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Model Reduction of Stochastic and Deterministic Continuous Time Linear Systems (Lypunov Equation, Schawrz Approximation).

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MODEL REDUCTION OF STOCHASTIC AND DETERMINISTIC CONTINUOUS TIME LINEAR SYSTEMS

The Louisiana State University and Agricultural and Mechanical Col. PH.D. 1986

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MODEL REDUCTION OF STOCHASTIC AND DETERMINISTIC CONTINUOUS TIME LINEAR SYSTEMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
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Doctor of Philosophy

in

The Electrical and Computer Engineering Department

by
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ABSTRACT

Two new approaches for reducing the order of large scale continuous time systems are presented. The first approach is a modified version of the aggregation technique. It uses the matching properties of the steady state output covariance and the Markov parameters of the high order system to those of the r^{th} order model. This approach also uses a new algorithm derived for the computation of the controllability and observability Grammians, to produce controllable, observable and stable low order models. There is no unique method available for evaluating the aggregation matrix or the matrix relating the system state vector to the model state vector. A procedure, based on the singular value decomposition of the controllability Grammian of the system, is provided for the computation of the aggregation matrix. This approach is also extended to design low order deterministic continuous time varying linear models.

The second approach in this dissertation is an improved version of the Schwarz approximation. It uses the Schwarz canonical form to insure the stability of the model, the impulse response energy to determine the order of the model, the singular values or the second order modes of the controllability Grammian to select the state variables to be retained in the model, and the time moment matching properties to reduce the steady state errors between the response of the model and that of the system.

This dissertation also introduces an algorithm for the computation of the frequency responses of the system and/or the model. This algorithm does not require any complex arithmetic, which is a major problem for most compliers; it is simple and easy to implement in a small computer.

CHAPTER I

INTRODUCTION

1.1. Motivation and Perspectives

In the last two decades considerable attention has been focused on the problem of approximating a higher order system by a lower order model. Low order models are desirable for a number of reasons, including the following:

(1) to achieve simpler simulation of the process,

(2) to reduce the computational effort for obtaining optimal and adaptive controllers.

These motives are particularly applicable to those on-line and off-line designs where iterative design needs to be performed.

The order reduction problem is of particular interest in areas such as power systems, aircraft maneuvering, chemical processes, nuclear reactors, biologic, economic and management systems. The order reduction problem consists of two major areas:

(1) model reduction

(2) controller reduction

The model reduction problem is essentially the simplification of high order systems. Various reduction methodologies have been proposed, namely, aggregation [7], output and equation error minimization [40, 41, 95, 106], continued fraction [140], Pade [109] and Routh [61] approximations, component cost analysis [126], Hankel norm approximations [70], balancing methods [119], partial realizations [46, 59, 112, 132] and identification techniques [12, 15, 20, 78, 107]. The reduction criteria used also ranges over a variety of measures.

The most commonly used methods [71, 122, 137] can be divided in three groups:

(1) Using a model reduction scheme to determine the r^{th} order model, then design its associated r^{th} order controller (open loop design).

(2) After reducing the system to order m, where $r < m < n$, a parameter optimization approach is used to give the optimal r^{th} order controller.

(3) An n^{th} order controller is applied to the system, then a closed loop model reduction technique is used to determine a reduced order closed loop model (closed loop design).

This dissertation is mainly concerned with the open loop design problem, and two methods of model reduction of stochastic and deterministic continuous time linear systems are introduced.

1.2. Contribution of This Work

Two different approaches to model reduction are developed in this research. The first one is based on the matching properties of the steady state output covariances and the Markov parameters of the high order system to those of the r^{th} order model. A new algorithm is developed for the computation of the controllability and observability Grammians. This may be used to derive controllable and observable stable low order models.

This approach also combines the aggregation and state feedback concepts to synthesize low order models. It is noted that the aggregation method, as it is usually employed, has the following drawbacks:

(1) The reduced model is not guaranteed to be stable.

(2) There is no unique method available for evaluating the aggregation matrix.

(3) The aggregation method ignores the state contribution to the performance of the model.

In addition to the use of state feedback to insure model stability, this approach uses a procedure based on the singular value decomposition of the controllability Grammian, to compute the aggregation matrix. The extension of this approach to design low order deterministic continuous time varying, uniformly controllable models is straightforward.

The last approach in this dissertation is an adaptation of the Schwarz approximation. It uses the impulse response energy to find the order of the model and the singular values or second order modes of the controllability Grammian to select the state variables to be retained in the model. Finally, an algorithm is introduced for the computation of the frequency responses of the system and the model. This algorithm is simple, it does not require any complex arithmetic which is a major problem for most compilers, and it is easy to implement in a small computer.

1.3. Organization of This Work

This dissertation is organized as follows: Chapter II provides an overview of the existing techniques of model reduction. Chapter III introduces the mathematical preliminaries such as steady state output covariance, Markov parameters, controllability and

observability Grammians to be used in the later chapters.

Chapter IV discusses a new method which combines the aggregation and state feedback concepts, its application to continuous time invariant linear systems with uncertainty (e.g., stochastic input), and its extension to continuous time varying deterministic, uniformly controllable linear systems. Chapter V describes the Routh and Schwarz approximations, their link, and develops a new approach which is a modified version of the Schwarz approximation. Chapters IV and V each present a numerical example to demonstrate the validity of the preceding results.

CHAPTER II

OVERVIEW OF THE ORDER REDUCTION TECHNIQUES

In this section, we will review the development of model reduction during the last two decades. Davison [34] proposed one of the first order reduction techniques. Chidambara [27, 28] showed that this method does not ensure steady state agreement between the dominant states of the original system and the reduced model. Further argument [28, 29] led to several variations of Davison's original scheme which we call here Chidambara's first method, Chidambara's second method and Davison's first method. Later, Davison [35] introduced a second modified method.

Independently, Marshall [84] presented a technique which preserves the steady state of the original system by exciting the modes in the reduced model differently from those of the original system. Chidambara [30] introduced two modal techniques, which are optimal in the sense that they minimize deviations between the original system and the reduced model, but these approaches do not provide an explicit formulation for the reduced model. Fossard [48] proposed a modification to Davison's original

technique which results both in initial and final (steady state) agreement, for unit step inputs, between the original system and the reduced model. Wilson [138] and Bonvin et al. [18] showed that Chidambara's first method is equivalent to Davison's first approach and Chidambara's second technique is similar to Marshall's method.

Litz [80, 81] presented a new modal technique which is optimal in the sense that the integral of the square of the errors between the dominant states in the system and the reduced model is minimized. Unlike Chidambara's technique [30], this new approach gives an explicit formulation for the model. The practical uses of the previous methods are not trivial, since the relationship between the states of the system and those of the model do not appear explicitly.

To overcome this difficulty, Aoki [5, 6] introduced the aggregation method. This approach has the advantage of preserving some of the internal structural properties of the system which are useful not only in analysis, but also in deriving suboptimal state feedback control. The concept of aggregation was further exploited by others [10, 56, 100, 102, 123, 125, 129].

Lindner et al. [79], Medanic et al. [85] and Tse et al. [130, 131] suggested the idea of chained aggregation

which is based on the generalized Hessenberg representation (GHR), and the order reduction is performed on the GHR by coordinate truncation, neglecting the feedback couplings from the truncated subsystem and using a generalized QL algorithm to transform the GHR into another representation from which the reduced model is derived. This way of reduction may destroy the basic idea of chained aggregation.

To avoid this problem, Kwong [72, 73] used an orthogonal transformation, which is equivalent to the Rayleigh-Ritz procedure, to compute an optimal approximate aggregation, which is similar to the one introduced by Aoki [7].

Since the aggregation matrix used by Aoki [6, 7]. is arbitrary, it is therefore possible to select it in an optimal manner by minimizing a performance criterion. Such an approach to optimal aggregation has been developed by Siret et al. [124]. It has been shown [39,53,54,87] that the reduced models given by optimal properties [89,90,91] are optimally aggregated models and Davidson's [34], Marshall's [84] and Chidambara's models are special cases of aggregated models. It is known [123] that the reaction of a system to an arbitrary initial state cannot be described by a simple model, except through the aggregation concept.

In practice, the arbitrary initial states are considered in many situations as disturbances in the states

of the physical system and are not so arbitrary. Davison's aggregation scheme [34,35,36] is based on the preservation of the smallest eigenvalues, hence dominant time constants of the original system, in the reduced model. This implies that the response of the reduced model will be similar to the response of the original system, since the effect of the neglected eigenvalues on the system response is important only at the transient part of the response. However, the retained eigenvalues affect the entire response and determine the type of the response of the system. This method assumes that the original system is asymptotically stable and completely controllable. Furthermore, it is computationally cumbersome, since it requires the evaluation of the eigenvalues and eigenvectors of the higher order system.

Anderson [4] presented a method in which the eigenvalues are not, in general, retained. The approach consists of fitting the output data of the original system to a reduced model by the least squares technique. However, this method can give rise to unstable models unless the model's structure is constrained. Most often the minimization of some objective function involving the approximation error is studied using the impulse response. The parameters of the reduced model are then obtained either from the necessary conditions of optimality or by means of a search algorithm.

Meier et al. [86] considered the problem of synthesizing the reduced model transfer function for a single input, single output system by minimizing the mean square error between the output of the system and that of the model. The approximation approach consists of finding optimum residues and poles of the reduced model transfer function, then developing a set of necessary conditions involving the poles only. The solution yields a set of nonlinear equations for the poles and zeroes of the optimum reduced model transfer function.

Riggs et al. [105, 106] have used similar approaches to derive new reduction methods. Wilson [136] formulated, for the multi-input multi-output problem, an algorithm which is based on the solution of a Lyapunov matrix equation and requires optimization of the individual parameters occurring in the particular canonical form chosen. The application of Wilson's algorithm to single input, single output is rather cumbersome and time consuming. Similar results have been reported in [8, 9, 60, 141]. Some results concerning the unit step response are also available such as [35, 52, 75]. However, the method used in [52] requires the transformation of the system to a diagonal representation and then assumes that the order of the reduced model is equal to the number of outputs. It also uses the dynamic and static criteria, which leads to an incorrect steady state response.

Siret et al. [125] extended Wilson's approach [135] to a class of polynomial inputs, to define an optimum

reduced model with the same steady state output as the system. This is obtained by expressing the constraint on the asymptotic stability behavior of the two models as a constraint on the structure of the reduced model, which is easier to take into account. The problem is then reduced to a problem of minimization without constraint. This method is also applicable to systems which have unstable modes.

For a different error criteria, Sinha et al. [121] proposed a computer program for obtaining the optimum reduced model based on arbitrary polynomial inputs. In this approach, all of the coefficients in the reduced model transfer function are optimized by using a search routine. They have applied this technique to minimize the maximum deviation between two models. However, it was shown in [136] that this optimization procedure would be numerically manageable only for reducing systems of low order and would be inefficient for the mean square error criterion.

The most common characteristic of the previous methods is that they minimize an integrated quadratic error between the original system and the reduced model. This minimization is used as a measure of the performance of modelling of the methods. The parameters of the reduced model are determined so that the integral quadratic error is minimized. However, this procedure requires solving nonlinear equations and thus the calculation is quite

difficult. Another major drawback of these approaches is that the canonical structure of the reduced order model has to be constrained in some manner. To avoid this problem, Obinata et al. [95,96] suggested a different approach. Instead of minimizing the output error, they minimized the equation error. This minimization leads to a linear problem. But, instead of the original system matrices, the input and output vectors have been used.

Eitelberg [41,42] using the same idea, found a simpler formulae where only the matrices of the original system are needed. These methods require the stability of the original system, and it is not clear that the reduced models obtained by these methods can always be asymptotically stable and completely controllable, even when the original system has these properties (see Appendix A).

Since the original system is often identified in the frequency domain and then subsequently converted to the time domain, many techniques have been derived in the frequency domain. Among them is the Pade approximation procedure, in which the frequency response of the system at low and/or high frequencies is matched with that of the model by essentially retaining the first few terms of the Taylor series expansion of the system frequency response about the origin and/or infinity. Other methods using partial sequences of the Markov parameters and/or the time moments of the system frequency response were proposed [19, 50, 108, 109, 110].

Chen et al. [23,24] presented a method based on the continued fraction concept. Their form of continued fraction expansion resembles multiple feedback loops and feedforward paths with blocks corresponding to quotients. As the quotients get lower and lower in position or equivalency and the blocks develop more and more loops, they have less and less significance as far as the overall system is concerned. Therefore, truncating the continued fraction after some terms is equivalent to ignoring the inner, less important loops. In this respect, three different reduced models are available through the use of the three Cauer forms. It was shown [20] that the continued fraction method and the moments matching method may give similar results and that they may both yield unstable models even for stable systems.

To improve the moments matching approach, Shamash [110] used the generalized Pade approximation to retain specified (dominant) modes in the reduced model. Similarly, Shieh et al. [115,116] used the dominant poles concept associated with the continued fraction approach to get stable reduced models. But the concept of dominant poles requires the determination of the dominant poles first before the simplification is carried out, thus for a high order system, the computation is tedious and for some systems, it is not easy to define the dominant poles.

Chen et al. [25, 26] suggested a different method called the stability equations technique. The simplifying procedure is simple and only two equations with one-half of the order of the system need to be factored. All the reduced models are guaranteed to be stable. But the method cannot be applied directly to reduce the transfer function of nonminimum phase systems.

Another method which has similar problems was introduced in the order reduction literature [43, 143]. It was called the norm approximation method. To solve this problem, Chen et al. [26] proposed an approach which combines the advantages of the stability equations and the continued fractions methods and it consists of three steps. The first step is to reduce the denominator of the transfer function by the stability equations method. The second step is to get the partial quotients by the algorithm of the continued fraction method, and the last step is to discard the unstable partial quotients and to reconstruct the reduced model of which the denominator is obtained in the first step.

Similarly, utilizing the Pade approximately and a continued fraction expansion, Shamash [113] derived a technique which produces stable models and has many useful properties such as computational simplicity. Fitting of the time moments and the steady state value of the original system is similar to that of the reduced model.

In the same domain as the previous methods, Hutton et al. [61] introduced an approach called the Routh-Hurwitz

approximant method which does not involve the computation of the eigenvalues of the original system. It involves no optimization routines and also guarantees stability because it is based on the Routh-Hurwitz stability table together with the fitting of the moments. Shamash [114], using examples, argued that the locations of the poles of the models from Hutton's method, are completely dependent on the locations of the poles of the system.

As mentioned before, for these methods to work, the time domain description is transferred to its equivalent frequency domain representation and the simplification is carried out in the frequency domain. The reduced model is then transferred back into the time domain. The time domain realization of the reduced model from its transfer function is not an easy task.

A method free of this and all of the previous problems is the singular perturbation approximation. A singular perturbation theory provides a systematic approach to deal with the modelling, analysis and control of dynamic systems in which the physical separation of modes is recognized in terms of a small, positive parameter multiplying a part of the system equations. This part is called the fast part of the system. The singular

perturbation approximation reduces the system order by first neglecting the fast phenomena. It then improves the approximation by reintroducing their effects as boundary layer corrections calculated in time scales. In this case, the system is temporarily decoupled into two lower order subsystems which represent the slow and fast part of the system [31, 65]. In an asymptotically stable system, the fast modes corresponding to large eigenvalues are important only during a short time period [64, 65, 66, 81]. Hence, a matrix norm is given under which the large eigenvalues of a two-time scale system will be sufficiently separated from the small eigenvalues.

It has been shown [103, 104] that the Routh approximation can be treated as an approximate aggregate technique and it has been argued by Hickin et al. [55] and Vittal Rao et al. [103], that aggregation as well as the Routh approximation can be used to express any system into its slow and fast subsystems. This argument is based on the assumption that most systems possess the two time scale property, namely the eigenvalues of the system can be clustered into two groups of dominant and nondominant modes.

Several researchers have viewed the order simplification problem as a minimal realization problem and have obtained partial results [46, 58, 112]. The Hankel matrix [57] is formed with the Markov parameters, then the reduction of

the Hankel matrix is determined with respect to the Hermite norm and a partial realization is obtained.

Such methods typically match some of the generalized Markov parameters instead of all of them as is done in minimal realization. Then a combination of the aggregation method with the partial realization gives an order reduction technique which has the advantage of matching Markov parameters as well as ensuring stability. The chained aggregation method [129,131] also belongs to this class of methods since it discards the weakly observable part of the system as a way of simplifying the large scale system.

In practice, it is more often desirable to design a control system of minimal size and yet controllable and observable. The most common features of the above techniques are that the approximation problem is approached by observing that the dynamical behavior of a physical system may, on occasion, be dominated by the dynamics of a particular subsystem. Therefore, most of the modal approaches, with the exception of the singular perturbation method, suffer from the drawback that the physical significance of the state is lost. This leads to some difficulties when the model is a part of the system. Thus, before any modal technique is used, several key decisions should be made, i.e.:

(1) The most important response modes of the system,

(2) An appropriate size (order for the reduced model.

(3) The most sensitive state variables which are to be retained in the reduced model.

Even with the singular perturbation method, there is no clear-cut technique which can help to decide on the structure of the reduced model. In [44, 83], it was noted that the quality of the aggregated model depends on the contribution of the neglected modes and an upper bound on the error incurred was introduced by simplifying the high order system. For the Routh approximation method [61], error functions are easily evaluated in terms of impulse energy error.

In [98, 99, 133, 134], a scale invariant criterion has been proposed for picking out the state which significantly participates in the construction of the model. The main idea behind this approach is the development of a participation matrix for evaluating the relative importance of each state. However, this method does not take into account the couplings between the states and the various inputs and outputs. The practical importance of this criterion is thus limited.

Lastman et al. [77] proposed another criterion for estimating the effective participation of each state by evaluating its contribution to the total impulse response energy at the output of the system. With this criterion, a method of simplification combining the singular perturbation and aggregation schemes was presented. This idea originated in [61], where the impulse response energy was

utilized for the selection of the order of the reduced model. The same concept can be found in [144], for selecting the eigenvalues to be retained in the reduced model.

A method based on applying unit impulses separately to each component of the input was proposed by Commault [32], to obtain reduced aggregated models. Siret et al. [125] also suggested a procedure of mode selection for the aggregation method based on the impulse response. Litz [80] determined the dominant modes by the use of unit step functions at each component of the input, the step functions are then used to obtain the model by approximating the nondominant modes. He analyzed the couplings between the inputs and the states through the various modes and proposed a dominance measure for the sensitive states. Bonvin et al. [17] extended Litz's method to any kind of inputs.

The operation of selecting within the system state space, a subspace of completely controllable and observable states as was done by Wonham [139]. In other words, the subspace is responsible for the whole of the input-output response. Then the simplification can be approached by relaxing controllability and observability concepts and a dominant subsystem in terms of strongly controllable and observable subspaces. This technique does not necessarily require a state variable description as a starting point, but input-output representation as well.

In [59], Ho et al. related the geometric structure of the controllable/observable subspace to the algebraic

structure of the impulse Hankel matrix. Several methods for representing linear systems through analysis of the rank or the dependence relation of the system Hankel matrix were proposed. However, due to a lack of efficient and robust numerical techniques for rank analysis, they proved unsuccessful for identification and not promising for solving the approximation problem. It has been shown in [21,93,94] that appropriate techniques in linear algebra such as orthogonal factorization and singular value decomposition play a significant role in the analysis of the structure of dynamic systems, and consequently, in the determination of reduced models for approximating complex systems.

The use of singular values (second order modes) as opposed to eigenvalues or primary modes of the system first appeared in the work of Zeigler et al. [143]. Using a similar approach, Moore [92] introduced the internally balanced realization concept. He showed that from the internally balanced realization, an asymptotically stable approximate realization can be obtained by neglecting those state variables which correspond to very small second order modes. Kung [69] approached the reduction problem from a realization perspective and proposed an order reduction scheme based on the singular value decomposition of the Hankel matrix associated with the system. He also provided stability results and an error analysis.

Pernebo et al. [97] presented a method with much stronger stability results based on Moore's balanced realization. Silverman et al. [119] showed that the singular values of the Hankel matrix are indeed the second order modes and the approaches of Moore [93] and Kung [69] lead to the same reduced models. Lastman et al. [76] showed that the aggregation technique may, in some cases, give better results than the balance realization method.

In general, the approximation of the Hankel matrix based on the singular value decomposition will not have a Hankel structure. To overcome this problem using the work of Adamjan et al. [1], in [12, 70], Kung et al. developed an optimal Hankel norm approximate approach. Glove [51] developed a new state space representation which he used to give a balancing-based state space algorithm for computing the multivariable Hankel-norm approximant. Desai et al. [37, 38] extended the concept of internally balanced realization to stochastic realization of reduced order models, controllers and observers. Enns [45] derived an error bound for models obtained from internally balanced realizations. The bound is that the infinity norm of the frequency response of the difference between the system and the model is bounded by twice the sum of the balanced Grammian's singular values that correspond to the truncated states of the balanced realizations.

Concurrent with the above work, Skelton et al. [126, 128] introduced the component cost analysis approach for

approximating stochastic systems. In the generalized form, Skelton et al. [127, 142] have referred to it as the covariance equivalent realization approach. This method gives an algorithm for obtaining a reduced order stochastic realization such that a certain specified number of output covariance parameters for the reduced model match the output covariance parameters for the reduced model. Their approach can also be used for controller order reduction.

Hyland et al. [62] found necessary conditions for optimality of a continuous time stochastic linear system in the form of two Riccati and two Lyapunov equations coupled by an oblique projection. They also used this projection to compare Moore [93], Skelton [126] and Wilson [135] approaches.

Motivated by the work of Akaike [3] on predictor spaces, canonical variables and stochastic realizations, Fujishige et al. [49], Baram et al. [14, 15], Desai et al. [37] and Larimore [78] derived algorithms for obtaining approximate stochastic realizations. The reduction scheme is based on canonical correlation coefficients which can also be viewed as generalized singular values of a normalized matrix.

CHAPTER III

STEADY STATE OUTPUT COVARIANCES AND MARKOV PARAMETERS OF CONTINUOUS TIME LINEAR SYSTEMS

3.1. Introduction

In this chapter, we will introduce the mathematical material that will be used in later chapters for the derivation of model reduction techniques of asymptotically stable, completely controllable and observable, large scale linear continuous time systems.

3.2. Mathematical Background

Consider the stochastic, asymptotically stable, completely controllable and observable, stationary, continuous time system

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{3.1a}$$

$$y(t) = Cx(t) \tag{3.1b}$$

$$x(t_0) = x_0$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^\ell$ and $w \in \mathbb{R}^m$.

The initial condition x_0 is assumed to be a zero mean Gaussian random vector independent of $w(t)$. $w(t)$ is a zero mean white noise process with intensity Q. Q is a symmetrix positive definite matrix. Then

$$E\{w(t)\} = 0, \quad \text{and} \quad E\{w(t)w^T(\tau)\} = Q\delta(t-\tau) \tag{3.2}$$

where $E\{\cdot\}$ denotes the statistical expectation.

