^k$ . Thus, the reduced-order system  $\hat{\Sigma}_{BC,k}$  satisfies (44), such that the result follows directly from Theorem 11 and Corollary 9.  $\square$

A similar result can be obtained for the generalized incremental controllability function for the reduced-order system.

*Lemma 16:* Let (2) be a generalized incrementally balanced realization of the system  $\Sigma_{BC}$  and let  $\hat{\Sigma}_{BC,k}$  as in (43) be a reduced-order system obtained by generalized incremental balanced truncation. Then, the function  $\tilde{E}_c^1$  as in (37) is a generalized incremental controllability function for  $\hat{\Sigma}_{BC,k}$ .

*Proof:* The proof follows the ideas as in the proof of Lemma 15, hereby exploiting Theorem 12.  $\square$

The results in Lemmas 15 and 16 can be considered as counterparts of Lemmas 5 and 6, that deal with the energy functions for the reduced-order system obtained by incremental balanced truncation. Similarly, stability properties of the reduced-order model  $\hat{\Sigma}_{BC,k}$  as obtained by generalized incremental balanced truncation can be obtained by exploiting the line of reasoning of Theorem 7, leading to the following result.

*Theorem 17:* Let (2) be a generalized incrementally balanced realization of the system  $\Sigma_{BC}$  and let  $\hat{\Sigma}_{BC,k}$  as in (43) be a reduced-order system obtained by generalized incremental balanced truncation. Then, the following statements hold:

- 1) Any state trajectory  $\xi(\cdot)$  of the reduced-order system (43) with initial condition  $\xi(0) = \xi_0 \in \mathbb{R}^k$  and input function  $u(\cdot) \in \mathcal{L}_2^m([0, \infty))$  is bounded for all  $t \geq 0$ ;
- 2) The reduced-order nonlinear system  $\hat{\Sigma}_{BC,k}$  is incrementally stable for the class of inputs  $\mathcal{L}_2^m([0, \infty))$ ;
- 3) For any bounded input function  $u(\cdot) \in \mathcal{L}_2^m([0, \infty))$  and initial conditions  $\xi(0) = \xi_0$ ,  $\bar{\xi}(0) = \bar{\xi}_0$ , the outputs corresponding to the state trajectories  $\xi(\cdot)$  and  $\bar{\xi}(\cdot)$  converge to each other, i.e.

$$\lim_{t \rightarrow \infty} |C_1 \xi(t) - C_1 \bar{\xi}(t)| = 0. \quad (45)$$

*Proof:* The statements are proven separately.

1. *Boundedness.* When the inequality (31) describing the generalized incremental controllability function is considered in balanced coordinates, substitution of  $x_1 = \xi$ ,  $\bar{x}_1 = \bar{\xi}$  and  $x_2 = \bar{x}_2 = 0$  yields

$$\begin{aligned} & (\xi + \bar{\xi})^T \Sigma_1^{-1} (f_1(\xi, 0) + f_1(\bar{\xi}, 0)) \\ & \leq -\frac{1}{2} (\xi + \bar{\xi})^T \Sigma_1^{-1} B_1 B_1^T \Sigma_1^{-1} (\xi + \bar{\xi}) \end{aligned} \quad (46)$$

which characterizes the generalized incremental controllability function  $\tilde{E}_c^1$  (see (37)) for the reduced-order system  $\hat{\Sigma}_{BC,k}$ . Then, the time-derivative of  $\tilde{E}_c^1$  yields

$$\begin{aligned} \dot{\tilde{E}}_c^1(\xi, \bar{\xi}) &= 2(\xi + \bar{\xi})^T \Sigma_1^{-1} (f_1(\xi, 0) + B_1 u + f_1(\bar{\xi}, 0) + B_1 \bar{u}), \\ &\leq -(\xi + \bar{\xi})^T \Sigma_1^{-1} B_1 B_1^T \Sigma_1^{-1} (\xi + \bar{\xi}) \\ &\quad + 2(\xi + \bar{\xi})^T \Sigma_1^{-1} B_1 (u + \bar{u}), \end{aligned} \quad (47)$$

$$= |u + \bar{u}|^2 - |(u + \bar{u}) - B_1^T \Sigma_1^{-1} (\xi + \bar{\xi})|^2 \quad (48)$$

where  $\xi(\cdot)$  is the solution of (43) corresponding to initial condition  $\xi(0) = \xi_0$  and input function  $u(\cdot)$ . Similarly,  $\bar{\xi}(\cdot)$  represents the solution of (43) corresponding to initial condition  $\bar{\xi}(0) = \bar{\xi}_0$  and input function  $\bar{u}(\cdot)$ . In (47), the inequality (46) is used. Integration of (48) yields

$$\tilde{E}_c^1(\xi(t), \bar{\xi}(t)) - \tilde{E}_c^1(\xi_0, \bar{\xi}_0) \leq \int_0^t |u(\tau) + \bar{u}(\tau)|^2 d\tau \quad (49)$$

which is of the same form as (61) in Theorem 7. As a result, the remainder of the proof follows that of the first item of Theorem 7, where it is noted that, due to the quadratic form of the generalized incremental controllability function, the function  $V$  in the proof of Theorem 7 is radially unbounded.

2. *Incremental stability* and 3. *Output convergence.* The proof follows the same line of reasoning of the second and third item of Theorem 7.  $\square$

As for model reduction using incremental balanced truncation, an error bound can be found for the reduced-order model obtained by generalized incremental balanced truncation. Here, due to the structure in the generalized energy function as (35), this error bound is not limited to a one-step reduction. This is formalized in the following theorem.

*Theorem 18:* Let (2) be a generalized incrementally balanced realization of the system  $\Sigma_{BC}$ . In addition, let  $\hat{\Sigma}_{BC,k}$  as in (43) be a reduced-order system obtained by generalized incremental balanced truncation such that the order  $k$  satisfies  $\sum_{i=1}^r m_i = k$  for some  $r$ . Then, for trajectories  $x(\cdot)$  and  $\xi(\cdot)$  of  $\Sigma_{BC}$  and  $\hat{\Sigma}_{BC,k}$ , respectively, for a common input function  $u(\cdot) \in \mathcal{L}_2^m([0, \infty))$  and initial conditions  $x(0) = 0$  and  $\xi(0) = 0$ , respectively, the corresponding outputs  $y(\cdot)$  and  $\hat{y}(\cdot)$  satisfy the error bound

$$\|y - \hat{y}\|_2 \leq \left( 2 \sum_{i=r+1}^q \sigma_i \right) \|u\|_2 \quad (50)$$

with  $\sigma_i$  as in Definition 10.