Let $\phi(t,\tau)$ be the state transition matrix of the system in equation (3.1). The solution of equation (3.1a) is then

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t,\xi)Bw(\xi)d\xi. \tag{3.3}$$

Let $K(t_1,t_2) = E\{x(t_1)x^T(t_2)\}$ denote the state covariance matrix. Then using standard arguments [47, 71] and equations 3.2 and 3.3, it follows that

$$K(t_1,t_2) = \phi(t_1,t_0)K_0\phi^T(t_2,t_0) + \int_{t_0}^{t} \phi(t_1,\xi)BQB^T\phi^T(t_2,\xi)d\xi \tag{3.4}$$

where $E\{x_0x_0^T\} = K_0$.

Now let $t_1 = t+\tau$ and $t_2 = t$. Then equation (3.4) yields

$$K(t+\tau,t) = \phi(t+\tau,t_0)K_0\phi^T(t,t_0) + \int_{t_0}^{t} \phi(t+\tau,\xi)BQB^T\phi^T(t,\xi)d\xi$$

The system (3.1) is a continuous time invariant linear system, thus

$$\phi(t_1,t_2) = e^{A(t_1-t_2)}$$

and

$$K(t+\tau,t) = e^{A(t+\tau-t_0)}K_0e^{A^T(t-t_0)} +$$

$$+ \int_{t_0}^{t} e^{A(t+\tau-\xi)}BQB^Te^{A^T(t-\xi)}d\xi$$

or

$$K(t+\tau,t) = e^{A\tau}[e^{A(t-t_0)}K_0e^{A^T(t-t_0)}$$

$$+ \int_{t_0}^{t} e^{A(t-\xi)}BQB^Te^{A^T(t-\xi)}d\xi]$$

thus

$$K(t+\tau,t) = e^{A\tau}K(t,t) \tag{3.7}$$

where

$K(t,t) = E\{x(t)x^T(t)\}$ is the state covariance matrix and it satisfies the matrix differential equation [47, 71]

$$\dot{K}(t,t) = AK(t,t) + K(t,t)A^T + BQB^T \tag{3.6}$$

$$K(t_0,t_0) = E\{x_0x_0^T\} = K_0$$

The autocorrelation matrix of the output $y(t)$ in the presence of the zero mean x_0 is denoted as $R(t+\tau,t)$. Since we may readily induce the properties of

$R(t+\tau,t) = E\{y(t+\tau)y^T(t)\}$, namely

$$R(t+\tau,t) = Ce^{A\tau}K(t,t)C^T \tag{3.7}$$

It has been shown in [127, 142] that, in steady state $R = \lim_{t\to\infty} R(t,t)$ and $K = \lim_{t\to\infty} K(t,t)$, thus equations (3.4)

and (3.7) becomes

$$0 = AK + KA^T + BQB^T \tag{3.8}$$

and

$$R(\tau) = \lim_{t\to\infty} R(t+\tau,t) = Ce^{A\tau}KC^T \tag{3.9}$$

The limit denotes the asymptotic approximation. Note that the steady state conditions are equivalent to the fact that y(t) is a stationary process, and since for stationarity, $E\{y(t+\tau,t)y^T(t)\} = R(t+\tau,t) = R(t+\tau-t) = R(\tau)$ and from [127, 142]

$$R_j(\tau) = \lim_{t\to\infty} \frac{d^jR(t+\tau,t)}{d\tau^j} = Ce^{A\tau}A^jKC^T \tag{3.10}$$

then

$$R_j = \lim_{\tau\to\infty} R_j(\tau) = \lim_{\tau\to\infty} Ce^{A\tau}A^jKC^T = CA^jKC^T. \tag{3.11}$$

Furthermore,

$$R(\tau) = \lim_{t\to\infty} R(t+\tau,t) = \sum_{\ell=0}^{\infty} CA^\ell KC^T\left(\frac{\tau^\ell}{\ell!}\right)$$

thus,

$$R(\tau) = \sum_{\ell=0}^{\infty} R_\ell \left(\frac{\tau^\ell}{\ell!}\right) \tag{3.12}$$

where

$$R_\ell = CA^\ell KC^T \tag{3.13}$$

The R_ℓ's are called the steady state covariance parameters.

Since system (3.1) is completely controllable and observable, its transfer function is given by the following rational function:

$$H(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \ldots + \beta_n}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n} \tag{3.14}$$

where the α_i's and β_i's are constant scalars and matrices respectively, H(s) can be expanded into an infinite power series about $s = \infty$ (Laurent series), of descending power of s. Thus,

$$H(s) = H_1 s^{-1} + H_2 s^{-2} + H_3 s^{-3} + \ldots \tag{3.15}$$

H(s) can also be expressed in terms of the system parameters A, B and C:

$$H(s) = C(sI-A)^{-1}B = s^{-1}C(I-s^{-1}A)^{-1}B$$

so

$$H(s) = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \ldots \qquad (3.16)$$

The comparison of equations (3.15) and (3.16) yields

$$H_\ell = CA^{\ell-1}B \quad \text{for} \quad \ell = 1, 2, 3, \ldots \qquad (3.17)$$

The H_ℓ's are called the Markov parameters. They are also computed recursively using α_ℓ and β_ℓ as follows:

$$H_1 = \beta_1$$

$$H_2 = \beta_2 - \alpha_1 H_1$$

$$\vdots$$

$$H_\ell = \beta_\ell - \alpha_1 H_{\ell-1} - \alpha_2 H_{\ell-2} - \cdots - \alpha_{\ell-1} H_1, \quad 1 \leq \ell \leq n$$

$$\vdots$$

and

$$H_{n+j} = \alpha_1 H_{n+j-1} - \alpha_2 H_{n+j-2} - \cdots \alpha_n H_j, \quad 1 \le j \le n$$

In short,

$$H_\ell = \begin{cases} \beta_\ell - \sum_{j=1}^{\ell-1} \alpha_{\ell-j} H_j & \text{for } \ell \le n \\ -\sum_{j=1}^{\ell-1} \alpha_{\ell-j} H_j & \text{for } \ell > n \text{ with } \alpha_\ell = 0 \text{ for } \ell < n \end{cases} \tag{3.18}$$

We will establish a relationship between R_ℓ, α_ℓ, and H_ℓ or β_ℓ. To do so, we introduce several matrices. In the following, N is 2nx2n nilpotent, $N^{2n}=0$, P is nxn lower triangular, S, S_1 and S_2 are nxn, 2nx2n and 2nx2n diagonal.

$$N = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{2nx2n}, \quad P = \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \alpha_1 & \cdots & 0 & 0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \alpha_{n-1} & \cdots & \alpha_1 & 1 \end{bmatrix}_{nxn}$$

$$S = \text{diag}\{(-1)^{n-1}, (-1)^{n-2}, \ldots, -1, 1\}$$
$$S_1 = \text{diag}\{(-1)^{n-1}, (-1)^{n-2}, \ldots, -1, 1, 0, \ldots 0\}$$
$$S_2 = \text{diag}\{(-1)^{n}, (-1)^{n-1}, \ldots, -1, 1, 0 \ldots, 0\}$$

and

$$\Omega = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ \cdot & \beta_1 & \cdots & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \beta_n & \cdot & & \cdot \\ 0 & \beta_n & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & \beta_n \end{bmatrix}^T, \quad \overline{\Omega} = \begin{bmatrix} \beta_1^T & 0 & \cdots & 0 \\ \cdot & \beta_1^T & \cdots & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \beta_n^T & \cdot & & \cdot \\ 0 & \beta_n^T & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & \beta_n^T \end{bmatrix}^T$$

$$R = [R_0^T, R_1^T, \ldots, R_{n-1}^T]^T_{\ell x n\ell}$$

$$H = [H_1^T, H_2^T, \ldots H_n^T]^T_{\ell x nm}$$

$$\overline{R} = [R_0, R_1, \ldots, R_{n-1}]^T_{\ell x n\ell}$$

$$\overline{H} = [H_1, H_2, \ldots, H_n]^T_{\ell x nm}$$

$$E = [0, I, 0, \ldots 0]^T_{\ell x nm}$$

$$P_1 = [P^T, 0]^T_{nx2n}$$

Proposition 3.1. The identity relating R_ℓ, α_ℓ and H_ℓ or β_ℓ is given by:

$$\sum_{\ell=1}^{n} \alpha_\ell N^\ell [(-1)^{n-\ell} P_1 R + S_1 P_1 \bar{R}]$$

$$= \sum_{\ell=0}^{n} (-1)^{n-\ell} \sum_{j=0} \alpha_j H_{\ell-j} N^\ell P_1 \bar{H} = \Omega S_2 \bar{\Omega} E \tag{3.19}$$

where $\alpha_0 = 1$.

In order to prove Proposition 3.1, we should note that since by assumption, the system (3.1) is asymptotically stable and completely controllable and observable, then there exists a symmetric positive definite matrix K such that

$$AK + KA^T + BQB^T = 0 \tag{3.20}$$

For simplicity of computation, let the weighting matrix Q be the identity matrix so that

$$AK + KA^T + BB^T = 0. \tag{3.21}$$

Now multiply equation (3.21) by -1, add and subtract from it sK, yields

$$(sI-A)K + K(-sI-A^T) - BB^T = 0.$$

Pre- and post-multiply this equation by $C(sI-A)^{-1}$ and $(-sI-A^T)^{-1}C^T$ gives

$$C(sI-A)^{-1}KC^T + CK(-sI-A^T)^{-1}C^T$$

$$- C(sI-A)^{-1}BB^T(-sI-A^T)^{-1}C^T = 0$$

or

$$C(sI-A)^{-1}KC^T + CK(-sI-A^T)^{-1}C^T$$

$$= C(sI-A)^{-1}BB^T(-sI-A^T)^{-1}C^T. \tag{3.22}$$

Lemma 3.1. The equation (3.22) can be expressed in terms of R_ℓ, α_ℓ and H_ℓ as follows:

$$\sum_{\ell=0}^{n} (-1)^{n-\ell}\alpha_\ell s^{n-\ell} \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} \alpha_j R_{\ell-j-1} s^{n-\ell}$$

$$+ \sum_{\ell=0}^{n} \alpha_\ell s^{n-\ell} \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} (-1)^{n-\ell} \alpha_j R^T_{\ell-1-j} s^{n-\ell}$$

$$= \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} \alpha_j H_{\ell-j} s^{n-\ell} \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} (-1)^{n-\ell} \alpha_j H^T_{\ell-j} s^{n-\ell} \qquad (3.23)$$

Recall that:

$$(sI-A)^{-1} = \frac{adj(sI-A)}{det(sI-A)} = \frac{1}{p(s)} adj(sI-A)$$

where

$$p_1(s) = det(sI-A) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$$

$$p_1(s) = \sum_{\ell=0}^{n} \alpha_\ell s^{n-\ell} \qquad \text{with } \alpha_0 = 1$$

and

$$adj(sI-A) = M_1 s^{n-1} + M_2 s^{n-2} + \ldots M_{n-1} s + M_n$$

with

$$M_1 = I$$

$$M_2 = A + \alpha_1 I$$

$$\vdots$$

$$M_\ell = A^{\ell-1} + \alpha_1 A^{\ell-2} + \cdots + \alpha_{\ell-2} A + \alpha_{\ell-1} I \ , \quad \ell = 1, \ldots, n$$

$$M_{n+1} = 0 \ .$$

$$\text{Similarly} \quad (-sI-A^T)^{-1} = \frac{\text{adj}(-sI-A^T)}{\det(-sI-A^T)}$$

and noting that

$$p_2(s) = \det(-sI-A^T) = (-1)^n \det(sI+A^T)$$

we have

$$p_2(s) = \sum_{\ell=0}^{n} (-1)^{n-\ell} \alpha_\ell s^{n-\ell} \ .$$

Thus,

$$\text{adj}(-sI-A^T) = \sum_{\ell=1}^{n} (-1)^{n-\ell} M_\ell^T s^{n-\ell}$$

where M_ℓ^T is the transpose of M_ℓ.

Therefore, equation (3.22) can be rewritten as follows:

$$\frac{1}{p_1(s)} = \sum_{\ell=1}^{n} CM_\ell KC^T s^{n-\ell} + \frac{1}{p_2(s)} \sum_{\ell=1}^{n} (-1)^{n-\ell} CKM^T C^T s^{n-\ell}$$

$$= \frac{1}{p_1(s)p_2(s)} \sum_{\ell=1}^{n} CM_\ell Bs^{n-\ell} \sum_{\ell=1}^{n} (-1)^{n-\ell} B^T M^T C^T s^{n-\ell}$$

or

$$p_2(s) \sum_{\ell=1}^{n} CM_\ell KC^T s^{n-\ell} + p_1(s) \sum_{\ell=1}^{n} (-1)^{n-\ell} CKM^T C^T s^{n-\ell} \tag{3.24}$$

$$= \sum_{\ell=1}^{n} CM_\ell Bs^{n-\ell} \sum_{\ell=1}^{n} (-1)^{n-\ell} B^T M^T C^T s^{n-\ell}.$$

Using the above form of M_ℓ, we have

$$CM_\ell KC^T = \sum_{j=0}^{\ell-1} \alpha_j CA^{\ell-1-j} KC^T \qquad \alpha_0 = 1 \tag{3.25}$$

and

$$CM_\ell B = \sum_{j=0}^{\ell-1} \alpha_j CA^{\ell-1-j} B \qquad \alpha_0 = 1. \tag{3.26}$$

Using equations (3.13) and (3.17), the last two equations become

$$CM_{\ell}KC^{T} = \sum_{j=0}^{\ell-1} \alpha_j R_{\ell-1-j} \tag{3.27}$$

and

$$CM_{\ell}B = \sum_{j=0}^{\ell-1} \alpha_j H_{\ell-j}. \tag{3.28}$$

Similarly,

$$CKM_{\ell}^{T}C^{T} = \sum_{j=0}^{\ell-1} \alpha_j R_{\ell-1-j}^{T} \tag{3.29}$$

and

$$B^{T}M_{\ell}^{T}C^{T} = \sum_{j=0}^{\ell-1} \alpha_j H_{\ell-j}^{T}. \tag{3.30}$$

Thus, equation (3.23) becomes

$$\sum_{\ell=0}^{n} (-1)^{n-\ell}\alpha_{\ell}s^{n-\ell} \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} \alpha_j R_{\ell-j-1}s^{n-\ell} +$$

$$+ \sum_{\ell=0}^{n} \alpha_\ell s^{n-\ell} \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} (-1)^{n-\ell} \alpha_j R^T_{\ell-1-j} s^{n-\ell}$$

$$= \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} \alpha_j H_{\ell-j} s^{n-\ell} \sum_{\ell=1}^{n} \sum_{j=0}^{\ell-1} (-1)^{n-\ell} \alpha_j H^T_{\ell-j} s^{n-\ell}.$$

This proves Lemma 3.1.

Note that if P(s) and Q(s) are two matrix polynomials of different orders but the same dimensions, then

$$P(s) = \sum_{j=0}^{k} P_j s^{k-j} \quad \text{and} \quad Q(s) = \sum_{j=0}^{\ell} Q_j s^{\ell-j} .$$

Their product G(s) is given by

$$G(s) = P(s)Q(s) = \sum_{j=0}^{m} G_j s^{m-j} \quad \text{with } m = \ell+k. \tag{3.31}$$

The matrix coefficients G_j's are determined from the following product:

$$\begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_k \\ \vdots \\ G_m \end{bmatrix} = \begin{bmatrix} P_0 & & & 0 \\ P_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & P_0 \\ P_k & & \ddots & P_1 \\ \vdots & \ddots & & \vdots \\ 0 & & & P_k \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_\ell \end{bmatrix}. \qquad (3.32)$$

Proof of Proposition 3.1. Applying equations (3.31) and (3.32) to the equation (3.23), and using the previous matrix notation and

$$M_r = \begin{bmatrix} R_0 & 0 & \cdots & 0 \\ R_1+\alpha_1 R_0 & R_0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ R_{n-1}+\ldots+\alpha_{n-1}R_0 & R_{n-2}+\ldots+\alpha_{n-2}R_0 & & 0 \\ 0 & R_{n-1}+\ldots+\alpha_{n-1}R_0 & \cdots & R_0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & R_{n-1}+\ldots+\alpha_{n-1}R_0 \end{bmatrix}$$

$$M_h = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ H_2+\alpha_1 H_1 & H_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ H_n+\ldots+\alpha_n H_1 & H_{n-1}+\ldots+\alpha_{n-1} H_1 & \cdots & 0 \\ 0 & H_n+\ldots+\alpha_n H_1 & \cdots & H_1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & H_n+\ldots+\alpha_n H_1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} \alpha_0 I \\ \alpha_1 I \\ \vdots \\ \alpha_{n-1} I \\ \alpha_n I \end{bmatrix}, \qquad L_2 = \begin{bmatrix} (-1)^n \alpha_0 I \\ (-1)^{n-1} \alpha_1 I \\ \vdots \\ -\alpha_{n-1} I \\ \alpha_n I \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0 \\ (-1)^{n-1} H_1^T \\ (-1)^{n-2}(H_2^T + \alpha_1 H_1^T) \\ \vdots \\ H_n^T + \ldots + {}_nH_1^T \end{bmatrix}, \quad S_3 = \begin{bmatrix} (-1)^n I & 0 & \cdots & 0 & 0 \\ 0 & (-1)^{n-1} I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$M_t = \begin{bmatrix} (-1)^{n-1} R_0^T & 0 & \cdots & 0 \\ (-1)^{n-2}(R_1^T + \alpha_1 R_0^T) & (-1)^{n-1} R_0^T & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ R_{n-1}^T + \ldots + \alpha_{n-1} R_0^T & R_{n-2}^T + \ldots + \alpha_{n-2} R_0^T & \cdots & 0 \\ 0 & R_{n-1}^T + \ldots + \alpha_{n-1} R_0^T & \cdots & (-1)^{n-1} R_0^T \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & R_{n-1}^T + \ldots + \alpha_{n-1} R_0^T \end{bmatrix}$$

We have

$$M_r L_2 + M_t L_1 = M_h L_3 = \Omega S_3 \bar{\Omega} E. \tag{3.33}$$

After some algebraic manipulation, equation (3.32) becomes:

$$\text{(a)} \quad \sum_{\ell=0}^{n} \alpha_\ell N^\ell [(-1)^{n-\ell} P_1 R + S_1 P_1 \bar{R}] = \sum_{\ell=0}^{n} (-1)^{n-\ell} \sum_{j=0}^{\ell-1} \alpha_j H_{\ell-j} N^\ell P_1 \bar{H}$$

$$= \Omega S_2 \bar{\Omega} E.$$

This finishes the proof of Proposition 3.1.

Our results can be further simplified. For this we define

$$D_1 = \begin{bmatrix} (-1)^n \alpha_0 I & 0 & \cdots & 0 & 0 \\ (-1)^{n-1} \alpha_1 I & (-1)^n \alpha_0 I & \cdots & 0 & 0 \\ (-1)^{n-2} \alpha_2 I & (-1)^{n-1} \alpha_1 I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_n I & -\alpha_{n-1} I & \cdots & 0 & 0 \\ 0 & \alpha_n & \cdots & 0 & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & (-1)^{n-1} \alpha_1 I & (-1)^n \alpha_0 I \end{bmatrix}_{2n\ell \times 2n\ell}$$

$$D_2 = \begin{bmatrix} (-1)^{n-1}\alpha_0 I & 0 & \cdots & 0 & 0 \\ (-1)^{n-1}\alpha_1 I & (-1)^{n-2}\alpha_0 I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^{n-1}\alpha_n I & (-1)^{n-2}\alpha_{n-1} I & \cdots & 0 & 0 \\ 0 & (-1)^{n-2}\alpha_n I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & -\alpha_1 I & \alpha_0 I \end{bmatrix}_{2n\ell \times 2n\ell}$$

Since $\beta_\ell = \sum_{j=0}^{\ell-1} \alpha_j H_{\ell-j}$, is an ℓxm matrix, then

$$D_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ (-1)^{n-1}\beta_1 & 0 & \cdots & 0 & 0 \\ (-1)^{n-2}\beta_2 & (-1)^{n-1}\beta_1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \beta_n & \beta_{n-1} & \cdots & 0 & 0 \\ 0 & \beta_n & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & (-1)^{n-1}\beta_1 & 0 \end{bmatrix}_{2nm \times 2nm},$$

$$D_4 = [\beta_1, \beta_2, \ldots, \beta_n, 0, \ldots, 0]^T_{\ell x 2nm}$$

and R_ℓ is an $\ell x \ell$ matrix. Then equation (3.19) becomes

$$D_1 P_1 R_1 + D_2 P_1 \bar{R} = D_3 D_4.$$

We now consider a special case of this equation. Suppose that system (3.1) is a single input, single output system that is $m = \ell = 1$, then R_j, H_j and β_j are scalars and

$$R_j^T = CK(A^T)^j C^T = (CA^j K C^T)^T = R_j \qquad \text{for } j = 0, 1, \ldots, n-1$$

$$H_\ell^T = B^T (A^T)^{\ell-1} C^T = (CA^{\ell-1}B)^T = H_\ell \qquad \text{for } \ell = 1, 2, \ldots, n$$

and

$$\beta_k^T = \sum_{j=0}^{k-1} \alpha_j H_{k-j}^T = \sum_{j=0}^{k-1} \alpha_j H_{k-j} \qquad \text{for } k = 1, 2, \ldots, n.$$

Hence, equation (3.33) reduces to the following form:

$$D_1 P_1 R + D_2 P_1 R = D_3 D_4.$$

Let $D_5 = (D_1 + D_2) P_1$, then $D_5 R = D_3 D_4$ or $(D_5^T D_5) R = D_5^T D_3 D_4$

and $R = (D_5^T D_5)^{-1} D_5 D_3 D_4$.

We shall now compute some of the elements of R for different values of n.

(a) $n = 2$

$$R_0 = \frac{\alpha_1^2 + \alpha_2}{2\alpha_1\alpha_2} H_1^2 + \frac{1}{2\alpha_1\alpha_2} H_2^2 + \frac{1}{\alpha_2} H_1 H_2$$

$$R_1 = -\frac{1}{2} H_1^2$$

and

$$\alpha_2 R_0 + \alpha_1 R_1 = \frac{\alpha_2}{2\alpha_1} H_1^2 + \frac{1}{2\alpha_1} H_2^2 + H_1 H_2$$

(b) $n = 3$

$$R_0 = \frac{\alpha_1\alpha_2^2 + \alpha_3(\alpha_1^2 - \alpha_2)}{2\alpha_3(\alpha_1\alpha_2 - \alpha_3)} H_1^2 + \frac{\alpha_1^3 + \alpha_3}{2\alpha_3(\alpha_1\alpha_2 - \alpha_3)} H_2^2$$

$$+ \frac{\alpha_1}{2\alpha_3(\alpha_1\alpha_2 - \alpha_3)} H_3^2 + \frac{\alpha_1^2\alpha_2}{\alpha_3(\alpha_1\alpha_2 - \alpha_3)} H_1 H_2$$

$$+ \frac{1}{\alpha_3} H_1 H_3 + \frac{\alpha_1^2}{\alpha_3(\alpha_1\alpha_2 - \alpha_3)} H_2 H_3$$

$$R_1 = -\frac{1}{2} H_1^2$$

$$R_2 = \frac{\alpha_1\alpha_3}{2(\alpha_3 - \alpha_1\alpha_2)} H_1^2 + \frac{\alpha_1^2 + \alpha_2}{2(\alpha_3 - \alpha_1\alpha_2)} H_2^2$$

$$+ \frac{1}{2(\alpha_3 - \alpha_1\alpha_2)} H_3^2 + \frac{\alpha_1\alpha_2}{\alpha_3 - \alpha_1\alpha_2} H_1H_2$$

$$+ \frac{\alpha_1}{\alpha_3 - \alpha_1\alpha_2} H_2H_3$$

and

$$\alpha_3R_0 + \alpha_2R_1 + \alpha_1R_2 = -\frac{1}{2} H_2^2 + H_1H_3$$

(c) $n = 4$, $R_3 = \frac{1}{2} H_2^2 - H_1H_3$.

Therefore, for $n = 3$, $\alpha_3R_0 + \alpha_2R_1 + \alpha_1R_2 = -R_3$, independently of the structures of the system matrices A, B and C.

Similarly for $n = 6$,

$$R_5 = -\frac{1}{2} H_3^2 + H_2H_3 - H_1H_5$$

and for $n = 8$,

$$R_7 = \frac{1}{2} H_4^2 - H_3H_5 + H_2H_6 - H_1H_7.$$

In general, for $j = 2\ell + 1$ and $\ell = 0, 1, \ldots (n-1)/2$,

$$R_j = R_{2\ell+1} = (-1)^{\ell+1} \frac{1}{2} H^2_{\ell+1} + \sum_{i=0}^{\ell-1} (-1)^{\ell-i} H_{\ell-i} H_{\ell+i+2} \tag{3.36}$$

However, for j even or j odd and j>n, R_j is a nonlinear function of α_ℓ, H_ℓ or β_ℓ.

NOTE: Although we do not take this up here, the R_j's can also be expressed in terms of the Routh array elements of the characteristic polynomial of the system.

In the next section a realization method, based on the Kalman and Ho realization [22], is proposed. It will be called partial realization because the system parameters A, B and C are not entirely derived from the system transfer function.

3.3. Partial Realization

Suppose that the matrices of the asymptotically stable, completely controllable and observable system are given by

$$A = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \\ -\alpha_n I & -\alpha_{n-1} I & \cdots & -\alpha_1 I \end{bmatrix}_{qnxqn},$$

$$B = \begin{bmatrix} H_1 \\ \vdots \\ H_{n-1} \\ H_n \end{bmatrix}_{qnxq} \qquad \text{and } C = [I, 0, \ldots, 0]_{qxnq} \tag{3.37}$$

since

$$A^{\ell}B = [H_{\ell+1}^T, \ldots H_{n+\ell}^T]^T, \quad \ell = 0, 1, \ldots, n-1$$

The controllability matrix is found to be

$$[B, AB, \ldots, A^{n-1}B] = \begin{bmatrix} H_1 & \cdots & H_n \\ \vdots & & \vdots \\ H_n & & H_{2n-1} \end{bmatrix}$$

which is recognized as a Hankel matrix. Similarly,

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = I, \quad \text{and since } R_{\ell} = CA^{\ell}KC^T, \text{ then}$$

$$\begin{bmatrix} R_0 \\ R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} KC^T = \begin{bmatrix} K_{11} \\ K_{12} \\ \cdot \\ \cdot \\ \cdot \\ K_{1n} \end{bmatrix}$$

and

$$A \begin{bmatrix} R_0 \\ R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-1} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{bmatrix} \quad \text{and} \quad A^{\ell} \begin{bmatrix} R_0 \\ R_1 \\ \cdot \\ \cdot \\ \cdot \\ R_{n-1} \end{bmatrix} = \begin{bmatrix} R_{\ell} \\ R_{\ell+1} \\ \cdot \\ \cdot \\ \cdot \\ R_{n+\ell-1} \end{bmatrix} \tag{3.38}$$

thus

$$A \begin{bmatrix} R_0 & \cdots & R_{n-1} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ R_{n-1} & \cdots & R_{2n-2} \end{bmatrix} = \begin{bmatrix} R_1 & \cdots & R_n \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ R_n & \cdots & R_{2n-1} \end{bmatrix}.$$

Now let

$$T = \begin{bmatrix} R_0 & \cdots & R_{n-1} \\ \vdots & & \vdots \\ R_{n-1} & \cdots & R_{2n-2} \end{bmatrix}_{qnxpn}, \quad \tilde{T} = \begin{bmatrix} R_1 & \cdots & R_n \\ \vdots & & \vdots \\ R_n & \cdots & R_{2n-1} \end{bmatrix}_{qnxpn}$$

and

$$\tilde{A} = \begin{bmatrix} 0 & \cdots & -\alpha_n I \\ I & \cdots & -\alpha_{n-1} I \\ \vdots & & \vdots \\ 0 & \cdots & -\alpha_1 I \end{bmatrix}_{pnxpn}.$$

then $AT = \tilde{T} = T\tilde{A}$, and

$$A^{\ell}T = T\tilde{A}^{\ell} \quad \text{for } \ell = 0, 1, 2, \ldots \tag{3.39}$$

Observe that the left upper corner element of $A^{\ell}T = T\tilde{A}^{\ell}$ is R_{ℓ}. Let $I_{q,qn} = [I_q \vdots 0]$, where I_q is the qxq identity matrix and 0 is the qxq(n-1) zero matrix. Since R_{ℓ} is a matrix, then R_{ℓ} can be removed from $A^{\ell}T = T\tilde{A}^{\ell}$ as

$$R_{\ell} = I_{q,qn}A^{\ell}TI_{p,pn}^{T} = I_{q,qn}T\tilde{A}^{\ell}I_{p,pn}^{T} \tag{3.40}$$

for $\ell = 0, 1, 2, \ldots$

Note that the matrix T can also be written as follows

$$T = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} [KC^T, AKC^T, \ldots, A^{n-1}KC^T]$$

$$\text{or } T = [KC^T, AKC^T, \ldots, A^{n-1}KC^T] \tag{3.41}$$

To determine the partial realization of the system described by the parameters of equation (3.36), we must find the singular value decomposition of the matrix T. Suppose there exists qnxqn and pnxpn unitary matrices P and Q such that

$$T = P \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} Q \tag{3.42}$$

where $\Sigma = \text{diag}\{\lambda_1, \lambda_2, \ldots \lambda_m\}$ and λ_ℓ for $\ell = 1, \ldots, m$ are the positive square roots of the eigenvalues of T*T.

Let $P = [P_1 \; P_2]$ and $Q = \begin{bmatrix} Q_1 \\ \cdots \\ Q_2 \end{bmatrix}$

then $T = P_1 \Sigma Q_1$, rank $T = m$

$$T = P_1 \Sigma^{1/2} \Sigma^{1/2} Q_1 = VU \tag{3.43}$$

where $\Sigma^{1/2} = \text{diag}\{\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}\}$

$V = P_1 \Sigma^{1/2}$ is a qnxm matrix

and

$U = \Sigma^{1/2} Q_1$ is a mxpn matrix.

Let us define $V^+ = \Sigma^{-1/2} P_1$ and $U^+ = Q_1^T \Sigma^{-1/2}$. (3.44)

V^+ and U^+ are called pseudoinverses of V and U.

Since $P_1^T P_1 = I_m$ and $Q_1 Q_1^T = I_m$, then

$$V^+ V = \Sigma^{-1/2} P_1^T P_1 \Sigma^{1/2} = I_m \tag{3.45}$$

$$UU^+ = \Sigma^{1/2} Q_1 Q_1^T \Sigma^{-1/2} = I_m \tag{3.46}$$

We can now establish the following proposition:

Proposition 3.2. The partial realization of the system (3.37) can be accomplished using the singular value decomposition as follows:

$$F = V^{+}\tilde{T}U^{+} \tag{3.47a}$$

$$H = \text{first } q \text{ rows of } I_{q,qn}V \tag{3.47b}$$

$$G = \text{first } p \text{ columns of } [H_1^T, \ldots, H_n^T]^T \tag{3.47c}$$

such that $T = VU$ and rank T = rank V = rank U,
where V is the same observability matrix

$$U = [KH^T, \ldots, F^{n-1}KH^T] \quad \text{and} \quad \tilde{T} = VFU$$

Proof:

Let $T^{+} = U^{+}V^{+}$

then we have, by using (3.43), (3.45) and (3.46):

$$TT^{+}T = VUU^{+}B^{+}VU = VU = T$$

$$F = V^{+}\tilde{T}U^{+}$$

$$F^2 = (V^{+}\tilde{T}U^{+})^2 = (V^{+}ATU^{+})^2$$

$$= (V^{+}ATU^{+})(V^{+}T\tilde{A}U^{+}) = V^{+}ATU^{+}V^{+}T\tilde{A}U^{+}$$

$$= V^{+}ATT^{+}T\tilde{A}U^{+} = V^{+}AT\tilde{A}U^{+}$$

$$= V^{+}A^2TU^{+},$$

In general, $F^{\ell} = V^{+}A^{\ell}TU^{+}$ for $\ell = 1, 2, 3, \ldots$ (3.48)

$$\text{and } R_\ell = I_{q,qn} A^\ell T T^T_{p,pn}$$

$$= I_{q,qn} A^\ell T T^+ T I^T_{p,pn}$$

$$= I_{q,qn} T \tilde{A}^\ell T^+ T I^T_{p,pn}$$

$$= I_{q,qn} T T^+ T \tilde{A}^\ell T^+ T I^T_{p,pn}$$

$$= I_{q,qn} T T^+ A^\ell T T^+ T I^T_{p,pn}$$

$$= \underbrace{I_{q,qn}}_{H} \underbrace{V U U^+ V^+}_{I_n} \underbrace{A^\ell T U^+ V^+}_{F^\ell} \underbrace{V U}_{I_n} \underbrace{I^T_{p,pn}}_{K H^T}$$

$$\text{Thus, } R_\ell = H F^\ell K H^T \quad \text{for } \ell = 0, 1, 2, \ldots \tag{3.49}$$

Note that the decomposition of $T = VU$ is not unique and the realization with $V^T V = U U^T$ is called stochastic internally balanced. Due to the chosen structures of the matrices F, G and H, this realization is controllable and observable. However, it may or may not be stable because the derivation of the system matrices F, G and H does not involve the system transfer function. If the realization is not minimal different choices of the matrices P and Q must be made. In the coming section, we will study different algorithms for the derivation of the Lyapunov matrix.

3.4. Computation of the Controllability and Observability Grammians

We introduce a method for the computation of the Lyapunov matrices. This method is based on matrix inversion. Let K_c and K_o be controllability and observability Grammians respectively. The matrix K_c and K_o are nxn symmetric definite matrices satisfying the following Lyapunov equations:

$$AK_c + K_cA^T = -BB^T \tag{3.50a}$$

and

$$A^TK_o + K_oA = -C^TC \tag{3.50b}$$

It is known [22, 63] that if the pairs $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{A})$ are controllable and observable respectively, and the system state matrix $\bar{A}$ has distinct eigenvalues, then there exists a nxn nonsingular similarity transformation matrix P such that the matrix $P\bar{A}P^{-1}$ has the companion form. From equations (3.50) and (3.51), we have

$$\bar{A}\bar{K}_c + \bar{K}_c\bar{A}^T = -\bar{B}\bar{B}^T$$

$$\bar{A}^T\bar{K}_o + \bar{K}_o\bar{A} = -\bar{C}^T\bar{C}$$

Applying the similarity transformation matrix P to these equations, we get

$$P\bar{A}P^{-1}P\bar{K}_cP^T + P\bar{K}_cP^T(P^{-1})^T\bar{A}^TP^T = - P\bar{B}\bar{B}^TP^T$$

$$(P^{-1})^T\bar{A}^TP^T(P^{-1})^T\bar{K}_oP^{-1} + (P^{-1})^T\bar{K}_oP^{-1}P\bar{A}P^{-1} = - (P^{-1})^TC^TCP^{-1}$$

thus

$$AK_c + K_cA^T = - BB^T$$

$$A^TK_o + K_oA = -C^TC$$

where

$$A = P\bar{A}P^{-1}, \quad B = P\bar{B}, \quad C = \bar{C}P^{-1} \tag{3.51}$$

$$K_c = P\bar{K}_cP^T \quad \text{and} \quad K_0 = (P^{-1})^T\bar{K}_0P^{-1} \tag{3.52}$$

Suppose that the A, B and C are real matrices of companion form, namely

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_1 \end{bmatrix} \tag{3.53}$$

$$B = [b_1, \ldots, b_n]^T$$

$$C = [C_1, \ldots, C_n]$$

Then the controllability and observability Grammians are determined as follows:

Controllability Grammian: From equation (3.50a) K_c and BB^T are nxn symmetric matrices. Therefore, they can be replaced by arrays with n(n+1)/2 elements each

$$V_n = \begin{bmatrix} k_{11} \\ \vdots \\ k_{1n} \\ k_{22} \\ \vdots \\ k_{2n} \\ \vdots \\ k_{nn} \end{bmatrix}_{\frac{n(n+1)}{2}\ x1}, \quad W_n = \begin{bmatrix} q_{11} \\ \vdots \\ q_{1n} \\ q_{22} \\ \vdots \\ q_{2n} \\ \vdots \\ q_{nn} \end{bmatrix} = \begin{bmatrix} b_1b_1 \\ \vdots \\ b_1b_n \\ b_2b_2 \\ \vdots \\ b_2b_n \\ \vdots \\ b_nb_n \end{bmatrix}_{\frac{n(n+1)}{2}\ x1} \tag{3.54a}$$

and let

$$A_n = \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix}_{nxn}.$$

$$E_n = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{n-1}_{n \times \frac{n(n-1)}{2}} \qquad (3.54b)$$

and

$$F_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & -\alpha_n & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\alpha_n & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\alpha_n & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -2\alpha_n \end{bmatrix}_{\frac{n(n-1)}{2} \times n}$$

Let

$$Z_n = \begin{bmatrix} A_n & E_n \\ F_n & Z_{n-1} \end{bmatrix} \frac{n(n+1)}{2} \times \frac{n(n+1)}{2} \tag{3.55}$$

and $Z_m V_n = -W_n$ or

$$V_n = -Z_n^{-1} W_n \tag{3.56}$$

where $Z_1 = -2\alpha_1$ and

$$Z_2 = \begin{bmatrix} 0 & 2 & 0 \\ -\alpha_2 & -\alpha_1 & 1 \\ 0 & -2\alpha_2 & -2\alpha_1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} k_{11} \\ k_{12} \\ k_{22} \end{bmatrix}, \quad W_2 = \begin{bmatrix} q_{11} \\ q_{12} \\ q_{22} \end{bmatrix}$$

$$Z_3 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & -\alpha_3 & 0 & -\alpha_2 & -\alpha_1 & 1 \\ 0 & 0 & -2\alpha_3 & 0 & -2\alpha_1 & -2\alpha_2 \end{bmatrix},$$

$$V_3^T = [k_{11}, k_{12}, k_{13}, k_{22}, k_{23}, k_{33}]$$

$$W_3^T = [q_{11}, q_{12}, q_{13}, q_{22}, q_{23}, q_{33}]$$

Since A_n and Z_{n-1} are square matrices and A_n is nonsingular, $\det A_n = (-1)^n \alpha_n$, $\alpha_n \neq 0$, and

$$A_n^{-1} = \begin{bmatrix} -\frac{\alpha_{n-1}}{2\alpha_n} & -\frac{\alpha_{n-2}}{\alpha_n} & \cdots & -\frac{\alpha_1}{\alpha_n} & -\frac{1}{\alpha_n} \\ \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3.57}$$

Then Z_n^{-1} exists and is determined as follows:

$$Z_n^{-1} = \begin{bmatrix} A_n & E_n \\ F_n & Z_{n-1} \end{bmatrix}^{-1}$$

$$Z_n^{-1} = \begin{bmatrix} A_n^{-1} + A_n^{-1}E_n\Delta_n^{-1}F_n & -A_n^{-1}E_n\Delta_n^{-1} \\ -\Delta_n^{-1}F_nA_n^{-1} & \Delta_n^{-1} \end{bmatrix} \tag{3.58}$$

where Δ_n is the Schur complement of A_n and is given by

$$\Delta_n = Z_{n-1} - F_n A_n^{-1} E_n. \tag{3.59}$$

Using a transparent identity

$$\Delta_n^{-1} = Z_{n-1}^{-1} + Z_{n-1}^{-1} F_n (A_n - E_n Z_{n-1}^{-1} F_n)^{-1} E_n Z_{n-1}^{-1} \tag{3.60}$$

where the recursive nature of equation (3.60) yields

$$Z_1^{-1} = -\frac{1}{2\alpha_1}$$

and

$$Z_2^{-1} = \begin{bmatrix} -\frac{\alpha_1^2 + \alpha_2}{2\alpha_1\alpha_2} & -\frac{1}{\alpha_2} & -\frac{1}{2\alpha_1\alpha_2} \\ \frac{1}{2} & 0 & 0 \\ -\frac{\alpha_2}{2\alpha_1} & 0 & -\frac{1}{2\alpha_1} \end{bmatrix}$$

as the first two terms.

The Observability Grammian can be obtained in a similar fashion, using the duality principle [22, 63].

3.4.3. Observations

(1) Δ_n is different from Z_{n-1}, because the left bottom part of Δ_n is different from F_{n-1}.

(2) Equations (3.54), (3.56), and (3.58) confirm the statement of equation (3.35).

(3) Despite the amount of storage required for the computation of the matrices Z_n^{-1}, this method has proven to be very efficient.

3.5. Conclusion

This chapter presents the necessary mathematics needed for the computation of the steady state output covariances and the Markov parameters. It also introduces a procedure for solving the Lyapunov equation. While this procedure requires a certain amount of computer storage, it is found to be as efficient as any other existing method. In the next chapter, the aggregation and the state feedback concepts will be combined to give a modified aggregation technique.

CHAPTER IV

ORDER REDUCTION BY STATE FEEDBACK AND AGGREGATION

4.1. Problem Formulation and Mathematical Background

In this chapter, we introduce topics relative to aggregated modeling. These results are subsequently used in an order reduction technique of systems with uncertainties.

Consider the stochastic, continuous dynamic system described by the state space equations

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t),\ x(t_0) = x_0 \qquad (4.1a)$$

$$y(t) = Cx(t) \qquad (4.1b)$$

where $x(t)$, $u(t)$, $w(t)$ and $y(t)$ are n, m, q, and p triples respectively. The initial condition x_o is assumed to be a zero mean Gaussian random vector independing of $w(t)$. $w(t)$is a zero mean white noise process with intensity W. A, B, G and C are constant matrices of compatible dimensions. (A,B), (A,G) are completely controllable and (C,A) is completely observable. We want to find an aggregated model of the form

$$\dot{x}_r(t) = A_m x_r(t) + B_r u(t) + G_r w(t) \qquad (4.2a)$$

$$y_r(t) = C_r x_r(t) \qquad (4.2b)$$

where $x_r(t) \in \mathbb{R}^r$ is the aggregated state vector, and $y_r(t) \in \mathbb{R}^s$ is the output response of the aggregated model. A_m, B_r, G_r and C_r are constant matrices of compatible dimensions.

For a given class of inputs $u(t)$, $w(t)$, the model outputs $\{y_r(\cdot)\}$ are to be satisfactory approximations to the original system outputs $\{y(\cdot)\}$.

Note that the model (4.2) has order r such that $\max\{q,m\} \leq r < n$. To establish a link between the system and the model, let

$$u(t) = Sx(t) \qquad (4.3)$$

be a state feedback law. The system (4.1) becomes

$$\dot{x}(t) = (A+BS)x(t) + Gw(t) \qquad (4.4a)$$

$$y(t) = Cx(t). \qquad (4.4b)$$

Lemma 4.1: The system (4.4) is controllable if, and only if the pair ((A+BS),G) is completely controllable.

Proof: It is known [22] that the state feedback preserves the controllability of the original system, then ((A+BS),B) is controllable if and only if (A,B) is controllable.

Assume that the pairs (A,B) and (A,G) are controllable.

Note that

$$(A+BS)G = AG + BSG = [I,BS] \begin{bmatrix} AG \\ G \end{bmatrix}$$

$$(A+BS)^2G = A^2G + BSG + ABSG + BSBSG$$

$$= [I,\ BS,\ (A+BS)GS] \begin{bmatrix} A^2G \\ AG \\ G \end{bmatrix}.$$

In general

$$(A+BS)^iG = [I,\ BS,\ \ldots,\ (A+BS)^{i-2}BS,\ (A+BS)^{i-1}BS] \begin{bmatrix} A^iG \\ A^{i-1}G \\ \vdots \\ AG \\ G \end{bmatrix}$$

thus

$$[G,\ (A+BS)G,\ \ldots,\ (A+BS)^{n-1}G] =$$

$$[I,\ BS,\ \ldots,\ (A+BS)^{n-2}BS] \begin{bmatrix} G & AG & \ldots & A^{n-1}G \\ 0 & G & \ldots & A^{n-2}G \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & G \end{bmatrix}$$

Hence, the pair (A+BS,G) is controllable if (A+BS,BS) and (A,G) are controllable.

Now suppose that there exists a nxr linear transformation T such that

$$AT - TA_r = -BST \tag{4.5}$$

where A_r is the rxr state matrix of the reduced model

$$\dot{x}_r(t) = A_r x_r(t) + G_r w(t) \tag{4.6a}$$

$$y_r(t) = C_r x_r(t) \tag{4.6b}$$

where C_r and G_r are as in equation (4.2), and

$$A_r = A_m + B_r ST \tag{4.7}$$

Let P be a mxr constant matrix such that

$$ST = PC_r. \tag{4.8}$$

From equation (4.5), we have

$$(A+BS)T = TA_r \quad \text{or} \quad T^T(A+BS)T = T^T TA_r.$$

Thus, T has a left pseudo-inverse, denoted by

$$T^+ = (T^T T)^{-1} T^T \tag{4.9}$$

where the matrices T^T, $T^T T$ and T^+ have dimensions rxn, rxr and rxn respectively, and the following properties [22, 63] hold:

(1) $T^{+}T = I_r$

(2) $(TT^{+})^{T} = TT^{+}$, (orthogonality)

(3) $(TT^{+})^{2} = TT^{+}$, (idempotency)

(4) $T^{+}TT^{+} = T^{+}$

(5) $TT^{+}T = T$.

Using these properties, and the definition (4.9) we get

$$T^{+}(A+BS)T = A_r \tag{4.10}$$

and let $x_r(t) = T^{+}x(t)$. (4.11)

Using equation (4.11), the equivalence between the system (4.4) and the model (4.2) is satisfied provided that the conditions

(1) $AT - TA_r = -BPC_r$

(2) $G_r = T^{+}G$ (4.12)

(3) $x_r(0) = T^{+}x(0)$.

Note that the model matrix A_r obtained in (4.10) is an approximate solution to (4.4a) and it depends on the matrix T.

4.2. Properties of the Model Matrix

More often, it is convenient to select the dynamic structure of the model (4.2) to reflect a significant portion of the dynamic system (4.4) in an appropriate sense. In this section, we will outline the most important properties of the model matrix A_r.

Proposition 4.1 [125]: When an aggregation matrix T exists,

$$\partial(A_r) \subset \partial(A+BS) \tag{4.13}$$

where $\partial(A)$ denotes the spectrum of A.

We now state a theorem which gives a necessary and sufficient condition for equation (4.5) to have a unique solution for A_r.