*Proof:* The proof is similar to that of Theorem 8. However, the generalized energy functions  $\tilde{E}_o$  and  $\tilde{E}_c$  are used rather than the incremental observability and incremental controllability functions. Then, the error bound can be obtained by repeated one-step reductions, hereby removing the state components corresponding to a single parameter  $\sigma_i$  as in (35). Here, since the multiplicity  $m_i$  of the discarded parameter  $\sigma_i$  might be larger than one, multiple state components might be removed in a single step. In fact, the quadratic form of the generalized incremental observability and generalized incremental controllability functions as in (36), (37), as well as the fact that they are characterized by the same matrix  $\Sigma$ , ensures that the condition (20) in Definition 7 is satisfied, even when multiplicities are considered.

Thus, when reducing  $\Sigma_{BC}$  to  $\hat{\Sigma}_{BC, n-m_q}$ , the result (27) in Theorem 7 holds with  $\rho = \sigma_q$ . Since the remaining parts of the generalized energy function  $\tilde{E}_o^1$  and  $\tilde{E}_c^1$  are again generalized energy functions for the reduced-order system (see Lemmas 15 and 16), this procedure can be repeated to remove more state components, eventually leading to the reduced-order system  $\hat{\Sigma}_{BC,k}$ . Then, the triangle inequality leads to the error bound (50).  $\square$

Due to the quadratic form of the generalized incremental observability and generalized incremental controllability function, the error bound (50) is of the same form as the error bound for linear systems as derived in [10] and [14] (see also [2]).

*Remark 9:* The error bound (50) is dependent on (the sum of) the discarded parameters  $\sigma_i$ ,  $i \in \{r+1, \dots, q\}$ . Thus, given the matrices  $\hat{Q}$  and  $\hat{R}$  describing the generalized incremental

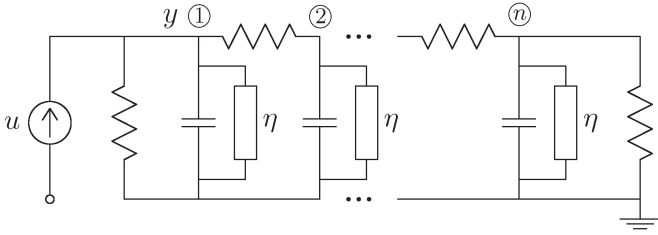


Fig. 1. Electronic circuit with nonlinear resistors  $\eta$ .

observability and generalized incremental controllability function, the ordering of the parameters as in Definition 10 gives the tightest error bound. However, since the generalized energy functions are not unique (see Remark 7), solutions to (30) and (31) might be sought for which the eigenvalues of  $\tilde{Q}\tilde{R}^{-1}$  are minimized. Here, a tractable approach might be obtained when the generalized energy functions are considered individually, hereby minimizing the eigenvalues of  $\tilde{Q}$  and maximizing the eigenvalues of  $\tilde{R}$ . It is noted that this observation corresponds to the intuition discussed in Remark 7.  $\triangleleft$

*Remark 10:* As stressed before, the generalized incremental observability and generalized incremental controllability function represent storage functions for the auxiliary systems  $\Sigma^o$  and  $\Sigma^c$  and corresponding supply rates (see Theorems 1 and 2). In addition, they are chosen to be of quadratic form in order to guarantee the existence of a (linear) coordinate transformation putting the system in a (generalized) incrementally balanced form, which satisfies the statements in Definition 7.

However, the ideas presented in this section may be extended to a larger class of storage functions. In particular, any (nonlinear) functions  $\tilde{E}_o$  and  $\tilde{E}_c$  for which a coordinate transformation can be found such that the transformed system satisfies the conditions in Definition 7 can be exploited. By increasing the class of generalized incremental energy functions beyond that of quadratic functions, tighter bounds on the incremental observability and incremental controllability function may be obtained, potentially leading to a more accurate reduced-order model and tighter error bounds.  $\triangleleft$

## VI. EXAMPLE

The reduction method of generalized incremental balanced truncation is illustrated by means of application to an example of a nonlinear electronic circuit. This example is taken from [26] and depicted in Fig. 1. Here, it is noted that no conditions for the preservation of stability and no error bound are given in [26]. In the electronic circuit, the resistors and capacitors have unit resistance and capacitance, respectively, whereas the input  $u \in \mathbb{R}$  represents the source current. Next, the output  $y \in \mathbb{R}$  is taken as the voltage at node 1.

The nonlinearity is due to the presence of nonlinear resistors, which are modeled by the relation  $i = \eta(v)$  with  $v$  the voltage across the resistor and  $i$  the resistor current. The function  $\eta(v)$  is assumed to be nondecreasing and satisfies  $\eta(0) = 0$  and the symmetry condition  $\eta(v) = -\eta(-v)$ . This leads to a model  $\Sigma_{BC}$  of the form (2), where the vector field  $f$  is given as

$$f(x) = Ax + \varphi(x) \quad (51)$$

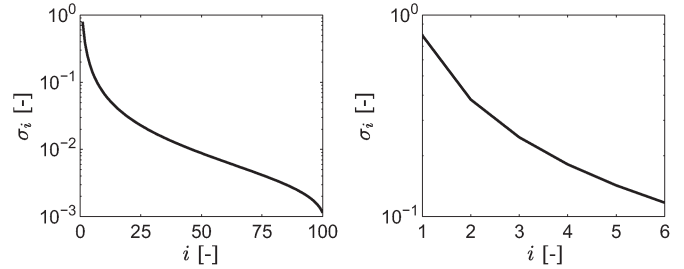


Fig. 2. Parameters  $\sigma_i$  characterizing the generalized incrementally balanced realization (left) and a zoomed version (right).

with

$$A = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}, \quad \varphi(x) = - \begin{bmatrix} \eta(x_{(1)}) \\ \vdots \\ \eta(x_{(n)}) \end{bmatrix}. \quad (52)$$

Here,  $x_{(i)}, i \in \{1, 2, \dots, n\}$ , denote the components of the state  $x \in \mathbb{R}^n$  (with  $n = 100$ ), which represent the voltages at the nodes indicated with ① to ② in Fig. 1. Next, the input and output matrices in (2) read  $B^T = C = [1 \ 0 \ \dots \ 0]$ .