Theorem 4.1 [117]: A necessary and sufficient condition for equation (4.5) to have a unique solution is that

$$\text{Range}\{T^T(A+BS)^T\} \subseteq \text{Range}\{T^T\} \tag{4.14}$$

If, in addition, rank $\{T\}$ = rank $\{(A+BS)T\}$ = r, then

$$\text{Null}\{T\} = \text{Null}\{(A+BS)T\} \tag{4.15}$$

Observe that equation (4.15) is satisfied, if (A+BS) is nonsingular and if all the conditions of the theorem are achieved, then equation (4.10) is the

desired solution. Note also that condition (4.15) implies that the pair $\{(A+BS),T\}$ is not controllable, since

$$\text{Null } \{T\} = \text{Null } \{(A+BS)T\}$$

and in general, Null $\{T\}$ = Null $\{(A+BS)^k T\}$ for any $k \geq 1$, which means that the matrix $\{T,(A+BS)T, \ldots, (A+BS)^{n-1}T\}$ has rank $r < n$ and so the pair $\{(A+BS),T\}$ is not controllable.

4.3. Computation of the Matrix T

Since $x_r(t) = T^+x(t)$, then let the n-dimensional row vectors of T^+ be $\{(t_i^+)\}$, $i = 1, \ldots, r$, thus we can express T^+ as

$$T^+ = [(t_1^+), (t_2^+), \ldots, (t_r^+)]^T$$

Then the ith component of $x_r(t)$ is $x_{r_i}(t)$ and is given by $(t_i^+)\, x(t)$ which means that the $x_{r_i}(t)$ is the weighted sum of those components of x(t). If we select the elements of T^+ so that T^+ has at most one entry in each column, then n components of x can be grouped into, at most, r separate clusters. Thus, the vectors $\{(t_i^+)\}$, for $1 \leq i \leq r$, are mutually orthogonal, which means that t^+ has maximal rank. This method constitutes a technique for determining the matrices T^+ and T because

$$T^{+} = (T^{T}T)^{-1}T^{T} \qquad \text{and} \qquad (T^{+})^{T} = T(T^{T}T)^{-1}$$

then

$$[T^{+}(T^{+})^{T}]^{-1} = T^{T}T \qquad \text{and} \qquad T = (T^{+})^{T}[T^{+}(T^{+})^{T}]^{-1}.$$

This technique consists of projecting the state vector $x(t)$ into an arbitrarily chosen r-dimensional subspace.

An alternative technique for determining the matrix T can be developed by considering the observability matrices of the system (4.4) and the model (4.6).

It has been shown in [22, 63] that a state feedback law of the form (4.3) preserves the observability of the system (4.1). Therefore, the pair (C,A+BS) is observable if the pair (C,A) is observable.

Define

$$R_s = \begin{bmatrix} C \\ C(A+BS) \\ . \\ . \\ . \\ C(A+BS)^{r-1} \end{bmatrix} \qquad \text{and} \qquad R_r = \begin{bmatrix} C_r \\ C_r A_r \\ . \\ . \\ . \\ C_r A_r^{n-1} \end{bmatrix}$$

where

$$(A+BS)T = TA_r$$

then

$$(A+BS)^2T = (A+BS)TA_r = TA_r^2.$$

In general

$$(A+BS)^iT = TA_r^i \quad \text{for} \quad i = 0, 1, 2, \ldots \tag{4.16}$$

and hence

$$CT = C_r$$

$$C(A+BS)\,T = CTA_r = C_rA_r$$

$$C(A+BS)^2T = C(A+BS)TA_r = CTA_r^2 = C_rA_r^2.$$

Thus

$$C(A+BS)^iT = C_rA_r^i \quad \text{for } i = 0, 1, 2, \ldots$$

and

$$R_sT = R_r.$$

Therefore

$$T = R_s^+R_r$$

where

$$R_s^+ = (R_s^TR_s)^{-1}R_s^T.$$

However, from the previous argument, if the pair $(C, A+B_1S)$ is observable, then $R_s^+ = R_s^{-1}$.

By specifying, $A_r = \text{diag}\{\lambda_1, \ldots, \lambda_r\}$ and choosing C_r so that the pair (C_r, A_r) is observable, then rank $R_r = r$.

A technique based on the selection of the matrices P and C_r such that the matrices A and A_r have no common eigenvalues and the pair (C_r, A_r) is observable, will now be introduced to compute the matrix T.

The necessary condition for the existence of a matrix T in $AT - TA_r = -BPC_r$, are that the pair (A,B) is controllable and the pair (C_r, A_r) is observable. Clearly from the Cayley Hamilton theorem $p(A) = 0$. Now, if λ_i is an eigenvalue of A_r, then $p(\lambda_i)$ is an eigenvalue of $p(A_r)$ and because A and A_r have no common eigenvalues, then $p(\lambda_i) \neq 0$ for all λ_i of A_r and $\det p(A_r) \neq 0$. Hence, $p(A_r)$ is nonsingular.

The substitution of $TA_r = AT + BPC_r$ into $A^2T - TA_r^2$ gives

$$A^2T - TA_r^2 = A^2T-(AT+BPC_r)A_r = A^2T-ATA_r-BPC_rA_r$$

$$= A(AT-TA_r) - BPC_rA_r = A(-B_1PC_r)-BPC_rA_r$$

$$= ABPC_r-BPC_rA_r.$$

Proceeding in a similar manner, we get the following equations:

$$IT - TI = 0$$

$$AT - TA_r = -BPC_r$$

$$A^2T - TA_r^2 = -ABPC_r - BPC_rA_r \qquad (4.17)$$

$$\vdots$$

$$A^nT - TA_r^n = -\sum_{j=0}^{n-1} A^{n-1-j}BPC_rA_r^j.$$

Now, multiplying the first equation by α_n, the second by α_{n-1}, . . ., the last by 1 and then sum them up, let

$$q(s) = \det(sI-A_r) = s^r + \xi_1 s^{r-1} + \ldots + \xi_r$$

then

$$q(A_r) = A_r^r + \xi_1 A_r^{r-1} + \ldots + \xi_r I_r = 0$$

and let the (i,j)th element of the (n-r+1)xr constant matrix V be defined as

$$v_{1j} = -\xi_j$$

$$v_{ij} = v_{i-1,1}v_{1j} + v_{i-1,j+1}$$

for $i = 2, \ldots, n-r+1, \quad j = 1, \ldots, r.$

Define the (n-r+1)xr constant matrix u as follows:

$$U = VJ$$

where

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{rxr}$$

such that

$$\begin{bmatrix} A_r^r \\ A_r^{r+1} \\ \vdots \\ A_r^n \end{bmatrix} = U \begin{bmatrix} I \\ A_r \\ \vdots \\ A_r^{r-1} \end{bmatrix}. \qquad (4.19)$$

Now let

$$R_B = [B, AB, \ldots, A^{n-1}B]$$

$$R_C = [C_r^T, A_r^T C_r^T, \ldots, (A_r^{r-1})^T C_r^T]^T$$

and

$$L = \begin{bmatrix} \alpha_{n-1}I_r & \cdots & \alpha_1 I_r & I_r \\ \alpha_{n-2}I_r & \cdots & I_r & 0 \\ \vdots & & \vdots & \vdots \\ \alpha_1 I_r & \cdots & 0 & 0 \\ I_r & \cdots & 0 & 0 \end{bmatrix}.$$

After some algebraic manipulations, we have

$$p(A)T - Tp(A_r) = -R_B P L \begin{bmatrix} I \\ \cdots \\ U \end{bmatrix} R_C \tag{4.20}$$

Noting that $p(A) = 0$, and $\det\{p(A_r)\} \neq 0$, we arrive at

$$T = R_B P L \begin{bmatrix} I \\ \cdots \\ U \end{bmatrix} R_C (p(A_r))^{-1}.$$

Since

$$p(A_r) = A_r^n + \alpha_1 A_r^{n-1} + \cdots + \alpha_{n-r} A_r^r + \cdots + \alpha_{n-1} A_r^r + \alpha_n I_r$$

we may write

$$p(A_r) = [\alpha_{n-r} I_r, \ldots \alpha_1 I_r, I_r] \begin{bmatrix} A_r^r \\ \cdot \\ \cdot \\ \cdot \\ A_r^n \end{bmatrix} +$$

$$+ [\alpha_{n-r+1} I_r, \ldots, \alpha_n I_r] \begin{bmatrix} A_r^{r-1} \\ \cdot \\ \cdot \\ \cdot \\ I_r \end{bmatrix} .$$

We then have

$$p(A_r) = [\alpha_{n-r} I_r, \ldots, \alpha_1 I_r, I_r] V \begin{bmatrix} A_r^{r-1} \\ \cdot \\ \cdot \\ \cdot \\ I_r \end{bmatrix} +$$

$$+ [\alpha_{n-r+1} I_r, \ldots, \alpha_n I_r] \begin{bmatrix} A_r^{r-1} \\ \cdot \\ \cdot \\ \cdot \\ I_r \end{bmatrix} .$$

$$p(A_r) = \{[\alpha_{n-r}I_r, \ldots, \alpha_1 I_r]V + [\alpha_{n-r+1}I_r + \ldots, \alpha_n I_r]\} \begin{bmatrix} A_r^{r-1} \\ \cdot \\ \cdot \\ \cdot \\ I_r \end{bmatrix}$$

or

$$p(A_r) = [\eta_1 I_r, \ldots, \eta_r I_r] \begin{bmatrix} A_r^{r-1} \\ \cdot \\ \cdot \\ \cdot \\ I_r \end{bmatrix} \tag{4.21}$$

where

$$\eta_j = \sum_{\ell=1}^{n-r+1} \alpha_{n-r+1-\ell} v_{\ell j} + \alpha_{n-r+j}, \tag{4.22}$$

for $j = 1, \ldots, r$ and $\alpha_0 = 1$

and

$$p(A_r) = \eta_1 A_r^{r-1} + \ldots + \eta_n I_r$$

Therefore, if all the α_i's are different from zero, then the η_i's are also different from zero and $(p(A_r))^{-1}$ exists.

Note that since $x_r(t)$ has lower dimension than $x(t)$, the system description given in (4.6) is nonminimal. This situation occurs when there are (n-r) pole zero cancellations. The aggregation matrix considered here is restricted to those creating zeros in the input/output relationship that cancels (n-r) poles of the transfer function. There cancelled poles are precisely the eigenvalues of (A+BS), that are not retained in A_r. Let the aggregation error e(t) be defined as

$$e(t) = x(t) - Tx_r(t) \tag{4.23}$$

Thus, from equations (4.2) and (4.6), the dynamics of the error are given by

$$\dot{e}(t) = (A+BS)e(t) + [TA_r - (A+BS)x]x_r(t) \tag{4.24}$$

This equation is obtained from the differentiation of equation (4.23) and the substitution of equations (4.2) and (4.6) in the resulting equation, that is

$$\begin{aligned}\dot{e}(t) &= \dot{x}(t) - T\dot{x}_r(t) \\ &= (A+BS)x(t) + Gw(t) - T[A_r x_r(t) + G_r w(t)] \\ &= (A+BS)x(t) + Gw(t) - TA_r x_r(t) - TG_r w(t)\end{aligned}$$

If $G_r = T^+G$ or $G = TG_r$

then

$$\dot{e}(t) = (A+BS)x(t) - TA_r x_r(t)$$

$$= (A+BS)x(t) - (A+BS)Tx_r(t) - TA_r x_r(t) + (A+BS)Tx_r(t)$$

$$= (A+BS)[x(t)-Tx_r(t)] + [(A+BS)T - TA_r]x_r(t)$$

$$= (A+BS)e(t) + [(A+BS)T - TA_r]x_r(t)$$

This equation yields $\dot{e}(t) = (A+BS)e(t)$ provided that $(A+BS)T = TA_r$ is satisfied, then $e(0) = 0$ and the case of perfect aggregation $\dot{e}(t) = (A+BS)e(t)$ implies that $e(t) = 0$ for all $t \geq 0$. However, if $x_r(0) \neq T^+x(0)$ since S is chosen so that $(A+BS)$ is asymptotically stable, then $e(t) = 0$ as t goes to infinity.

4.4. Steady State Covariance Matrix and Markov Parameters Matching

In this section, we will show that the model steady state covariance and Markov parameters are automatically matched to those of the system.

Let the transfer function of the system (4.1) be represented as follows:

$$H(s) = [H_1(s), H_2(s)] \qquad (4.25)$$

where

$$H_1(s) = C(sI-A)^{-1}B \quad \text{and} \quad H_2(s) = C(sI-A)^{-1}H.$$

The expansions of $H_1(s)$ and $H_2(s)$ in power series about infinity are

$$H_1(s) = CBS^{-1} + CABs^{-2} + \ldots + CA^iBs^{-(i+1)} + \ldots$$

and

$$H_2(s) = CGs^{-1} + CAGs^{-2} + \ldots + CA^iGs^{-(i+1)} + \ldots$$

Let $H_{i1} = CA^{i-1}B$ and $H_{i2} = CA^{i-1}G$. The Markov parameters of the system (4.1) are given by

$$H_i = [H_{i1}, H_{i2}] \quad \text{for } i = 1, 2, \ldots \tag{4.26}$$

and

$$H(s) = \sum_{i=1}^{\infty} H_i s^{-i} = \sum_{i=1}^{\infty} [H_{i1}, H_{i2}]s^{-i}.$$

To determine the steady state covariance matrix parameters of the system (4.1), let

$$D = [B, G] \quad \text{and} \quad Q = \begin{bmatrix} I & O \\ O & W \end{bmatrix}$$

then the Lyapunov matrix for the system (4.1) is

$$K_1 = \int_0^{\infty} e^{At} DQD^T e^{A^T t} dt \tag{4.28}$$

$$= \int_0^{\infty} e^{At} BB^T e^{A^T t} dt + \int_0^{\infty} e^{At} GWG^T e^{A^T t} dt.$$

Using familiar techniques, we have

$$AK_1 + K_1A^T + DQD^T = 0 \tag{4.29}$$

or

$$AK_1 + K_1A^T + BB_1^T + GWG^T = 0$$

From equation (3.13), the steady state covariance parameters for the system (4.1) are

$$R_i = CA^iK_1C^T \quad \text{for } i = 0, 1, 2, \ldots \tag{4.30}$$

Similarly, the Lyapunov matrix for the system (4.4) is given by

$$K_2 = \int_0^\infty e^{(A+BS)t}GWG^Te^{(A+BS)^Tt}dt \tag{4.31}$$

and satisfies

$$(A+BS)K_2 + K_2(A+BS)^T + GWG^T = 0 \tag{4.32}$$

or

$$AK_2 + K_2A^T + BSK_2 + K_2S^TB^T + GWG^T = 0.$$

The steady state covariance parameters for the system (4.4) are then computed by

$$\bar{R}_i = C(A+BS)^iK_2C^T. \tag{4.33}$$

Let $C = C_rT^+$ or $C_r = CT$, then the condition (4.8) becomes

$$S = PC. \tag{4.34}$$

Now we can state the following proposition.

Proposition 4.2: The steady state covariance parameters of the system (4.4) can be formulated as follows:

$$\bar{R}_i = CA^iK_2C^T + \sum_{j=0}^{i-1} H_{i-j,1}P\bar{R}_j \quad \text{for } i = 0, 1, \ldots \tag{4.35}$$

and $H_{\ell,1} = 0$ for $\ell \leq 0$.

Proof: Since $\bar{R}_i = C(A+BS)^iK_2C^T$ and $S = PC$, then

$$\bar{R} = CK_2C^T$$

$$\bar{R}_0 = CK_2C^T$$

$$\bar{R}_1 = C(A+BS)K_2C^T = CAK_2C^T + CBSK_2C^T$$

$$= CAK_2C^T + H_{11}PCK_2C^T = CAK_2C^T + H_{11}P\bar{R}_0$$

$$\bar{R}_2 = C(A+BS)^2K_2C^T = CA^2K_2C^T + CABSK_2C^T +$$

$$CBSAK_2C^T + CBSBSK_2C^T$$

$$= CA^2K_2C^T + H_{21}PCJ_2C^T + H_{11}PCAK_2C^T + H_{11}PCBSK_2C^T$$

$$= CA^2K_2C^T + H_{21}P\bar{R}_0 + H_{11}PC(A+BS)K_2C^T$$

$$= CA^2K_2C^T + H_{21}P\bar{R}_0 + H_{11}P\bar{R}_1.$$

Proceeding in a similar manner, we have

$$\bar{R}_i = CA^iK_2C^T + \sum_{j=0}^{i-1} H_{i-j,1}P\bar{R}_j.$$

Let

$$K_3 = \int_0^\infty e^{A_r t}G_r W G_r^T a A_r^T t dt \tag{4.36}$$

be the state covariance matrix of the model (4.6), such that

$$A_rK_3 + K_3A_r^T + G_rWG_r^T = 0. \tag{4.37}$$

The steady state covariance parameters of the model (4.6) are then equal to

$$\bar{\bar{R}}_i = C_rA_r^iK_3C_r^T. \tag{4.38}$$

Note that $G_r = T^+G$ and

$$T^+(A+BS) = A_rT^+ \tag{4.39}$$

since $T^+T = I_r$, then the post-multiplication of the equation (4.39) by T, yields the expression given by equation (4.9). Thus

$$T^+(A+BS)^i = A_r^iT^+ \quad \text{for } i = 0, 1, 2, \ldots \tag{4.40}$$

Lemma 4.2: The expressions (4.32) and (4.37) are equivalent if and only if $K_3 = T^+K_2(T^+)^T$. (4.41)

Proof: The proof of this Lemma consists of two parts.

(a) Necessary part: If the expressions (4.32) and (4.37) are equivalent, then the pre- and post-multiplication of equation (4.31) by T^+ and $(T^+)^T$ respectively yield

$$T^+(A+BS)K_2(T^+)^T + T^+K_2(A+BS)^T(T^+) + T^+GWG^T(T^+)^T = 0.$$

Using equation (4.38) and $G_r = T^+G$, we get

$$A_rT^+K_2(T^+)^T + T^+K_2(T^+)^TA_r^T + G_rWG_r^T = 0.$$

Subtracting this equation from the equation (4.32), we have

$$A_r[K_3 - T^+K_2(T^+)^T] + [K_3 - T^+K_2(T^+)^T]A_r^T = 0$$

The pre- and post-multiplication of this equation by e^{A_rt} and $e^{A_r^Tt}$ respectively leads to

$$e^{A_rt}\{A_r[K_3-T^+K_2(T^+)^T] + [K_3-T^+K_2(T^+)^T]A_r^T\}e^{A_r^Tt} = 0.$$

This equation can also be written as follows:

$$\frac{d}{dt}\{e^{A_rt}[K_3 - T^+K_2(T^+)^T]e^{A_r^Tt}\} = 0$$

which implies that $e^{A_rt}[K_3 - T^+K_2(T^+)^T]e^{A_r^Tt}$ is a constant for all t. In particular, the integration of the differential equation from t = 0 to t =ξ gives:

$$K_3 - T^+K_2(T^+)^T = e^{A_r\xi}[K_4 - T^+K_2(T^+)^T]e^{A_r^T\xi}.$$

Now letting $\xi \to \infty$ and using the fact that A_r is a stable matrix, we obtain

$$K_3 - T^+K_2(T^+)^T = 0 \quad \text{or} \quad K_3 = T^+K_2(T^+)^T.$$

(b) Sufficient part: If $K_3 = T^+K_2(T^+)^T$, then the substitution of equation (4.41) into equation (4.37) gives

$$A_rT^+K_2(T^+)^T + T^+K_2(T^+)^TA_r^T + G_rWG_r^T = 0.$$

Using equation (4.38) and $G_r = T^+G$, we have

$$T^+(A+BS)K_2(T^+)^T + T^+K_2(A+BS)^T(T^+)^T + T^+GWG^T(T^+)^T = 0.$$

Thus,

$$T^+[(A+BS)K_2 + K_2(A+BS)^T + GWG^T](T^+)^T = 0$$

or

$$(A+BS)K_2 + K_2(A+BS)^T + GWG^T = 0.$$

Therefore, the equations (4.32) and (4.37) are equivalent.

Using expressions (4.40) and (4.41), we can relate the steady state covariance parameters of the model (4.6) to those of the system (4.4) as follows.

Proposition 4.3: If $K_3 = T^+K_2(T^+)^T$, then

$$\bar{R}_i = \bar{\bar{R}}_i \quad \text{for } i = 0, 1, 2, \ldots . \tag{4.42}$$

Proof: Since the steady state covariance parameters of the system (4.4) are given by $\bar{R}_i = C(A+BS)^iK_2C^T$, then,

using the equations $C = C_r T^+$ and (4.40), we have

$$\bar{R}_i = C_r T^+ (A+BS)^i K_2 (T^+)^t C_r^T = C_r A_r^i T + K_2 (T^+)^T C_r^T.$$

From equations (4.41) and (4.38), we have

$$\bar{R}_i = C_r A_r^i K_3 C_r^T = \bar{\bar{R}}_i.$$

The Markov parameters of the system (4.4) are determined by the following equation

$$\bar{H}_i = C(A+BS)^{i-1} G. \tag{4.43}$$

Proposition 4.4: $\bar{H}_i$ can be expressed in terms of the Markov parameters (H_{i1}, H_{i2}) of the system (4.1), as follows:

$$\bar{\bar{H}}_i = H_{i2} + \sum_{j=1}^{i-1} H_{i-j,1} P \bar{H}_j \qquad \text{for } i = 1, 2, \ldots \tag{4.44}$$

and $H_{\ell 1} = 0 \quad$ for $\ell \leq 0$.

Proof: Since $H_{i1} = CA^{i-1}B$, $H_{i2} = CA^{i-1}G$ and $S = PC$, then

$$\bar{H}_1 = CG = H_{12}$$

$$\bar{H}_2 = C(A+BS)G = CAG + CBSG = H_{22} + H_{11}PCG$$

$$= H_{22} + H_{11}PH_{12} = H_{22} + H_{11}P\bar{H}_1$$

$$\bar{H}_3 = C(A+BS)^2G + CA^2G + CABSG + CBSAG + CBSBSG$$

$$= CA^2G + CABPCG + CBPC(A+BS)G$$

$$= H_{32} + H_{21}P\bar{H}_1 + H_{11}P\bar{H}_2.$$

Proceeding in a similar fashion, we get

$$\bar{\bar{H}}_i = H_{i2} + \sum_{j=1}^{i-1} \tilde{H}_{i-j,1}P\bar{H}_j$$

Similarly, let the Markov parameters of the model (4.6) be defined as

$$\bar{\bar{H}}_i = C_r A_r^{i-1} G_r \qquad \text{for } i = 1, 2, \ldots \tag{4.45}$$

Proposition 4.5: The equations (4.44) and (4.45) are equivalent and $\bar{\bar{H}}_i = \bar{H}_i$ for $i = 1, 2, \ldots$

Proof: From equation (4.17), we have

$$TA_r^i = A^iT + \sum_{j=0}^{i-1} A^{i-j-1}BPC_rA_r^j$$

then

$$A_r^i = T^+A^iT + \sum_{j=0}^{i-1} T^+A^{i-j-1}BPC_rA_r^j. \tag{4.46}$$

Replacing i by i-1 gives

$$A_r^{i-1} = T^+A^{i-1}T + \sum_{j=0}^{i-2} T^+A^{i-j-2}BPC_rA_r^j .$$

Substituting j-1 instead of j yields

$$A_r^{i-1} = T^+A^{i-1}T + \sum_{j=1}^{i-1} T^+A^{i-j-1}BPC_rA_r^{j-1}. \qquad (4.47)$$

Pre- and post-multiplying equation (4.47) by C_r and G_r respectively, we get

$$C_rA_r^{i-1}G_r = C_rT^+A^{i-1}TG_r + \sum_{j=1}^{i-1} C_rT^+A^{i-j-1}BPC_rA_r^{j-1}G_r.$$

Using the expressions $C = C_rT^+$ and $G = TG_r$, we have

$$\bar{\bar{H}}_i = CA^{i-1}G + \sum_{j=1}^{i-1} CA^{i-j-1}BPC_rA_r^{j-1}G_r$$

or

$$\bar{\bar{H}}_i = H_{i2} + \sum_{j=1}^{i-1} H_{i-j,1}P\bar{H}_j.$$

This expression is equal to the one given by equation (4.44). Therefore, $\bar{\bar{H}}_i = \bar{H}_i$.

Let the transfer function of the model (4.2) be

$$\tilde{H}(s) = [\tilde{H}_1(s), \tilde{H}_2(s)] \qquad (4.48)$$

where

$$\tilde{H}_1(s) = C_r(sI_r - A_m)^{-1}B_r$$

and

$$\tilde{H}_2(s) = C_r(sI_r - A_m)^{-1}G_r.$$

The power series expansions of $\tilde{H}_1(s)$ and $\tilde{H}_2(s)$ about infinity result in the following expressions:

$$\tilde{H}_1(s) = C_rB_r s^{-1} + C_rA_mB_r s^{-2} + \ldots + C_rA_m^{i-1}B_r s^{-i} + \ldots$$

$$\tilde{H}_2(s) = C_rG_r s^{-1} + C_rA_mG_r s^{-2} + \ldots + C_rA_m^{i-1}G_r s^{-i} + \ldots$$

Now let

$$\tilde{H}_{i1} = C_rA_m^{i-1}B_r \quad \text{and} \quad \tilde{H}_{i2} = C_rA_m^{i-1}G_r. \tag{4.49}$$

The Markov parameters of the model (4.2) are thus given by

$$\tilde{H}_i = [\tilde{H}_{i1}, \tilde{H}_{i2}] \quad \text{for } i = 1, 2, \ldots \tag{4.50}$$

and

$$\tilde{H}(s) = \sum_{i=1}^{\infty} \tilde{H}_i s^{-i} = \sum_{i=1}^{\infty} [\tilde{H}_{i1}, \tilde{H}_{i2}] s^{-i}. \tag{4.51}$$

Using equation (4.7), we can derive an expression similar to the one given by equation (4.17) that is

$$A_r^i = A_m^i + \sum_{j=0}^{i-1} A_m^{i-j-1} B_r P C_r A_r^j. \tag{4.52}$$

Furthermore, from equations (4.5), (4.7), and (4.34), we have

$$(A+BS)T = TA_r, \qquad A_r = A_m + B_r PC_r$$

$$ST = PC_r \quad \text{and} \quad B = TB_r.$$

Then $AT + BST = T(A_m + B_r PC_r) = TA_m + TB_r PC_r$

$$= TA_m + BST$$

and $AT = TA_m$ so

$$A^i T = TA_m^i \quad \text{and} \quad T^+ A^i T = A_m^i \quad \text{for } i = 0, 1, 2, \ldots \tag{4.53}$$

We now establish a link between the Markov parameters of the model (4.2) and those of the original system (4.1).

Proposition 4.6: $\tilde{H}_i = H_i$ for $i = 0, 1, 2, \ldots$ (4.54)

Proof: From equation (4.49), the Markov parameters of the model (4.6) are given as: $\tilde{H}_i = [\tilde{H}_{i1}, \tilde{H}_{i2}]$ where $\tilde{H}_{i1} = C_r A_m^{i-1} B_r$ and $\tilde{H}_{i2} = C_r A_m^{i-1} G_r$. Since $C_r T^+ = C$, $TB_r = B$ and $TG_r = G$, using equation (4.53), we obtain $\tilde{H}_{i1} = C_r T^+ A^{i-1} TB_r = CA^{i-1}B$ and $\tilde{H}_{i2} = C_r T^+ A^{i-1} TG_r = CA^{i-1}G$. Thus, equation (4.26) leads to $\tilde{H}_{i1} = H_{i1}$ and $\tilde{H}_{i2} = H_{i2}$ and $\tilde{H}_i = [\tilde{H}_{i1}, \tilde{H}_{i2}] = [H_{i1}, H_{i2}] = H_i$.

Therefore $\tilde{H}_i = H_i$ for $i = 0, 1, 2, \ldots$

To determine the steady state covariance parameters of the model (4.6), let

$$D_r = [B_r,\ G_r] \quad \text{and} \quad Q = \begin{bmatrix} I & O \\ O & W \end{bmatrix}.$$

The Lyapunov matrix is then given by:

$$K_4 = \int_0^\infty e^{A_m T} D_r Q D_r^T e^{A_m^T t} dt$$

$$= \int_0^\infty e^{A_m t} B_r B_r^T e^{A_m^T t} dt + \int_0^\infty e^{A_m t} G_r W G_{r2}^T e^{A_m^T t} dt$$

and satisfies

$$A_m K_4 + K_4 A_m^T + D_r Q D_r^T = 0 \tag{4.55}$$

$$A_m K_4 + K_4 A_m^T + B_r B_r^T + G_r W G_r^T = 0.$$

The steady state covariance parameters are

$$\tilde{R}_i = C_r A_m^i K_4 C_r^T . \tag{4.56}$$

Using the equations (4.52) and (4.53) and the property $T^+T = I_r$, we have $T^+A = A_m T^+$ and $T^+A^i = A_m^i T^+$ for $i = 0, 1, 2, \ldots$ The following Lemma determines the link between the Lyapunov equation of the model (4.2) and that of the original system (4.1). (4.57)

Lemma 4.3: The expressions (4.29) and (4.55) are equivalent if and only if $K_4 = T^+ K_1 (T^+)^T$. (4.58)

The proof of the Lemma 4.3 is similar to the proof of Lemma 4.2, and therefore, it is omitted here.

Using the expressions (4.57) and (4.58) we can relate the steady state covariance parameters of the model (4.2) to those of the original system (4.1).

Proposition 4.7: If $K_4 = T^+ K_1 (T^+)^T$, then

$$\tilde{R}_i = R_i \quad \text{for} \quad i = 0, 1, 2, \ldots \tag{4.59}$$

The arguments for proof of this proposition are similar to those given in the proof of Proposition 4.3.

Up to this point, we have seen two ways to derive the reduced order model:

(a) by applying the aggregation method directly to the original high order system, and

(b) by using state feedback and then applying the aggregation to the resulting system.

The first method is straightforward, but it requires more computation to determine the Lyapunov matrix. By using the state feedback concept, the second method ensures the stability of the reduced model. Note that if B_r or P are equal to zero, the two methods are similar. From now on, we will use the second method.

4.5. Selection of the Aggregation Matrix

In a comparative study done by Hyland et al. [62] it was shown that the major differences among Moore [93]

Skelton [126] and Wilson [135] methods are in the definitions of the aggregation matrix and its pseudo-inverse.

From equation (4.41), we have

$$K_3 = T^{+}K_2(T^{+})^T$$

where K_2 and K_3 are real symmetric positive definite matrices. Their singular value decompositions [22] are

$$K_2 = U_2 D_2 U_2^T \tag{4.60}$$

and

$$K_3 = U_3 D_3 U_3^T \tag{4.61}$$

where D_2 and D_3 are nxn and rxr diagonal matrices and U_2 and U_3 are nxn and rxr orthogonal matrices.

$$U_2U_2^T = U_2^TU_2 = I_n \quad \text{and} \quad U_3U_3^T = U_3^TU_3 = I_r.$$

Then, the pre- and post-multipliation of equation (4.60) by U_2^T and U_2 respectively yields

$$D_2 = U_2^TK_2U_2 \tag{4.62}$$

$$= \text{diag}\{\lambda_1, \lambda_2, \ldots \lambda_n\}$$

where the λ_i's are the singular values of K_2 and satisfy

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0.$$

Repeating the previous operations with equation (4.61) U_3^T and U_3, we obtain

$$D_3 = U_3 K_3 U_3^T \tag{4.63}$$

$$= \text{diag}\{\mu_1, \mu_2, \ldots \mu_r\}$$

where the μ_i's are the singular values of K_3 and satisfy

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r \geq 0.$$

Since the controllability Grammian K_2 determines the effects of the system inputs on the system state variables, then assume that $\lambda_r >> \lambda_{r+1}$ and $\lambda_{r+1} \geq \lambda_{r+2} \geq \cdots \geq \lambda_n$.

Let L be a rxn matrix defined as follows:

$$L = [I_r : 0] \tag{4.64}$$

such that $LL^T = I_r$. Let

$$D_3 = \text{diag}\{\lambda_1, \ldots, \lambda_r\}$$

that is $\mu_i = \lambda_i$ for $i = 1, 2, \ldots, r$

then

$$D_3 = LD_2L^T. \tag{4.65}$$

The substitution of equation (4.65) into (4.63) results in

$$D_3 = LU_2^T K_2 U_2 L^T \tag{4.66}$$

and the substitution of equation (4.66) into (4.61) leads to

$$K_3 = U_3 L U_2^T K_2 U_2 L^T U_3^T. \tag{4.67}$$

By comparing equations (4.41) and (4.66), we have

$$T^+ = U_3 L U_2^T.$$

Since

$$T = (T^+)^T [T^+ (T^+)^T]^{-1}$$

it follows that

$$T = U_2 L^T U_3^T. \tag{4.68}$$

T satisfies all the properties stated in section 4.2.

To determine the orthogonal matrix U_3, let

$$K_r = L K_2 L^T \tag{4.69}$$

then K_r is a real symmetric positive definite matrix, because it is the rxr upper left submatrix of K_2. It can be represented [22, 119] as follows:

$$K_r = U_r D_r U_r^T \tag{4.70}$$

Letting $U_3 = U_r$, equation (4.67) becomes

$$T = U_r L^T U_3^T. \tag{4.71}$$

4.5.1. Observations

(a) Let K_{c2} and K_{o2} be the system controllability and observability Grammians, and let K_{c3} and K_{o3} be the model controllability and observability Grammians. Then using equations (3.59), (3.60) and (4.41) we have

$$K_{c3} = T^{+}K_{c2}(T^{+})^{T} \tag{4.72}$$

$$K_{o3} = T^{T}K_{o2}T. \tag{4.73}$$

The real symmetric positive definite matrices, K_{c2}, K_{o2}, K_{c3} and K_{o3} can be written [22] as

$$K_{c2} = U_{c2}D_{c2}U_{c2}^{T}$$

$$K_{02} = U_{o2}D_{o2}U_{o2}^{T}$$

$$K_{c3} = U_{c3}D_{c3}U_{c3}^{T}$$

$$K_{03} = u_{o3}D_{o3}U_{o3}^{T}$$

where U_{c2}, U_{o2}, U_{c3} and U_{o3} are orthogonal matrices with appropriate dimensions, and D_{c2}, D_{o2}, D_{c3} and D_{o3} are diagonal matrices.

Define

$$H_r = D_{o2}^{1/2}U_{o2}^{T}U_{o2}D_{c2}^{1/2\ T}$$

then the singular value decomposition of H_r is

$$H_r = V_r\Sigma_r W_r$$

such that $V_r^T V_r = I_n$ and $W_r W_r^T = I_r$.

if T (respectively T^+) is chosen to be:

(1) $T = U_{c2} D_{c2}^{1/2} L^T W_r^T$, $(T^+ = W_r L D_{c2}^{-1/2} U_{c2}^T)$

it follows that $K_{c3} = I_r$ and $K_{o3} \neq I_r$. The model is then said to be input-normal.

(2) $T = U_{o2} D_{o2}^{-1/2} V_r L^T$, $(T^+ = L V_r^T D_{o2}^{1/2} U_{o2}^T)$

it follows that $K_{c3} \neq I_r$ and $K_{03} = I_r$. The model is then said to be output-normal.

(b) We now review the ways to determine the state feedback matrix S of equation (4.3) or the matrix P of equation (4.34).

Recall $p(s) = \det(sI-A)$

$$= s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n.$$

Now let $A_f = A+BS$, and following equation (4.34), set S=PC, we then have

$$p_f(s) = \det(sI-A_f)$$

$$= s^n + \bar{\alpha}_1 s^{n-1} + \ldots + \bar{\alpha}_n.$$

We consider separately the single variate and multivariate cases.

(1) Single input single output system. In this case P is a scalar and $p_f(s)$ is given by

$$p_f(s) = \det(sI-A-BPC) = \det[(sI-A)-BPC]$$

$$= \det\{(sI-A)[I-(sI-A)^{-1}BPC]\}$$

$$= \det(sI-A)[1-PC(sI-A)^{-1}B]$$

$$= p(s)-PCadj(sI-A)B$$

where

$$adj(sI-A) = F_1 s^{n-1} + F_2 s^{n-2} + \ldots + F_n$$

and

$$F_1 = I$$

$$F_{i+1} = AF_i + \alpha_i I$$

$$\alpha_i = -\frac{1}{i}tr(AF_i) \quad \text{for} \quad i = 1, \ldots, n$$

$$F_{n+1} = 0.$$

This is the so-called Faddeva algorithm [22, 63].

By subtraction, we have

$$p_f(s) - p(s) = -P[CF_1Bs^{n-1} + CF_2Bs^{n-2} + \ldots + CF_nB]$$

so

$$[(\bar{\alpha}_1-\alpha_1),\ldots,(\bar{\alpha}_n-\alpha_n)] \begin{bmatrix} s^{n-1} \\ \vdots \\ 1 \end{bmatrix} = -P[CF_1B,\ldots,CF_nB] \begin{bmatrix} s^{n-1} \\ \vdots \\ 1 \end{bmatrix}$$

or

$$[(\bar{\alpha}_1-\alpha),\ldots,(\bar{\alpha}_n-\alpha_n)] = -P[CF_1B,\ldots,CF_nB].$$

In this case, P is chosen so that the roots of $p_f(s)$ are in the desired locations.

(2) Multi-input, multi-output system. In order to investigate this case, we will review the following [22, 63]:

Definition 4.1: A matrix A_f is called cyclic if its characteristic polynomial is equal to its minimal polynomial.

Definition 4.2: A_f is cyclic if and only if the Jordan canonical form of A_f has one and only one Jordan block associated with each distinct eigenvalue.

Lemma 4.4: If {A,B,C} is irreducible, then for almost any mxr real constant matrix P, the eigenvalues of A+BPC are distinct and consequently A_f = A+BPC is cyclic. Since A+BPC is cyclic, then the matrix P can be written as the product of two vectors and the single input single output technique is used to compute these vectors.

4.6. Illustrative Example

To illustrate this technique, the following example was contrived.

Recall system (4.1), namely

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t)$$

$$y(t) = Cx(t)$$

Let us choose

$$A = \begin{bmatrix} -1.0 & 0.0 & 0.01 & 0.05 & 0.25 \\ 0.0 & -4.0 & 0.0 & 0.45 & 0.1 \\ -0.088 & 0.2 & -5.0 & 0.0 & 0.22 \\ 1.0 & 0.0 & 0.075 & -4.0 & 0.05 \\ 0.11 & 0.2 & 1.0 & 0.44 & -3.0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1.0 \\ 0.0 \\ 0.5 \\ 2.0 \\ 1.0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1.5 \\ 1.0 \\ 1.05 \\ 2.65 \\ 1.0 \end{bmatrix}$$

$$C = [-0.61249, -0.33522, 2.18751, 0.2333, -0.24787].$$

The pairs (A,B) and (A,G) are controllable and the pair (C,A) is observable. The system is stable, its characteristic polynomial is

$$p(s) = s^5+17s^4+110.66138s^3+339.52548s^2+476.62259s+228.38203$$

and its eigenvalues are

$$\lambda_1 = -5.09152$$
$$\lambda_2 = -3.92956$$
$$\lambda_3 = -4.11341$$
$$\lambda_4 = -2.91282$$
$$\lambda_5 = -0.9527$$

Following the development of page 81, we take $ST = PC_r$, or

$$S = PC,$$

where S, C are 1x5 vectors and P is scalar used to allocate the eigenvalues of the system with state matrix $A_f = A+BS$.

Let P = 1, then S = C and

$$A_f = \begin{bmatrix} -1.61249 & -0.33522 & 2.19751 & 0.2833 & 0.00213 \\ 0.0 & -4.0 & 0.0 & 0.45 & 0.1 \\ -0.39425 & 0.03239 & -3.90624 & 0.11665 & 0.09606 \\ -0.22499 & -0.67044 & 4.45002 & -3.5334 & -0.44574 \\ -0.50249 & -0.13522 & 3.18751 & 0.6733 & -3.24787 \end{bmatrix}.$$

The characteristic polynomial of A_f is

$$p_f(s) = s^5+16.3s^4+105.12s^3+335.5s^2+531.00563+334.86469$$

The eigenvalues are

$$\lambda_1 = -4.05-j0.15$$

$$\lambda_2 = -4.05+j0.15$$

$$\lambda_3 = -3.5$$

$$\lambda_4 = -2.35-j0.55$$

$$\lambda_5 = -2.35+j0.55\,.$$

Therefore, the system

$$\dot{x}(t) = A_f x(t) + Gw(t)$$

$$y(t) = Cx(t)$$

is stable.

The pair (A_f,G) is controllable and the pair (C,A_f) is observable. The real symmetric positive definite matrix K_2 satisfying the Lyapunov equation

$$A_f K_2 + K_2 A_f^T + GWG^T = 0, \quad \text{with} \quad W = I_n$$

is given by

$$K_2 = \begin{bmatrix} 1.20249 & 0.4340 & 0.29809 & 1.07033 & 0.62757 \\ 0.4349 & 0.18513 & 0.14681 & 0.47672 & 0.25996 \\ 0.29809 & 0.14681 & 0.12882 & 0.39028 & 0.19978 \\ 1.07093 & 0.47672 & 0.39028 & 1.24231 & 0.66828 \\ 0.62757 & 0.25996 & 0.19978 & 0.66828 & 0.38064 \end{bmatrix}$$

or $K_2 = U_2 D_2 U_2^T$

where

$$U_2 = \begin{bmatrix} 0.61084 & -0.755 & 0.19918 & -0.00083 & 0.13105 \\ 0.24899 & 0.10496 & 0.14724 & -0.54041 & -0.78311 \\ 0.19043 & 0.34083 & 0.31214 & -0.62695 & 0.59756 \\ 0.63486 & 0.54632 & 0.16096 & 0.51939 & -0.05308 \\ 0.35436 & 0.06578 & -0.90294 & -0.21245 & 0.09832 \end{bmatrix}$$

and

$D_2 = \text{diag}\{2.94978, 0.17786, 0.01162, 0.00013, 0.000001\}$.

Now let

$$P = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}$$

Using equations (4.67), (4.68) and (4.70), we have

$$T = \begin{bmatrix} 0.88016 & -0.31726 & -0.32787 \\ 0.20689 & 0.01718 & 0.22714 \\ 0.08593 & -0.02155 & 0.49195 \\ 0.38626 & 0.35146 & 0.67431 \\ 0.16105 & 0.88038 & -0.37973 \end{bmatrix}.$$

Since $A_r = T^+ A_f T$ and $G_r = T^+ G_2$, then

$$A_r = \begin{bmatrix} -1.84345 & -0.49498 & 1.74063 \\ -0.51341 & -3.00001 & 2.42753 \\ -0.82095 & -0.1422 & -2.92501 \end{bmatrix}$$

and

$$G_r = \begin{bmatrix} 2.802 \\ 1.33039 \\ 1.65908 \end{bmatrix}.$$

The characteristic polynomial of the matrix A_r is found to be

$$p_r(s) = s^3+7.768467s^2+21.217558s+19.242783$$

and its roots are

$$\mu_1 = -2.91067-j1.187722$$
$$\mu_2 = -2.91067+j1.187722$$
$$\mu_3 = -1.947114 .$$

The pair (A_r, G_r) is controllable. The parameters of the reduced model

$$\dot{x}_r(t) = A_m x_r(t)+B_r u(t)+ G_r w(t)$$

$$y_r(t) = C_r x_r(t)$$

where

$$A_m = A_r - B_r ST$$

$$B_r = T^+ B$$

$$C_r = CT$$

are

$$A_m = \begin{bmatrix} -1.15595 & -0.50462 & -0.95578 \\ -0.04862 & -3.00652 & 0.60459 \\ -0.49251 & -0.1468 & -4.21317 \end{bmatrix}$$

$$B_r = \begin{bmatrix} 1.8567 \\ 1.25525 \\ 0.877 \end{bmatrix} \quad \text{and} \quad C_r = [-0.37028,\ 0.00519,\ 1.45226].$$

The pairs (A_m, B_r) and (A_m, G_r) are controllable. The pair (C_r, A_m) is observable. The characteristic polynomial of A_m is

$$p_m(s) = s^3 + 8.375642s^2 + 20.606097s + 13.082983.$$

Its roots are

$$\mu_1 = -3.139515$$

$$\mu_2 = -4.257288$$

$$\mu_3 = -0.978839$$

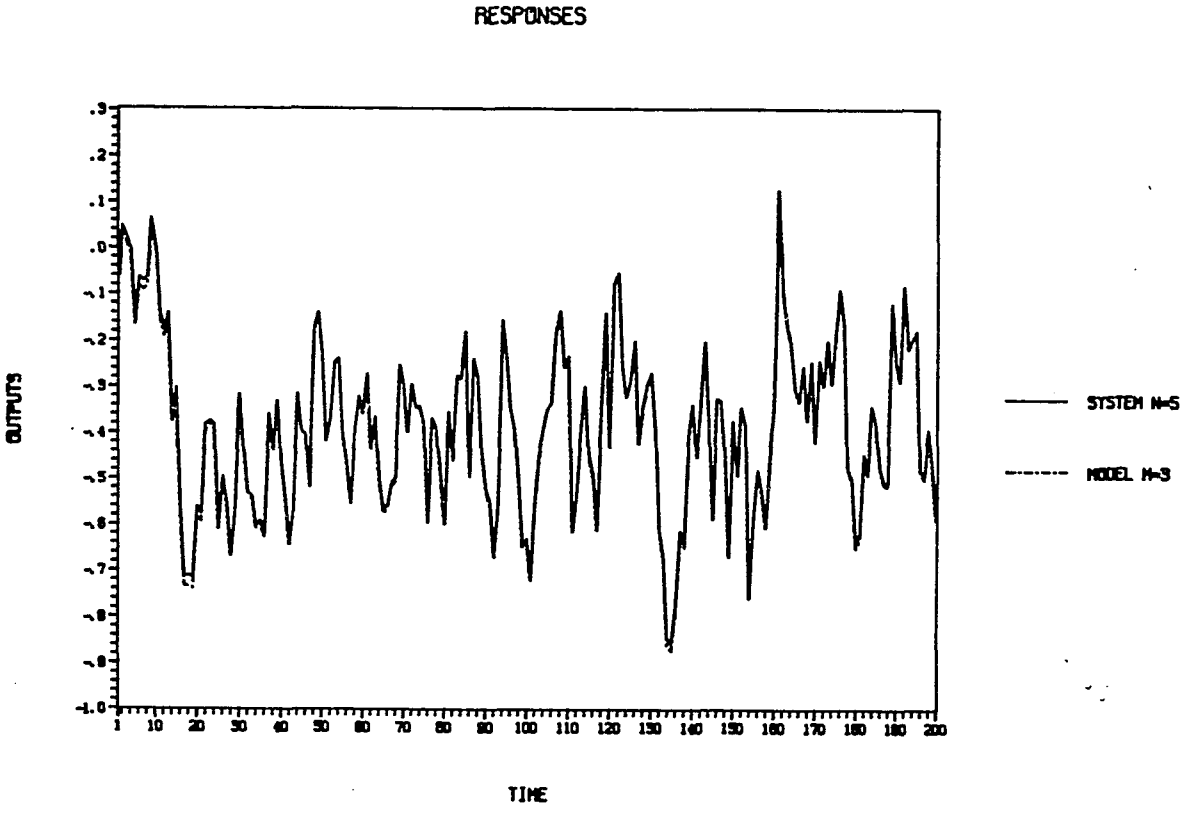


Figure 4.1. Output Responses of the 5th Order System and the 3rd Order Model to Unit Step and White Noise Inputs

Therefore, the model is stable.