In order to find a reduced-order approximation of  $\Sigma_{BC}$ , solutions  $\tilde{Q}$  to (30) and  $\tilde{R}$  to (31) are sought, characterizing the generalized incremental observability function  $\tilde{E}_o$  and generalized incremental controllability function  $\tilde{E}_c$ , respectively. By choosing  $\tilde{Q}$  in (30) as a diagonal matrix  $\tilde{Q} = \text{diag}\{\tilde{q}_1, \dots, \tilde{q}_n\}$  (with  $\tilde{q}_i > 0$ ), the structure of the nonlinearity  $\varphi$  in (51) as well as the fact that  $\eta$  is nondecreasing guarantee that

$$(x - \bar{x})^T \tilde{Q} (\varphi(x) - \varphi(\bar{x})) \leq 0 \quad (53)$$

for all  $x, \bar{x} \in \mathbb{R}^n$ . Then, (30) reduces to the linear matrix inequality

$$A^T \tilde{Q} + \tilde{Q} A \preceq -C^T C \quad (54)$$

for which efficient solvers exist. The same ideas can be exploited for the generalized incremental controllability function, such that  $\tilde{R} = \text{diag}\{\tilde{r}_1, \dots, \tilde{r}_n\}$ . In fact, as the matrices in (51) satisfy  $A = A^T$  and  $B = C^T$ , it follows that  $\tilde{R}$  can be chosen as  $\tilde{R}^{-1} = \tilde{Q}$ . Solving (54), while minimizing the trace of  $\tilde{Q}$  to obtain a tight error bound (see Remark 9), leads to  $\tilde{Q} = \tilde{R}^{-1} = \Sigma$ , where the entries  $\sigma_i$  of the diagonal matrix  $\Sigma = \text{diag}\{\sigma_1, \dots, \sigma_n\}$  satisfy  $\sigma_i \geq \sigma_{i+1}, i \in \{1, \dots, n-1\}$ , and are depicted in Fig. 2.

Then, Theorems 11 and 12 give the generalized incremental energy functions. Moreover, Definition 10 implies that  $\Sigma_{BC}$  as in (2) with  $f$  as in (51) is a generalized incrementally balanced realization in the original coordinates  $x$ . Consequently, the importance of the state components can be interpreted in terms of the nodes in Fig. 1, such that truncation corresponds to the removal of the rightmost nodes in Fig. 1. Based on the parameters  $\sigma_i$  in Fig. 2, the reduction order is chosen as  $k = 4$ , leading to a reduced-order system  $\tilde{\Sigma}_{BC,4}$  as in (43).

By Theorem 17, this reduced-order system has bounded trajectories and is incrementally stable for all input functions

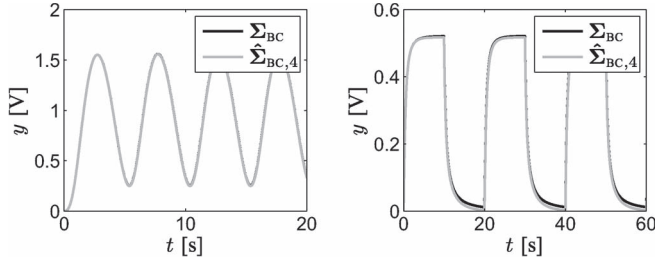


Fig. 3. Comparison of the high-order system  $\Sigma_{BC}$  and the reduced-order approximation  $\tilde{\Sigma}_{BC,4}$  for nonlinearity  $\eta(v) = \text{sign}(v)v^2$  and external input  $u(t) = (5/2)(1 - \cos(2\pi(1/5)t))$  (left) and  $u(t) = (1/2)(\text{sign}(\sin(2\pi(1/20)t)) + 1)$  (right).

TABLE I  
ERROR BOUND  $\varepsilon$ , INPUT SIGNAL NORM  $\|u\|_2$ , AND ABSOLUTE AND RELATIVE OUTPUT ERRORS  $\|y - \hat{y}\|_2$  AND  $\|y - \hat{y}\|_2/\|u\|_2$  FOR THE SIMULATIONS IN FIGS. 3 AND 4

	$\varepsilon$	$\ u\ _2$	$\ y - \hat{y}\ _2$	$\frac{\ y - \hat{y}\ _2}{\ u\ _2}$
Figure 3 (left)	3.402	30.69	$5.872 \cdot 10^{-2}$	$1.918 \cdot 10^{-3}$
Figure 3 (right)	3.402	7.069	$7.629 \cdot 10^{-2}$	$1.079 \cdot 10^{-2}$
Figure 4 (left)	3.402	0.245	$2.904 \cdot 10^{-2}$	$1.186 \cdot 10^{-1}$
Figure 4 (right)	0.543	0.245	$2.271 \cdot 10^{-2}$	$9.272 \cdot 10^{-2}$

in the class  $\mathcal{L}_2^1([0, \infty))$ . In addition, Theorem 18 provides an error bound with  $\varepsilon = 2 \sum_{i=5}^{100} \sigma_i = 3.402$ .

To evaluate the quality of the reduced-order model obtained by generalized incremental balanced truncation, the results of time simulations are depicted in Fig. 3. Here, the nonlinearity  $\eta$  is chosen as in [26], such that  $\eta(v) = \text{sign}(v)v^2$ . From Fig. 3, it is clear that the reduced-order system provides a good approximation of the original high-order system. This can also be concluded from Table I, which shows the error bound  $\varepsilon$ , the norm on the input signal  $\|u\|_2$  and the output error  $\|y - \hat{y}\|_2$  in its first three columns. The fourth column gives the ratio between the output error and norm on the input signal, specifying the actual relative error. By comparing this relative error to the error bound, it can be seen that the error bound is conservative for the input functions in Fig. 3. Here, it is remarked that the input functions  $u$  are set to satisfy  $u(t) = 0$  for all  $t \geq 100$ . As a result, both  $u$  and the corresponding output error  $y - \hat{y}$  are in  $\mathcal{L}_2^1([0, \infty))$  and the error bound applies.