Using the Fourth order Runge Kutta routine, the outputs of the 5th order system (4.1) and the 3rd order model for this example are plotted in Figure 4.1, where u(t) is the unit step, and w(t)is a Gaussian white noise. As it is seen from Figure 4.1, the model output follows the system output perfectly.

4.7. Order Reduction of Deterministic, Uniformly Controllable, Continuous, Time Varying System

Consider the n-dimensional linear time varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4.74a}$$

$$y(t) = C(t)x(t) \tag{4.74b}$$

where A(t), B(t) and C(t) are nxn, nxp and qxn continuously differentiable matrix functions of t.

To reduce the order of this system using the aggregation method, we must find an aggregation matrix T which satisfies conditions similar to (4.9) or an algebraic transformation similar to T. However, the existence of algebraic transformations are related directly to the requirements of controllability and observability. Furthermore, these conditions can be specified in terms of matrices which do not depend on the knowledge of the state transition matrix

which is known to play an important role in the theory of controllability and observability [22].

Let

$$R_c(t) = [d_1(t) \ . \ . \ ., \ d_n(t)] \tag{4.75}$$

where

$$\begin{aligned} d_1(t) &= B(t) \\ d_{i+1}(t) &= \dot{d}_i(t) - A(t)d_i(t) \\ \dot{d}_i(t) &= \frac{d}{dt}[d_i(t)] \quad \text{for } i = 1, \ . \ . \ ., \ n-1 \end{aligned} \tag{4.76}$$

We now review the following controllability theorems [118]:

Theorem 4.2: The system (4.74) is completely controllable on the interval $[t_0, t_1]$, if and only if the matrix $R_c(t)$ has rank n on the interval $[t_0, t_1]$.

Theorem 4.3: The system (4.73) is totally controllable (differentially controllable) on the interval $[t_0, t_1]$, if and only if it is completely controllable on every subinterval of $[t_0, t_1]$.

It is shown in [118] that a necessary and sufficient condition for total controllability is that the matrix $R_c(t)$ has rank n almost everywhere on $[t_0, t_1]$, (i.e., except for a subset of measure zero). Notice that neither complete nor even total controllability are

strong enough conditions to ensure the existence of a transformation needed in a number of system representation problems. These problems are characterized by the requirement that the matrix $R_c(t)$ has rank n everywhere in a given interval. Therefore, a new concept of controllability is introduced.

Definition 4.5 [118]: The system (4.74) is said to be uniformly controllable on the interval $[t_0, t_1]$ if the matrix $R_c(t)$ has a rank n everywhere on $[t_0, t_1]$.

Let

$$\bar{R}_c(t) = [d_2(t), \ldots, d_{n+1}(t)] \tag{4.77}$$

then on any interval where $R_c(t)$ has rank n, the A(t) matrix of the system (4.74) can be written as:

$$A(t) = [\dot{R}_c(t) - R_c(t)]R_c^+(t) \tag{4.78}$$

where

$$R_c^+(t) = R_c^T(t)[R_c(t)R_c^T(t)]^{-1}.$$

Equation (4.78) is obtained from the following equation:

$$\bar{R}_c(t) = -A(t)R_c(t) + \dot{R}_c(t)$$

Since $d_1(t) = B(t)$, then $R_c(t)$ is sufficient to determine the system (4.74) uniquely on any interval where (4.74) is uniformly controllable.

Before we introduce the system transformation, it is necessary to review the definition of equivalence.

Definition 4.6 [117]: Let $P(t)$ be an nxn nonsingular differentiable matrix, such that $z(t) = P(t)x(t)$. Then, it can be said that the system

$$\dot{z}(t) = A_p(t)z(t) + B_p(t)u(t) \tag{4.79a}$$

$$y(t) = C_p(t)z(t) \tag{4.79b}$$

where

$$A_p(t) = [P(t)A(t) + \dot{P}(t)]P^{-1}(t)$$

$$B_p(t) = P(t)B(t) \tag{4.80}$$

$$C_p(t) = C(t)P^{-1}(t)$$

is equivalent to the system (4.74) and $P(t)$ is an equivalent algebraic transformation.

Using this definition, the basic transformal property of the matrix $R_c(t)$ can be established.

Let $P(t)$ be the algebraic transformation given in the definition 4.6, then

$$\bar{R}_c(t) = P(t)R_c(t). \tag{4.81}$$

This equation is found by induction. If $\bar{d}_k(t) = P(t)d_k(t)$, then

$$\bar{d}_{k+1}(t) = -A_p(t)\bar{d}_k(t) + \dot{\bar{d}}_k(t)$$

$$= -[P(t)A(t)+\dot{P}(t)]P^{-1}(t)[P(t)d_k(t)]+$$

$$[\dot{P}(t)d_k(t)+P(t)\dot{d}_k(t)]$$

$$= -P(t)A(t)d_k(t)-\dot{P}(t)d_k(t)+\dot{P}(t)d_k(t)+$$

$$P(t)\dot{d}_k(t)$$

$$= P(t)[-A(t)d_k(t)+\dot{d}_k(t)] = P(t)d_{k+1}(t)$$

and

$$P(t)d_1(t) = P(t)B(t) = \bar{d}_1(t) = B_p(t).$$

The significance of equation (4.81) is that if an equivalent algebraic transformation between two uniformly controllable systems is known to exist, it is given as

$$P(t) = \bar{R}_c(t)R_c^{+}(t). \qquad (4.82)$$

4.7.1. Phase Variable Canonical Form

The technique for finding the phase canonical form using the transformation matrix which depends on the uniform controllability was first introduced by Silverman [118], and later on, improved by Ramaswami and Ramar [101]. The construction of the transformation matrix reveals that the uniform controllability is a necessary and sufficient condition for a nonsingular transformation matrix to exist.

Consider the single input system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4.83}$$

Let $P(t)$ be the nonsingular transformation matrix which reduces (4.84) to the phase variable canonical form.

$$\text{Let } z(t) = P(t)x(t) \tag{4.84}$$

then

$$\dot{z}(t) = A_p(t)z(t) + B_p(t)u(t) \tag{4.85}$$

where

$$A_p(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_n(t) & -\alpha_{n-1}(t) & \cdots & -\alpha_2(t) & -\alpha_1(t) \end{bmatrix},$$

$$B_p(t) = [0, \ldots, 0, 1]^T.$$

$A_p(t)$ and $B_p(t)$ are continuously differentiable matrices. Let $x_1(t), \ldots, x_n(t)$ and $z_1(t), \ldots, z_n(t)$ be the components of the vectors $x(t)$ and $z(t)$, respectively.

Let

$$z_1(t) = p_{11}(t)x_1(t) + p_{12}(t)x_2(t) + \ldots + p_{1n}(t)x_n(t) = p_1(t)x(t) \tag{4.86}$$

where $p_1(t)$ is given by

$$p_1(t) = [p_{11}(t), p_{12}(t), \ldots, p_{1n}(t)].$$

From equation (4.84) and the representation (4.85), we have

$$\dot{z}_1(t) = z_2(t).$$

Using equation (4.86), we get

$$z_2(t) = \dot{z}_1(t) = \dot{p}_1(t)x(t) + p_1(t)\dot{x}(t)$$

$$= [\dot{p}_1(t) + p_1(t)A(t)]x(t) + p_1(t)B(t)u(t).$$

Since the state variable $z_2(t)$ is set to be a function of the state variable $x(t)$ alone, we take $p_1(t)B(t) = 0$. So

$$z_2(t) = [p_1(t)A(t)+\dot{p}_1(t)]x(t) = p_2(t)x(t), \quad p_1(t)B(t) = 0.$$

Therefore, the row vector $p_2(t)$ is given by

$$p_2(t) = p_1(t)A(t) + \dot{p}_1(t).$$

Repeating the procedure, we obtain

$$z_3(t) = [p_2(t)A(t)+\dot{p}_2(t)]x(t) = p_3(t)x(t), \quad p_2(t)B(t)=0$$

$$\vdots$$

$$z_n(t) = [p_{n-1}(t)A(t)+\dot{p}_{n-1}(t)]x(t) =$$

$$p_n(t)x(t), \quad p_{n-1}(t)B(t) = 0.$$

The transformation matrix P(t) is then

$$P(t) = [p_1^T(t),\ p_2^T(t),\ \ldots,\ p_{n-1}^T(t),\ p_n^T(t)]^T$$

The rows of the matrix P(t) are to be computed from the recursive formula

$$p_{i+1}(t) = p_i(t)A(t) + \dot{p}_i(t) \tag{4.87}$$

for i = 1, . . ., n-1.

These rows should satisfy

$$p_1(t)B(t) = p_2(t)B(t) = \ldots = p_{n-1}(t)B(t) = 0. \tag{4.88}$$

However, from equation (4.84), we know that P(t)B(t) = $B_p(t)$, then

$$\begin{bmatrix} p_1(t)B(t) \\ \vdots \\ p_n(t)B(t) \\ p_n(t)B(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{4.89}$$

To construct the matrix P(t), one has to evaluate $p_1(t)$. Therefore, equation (4.89) leads to

$$p_1(t)B(t) = p_1(t)d_1(t) = 0 \quad \text{or} \quad d_1(t) = B(t)$$

then

$$\dot{p}_1(t)B(t) + p_1(t)\dot{B}(t) = 0 \quad \text{or} \quad \dot{p}_1(t)B(t) = -p_1(t)\dot{B}(t)$$

and

$$p_2(t)B(t) = [p_1(t)A(t)+\dot{p}_1(t)]B(t)$$

$$= p_1(t)[A(t)B(t)-\dot{B}(t)] = -p_1(t)d_2(t) = 0.$$

Thus

$$\dot{p}_1(t)d_2(t) = -p_1(t)\dot{d}_2(t) \quad \text{and} \quad \dot{p}_2(t)B(t) = -p_2(t)\dot{B}(t)$$

$$p_3(t)B(t) = [p_2(t)A(t)+\dot{p}_2(t)]B(t)$$

$$= p_2(t)[A(t)B(t)-\dot{B}(t)] = -p_2(t)d_2(t) = 0.$$

and

$$p_3(t)B(t) = -[p_1(t)A(t)+\dot{p}_1(t)]d_2(t)$$

$$= -p_1(t)[A(t)d_2(t)-\dot{d}_2(t)] = p_1(t)d_3(t) = 0$$

$$\vdots$$

$$p_n(t)B(t) = [p_{n-1}(t)A(t)+\dot{p}_{n-1}(t)]B(t) = -p_{n-2}(t)d_2(t)$$

$$= \ldots = (-1)^{n-1}p_1(t)d_n(t) = 1 \qquad (4.90)$$

where $d_1(t)$, $d_2(t)$, . . ., $d_n(t)$ are the columns of the controllability matrix $R_c(t)$ of the system (4.74)

$$R_c(t) = [d_1(t), \ldots, d_n(t)]$$

where $d_1(t) = B(t)$ and $d_{i+1}(t) = \dot{d}_i(t) - A(t)d_i(t)$ for $i = 1, \ldots, n-1$. Using equation (4.87), the representation (4.90) can be written as

$$p_1(t)[d_1(t), \ldots, d_n(t)] = (-1)^{n-1}[0, 0, \ldots, 0, 1]$$

or

$$p_1(t)R_c(t) = (-1)^{n-1}[0, 0, \ldots, 0, 1]$$

Therefore,

$$p_1(t) = (-1)^{n-1}[0, 0, \ldots, 0, 1]R_c^{-1}. \qquad (4.91)$$

Once $p_1(t)$ is known, we can form $P(t)$ using the recursive formula of equation (4.87), then we compute $A_p(t)$ as

$$A_p(t) = [P(t)A(t) + \dot{P}(t)]P^{-1}(t) \qquad (4.92)$$

Since we know the structure of $A_p(t)$ we need to compute the last row only. Thus,

$$[-\bar{\alpha}_n(t), -\bar{\alpha}_{n-1}(t), \ldots, -\bar{\alpha}_1(t)] = [p_n(t)A(t) + \dot{p}_n(t)]P^{-1}(t)$$

The proof of the nonsingularity of $P(t)$ can be found in [117].

4.7.2. System Transformation Using the State Feedback Concept

The controllability of the system (4.74) is invariant under any state feedback of the following form:

$$u(t) = v(t) + S(t)z(t) \tag{4.93}$$

where S(t) is a pxn state feedback continuous matrix function of t. From a previous argument, it is known that, if a single input, single output linear time varying system is uniformly controllable, it can be transformed to its state variable canonical form, and if the system is in its companion form, by an appropriate choice of the state feedback matrix S(t), it can be changed into a linear time invariant system, and the theory developed in the previous sections can be applied here without difficulty.

Note that the transformation (4.84) does not preserve the dynamic properties of the system (4.74). To see this fact, let

$$p(s,t) = \det(sI_n - A(t)) = s^n + \alpha_1(t)s^{n-1} + \ldots +$$

$$\alpha_{n-1}(t)s + \alpha_n(t)$$

and

$$p_p(s,t) = \det(sI_n - A_p(t))$$

$$= \det[sI_n - P(t)A(t)P^{-1}(t) - \dot{P}(t)P^{-1}(t)]$$

$$= \det\{P(t)[sI_n - A(t) - P^{-1}(t)\dot{P}(t)]P^{-1}(t)\}$$

$$= \det[sI_n - A(t) - P^{-1}(t)\dot{P}(t)]$$

which can also be written as

$$p_p(s,t) = s^n + \bar{\alpha}_1(t)s^{n-1} + \ldots + \bar{\alpha}_{n-1}(t)s + \bar{\alpha}_n(t)$$

where the coefficients $\bar{\alpha}_1(t),\ldots,\bar{\alpha}_n(t)$ are different from the coefficients $\alpha_1(t),\ldots,\alpha_n(t)$ of the original system (4.77).

Now substitute (4.93) into (4.79), and let

$$A_s = A_p(t) + B_p S(t) \tag{4.94}$$

we have

$$\dot{z}(t) = A_s z(t) + B_p v(t) \tag{4.95a}$$

$$y(t) = C_p z(t) \tag{4.95b}$$

and

$$A_s = P(t)A(t)P^{-1}(t) + \dot{P}(t)P^{-1}(t) + P(t)B(t)S(t) \tag{4.96}$$

where

$$A_s = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ -\xi_n & -\xi_{n-1} & & -\xi_2 & -\xi_1 \end{bmatrix}$$

The $\xi_1, \ldots, \xi_n$ are the constant coefficients of the characteristic polynomial of the new system (4.95).

The problem is to approximate the new system by a model of the following form

$$\dot{z}_r(t) = A_r z_r(t) + B_r v(t) \tag{4.97a}$$

$$y_r(t) = C_r z_r(t) \tag{4.97b}$$

where $z_r(t) \in \mathbb{R}^r$.

Therefore, we need to find an nxr aggregation constant matrix T such that the aggregation error

$$e(t) = z(t) - Tz_r(t) \quad \text{is zero for all } t \tag{4.98}$$

or $A_s T = TA_r$, $C_r = C_p T$, $B_r = T^+ B_p$ and $z_r(0) = T^+ x(0)$ then

$$[P(t)A(t)P^{-1}(t) + \dot{P}(t)P^{-1}(t) + P(t)B(t)S(t)]T = TA_r.$$

Let Q(t) be a pxq time varying matrix such that

$$S(t)T = Q(t)C_r$$

then

$$A(t)P^{-1}(t)T+P^{-1}(t)\dot{P}(t)P^{-1}(t)T+B(t)S(t)T = P^{-1}(t)TA_r. \quad (4.99)$$

Now let

$$M(t) = P^{-1}(t)T \quad (4.100)$$

be an nxr continuous time varying matrix with the following properties:

(1) $M^+(t) = [M^T(t)M(t)]^{-1}M^T(t)$

(2) $M^+(t)M(t) = I_r$

(3) $\dot{M}(t) = -P^{-1}(t)\dot{P}(t)M(t)$

(4) $\dot{M}^+(t)M(t) = M^+(t)P^{-1}(t)\dot{P}(t)M(t)$

(5) $M(t)M^+(t)M(t) = M(t)$

(6) $M^+(t)M(t)M^+(t) = M^+(t)$

(7) $[M(t)M^+(t)]^T = M(t)M^+(t)$

(8) $[M(t)M^+(t)]^2 = M(t)M^+(t)$.

$M(t)M^+(t)$ is an nxn orthogonal, idempotent matrix, thus it is a projection matrix, its complementary projection matrix is $(I_n - M(t)M^+(t))$.

Using equation (4.100) and the property 3, the equation (4.99) becomes

$$A(t)M(t)-\dot{M}(t)+B(t)S(t)T = M(t)A_r. \qquad (4.101)$$

In order to have a good approximation, the following conditions must be satisfied:

(1) $A(t)M(t)-M(t)A_r = \dot{M}(t)-B(t)Q(t)C_r$

where $S(t)T = Q(t)C_r$

(2) $B_r = M^+(t)B(t)$

(3) $z_r(0) = M^+(0)z(0).$

These conditions are similar to the ones given by equation (4.9). Note that, since $M(t) = P^{-1}(t)T$, then $M(t)$ is also dependent on the uniform controllability of the original system (4.74). The substitution of

$$M(t)A_r = A(t)M(t)-\dot{M}(t)+B(t)Q(t)C_r$$

into

$$A^2(t)M(t) - M(t)A_r^2$$

yields

$$A^2(t)M(t)-M(t)A_r^2 = A^2(t)M(t)-[A(t)M(t)-\dot{M}(t)+B(t)Q(t)C_r]A_r$$

$$= A^2(t)M(t)-A(t)M(t)A_r+\dot{M}(t)A_r-B(t)Q(t)C_rA_r$$

$$= A(t)[A(t)M(t)-M(t)A_r]+\dot{M}(t)A_r-B(t)Q(t)C_rA_r$$

$$= A(t)[\dot{M}(t)-B(t)Q(t)C_r]+\dot{M}(t)A_r-B(t)Q(t)C_rA_r$$

$$= A(t)[\dot{M}(t)-B(t)Q(t)C_r]+[\dot{M}(t)-B(t)Q(t)C_r]A_r.$$

Proceeding in a similar manner, we get the following equations

$$I_nM(t)-M(t)I_r = 0$$

$$A(t)M(t)-M(t)A_r = \dot{M}(t)-B(t)Q(t)C_r$$

$$A^2(t)M(t)-M(t)A_r^2 = A(t)[\dot{M}(t)-B(t)Q(t)C_r]+[\dot{M}(t)-B(t)Q(t)C_r]A_r$$

$$\vdots$$

$$A^i(t)M(t)-M(t)A_r^i = \sum_{j=0}^{i-1} A^{i-j-1}(t)[\dot{M}(t)-B(t)Q(t)C_r]A_r^j$$

then

$$A_r^i = M^+(t)A^i(t)M(t) - \sum_{j=0}^{i-1} M^+(t)A^{i-j-1}(t)[\dot{M}-B(t)Q(t)C_r]A_r^j \qquad (4.102)$$

Since the systems (4.95) and (4.97) are continuous time invariant linear systems, then let $\bar{\bar{H}}_i$'s be the Markov parameters of the reduced order time invariant model (4.98)

$$\bar{\bar{H}}_i = C_r A_r^{i-1} B_r \quad \text{for } i = 1, 2, 3, \ldots \tag{4.103}$$

and $\bar{H}_i$'s be the Markov parameters of the time invariant system (4.95)

$$\bar{H}_i = C_p A_s^{i-1} B_p \tag{4.104}$$

for $i = 1, 2, 3, \ldots$.

Since $A_s T = T A_r$, then

$$A_s^{i-1} T = T A_r^{i-1} \quad \text{or} \quad T^+ A_s^{i-1} T = A_r^{i-1} \tag{4.105}$$

for $i = 1, 2, 3, \ldots$,

while $C_p = C_r T^+$ or $C_r = C_p T$

and $B_p = T B_r$ or $B_r = T^+ B_p$.

The substitution of $C_p = C_r T^+$ and $B_p = T B_r$ into equation (4.104) yields

$$\bar{H}_i = C_p A_s^{i-1} B_p = C_r T^+ A_s^{i-1} T B_r$$

$$= C_r A_r^{i-1} B_r.$$

Hence $\bar{\bar{H}}_i = \bar{H}_i$. for i = 1, 2, 3, . . . (4.106)

The aggregation matrix T is determined in similar manner as in section 4.6.

Note that, in order for the equality $C_r = C_p T$ to exist, the following condition must be satisfied

$$C_p = C(t)P^{-1}(t)$$

so that

$$C_r = C_p T = C(t)P^{-1}(t)T$$

$$= C(t)M(t). \qquad (4.107)$$

Similarly, since $B_p = P(t)B(t) = TB_r$, we have

$$P^{-1}(t)TB_r = B(t), \quad \text{thus} \quad B_r = M^{+}(t)B(t). \qquad (4.108)$$

Let

$$A_m(t) = M^{+}(t)A(t)M(t) - M^{+}(t)\dot{M}(t)$$

$$= A_r - B_r Q(t) C_r \qquad (4.109)$$

be the state matrix of the reduced order time varying model given by

$$\dot{x}_r(t) = A_m(t)x_r(t)+B_m(t)u(t) \tag{4.110a}$$

$$y_r(t) = C_m(t)x_r(t) \tag{4.110b}$$

where

$$B_m(t) = B_r \quad \text{and} \quad C_m(t) = C_r \tag{4.111}$$

Equation (4.111) can be obtained by defining a new aggregation error as

$$e'(t) = x(t)-M(t)x_r(t) \tag{4.112}$$

then

$$\dot{e}'(t) = \dot{x}(t)-\dot{M}(t)x_r(t)-M(t)\dot{x}_r(t)$$

$$\dot{e}'(t) = A(t)x(t)+B(t)u(t)-\dot{M}(t)x_r(t)-M(t)A_m(t)x_r(t)$$

$$- M(t)B_m(t)u(t).$$

If equations (4.108) and (4.111) are satisfied, then

$$\dot{e}'(t) = A(t)x(t)-\dot{M}(t)x_r(t)-M(t)A_m(t)x_r(t)$$

$$= A(t)x(t)-A(t)M(t)x_r(t)+A(t)M(t)x_r(t)$$

$$-\dot{M}(t)x_r(t)-M(t)A_m(t)x_r(t)$$

$$= A(t)e'(t)+[A(t)M(t)-\dot{M}(t)-M(t)A_m(t)]x_r(t).$$

The dynamics of the error are given by

$$\dot{e}'(t) = A(t)e'(t)+[A(t)M(t)-\dot{M}(t)-M(t)A_m(t)]x_r(t) \quad (4.113)$$

If equation (4.110) is satisfied, equation (4.113) becomes

$$\dot{e}'(t) = A(t)e'(t)$$

Therefore, for a good approximation, the following conditions must be satisfied.

(1) $A(t)M(t) = M(t)A_m(t)+\dot{M}(t)$

(2) $B_m(t) = M^+(t)B(t), \quad C_m(t) = C(t)M(t)$

(3) $x_r(0) = M^+(0)x(0)$

Furthermore, since $z(t) = P(t)x(t)$ and $P^{-1}(t)$ exists, then the substitution of $x(t) = P^{-1}(t)z(t)$, $x_r(t) = z_r(t)$, and equations (4.98) and (4.100) yields

$$e'(t) = P^{-1}(t)z(t)-P^{-1}(t)Tz_r(t)=P^{-1}(t)[z(t)-Tz_r(t)]$$

$$e'(t) = P^{-1}(t)e(t) \quad (4.114)$$

The constant matrix A_s of the system (4.95) is stable, then $e(t) \to 0$ as $t \to \infty$. Therefore, if $P^{-1}(t)$ is bounded, then from equation (4.114), $e'(t) \to 0$ as $t \to \infty$.

Since the model (4.96) is a time invariant system, then its state transition matrix is given by

$$\phi_r(t,t_0) = e^{A_r(t-t_0)}. \quad (4.115)$$

Let $A_d(t)$ be an rxr continuous time varying matrix defined as

$$A_d(t) = -e^{-A_r(t-t_0)}B_rQ(t)C_re^{A_r(t-t_0)}. \tag{4.116}$$

and satisfying the following expression

$$\dot{\phi}_d(t,t_0) = A_d(t)\phi_d(t,t_0). \tag{4.117}$$

Proposition 4.8: The state transition matrix of the reduced time varying model (4.111) is given by

$$\phi_m(t,t_0) = e^{A_r(t-t_0)}\phi_d(t,t_0) \tag{4.118}$$

Proof: Since $\phi_m(t,t_0)$ is the state transition matrix of the model (4.111), then it satisfies the following condition

$$\dot{\phi}_m(t,t_0) = A_m(t)\phi_m(t,t_0). \tag{4.119}$$

Since

$$\phi_m(t,t_0) = e^{A_r(t-t_0)}\phi_d(t,t_0)$$

then

$$\dot{\phi}_m(t,t_0) = A_re^{A_r(t-t_0)}\phi_d(t,t_0)+e^{A_r(t,t_0)}\dot{\phi}_d(t,t_0)$$

$$= A_re^{A_r(t-t_0)}\phi_d(t,t_0)+e^{A_r(t-t_0)}A_d(t)\phi_d(t,t_0)$$

$$= A_re^{A_r(t-t_0)}\phi_d(t,t_0)+e^{A_r(t-t_0)}\{-e^{-A_r(t-t_0)}$$

$$B_rQ(t)C_re^{A_r(t-t_0)}\}\phi_d(t,t_0)$$

$$= A_re^{A_r(t-t_0)}\phi_d(t,t_0)-B_rQ(t)C_re^{A_r(t-t_0)}\phi_d(t,t_0)$$

$$= [A_r - B_r Q(t) C_r] e^{A_r (t-t_0)} \phi_d(t,t_0)$$

$$= A_m(t) \phi_m(t,t_0).$$

4.7.3. Stability of the Reduced Order Time Varying Model

Consider the n^{th} order linear time varying system of equation (4.74)

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t).$$

The response of the system is composed of two parts, the zero input response and the zero state response as

$$y(t) = C(t)\phi(t,t_0)x(t_0) + \int_{t_0}^{t} C(t)\phi(t,\tau)B(\tau)u(\tau)d\tau$$

where $\phi(t,t_0)$ is the state transition matrix of the system (4.73). To study the stability of the reduced order model (4.111), we will review the following definitions and theorems [22, 63].

Definition 4.7: A linear time varying system is said to be totally stable if and only if for any initial state and any bounded input the output as well as the state variables are bounded.

Theorem 4.4: A linear time varying system is totally stable if and only if $C(t)$ and $\phi(t,t_0)$ are bounded and

$$\int_{t_0}^{t} ||\phi(t,\tau)B(\tau)||d\tau \leq k < \infty$$

for any t_0 and for all $t \geq t_0$. In the above, any valid matrix norm can be used.

Theorem 4.5: The zero state $x(t) = \phi(t,t_0)x(t_0)$ of $\dot{x}(t) = A(t)x(t)$ is asymptotically stable at t_0 if and only if

$$||\phi(t,t_0)|| \leq k(t_0) < \infty$$

implies

$$||\phi(t,t_0)|| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

The zero state is uniformly asymptotically stable over $(0, \infty)$ if and only if there exists positive numbers k_1 and k_2 such that

$$||\phi(t,t_0)|| \leq k_1 e^{-k_2(t-t_0)}$$

for any $t_0 \geq 0$ and all $t \geq t_0$.

If the matrices B(t) and C(t) of the system (4.74) are bounded, uniform asymptotic stability implies total stability.

Suppose now that system (4.74) is totally stable. To investigate the stability of the model (4.111), we need to find an expression relating $\phi(t,t_0)$ of the system to $\phi_m(t,t_0)$ of the model (4.111). Before doing so, let $\phi_p(t,t_0)$ be the state transition matrix of the system (4.78), that is

$$\dot{\phi}_p(t,t_0) = A_p(t)\phi_p(t,t_0). \tag{4.120}$$

From equation (4.81), we have

$$A_p(t) = P(t)A(t)P^{-1}(t)+\dot{P}(t)P^{-1}(t)$$

and

$$\phi(t,t_0) = P^{-1}(t)\phi_p(t,t_0)P(t_0). \tag{4.121}$$

From equation (4.94), we get

$$A_s = A_p(t)+B_pS(t)$$

or

$$A_p(t) = A_s-B_pS(t) \tag{4.122}$$

Since the system (4.95) is a linear time invariant system, its state transition matrix is given as

$$\phi_s(t,t_0) = e^{A_s(t-t_0)} \tag{4.123}$$

and satisfies the expression

$$\dot{\phi}_s(t,t_0) = A_s\phi_s(t,t_0) \tag{4.124}$$

thus

$$\phi_p(t,t_0) = e^{A_s(t-t_0)}\phi_q(t,t_0) \tag{4.125}$$

and

$$\dot{\phi}_p(t,t_0) = A_p(t)\phi_p(t,t_0) \tag{4.126}$$

where $\phi_q(t,t_0)$ satisfies the following

$$\dot{\phi}_q(t,t_0) = A_q(t)\phi_q(t,t_0) \tag{4.127}$$

and

$$A_q(t) = -e^{-A_s(t-t_0)}B_pS(t)e^{A_s(t-t_0)}. \tag{4.128}$$

Equations (4.105), (4.114) and (4.122) give

$$A_sT = TA_r \quad \text{or} \quad A_s^iT = TA_r^i \quad \text{for } i = 0, 1, 2, \ldots$$

$$\phi_r(t,t_0) = e^{A_r(t-t_0)} = \sum_{i=0}^{\infty} A_r^i \frac{(t-t_0)^i}{i!}$$

then

$$T\phi_r(t,t_0) = Te^{A_r(t-t_0)} = \sum_{i=0}^{\infty} TA_r^i \frac{(t-t_0)^i}{i!}$$

$$= \sum_{i=0}^{\infty} A_s^i T \frac{(t-t_0)^i}{i!} = [\sum_{i=0}^{\infty} A_s^i \frac{(t-t_0)^i}{i!}]T$$

$$= e^{A_s(t-t_0)}T .$$

or

$$T\phi_r(t,t_0) = \phi_s(t,t_0)T \tag{4.129}$$

Post-multiplying equation (4.128) by T, leads to

$$A_q(t)T = -e^{-A_s(t-t_0)}B_p S(t) e^{A_s(t-t_0)}T.$$

Using equations (4.109), (4.114) and (4.129) and the condition

$$S(t)T = Q(t)C_r,$$

we have

$$A_q(t)T = -e^{-A_s(t-t_0)}TB_r S(t) T e^{A_r(t-t_0)}$$

$$= -Te^{-A_r(t-t_0)}B_r Q(t) C_r e^{A_r(t-t_0)}$$

$$= T[-e^{-A_r(t-t_0)}B_r Q(t) C_r e^{A_r(t-t_0)}]$$

thus

$$A_q(t)T = TA_d(t) \tag{4.130}$$

and hence

$$\phi_q(t,t_0)T = T\phi_d(t,t_0) \cdot \tag{4.131}$$

The substitution of equation (4.125) into (4.120) gives

$$\phi(t,t_0) = P^{-1}(t)e^{A_s(t-t_0)}\phi_q(t,t_0)P(t_0)$$

or

$$P(t)\phi(t,t_0) = e^{A_s(t-t_0)}\phi_q(t,t_0)P(t_0)$$

The pre-multiplication of equation (4.117) by T yields

$$\begin{aligned} T\phi_m(t,t_0) &= Te^{A_r(t-t_0)}\phi_d(t,t_0) \\ &= e^{A_s(t-t_0)}T\phi_d(t,t_0) \\ &= e^{A_s(t-t_0)}\phi_q(t,t_0)T \\ &= e^{A_s(t-t_0)}\phi_q(t,t_0)P(t_0)P^{-1}(t_0)T \\ &= P(t)\phi(t,t_0)P^{-1}(t_0)T. \end{aligned}$$

Thus

$$T\phi_m(t,t_0) = P(t)\phi(t,t_0)P^{-1}(t_0)T$$

or

$$P^{-1}(t)T\phi_m(t,t_0) = \phi(t,t_0)P^{-1}(t_0)T$$

$$M(t)\phi_m(t,t_0) = \phi(t,t_0)M(t_0)$$

and

$$\phi_m(t,t_0) = M^+(t)\phi(t,t_0)M(t_0) \qquad (4.132)$$

If $P(t)$ and $P^{-1}(t)$ are bounded for all $t\varepsilon(0,\infty)$, then $M(t)$ and $M^+(t)$ would also be bounded in $(0,\infty)$, and since the original system is assumed to be totally stable, then $\phi(t,t_0)$ is bounded. Therefore, $\phi_m(t,t_0)$ is also bounded. Furthermore, since B_m and C_m are constant matrices, then they are bounded and the reduced order model (4.111) is totally stable.

4.8. Conclusion

In this chapter we have studied the order reduction of stochastic linear time invariant systems or linear time invariant systems with uncertainty, and deterministic linear time varying systems. A modified version of the aggregation method has been proposed and an example has been given to justify the effectiveness of this method.

In the next chapter, we will consider the Routh and Schwarz simplification techniques for deterministic linear time invariant systems.

CHAPTER V

ROUTH AND SCHWARZ APPROXIMATIONS

5.1. Introduction

In the previous two chapters, we have examined order reduction techniques using state space framework. However, in control applications, more often, systems are designed in the frequency domain and then subsequently converted to the state space representation. In this chapter, we will focus our attention on the description of the reduction techniques that always yield stable models. These approximations use the Routh table and the properties of the Schwarz matrix.

5.2. Mathematical Background

Consider the n-dimensional linear continuous time invariant system, which is asymptotically stable, completely controllable and observable:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.1a}$$

$$y(t) = Cx(t) \tag{5.1b}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ -\alpha_n & -\alpha_{n-1} & & -\alpha_2 & -\alpha_1 \end{bmatrix}_{nxn}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}_{nx1} \tag{5.2}$$

and $C = [\beta_n, \beta_{n-1}, \ldots, \beta_1]_{1xn}$.

Let the characteristic polynomial of the matrix A be

$$p(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n \tag{5.3}$$

Since the system (5.1) is asymptotically stable, then for completeness we shall state the following theorems.[16, 22, 63].

Theorem 5.1 (Routh): Construct the Routh array $\{r_{ij}\}$ having initial two rows

$$\{r_{01}, r_{02}, r_{03}, \ldots\} = \{\alpha_0, \alpha_2, \alpha_4, \ldots\}$$
$$\{r_{11}, r_{12}, r_{13}, \ldots\} = \{\alpha_1, \alpha_3, \alpha_5, \ldots\} \tag{5.4a}$$

and subsequent rows defined by

$$r_{ij} = -\det\begin{bmatrix} r_{i-2,1} & r_{i-2,j+1} \\ r_{i-1,1} & r_{i-1,j+1} \end{bmatrix} / r_{i-1,1} \qquad (5.4b)$$

for $i = 2, 3, \ldots, n$

The polynomial p(s) is asymptotically stable if and only if all the first column elements r_{i1} for $i = 0, 1, \ldots, n$ are positive.

Theorem 5.2 (Hurwitz): Construct the nxn Hurwitz matrix

$$H = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_5 & \cdots & \alpha_{2n-1} \\ \alpha_0 & \alpha_2 & \alpha_4 & \cdots & \alpha_{2n-2} \\ 0 & \alpha_1 & \alpha_3 & \cdots & \alpha_{2n-3} \\ 0 & \alpha_0 & \alpha_2 & \cdots & \alpha_{2n-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \alpha_n \end{bmatrix} \qquad (5.5)$$

where $\alpha_k = 0$ for $k > n$.

The polynomial p(s) is asymptotically stable if and only if all the leading principal minors $H_1, H_2, \ldots, H_n$ of H are positive.

Theorem 5.3 (Margin of Stability): If $p(s)$ is asymptotically stable then the negative number $\sigma=\max_i\{\mathrm{Re}\{\lambda_i\}\}$, called the abscissa of stability of $p(s)$, has the following upper bounds

$$\sigma \leq -2^{-\varepsilon(n)}R^{-(n-1)(n+1)/2}r_{11}r_{21}\cdots r_{n1} \tag{5.6}$$

and

$$\sigma < -2^{-n}R^{-2(n-1)}r_{n1}r_{n-1,1} \tag{5.7}$$

where $R = 2\max_i|\alpha_i/\alpha_0|^{1/i}$

and $\varepsilon(n) = (-1)^{n(n-1)/2}$.

Proofs of these theorems can be found in [16, 22, 63]. The first two theorems are equivalent, as the following example for n = 3, shows

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{\alpha_0}{\alpha_1} & 1 & 0 \\ \frac{-\alpha_0\alpha_1}{\alpha_0\alpha_3-\alpha_1\alpha_2} & \frac{\alpha_1^2}{\alpha_0\alpha_3-\alpha_1\alpha_2} & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_0 & \alpha_2 & 0 \\ 0 & \alpha_1 & \alpha_3 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{21} & r_{22} \\ 0 & 0 & r_{31} \end{bmatrix}$$

This can be generalized to give

$$T_1 H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1,n} \\ 0 & r_{21} & r_{22} & \cdots & r_{2n-1} \\ 0 & 0 & r_{31} & \cdots & r_{3n-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & r_{n1} \end{bmatrix} \tag{5.8}$$

where T_1 is an nxn nonsingular matrix. The principal minors of H are

$$H_i = r_{11}\, r_{21} \cdots r_{i1} \quad \text{for} \quad i = 1, 2, \ldots, n \tag{5.9}$$

and

$$H_n = \alpha_n H_{n-1}.$$

The first column elements of the Routh array are then given as follows:

$$r_{11} = H_1, \quad r_{i1} = \frac{H_i}{H_{i-1}} \quad \text{for} \quad i = 2, \ldots, n \tag{5.10}$$

Now, let

$$T_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \frac{r_{n-1,2}}{r_{n-1,1}} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{r_{n-2,2}}{r_{n-2,1}} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ & & \cdots & 0 & 1 & 0 & 0 \\ & & \cdots & \frac{r_{32}}{r_{31}} & 0 & 1 & 0 \\ & & \cdots & 0 & \frac{r_{22}}{r_{21}} & 0 & 1 \end{bmatrix}_{nxn} \qquad (5.11)$$

be a similarity transformation matrix [3, 16], such that

$$z(t) = T_2 x(t). \qquad (5.12)$$

Then the system (5.1) becomes

$$\dot{z}(t) = A_s z(t) + B_s u(t) \qquad (5.13a)$$

$$y(t) = C_s z(t) \qquad (5.13b)$$

where
$$A_s = T_2 A T_2^{-1}$$
$$B_s = T_2 B \qquad (5.14)$$
$$C_s = C T_2^{-1}$$

and

$$A_s = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -\gamma_n & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -\gamma_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\gamma_3 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -\gamma_2 & -\gamma_1 \end{bmatrix}_{nxn} \tag{5.15}$$

A_s is called the Schwarz matrix [16].

We will show that $p(s) = \det(sI_n - A_s)$. To do so, let

$$D = \begin{bmatrix} s & -1 & 0 & \cdots & 0 & 0 \\ \gamma_n & s & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s & -1 \\ 0 & 0 & 0 & \cdots & \gamma_2 & s+\gamma_1 \end{bmatrix} \tag{5.16}$$

and D_k for $k = 1, 2, \ldots, n$ be the k^{th} leading principal minors of $\det(sI_n - A_s)$ and $D_n = \det(sI_n - A_s)$. Then the

expansion of D_n by its last column gives

$$D_n = (s + \gamma_1)D_{n-1} + \gamma_2 D_{n-2} \tag{5.17}$$

$$D_k = sD_{k-1} + \gamma_{n-k+2}D_{k-2} \tag{5.18}$$

for $k = 3, \ldots, n-1$.

Let $$D_{n-\ell} = d_{\ell 1}s^{n-\ell} + d_{\ell 2}s^{n-\ell-2} + \ldots + d_{\ell,j+1}s^{n-\ell-2j} + \ldots \tag{5.19}$$

for $\ell = 1, \ldots, n-1$.

Substitute equation (5.19) into (5.18) with $k = n-\ell$ and equate coefficients $s^{n-\ell-2j}$, to get

$$d_{\ell,j+1} = d_{\ell+1,j+1} + \gamma_{\ell+2}d_{\ell+2,j}.$$

Evaluate D_{n-1} and D_{n-2}, using equation (5.19). It follows that the first two rows of the Routh array $\{r_{ij}\}$ for D_n are the coefficients of $sD_{n-1} + \gamma_2 D_{n-2}$ and $\gamma_i D_{n-1}$, that is

$$\{r_{01}, r_{02}, r_{03}, \ldots\} = \{d_{11}, d_{12} + \gamma_2 d_{21}, d_{13} + \gamma_2 d_{22}, \ldots\}$$

$$\{r_{11}, r_{12}, r_{13}, \ldots\} = \{\gamma_1 d_{11}, \gamma_1 d_{12}, \gamma_1 d_{13}, \ldots\}.$$

The third row is obtained as

$$r_{2j} = -\frac{1}{\gamma_1 d_{11}} \det \begin{bmatrix} d_{11} & d_{1,j+1}+\gamma_2 d_{2j} \\ \alpha_1 d_{11} & \alpha_1 d_{1,j+1} \end{bmatrix}$$

$$= \gamma_2 d_{2j} \quad \text{for } j = 1, 2, \ldots$$

Then we can establish by induction [16], that in general

$$r_{i1} = \gamma_i r_{i-2,1} \tag{5.20}$$

$$r_{ij} = r_{i1} d_{ij} \qquad i \geq 1 . \tag{5.21}$$

The first column of the Routh array for $\det(sI_n - A_s)$ is

$$1, \gamma_1, \gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_4, \gamma_1\gamma_3\gamma_5, \gamma_2\gamma_4\gamma_6, \gamma_1\gamma_3\gamma_5\gamma_7, \ldots \tag{5.22}$$

We select the elements of the Schwarz matrix to be

$$\gamma_1 = H_1, \gamma_2 = \frac{H_2}{H_1}, \quad \gamma_3 = \frac{H_3}{H_1 H_2}$$

$$\gamma_k = \frac{H_{k-3} H_k}{H_{k-2} H_{k-1}} \tag{5.23}$$

for $k = 4, \ldots, n$

Then the sequence (5.22) becomes

$$1,\ H_1,\ \frac{H_2}{H_1},\ \frac{H_3}{H_1},\ \ldots,\ \frac{H_n}{H_{n-1}}\ . \tag{5.24}$$

It follows that with the choice made previously for the δ_i's the polynomial $p(s)$ and $\det(sI_n - A_s)$ are identical, since their Routh arrays have the same first columns. Therefore,

$$p(s) = \det(sI_n - A_s).$$

Let $G(s)$ be the transfer function of the system (5.1). Then

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \ldots + \beta_n}{\alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n}$$

$G(s)$ can be expanded in the continued fraction form:

$$G(s) = \cfrac{1}{1+\delta_1 s + \cfrac{1}{\delta_2 s + \cfrac{1}{\ddots + \cfrac{1}{\delta_n s}}}} \left\{\sigma_1 + \cfrac{1}{\delta_2 s + \cfrac{1}{\delta_3 s + \cfrac{1}{\ddots + \cfrac{1}{\delta_n s}}}} \left\{\sigma_2 + \ldots + \left(\frac{\sigma_n}{\delta_n s}\right)\ldots\right\}\right\} \tag{5.26}$$

where δ_i and σ_i for $i = 1, \ldots, n$, are scalars obtained from the following arrays:

$$r_{01} = \alpha_0 \qquad r_{02} = \alpha_2 \qquad r_{03} = \alpha_4 \quad . \ . \ .$$

$$r_{11} = \alpha_1 \qquad r_{12} = \alpha_3 \qquad r_{13} = \alpha_5 \quad . \ . \ .$$

and

$$r_{ij} = r_{i-2,j+1} - \delta_{i-1} r_{i-1,j+1} \tag{5.27}$$

for $i = 2, \ldots, n+1$ and $j = 1, 2, \ldots$

$$\delta_i = \frac{r_{i-1,1}}{r_{i,1}}$$

for $i = 1, \ldots, n$.

Similarly,

$$s_{01} = \beta_1, \qquad s_{02} = \beta_3, \qquad s_{03} = \beta_5, \ . \ . \ .$$

$$s_{11} = \beta_2, \qquad s_{12} = \beta_4, \qquad s_{13} = \beta_6, \ . \ . \ .$$

and

$$s_{ij} = s_{i-2,j+1} - \sigma_{i-1} r_{i-1,j+1} \tag{5.28}$$

for $i = 2, \ . \ . \ ., n+1$ and $j = 1, 2, \ldots$ while

$$\sigma_i = \frac{s_{i-1,1}}{r_{i,1}}$$

for $i = 1, 2, \ldots, n$

These arrays are called the δ- and σ-arrays [13, 61].