The conservatism in the error bound for the simulations in Fig. 3 has several causes. First, it is recalled that the error bound holds for all input functions (in  $\mathcal{L}_2^m([0, \infty))$ ) such that the quality of the error bound is dependent on the particular input function. This is illustrated by the simulation depicted in the left graph of Fig. 4 and the corresponding error in Table I. Here, the signal norms are obtained by truncating the input function at  $t = 1200$ . For this input function, a larger error is obtained (i.e., closer to the error bound) than in earlier simulations in Fig. 3. Nonetheless, the error bound remains conservative, as can be seen by comparing the first and fourth column in Table I.

The remaining conservatism is largely due to properties of the reduced-order model rather than the choice of the particular input function. First, the method of generalized balanced approximation is based on the computation of quadratic bounds on the incremental observability and incremental controllability functions rather than on these incremental energy functions

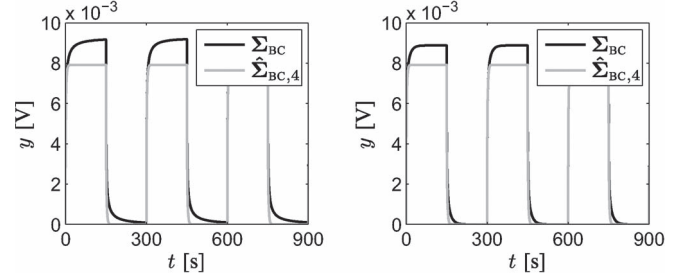


Fig. 4. Comparison of the high-order system  $\Sigma_{BC}$  of order  $n$  and the reduced-order approximation  $\Sigma_{BC,4}$  for nonlinearity  $\tilde{\eta}(v) = \text{sign}(v)v^2$  and external input  $u(t) = (10^{-2}/2)(\text{sign}(\sin(2\pi(1/300)t)) + 1)$ :  $n = 100$  (left) and  $n = 10$  (right).

themselves, which generally leads to a larger error bound (see also Remark 7). Moreover, in this particular example, the matrices  $\tilde{Q}$  and  $\tilde{R}$  characterizing the generalized incremental observability and generalized incremental controllability function, respectively, are chosen diagonal (in order to guarantee that the required inequalities hold for the nonlinearity  $\varphi$ , see (53)), which limits the flexibility in the optimization of these bounds. Next, it is noted that the proof of the error bound in Theorem 18 is based on the successive application of one-step reductions and the triangle inequality, where the latter is conservative.

To illustrate these aspects, the electronic circuit model in Fig. 1 is considered for order  $n = 10$ , hereby reducing the conservatism introduced by choosing the diagonal form for  $\tilde{Q}$  and  $\tilde{R}$ . Next, the relatively small reduction from  $n = 10$  to  $k = 4$  reduces the number of successive applications of the triangle inequality. The results for this case are depicted in the right graph of Fig. 4, whereas the error bound is given in the fourth row in Table I. As expected, the conservatism in the error bound obtained by Theorem 18 is significantly reduced in this case. Finally, it is noted that a tighter error bound can be obtained by recomputing the diagonal matrices  $\tilde{Q}$  and  $\tilde{R}$  after the states of the reduced-order model have been selected, hereby minimizing the trace of only the discarded part of the matrices.

## VII. CONCLUSION

The model reduction method of incremental balanced truncation is introduced in this paper. It differs from existing extensions of balanced truncation for linear systems to the nonlinear case in the definition of two novel incremental energy functions. These incremental observability and incremental controllability function replace the (non-incremental) observability and controllability function as the basis for a model reduction procedure, yielding the following advantages. First, the incremental observability function is directly related to the notion of observability for nonlinear systems. Second, the reduction procedure of incremental balanced truncation guarantees the preservation of stability properties for both zero and nonzero input functions. Third, an *a priori* error bound can be provided for the reduced-order model.

Moreover, an extension towards generalized incremental balanced truncation is introduced, which is based on the computation of bounds on the incremental energy functions rather than the incremental energy functions themselves. This increases the computational feasibility of the reduction at the cost of a

(potentially) increased error bound and potentially less accurate reduced-order model. This method is illustrated by means of application to an example of an electronic circuit.

## APPENDIX A

### A. Proof of Lemma 5

By the definition of an incrementally balanced realization (see Definition 7), the incremental observability function  $E_o$  exists and can be decomposed as in (18). By Theorem 1,  $E_o$  in partitioned coordinates is characterized by

$$\begin{aligned} & \frac{\partial E_o^1}{\partial x_1}(x_1, \bar{x}_1)f_1(x_1, x_2) + \frac{\partial E_o^2}{\partial x_2}(x_2, \bar{x}_2)f_2(x_1, x_2) \\ & + \frac{\partial E_o^1}{\partial x_1}(x_1, \bar{x}_1)f_1(\bar{x}_1, \bar{x}_2) + \frac{\partial E_o^2}{\partial \bar{x}_2}(x_2, \bar{x}_2)f_2(\bar{x}_1, \bar{x}_2) \\ & = -|\Delta h(x_1, x_2, \bar{x}_1, \bar{x}_2)|^2 - |l_o(x_1, x_2, \bar{x}_1, \bar{x}_2)|^2, \end{aligned} \quad (55)$$

$$\begin{aligned} & \frac{\partial E_o^1}{\partial x_1}(x_1, \bar{x}_1)g_1(x_1, x_2) + \frac{\partial E_o^2}{\partial x_2}(x_2, \bar{x}_2)g_2(x_1, x_2) \\ & + \frac{\partial E_o^1}{\partial x_1}(x_1, \bar{x}_1)g_1(\bar{x}_1, \bar{x}_2) + \frac{\partial E_o^2}{\partial \bar{x}_2}(x_2, \bar{x}_2)g_2(\bar{x}_1, \bar{x}_2) = 0 \end{aligned} \quad (56)$$

with  $\Delta h(x_1, x_2, \bar{x}_1, \bar{x}_2) = h(x_1, x_2) - h(\bar{x}_1, \bar{x}_2)$  and for some function  $l_o$ . The partitioning (18) implies that the partial derivatives in (55), (56) with respect to  $x_i$  and  $\bar{x}_i$  are independent of  $x_j$  and  $\bar{x}_j$ , with  $i, j \in \{1, 2\}$ ,  $i \neq j$ . Substitution of  $x_1 = \xi$ ,  $\bar{x}_1 = \bar{\xi}$  and  $x_2 = \bar{x}_2 = 0$  in (55), (56) gives

$$\begin{aligned} & \frac{\partial E_o^1}{\partial \xi}(\xi, \bar{\xi})f_1(\xi, 0) + \frac{\partial E_o^1}{\partial \bar{\xi}}(\xi, \bar{\xi})f_1(\bar{\xi}, 0) \\ & = -|h(\xi, 0) - h(\bar{\xi}, 0)|^2 - |l_o(\xi, 0, \bar{\xi}, 0)|^2, \end{aligned} \quad (57)$$

$$\frac{\partial E_o^1}{\partial \xi}(\xi, \bar{\xi})g_1(\xi, 0) + \frac{\partial E_o^1}{\partial \bar{\xi}}(\xi, \bar{\xi})g_1(\bar{\xi}, 0) = 0. \quad (58)$$

Here, the property  $(\partial E_o^2/\partial x_2)(0, 0) = (\partial E_o^2/\partial \bar{x}_2)(0, 0) = 0$  is used, which follows from the fact that  $x_2 = \bar{x}_2$  is a minimum for  $E_o^2$ . The partial differential (57), (58) are of the same form as (8), (9), such that Theorem 1 guarantees existence of the incremental observability function  $\hat{E}_o$  for the reduced-order system  $\hat{\Sigma}_{n-1}$ . It directly follows that (24) holds, since  $\hat{E}_o$  is the minimal solution  $E_o^1$  satisfying (57), (58) for some function  $l_o$ .

### B. Proof of Lemma 6

The statement can be proven along the lines of the proof of Lemma 5, hereby exploiting the partitioned form of (12) and (13) as

$$\begin{aligned} & \frac{\partial E_c^1}{\partial x_1}(x_1, \bar{x}_1)f_1(x_1, x_2) + \frac{\partial E_c^2}{\partial x_2}(x_2, \bar{x}_2)f_2(x_1, x_2) \\ & + \frac{\partial E_c^1}{\partial \bar{x}_1}(x_1, \bar{x}_1)f_1(\bar{x}_1, \bar{x}_2) + \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, \bar{x}_2)f_2(\bar{x}_1, \bar{x}_2) \\ & = -l_c^T(x_1, x_2, \bar{x}_1, \bar{x}_2)l_c(x_1, x_2, \bar{x}_1, \bar{x}_2), \end{aligned} \quad (59)$$

$$\begin{aligned} & \left[ \frac{\partial E_c^1}{\partial x_1}(x_1, \bar{x}_1)g_1(x_1, x_2) + \frac{\partial E_c^2}{\partial x_2}(x_2, \bar{x}_2)g_2(x_1, x_2) \right. \\ & \left. \frac{\partial E_c^1}{\partial \bar{x}_1}(x_1, \bar{x}_1)g_1(\bar{x}_1, \bar{x}_2) + \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, \bar{x}_2)g_2(\bar{x}_1, \bar{x}_2) \right] \\ & = -2l_c^T(x_1, x_2, \bar{x}_1, \bar{x}_2)W(x_1, x_2, \bar{x}_1, \bar{x}_2) \end{aligned} \quad (60)$$

with  $W$  satisfying (14). Substitution of  $x_1 = \xi$ ,  $\bar{x}_1 = \bar{\xi}$  and  $x_2 = \bar{x}_2 = 0$  leads to a characterization of a storage function for the reduced-order auxiliary system  $\hat{\Sigma}_{n-1}^c$  and supply rate  $|u + \bar{u}|^2$ , from which the result follows. Here,  $\hat{\Sigma}_{n-1}^c$  is defined similar to (10), i.e., it consists out of two copies of the one-step reduced-order system  $\hat{\Sigma}_{n-1}$ .

The incremental controllability function  $\hat{E}_c$  is not guaranteed to exist *a priori* as  $\hat{\Sigma}_{n-1}$  might not be reachable from the origin (see also Theorem 2). When it exists, the inequality (25) directly follows from dissipativity theory.

### C. Proof of Theorem 7

The items will be proven separately.

1. *Boundedness.* Even though  $E_c^1$  is not necessarily the controllability function for the reduced-order system  $\hat{\Sigma}_{n-1}$ , it follows from Lemma 6 that it is a storage function for the auxiliary system  $\hat{\Sigma}_{n-1}^c$  and supply rate  $|u + \bar{u}|^2$ . Here,  $\hat{\Sigma}_{n-1}^c$  consists of two copies of the reduced-order system  $\hat{\Sigma}_{n-1}$ , similar to the definition in (10). Then, the characterization of  $E_c^1$  in terms of the partitioned counterparts of (12)–(14) guarantee that  $E_c^1$  satisfies

$$\begin{aligned} E_c^1(\xi(t), \bar{\xi}(t)) - E_c^1(\xi_0, \bar{\xi}_0) &= \int_0^t |u(\tau) + \bar{u}(\tau)|^2 d\tau \\ & - \int_0^t \left| W(\xi, 0, \bar{\xi}, 0) \begin{bmatrix} u(\tau) \\ \bar{u}(\tau) \end{bmatrix} + l_c(\xi, 0, \bar{\xi}, 0) \right|^2 d\tau \end{aligned} \quad (61)$$

along trajectories of  $\hat{\Sigma}_{n-1}^c$ . Here,  $\xi_0$  and  $\bar{\xi}_0$  denote the initial conditions. Denote  $V(\xi) = E_c^1(\xi, \xi)$ . Then, given the fact that  $E_c(x, x) > 0$  for all  $x \neq 0$  and the partitioning in (19),  $V$  is positive definite, implying the existence of two class  $\mathcal{K}$  functions  $\nu_1, \nu_2$  such that

$$\nu_1(|\xi|) \leq V(\xi) \leq \nu_2(|\xi|) \quad (62)$$

see, e.g., [19]. Furthermore, let  $a$  be a positive number such that  $a < \lim_{r \rightarrow \infty} \nu_1(r)$ . Next, classes of initial conditions and input functions are defined as

$$\begin{aligned} \mathcal{X}_r &:= \{ \xi \in \mathbb{R}^{n-1} \mid \nu_2(|\xi|) \leq r \}, \\ \mathcal{U}_s &:= \left\{ u \in \mathcal{L}_2^m([0, \infty)) \mid \int_0^\infty |u(t)|^2 dt \leq \frac{s}{4} \right\}. \end{aligned} \quad (63)$$

Evaluation of trajectories of the auxiliary system  $\hat{\Sigma}_{n-1}^c$  for  $\xi_0 = \bar{\xi}_0 \in \mathcal{X}_r$  and  $u(\cdot) = \bar{u}(\cdot) \in \mathcal{U}_s$  gives the inequality

$$V(\xi(t)) \leq V(\xi_0) + 4 \int_0^t |u(\tau)|^2 d\tau \leq r + s \quad (65)$$

for all  $t \geq 0$ , as follows from (61). Then, by choosing  $r + s < a$ , it is clear from (62) that the state trajectory is bounded as  $|\xi(t)| \leq \nu_1^{-1}(r + s)$  for all  $t \geq 0$ , where the condition  $r + s < a$  ensures that the inverse of  $\nu_1$  indeed exists. If  $E_c(x, x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ ,  $V$  is radially unbounded and boundedness of

state trajectories holds for all initial conditions and all input functions in the class  $\mathcal{L}_2^m([0, \infty))$ .

2. *Incremental stability.* In this proof, the function  $E_o^1$  as in (18) will be used as a Lyapunov function for incremental stability. Since  $E_o(x, \bar{x}) > 0$  for all  $x, \bar{x} \in \mathbb{R}^n$ ,  $x \neq \bar{x}$ , it follows that there exists functions  $\alpha_1$  and  $\alpha_2$  of class  $\mathcal{K}$  such that

$$\alpha_1(|\xi - \bar{\xi}|) \leq E_o^1(\xi, \bar{\xi}) \leq \alpha_2(|\xi - \bar{\xi}|). \quad (66)$$

Then, let  $c^*$  be a positive constant such that  $c^* < \lim_{r \rightarrow \infty} \alpha_1(r)$  and define  $c := \alpha_2^{-1}(c^*)$ , where the inverse is guaranteed to exist due to the choice of  $c^*$ . The time-derivative of  $E_o^1$  along trajectories of the reduced-order auxiliary system  $\hat{\Sigma}_{n-1}^o$ , which is defined as the parallel interconnection of two copies of  $\hat{\Sigma}_{n-1}$  similar to (6), is given by

$$\dot{E}_o^1(\xi, \bar{\xi}) = -|h(\xi, 0) - h(\bar{\xi}, 0)|^2 - |l_o(\xi, 0, \bar{\xi}, 0)|^2 \leq 0. \quad (67)$$

This follows from (57) and (58), where the latter also shows that the equality (67) holds for any input function. When choosing the initial conditions  $\xi_0, \bar{\xi}_0 \in \mathbb{R}^{n-1}$  such that  $c_0 := |\xi_0 - \bar{\xi}_0| < c$ , it follows from (66) that  $E_o^1(\xi_0, \bar{\xi}_0) < \alpha_2(c_0)$ . Since, by (67),  $E_o^1$  is nonincreasing along trajectories of the auxiliary system  $\hat{\Sigma}_{n-1}^o$ , it follows that  $E_o^1(\xi(t), \bar{\xi}(t)) < \alpha_2(c_0)$ . Then, (66) gives

$$|\xi(t) - \bar{\xi}(t)| \leq \alpha_1^{-1}(E_o^1(\xi(t), \bar{\xi}(t))), \quad (68)$$

$$\leq \alpha_1^{-1} \circ \alpha_2(|\xi_0 - \bar{\xi}_0|) \quad (69)$$

for all  $|\xi_0 - \bar{\xi}_0| < c$  and where invertibility of  $\alpha_1$  is guaranteed since  $c_0 < c$ , such that  $\alpha_2(c_0) < c^*$ . Consequently, the function  $\alpha$  in Definition 4 is given as  $\alpha = \alpha_1^{-1} \circ \alpha_2$  and  $\hat{\Sigma}_{n-1}$  is incrementally stable.

3. *Output convergence.* The property (26) will be proven using Barbalat's lemma (see, e.g., [19]). Integration of (67), hereby using the fact that  $E_o(x, \bar{x})$  is nonnegative, leads to

$$\int_0^\infty |h(\xi(t), 0) - h(\bar{\xi}(t), 0)|^2 dt \leq E_o^1(\xi_0, \bar{\xi}_0) < \infty \quad (70)$$

which holds for any input function in  $\mathcal{L}_2^m([0, \infty))$  and all initial conditions  $\xi_0, \bar{\xi}_0$ . Here, it is noted that boundedness of  $E_o^1$  follows from boundedness of  $E_o$  in Definition 7. By assumption, the state trajectories  $\xi(\cdot)$  and  $\bar{\xi}(\cdot)$  are bounded and there exists a compact set  $\mathcal{B} \ni 0$  such that  $\xi(t) \in \mathcal{B}, \bar{\xi}(t) \in \mathcal{B}$  for all  $t \geq 0$ . By continuity of  $f$  and  $g$  as in (1) and boundedness of  $u$ , the function  $f_1(\xi, 0) + g_1(\xi, 0)u$  as in (23) is bounded for all (bounded)  $\xi$ . Consequently, the time derivatives of  $\xi(\cdot)$  and  $\bar{\xi}(\cdot)$  are bounded and  $\xi(\cdot)$  and  $\bar{\xi}(\cdot)$  are uniformly continuous. By continuity of the output equation  $h$ , the function  $|h(\xi, 0) - h(\bar{\xi}, 0)|^2$  is uniformly continuous in  $(\xi, \bar{\xi})$  on the compact set  $\mathcal{B} \times \mathcal{B}$ . Thus,  $|h(\xi(t), 0) - h(\bar{\xi}(t), 0)|^2$  is uniformly continuous in  $t$ , allowing for the application of Barbalat's lemma. This directly proves the desired result.

#### D. Proof of Theorem 8

The theorem will be proven by showing that the error system  $\Sigma - \hat{\Sigma}_{n-1}$  is dissipative with respect to the supply rate

$$s(u, y, \hat{y}) = (2\rho)^2|u|^2 - |y - \hat{y}|^2. \quad (71)$$

Herein, the candidate storage function  $V$  is introduced as

$$V(x_1, x_2, \xi) = E_o^1(x_1, \xi) + E_o^2(x_2, 0) + \rho^2(E_c^1(x_1, \xi) + E_c^2(x_2, 0)) \quad (72)$$

where  $E_o^i$ ,  $i \in \{1, 2\}$ , represent the partitioned incremental observability function as in (18). Similarly,  $E_c^i$ ,  $i \in \{1, 2\}$ , are the components of the partitioned incremental controllability function as in (18). By nonnegativity of the incremental energy functions  $E_o$  and  $E_c$  (see Definitions 5 and 6), it is easily observed that  $V(x_1, x_2, \xi) \geq 0$  for all  $x_1, \xi \in \mathbb{R}^{n-1}$ ,  $x_2 \in \mathbb{R}$ . Then, differentiation of (72) along trajectories of  $\Sigma$  as in (1) and  $\hat{\Sigma}_{n-1}$  as in (23) yields

$$\begin{aligned} \dot{V}(x_1, x_2, \xi) &= \frac{\partial E_o^1}{\partial x_1}(x_1, \xi) (f_1(x_1, x_2) + g_1(x_1, x_2)u) \\ &+ \frac{\partial E_o^1}{\partial \bar{x}_1}(x_1, \xi) (f_1(\xi, 0) + g_1(\xi, 0)u) \\ &+ \frac{\partial E_o^2}{\partial x_2}(x_2, 0) (f_2(x_1, x_2) + g_2(x_1, x_2)u) \\ &+ \rho^2 \frac{\partial E_c^1}{\partial x_1}(x_1, \xi) (f_1(x_1, x_2) + g_1(x_1, x_2)u) \\ &+ \rho^2 \frac{\partial E_c^1}{\partial \bar{x}_1}(x_1, \xi) (f_1(\xi, 0) + g_1(\xi, 0)u) \\ &+ \rho^2 \frac{\partial E_c^2}{\partial x_2}(x_2, 0) (f_2(x_1, x_2) + g_2(x_1, x_2)u). \end{aligned} \quad (73)$$

In order to rewrite the result in (73), the characterization of the incremental observability and incremental controllability function in partitioned coordinates (55), (56) and (59), (60) will be used. However, they cannot be applied to the form (73), since the partial derivatives with respect to the coordinate  $\bar{x}_2$  do not appear in (73). Thereto, the condition (20) is used, which follows from the assumption that  $\Sigma$  is an incrementally balanced realization according to Definition 7. In particular, (20) implies

$$\begin{aligned} \frac{\partial E_o^2}{\partial \bar{x}_2}(x_2, 0) (f_2(\xi, 0) + g_2(\xi, 0)u) + \\ \rho^2 \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, 0) (f_2(\xi, 0) + g_2(\xi, 0)u) = 0 \end{aligned} \quad (74)$$

such that the left-hand side of (74) can be added to (73). This allows for the application of (55), (56) and (59), (60), hereby using  $\bar{x}_1 = \xi$  and  $\bar{x}_2 = 0$ . This leads to

$$\begin{aligned} \dot{V}(x_1, x_2, \xi) &\leq -|h(x_1, x_2) - h(\xi, 0)|^2 \\ &- \rho^2 l_c^T(x_1, x_2, \xi, 0) l_c(x_1, x_2, \xi, 0) \\ &+ \rho^2 \left[ \frac{\partial E_c^1}{\partial x_1}(x_1, \xi) g_1(x_1, x_2) + \frac{\partial E_c^2}{\partial x_2}(x_2, 0) g_2(x_1, x_2) \right. \\ &\left. \frac{\partial E_c^1}{\partial \bar{x}_1}(x_1, \xi) g_1(\xi, 0) + \frac{\partial E_c^2}{\partial \bar{x}_2}(x_2, 0) g_2(\xi, 0) \right] \begin{bmatrix} u \\ u \end{bmatrix} \end{aligned} \quad (75)$$

where the inequality follows from discarding the term represented by the function  $l_o$  in the characterization of the incremental observability function in (55). Moreover, the input-dependent terms are written as a matrix multiplication. This allows for the application of (60) with  $\bar{x}_1 = \xi$  and  $\bar{x}_2 = 0$ , such that (75) can be rewritten as

$$\begin{aligned} \dot{V}(x_1, x_2, \xi) \leq & -|h(x_1, x_2) - h(\xi, 0)|^2 \\ & - \rho^2 l_c^T(x_1, x_2, \xi, 0) l_c(x_1, x_2, \xi, 0) \\ & - 2\rho^2 l_c^T(x_1, x_2, \xi, 0) W(x_1, x_2, \xi, 0) \begin{bmatrix} u \\ u \end{bmatrix}. \end{aligned} \quad (76)$$

Completion of the squares results in

$$\begin{aligned} \dot{V}(x_1, x_2, \xi) \leq & -|h(x_1, x_2) - h(\xi, 0)|^2 \\ & + \rho^2 \begin{bmatrix} u \\ u \end{bmatrix} W^T(x_1, x_2, \xi, 0) W(x_1, x_2, \xi, 0) \begin{bmatrix} u \\ u \end{bmatrix} \\ & - \rho^2 \left| W(x_1, x_2, \xi, 0) \begin{bmatrix} u \\ u \end{bmatrix} + l_c(x_1, x_2, \xi, 0) \right|^2. \end{aligned} \quad (77)$$

Finally, the application of (14) (with  $W(x, \bar{x}) = W(x_1, x_2, \bar{x}_1, \bar{x}_2)$  as before) gives

$$\begin{aligned} \dot{V}(x_1, x_2, \xi) \leq & -|h(x_1, x_2) - h(\xi, 0)|^2 + \rho^2 |2u|^2 \\ & - \rho^2 \left| W(x_1, x_2, \xi, 0) \begin{bmatrix} u \\ u \end{bmatrix} + l_c(x_1, x_2, \xi, 0) \right|^2, \\ \leq & (2\rho)^2 |u|^2 - |y - \hat{y}|^2. \end{aligned} \quad (78)$$

Hence, the function  $V$  as in (72) is indeed a storage function for the supply rate (71), which proves the bounded  $\mathcal{L}_2$  gain (see, e.g., [28]) of the output error.

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**Bart Besselink** received his M.Sc. degree (*cum laude*) in mechanical engineering from Eindhoven University of Technology, Eindhoven, The Netherlands, in 2008. In 2012, he received his Ph.D. degree from the same university for his thesis on model reduction techniques for nonlinear control systems.

He was a short-term Visiting Researcher at the Tokyo Institute of Technology, Tokyo, Japan, in 2012 and is, since October 2012, a Postdoctoral Researcher with the ACCESS Linnaeus Centre and

Department of Automatic Control, KTH Royal Institute of Technology, Stockholm, Sweden. His research interests include system theory and model reduction techniques for nonlinear dynamical systems and large-scale interconnected systems as well as applications in the field of intelligent transportation systems.



**Jacquelin M. A. Scherpen** (SM'03) received the M.Sc. and Ph.D. degrees in applied mathematics from the University of Twente, Enschede, The Netherlands, in 1990 and 1994, respectively.

She was with the Delft University of Technology, The Netherlands, from 1994 to 2006. Since September 2006, she has been a professor with the University of Groningen, Institute for Industrial Technology and Management of the Faculty of Mathematics and Natural Sciences, The Netherlands, where she was recently appointed as scientific director.

She has held visiting research positions at the University of Tokyo, Japan, Université de Compiègne, and SUPELEC, Gif-sur-Yvette, France, and the Old Dominion University, Norfolk, VA, USA. She was an Associate Editor for *International Journal of Robust and Nonlinear Control* (IJRNC) and is currently an Associate Editor of the *IMA Journal of Mathematical Control and Information*. Her current research interests include nonlinear model reduction methods, nonlinear control methods, modeling and control of physical systems with applications to electrical circuits, electromechanical systems and mechanical systems, and distributed optimal control applications to smart grids.

Dr. Scherpen was an Associate Editor of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. She is on the editorial board of the IJRNC.



**Nathan van de Wouw** (M'08) was born in 1970. He received the M.Sc. degree (with honors) and the Ph.D. degree in mechanical engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands, in 1994 and 1999, respectively.

From 1999 until 2014 he has been affiliated with the Department of Mechanical Engineering of the Eindhoven University of Technology as an assistant/associate professor. Nathan van de Wouw currently holds an adjunct full professor position at the University of Minnesota, Minneapolis. In 2000, he

has been working with Philips Applied Technologies, Eindhoven, and, in 2001, he was with the Netherlands Organization for Applied Scientific Research (TNO), Delft, The Netherlands. He has held positions as a visiting professor at the University of California at Santa Barbara, from 2006 to 2007, was at the University of Melbourne, Australia, from 2009 to 2010, and at the University of Minnesota, Minneapolis, in 2012 and 2013. He has published a large number of journal and conference papers and co-authored the books *Uniform Output Regulation of Nonlinear Systems: A convergent Dynamics Approach* (Birkhauser, 2005) and *Stability and Convergence of Mechanical Systems with Unilateral Constraints* (Springer-Verlag, 2008). He is an Associate Editor for *Automatica*. His current research interests are the analysis and control of nonlinear/non-smooth systems and networked control systems.

Dr. Van de Wouw is currently an Associate Editor for the IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY.



**Henk Nijmeijer** (F'00) was born in 1955. He received the M.Sc. degree and Ph.D. degree in mathematics from the University of Groningen, Groningen, the Netherlands, in 1979 and 1983, respectively.

From 1983 until 2000 he was affiliated with the Department of Applied Mathematics of the University of Twente, Enschede, the Netherlands. Since 1997, he was also part-time affiliated with the Department of Mechanical Engineering of the Eindhoven University of Technology, Eindhoven, the Netherlands. Since 2000, he is a full professor at

Eindhoven, and chairs the Dynamics and Control group. He has published a large number of journal and conference papers, and co-authored several books, including the "classical" *Nonlinear Dynamical Control Systems* (Springer, 1990), *Synchronization of Mechanical Systems* (World Scientific, 2003) *Dynamics and Bifurcations of Non-Smooth Mechanical Systems* (Springer-Verlag, 2004), *Uniform Output Regulation of Nonlinear Systems* (Birkhauser, 2005), and most recently *Parametric Resonance in Dynamical Systems* (Springer, 2012). He was Editor-in-Chief of the *Journal of Applied Mathematics* until 2009, Corresponding Editor of the *SIAM Journal on Control and Optimization*, and Board Member of the *International Journal of Control, Automatica, Journal of Dynamical Control Systems, International Journal of Bifurcation and Chaos, International Journal of Robust and Nonlinear Control, Journal of Nonlinear Dynamics*, and the *Journal of Applied Mathematics and Computer Science, and Hybrid Systems*. In the 2008 research evaluation of the Dutch Mechanical Engineering Departments the Dynamics and Control group was evaluated as excellent regarding all aspects (quality, productivity, relevance and viability). His research interests are in the broad scope of dynamics and control and their applications.

Dr. Nijmeijer received the IEE Heaviside premium in 1990 and is Honorary Knight of the "Golden Feedback Loop" (NTNU), since 2011.