Let T_3 be a similarity transformation matrix such that

$$\eta(t) = T_3 z(t) \tag{5.29}$$

where

$$T_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\delta_1} \\ 0 & 0 & \cdots & \frac{1}{\delta_1\delta_2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \frac{1}{\delta_1\cdots\delta_{n-1}} & \cdots & 0 & 0 \\ \frac{1}{\delta_1\cdots\delta_n} & 0 & \cdots & 0 & 0 \end{bmatrix}_{nxn} . \tag{5.30}$$

The system (5.13) becomes

$$\dot{\eta}(t) = A_r \eta(t) + B_r u(t) \tag{5.31a}$$

$$y(t) = C_r \eta(t) \tag{5.31b}$$

where

$$A_r = T_3 A_s T_3^{-1}$$

$$B_r = T_3 B_s$$

$$C_r = C_s T_3^{-1}$$

$$A_r = \begin{bmatrix} -\frac{1}{\delta_1} & -\frac{1}{\delta_1} & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{\delta_2} & 0 & -\frac{1}{\delta_2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\delta_{n-1}} & 0 & -\frac{1}{\delta_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\delta_n} & 0 \end{bmatrix}_{nxn}, \tag{5.32}$$

$$B_r = \begin{bmatrix} \frac{1}{\delta_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{nx1}$$

and

$$C_r = [\sigma_1, \sigma_2, \ldots, \sigma_n]_{1xn}.$$

From equations (5.20) and (5.27), we have

$$\gamma_1 = r_{11}, \quad \gamma_i = \frac{r_{i,1}}{r_{i-2,1}} \qquad \text{for} \quad i = 2, \ldots, n$$

and

$$\delta_i = \frac{r_{i-1,1}}{r_{i,1}} \quad \text{for} \quad i = 1, \ldots, n$$

Thus, $r_{01} = 1$, $r_{11} = \frac{1}{J_1}$ and $r_{i1} = \frac{1}{\delta_1 \cdots \delta_i}$

for $i = 2, \ldots, n$

and $\gamma_1 = \frac{1}{\delta_1}$

$$\gamma_i = \frac{1}{\delta_{i-1}\delta_i} \tag{5.33}$$

for $i = 2, \ldots, n$

The system (5.31) can be obtained directly from the system representation (5.1), using the transformation $T_4 = T_3 T_2$, or

$$T_4 = \begin{bmatrix}
 & & \cdots & 0 & \frac{r_{11}r_{23}}{r_{31}} & 0 & \frac{r_{11}r_{22}}{r_{21}} & 0 & r_{11} \\
 & & \cdots & \frac{r_{21}r_{33}}{r_{31}} & 0 & \frac{r_{21}r_{32}}{r_{31}} & 0 & r_{21} & 0 \\
 & & \cdots & 0 & \frac{r_{31}r_{42}}{r_{41}} & 0 & r_{31} & 0 & 0 \\
 & & \cdots & \frac{r_{41}r_{52}}{r_{51}} & 0 & r_{41} & 0 & 0 & 0 \\
 & & \cdots & 0 & r_{51} & 0 & 0 & 0 & 0 \\
 & & \cdots & r_{61} & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \frac{r_{n-5,1}r_{n-4,3}}{r_{n-4,1}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{r_{n-4,1}r_{n-3,3}}{r_{n-3,1}} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{r_{n-3,1}r_{n-2,3}}{r_{n-3,1}} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{r_{n-2,1}r_{n-1,2}}{r_{n-1,1}} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r_{n-1,1} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
r_{n,1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

5.3. Routh Approximation [61]

The Routh approximation is based on expanding the n^{th} order system transfer function in the continued fraction form (5.26). The continued fraction expansion (5.26) can be rewritten as follows:

$$G(s) = \sigma_1 f_1(s) + \sigma_2 f_1(s) f_2(s) + \ldots + \sigma_n f_1(s) \ldots f_n(s)$$

$$= \sum_{i=1}^{n} \sigma_i \prod_{j=1}^{i} f_j(s) \tag{5.34}$$

where $f_k(s)$, for $k = 2, 3, \ldots, n$ are determined by the following continued fraction:

$$f_k(s) = \cfrac{1}{\delta_k s + \cfrac{1}{\delta_{k+1} s + \cfrac{1}{\delta_{k+2} s + \cfrac{1}{\ddots + \cfrac{1}{\delta_n(s)}}}}} \tag{5.35}$$

and $f_1(s) = \dfrac{1}{1 + \delta_1 s}$.

The reduction process is done by truncating the expansion (5.34) and rearranging the retained terms as a rational transfer function. Truncating the continued fraction (5.35) after the k^{th} term and denoting it by $g_{j,k}(s)$, the k^{th} order model transfer function $R_k(s)$ is similar to (5.34) and is given as

$$R_k(s) = \sum_{i=1}^{k} \sigma_i \prod_{j=1}^{i} g_{j,k}(s) \tag{5.36}$$

where

$$g_{j,k}(s) = \cfrac{1}{\delta_{js} + \cfrac{1}{\delta_{j+1}s + \cfrac{1}{\delta_{\delta+2}s + \cfrac{1}{\ddots \; + \cfrac{1}{\delta_{k-1}s + \cfrac{1}{\delta_k s}}}}}}. \tag{5.37}$$

Let the numerator and denominator of $R_k(s)$ be $P_k(s)$ and $Q_k(s)$ respectively, such that

$$P_1(s) = \sigma_1, \qquad Q_1(s) = 1 + \delta_1 s$$

$$P_2(s) = \sigma_2 + \delta_2\sigma_1 s, \qquad Q_2(s) = 1 + \delta_2 s + \delta_1\delta_2 s^2$$

$$P_2(s) = (\sigma_1+\sigma_3) + \delta_3\sigma_2 s + \delta_2\delta_3\sigma_1 s^2,$$

$$Q_3(s) = 1 + (\delta_1+\delta_3)s + \delta_2\delta_3 s^2 + \delta_1\delta_2\delta_3 s^3$$

. . .

In general

$$P_k(s) = \delta_k s P_{k-1}(s) + P_{k-2}(s) + \sigma_k \tag{5.38}$$

$$Q_k(s) = \delta_k s Q_{k-1}(s) + Q_{k-2}(s) \tag{5.39}$$

for $k = 1, 2, \ldots,$

$$P_{-1}(s) = P_0(s) = 0 \quad \text{and} \quad Q_{-1}(s) = Q_0(s) = 1. \tag{5.40}$$

The equations (5.38), (5.39) and (5.40) with the δ- and σ-arrays are sufficient to determine a k^{th} order model. However, this model preserves high frequency characteristics, and for control applications, it is desirable to use the system reciprocal transfer function given by

$$\hat{G}(s) = \frac{1}{s}G(\frac{1}{s}) = \frac{\beta_n s^{n-1} + \beta_{n-1} s^{n-1} + \ldots + \beta_1}{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_0}. \tag{5.41}$$

The reduction process can then be summarized by the following algorithm.

5.3.1. Routh Approximation Algorithm

Step 1: Determine the reciprocal of the system transfer function $\hat{G}(s) = \frac{1}{s}G(\frac{1}{s})$.

Step 2: Construct the $\hat{\delta}$ - $\hat{\sigma}$ arrays corresponding to $\hat{G}(s)$.

Step 3: For a k^{th} order model use the recursive formulas (5.38) and (5.39), to determine $\hat{R}_k(s) = \hat{P}_k(s)/\hat{Q}_k(s)$.

Step 4: Reverse the coefficients of $\hat{P}_k(s)$ and $\hat{Q}_k(s)$ back to find $R_k(s) = P_k(s)/Q_k(s)$.

For a stable system, this algorithm gives a stable reduced order model. For an unstable G(s), a shift of the imaginary axis would provide a modified asymptotically stable transfer function

$$\hat{G}(s) = G(s+\sigma) \tag{5.42}$$

where the real positive parameter σ is chosen to be $\sigma > \mathrm{Re}\{\lambda_m\}$ where λ_m is the closed loop pole with the largest positive real part. However, it is not always easy to compute the closed loop pole. Therefore, it would be simpler to estimate σ so that $\hat{G}(s)$ is stable and then evaluate the new σ which is equal to the absolute value of the results of equations (5.6) and (5.7). The next step would be to find $R_k(s)$ as usual and finally shift back the imaginary axis to its original position providing

$$R_k(s) = \hat{R}_s(s-\sigma)$$

where $R_k(s)$ is the k^{th} approximant of the unstable system.

Another Routh-based model reduction scheme has been proposed by Krishnamurthy et al. [67, 68]. In this technique, the Routh δ - σ arrays for the numerator and denominator of an n^{th} order system are first constructed and then a desired reduced order model is obtained. Using the coefficients of the second and third rows, a polynomial

of $(n-1)^{th}$ order can be constructed. Similarly, an $(n-2)^{th}$ order polynomial can be constructed using the third and fourth rows of the σ-Routh array, etc. Although, this method guarantees that a stable system is reduced to a stable model. Singh [120] has shown that it is a nonunique procedure in that several systems can have the same reduced model.

In the next section, we will describe a method that overcomes the Routh approximation drawback.

5.4. Schwarz Approximation

This approximation method is based upon the Schwarz canonical form, to produce a stable reduced model from a stable system. It was originally presented by Arumugan et al. [11] as a state space method, then showed by Lal et al [74] to have moment matching properties, and later reformulated by Davidson et al. [33] as a frequency domain technique. Lucas et al. [82] established links between Routh and Schwarz approximations and showed that the Schwarz approximation can also be obtained from the continued fraction expansion of G(s). It does not, however, require any reciprocal transformation for the continued fraction expansion to obtain a reduced transfer function. This is due to the fact that the last k terms of the continued fraction expansion of G(s) are retained instead of the first k terms, as in the Routh approximation.

Hutton [61] has shown that a state space realization of G(s) is

$$\dot{\eta}(t) = \begin{bmatrix} -\frac{1}{\delta_1} & -\frac{1}{\delta_2} & 0 & \cdots & 0 & 0 \\ \frac{1}{\delta_2} & 0 & -\frac{1}{\delta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\delta_{n-1}} \\ 0 & 0 & 0 & \cdots & \frac{1}{\delta_n} & 0 \end{bmatrix} \eta(t) + \begin{bmatrix} \frac{1}{\delta_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(t) \qquad (5.43a)$$

$$y(t) = [\sigma_1, \sigma_2, \ldots, \sigma_n]u(t). \qquad (5.43b)$$

The block diagram realization of the state space representation (5.43) is shown in Figure 5.1.

The k^{th} order model is obtained by letting $\eta_j(t) = 0$ for $k < j \leq n$ to get

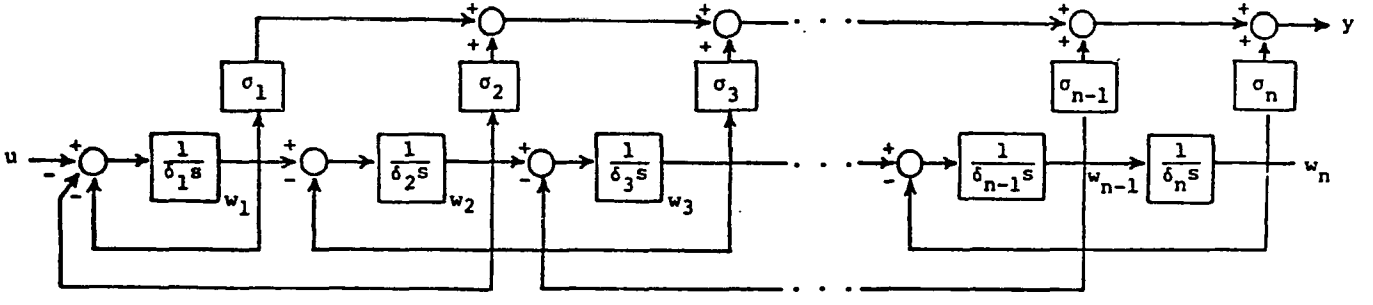


Figure 5.1. Block Diagram Realization of the Routh System.

$$\dot{\eta}_k(t) = \begin{bmatrix} -\frac{1}{\delta_1} & -\frac{1}{\delta_1} & 0 & \cdots & 0 & 0 \\ \frac{1}{\delta_2} & 0 & -\frac{1}{\delta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{\delta_{k-1}} \\ 0 & 0 & 0 & \cdots & \frac{1}{\delta_k} & 0 \end{bmatrix} \eta_k(t) + \begin{bmatrix} \frac{1}{\delta_1} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} U(t) \qquad (5.44a)$$

$$y_k(t) = [\sigma_1, \sigma_2, \ldots, \sigma_k]\eta_k(t) \qquad (5.44b)$$

where $\eta_k(t)$ approximates the first k δ- and σ- parameters and has the property of matching the first k Markov parameters of the system and the model. However, it is a high frequency approximation, since no reciprocal transformation has been used.

The Schwarz version begins with the realization

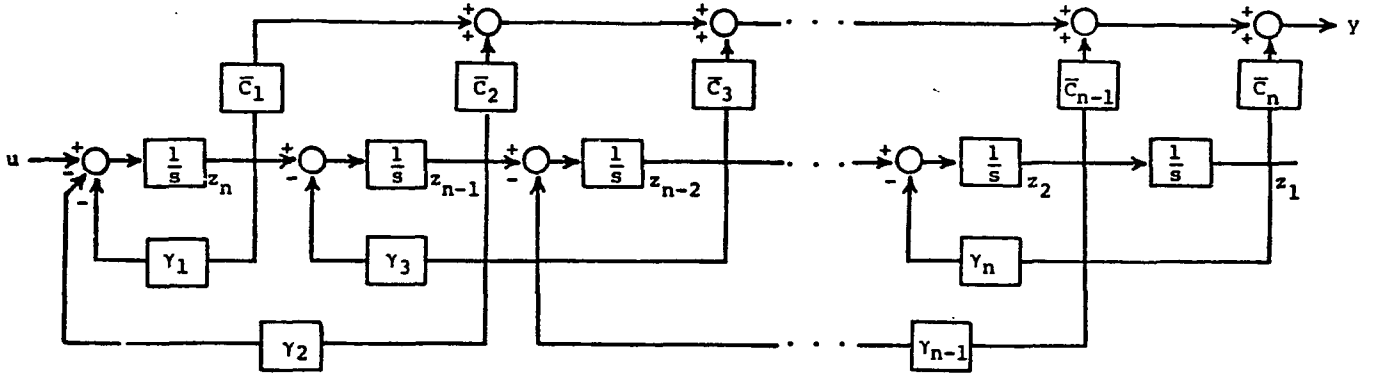
$$\dot{z}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -\gamma_n & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & -\gamma_2 & -\gamma_1 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (5.45a)$$

$$y(t) = [\bar{c}_n, \bar{c}_{n-1}, \ldots, \bar{c}_1] z(t) \qquad (5.45b)$$

where the γ_i's are given by the equation (5.33) and similarly, this state space representation has its associated block diagram realization shown in Figure 5.2.

In deriving this scheme, Lucas et al. [82] have found the k^{th} order model to be

$$\dot{z}_k(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ -\gamma_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -\gamma_{n-k+2} & -\dfrac{r_{n-k+1,1}}{r_{n-k-1,1}} \end{bmatrix} z_k(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k \end{bmatrix} u(t) \qquad (5.46a)$$



5.2. Block Diagram Realization of the Schwarz System.

$$y_k(t) = [\bar{c}_n, \bar{c}_{n-1}, \ldots, \bar{c}_{n-k+1}]\, z_k(t) \qquad (5.46b)$$

where

$$K = \frac{r_{01}}{r_{n-K-1}} = \frac{r_{01}}{r_{11}}\,\frac{r_{11}}{r_{21}} \cdots \frac{r_{n-k-2,1}}{r_{n-k-1,1}} = \delta_1\delta_2\cdots\delta_{n-k-1}. \qquad (5.47)$$

z_k approximates the first k components of z. Using equations (5.33) the system (5.46) becomes

$$\dot{z}_k(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \frac{-1}{\delta_n\delta_{n-1}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \frac{-1}{\delta_{n-k+2}\delta_{n-k+1}} & \frac{1}{\delta_{n-k+1}} \end{bmatrix} z_k(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K \end{bmatrix} u(t) \qquad (5.48)$$

$$y_k(t) = [\bar{c}_n, \bar{c}_{n-1}, \ldots, \bar{c}_{n-k+1}]\, z_k(t). \qquad (5.48b)$$

The approximation retains the last k δ-parameters. Hence, the model will have the low frequency characteristics of the system.

Using the following transformation

$$z_k(t)=\begin{bmatrix} 0 & 0 & \cdots & 0 & \delta_{n-k+1}\cdots\delta_n \\ 0 & 0 & \cdots & \delta_{n-k+1}\cdots\delta_{n-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \delta_{n-k+1}\delta_{n-k+2} & \cdots & 0 & 0 \\ \delta_{n-k+1} & 0 & \cdots & 0 & 0 \end{bmatrix} \eta_k(t) \qquad (5.49)$$

the Routh counterpart is found to be

$$\dot{\eta}_k(t)=\begin{bmatrix} -\frac{1}{\delta_{n-k+1}} & -\frac{1}{\delta_{n-k+1}} & \cdots & 0 & 0 \\ \frac{1}{\delta_{n-k+2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{1}{\delta_{n-1}} \\ 0 & 0 & \cdots & \frac{1}{\delta_n} & 0 \end{bmatrix} \eta(t) + \begin{bmatrix} \frac{1}{\delta_{n-k+1}} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(t) \qquad (5.50a)$$

$$y_k(t) = \frac{1}{k}\,[\sigma_{n-k+1}, \ldots, \sigma_n]\eta_k(t). \qquad (5.50b)$$

By comparing Equation (5.50) with (5.32) and (5.26), it can be seen that the transfer function of the model (5.50) is

$$G_k(s) = \cfrac{1}{1+\delta_{n-k+1}s+\cfrac{1}{\delta_{n-k+2}s+\cfrac{1}{\ddots+\cfrac{1}{\delta_n s}}}}\left\{\sigma_{n-k+1}+\cfrac{1}{\delta_{n-k+2}s+\cfrac{1}{\ddots+\left\{\cfrac{\sigma_n}{\delta_n s}\right\}\ldots}}\right\}. \tag{5.51}$$

Therefore, the Schwarz approximant can also be written as a continued fraction. It is not, however, necessary to invert this continued fraction to obtain the reduced transfer function. The denominators and numerators of the model transfer functions are generated by successive pairs of rows in the δ and σ-arrays respectively. However, by reformulating the δ and σ-arrays, it is possible to exhibit the reduced transfer function coefficients in single rows of each array. To determine the Schwarz approximation of G(s), Lucas [82] derived the so-called D and N tables which are different from the δ and σ-tables used in the Routh approximation.

Given

$$G(s) = \frac{\beta_1 s^{n-1}+\beta_2 s^{n-2}+\ldots+\beta_n}{\alpha_0 s^n+\alpha_1 s^{n-1}+\ldots+\alpha_n}$$

The D and N tables are:

D Table

$$\begin{array}{cccccc} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ \alpha_1 & 0 & \alpha_3 & 0 & \cdots & \\ \alpha_1' & \alpha_2' & \alpha_3' & \alpha_4' & \cdots & \\ \alpha_2' & 0 & \alpha_4' & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \end{array} \qquad (5.52)$$

where $\alpha_1' = \alpha_1 \qquad \alpha_2' = \alpha_2 - \frac{\alpha_0}{\alpha_1}\alpha_3$

$\alpha_3' = \alpha_3 \qquad \alpha_4' = \alpha_4 - \frac{\alpha_0}{\alpha_1}\alpha_5$

and so on.

N Table

$$\begin{array}{cccccc} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_n \\ \alpha_1 & 0 & \alpha_3 & 0 & \cdots & \\ \beta_2' & \beta_3' & \beta_4' & \beta_5' & \cdots & \\ \alpha_2' & 0 & \alpha_4' & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \end{array}$$

where $\beta_2' = \beta_2$ $\quad \beta_3' = \beta_3 - \frac{\beta_3}{\alpha_1}\alpha_3$

$\beta_4' = \beta_4$ $\quad \beta_5' = \beta_5 - \frac{\beta_1}{\alpha_1}\alpha_5$

and so on.

If each row and column containing a separating zero is omitted, we obtain the δ- and σ- tables associated with G(s). The D and N tables do not have to be completed to obtain the Lucas-Schwarz approximation. For example, the $(n-1)^{th}$ order Schwarz approximation of G(s) is

$$G_{n-1}(s) = \frac{\beta_2' s^{n-2} + \beta_3' s^{n-3} + \ldots + \beta_{n-2}'}{\alpha_1' s^{n-1} + \alpha_2' s^{n-2} + \ldots + \alpha_{n-1}'} \qquad (5.54)$$

5.4.1. Time Moment Matching Property

Most of the order reduction techniques do not match some of the first-time moments between the system and the model. This leads to steady state errors in the responses to step and polynomial inputs. The time moment matching property is essentially a match of time moments of the system impulse response to those of the model.

The n^{th} order transfer function of the system (5.15)

$$G(s) = \frac{\beta_1 s^{n-1}+\ldots+\beta_n}{s^n+\alpha_1 s^{n-1}+\ldots+\alpha_n}$$

can also be expressed in terms of the system impulse response g(t) as follows

$$G(s) = \int_0^\infty g(t)e^{-st}dt. \tag{5.55}$$

The power expansion of e^{-st} about s = 0, leads to

$$G(s) = \int_0^\infty g(t)e^{-st}dt = \int_0^\infty g(t)\{1-st+s^2\frac{t^2}{2!}+\ldots\}dt$$

$$= \int_0^\infty g(t)dt - s\int_0^\infty tg(t)dt + s^2\int_0^\infty \frac{t^2}{2!}g(t)dt+\ldots$$

$$= m_0 + m_1 s + m_2 s^2 + \ldots \tag{5.56}$$

where the constants m_i's are related to the time moments M_i's by the following relation

$$m_i = \frac{(-1)^i}{i!}\int_0^\infty t^i g(t)dt = \frac{(-1)^i}{i!}M_i \tag{5.57}$$

for i = 0, 1, . . .

Furthermore, from the direct division of equation (5.25) the constants m_i's are also given as follows:

$$m_o = \frac{\beta_n}{\alpha_n}$$

$$m_i = \frac{1}{\alpha_n}[\beta_{n-i} - \sum_{j=1}^{i} \alpha_{n-j} m_{i-j}] \qquad (5.58)$$

for i = 0, 1, . . .

To overcome the steady state errors in the responses to step and polynomial inputs, the Schwarz approximation may be modified to build in the time moments matching property. The denominator of the reduced transfer function is still computed from the D-table to insure stability. But the denominator is calculated by equating the coefficients of s^i in the power series expansion, for i = 0, 1, . . ., k-1, (for a k^{th} order model), between the system and the model transfer functions. This guarantees that the same steady state responses are given by the system and the model to step and polynomial inputs up to degree (k-1).

5.5. Modified Schwarz Approximation

Consider the n^{th} dimensional, asymptotical stable, completely controllable and observable system

$$\dot{z}(t) = A_s z(t) + B_s u(t)$$

$$\dot{y}(t) = C_s z(t)$$

where the matrices A_s, B_s and C_s are obtained from the equations (5.2), (5.14) and (5.15).

Let K be an nxn symmetric positive definite matrix

$$K = \int_0^\infty e^{A_s t} B_s B_s^T e^{A_s^T t} dt$$

and satisfying the Lypunov equation

$$A_s K + K A_s^T + B_s B_s^T = 0. \tag{5.59}$$

Using equations (3.50), (3.57) and (3.58), the expression (5.59) becomes:

$$\bar{Z} V = \bar{W} \tag{5.60}$$

where

$$\bar{Z} = \bar{Z}_n = \begin{bmatrix} \bar{A}_n & E_n \\ \bar{F}_n & \bar{Z}_{n-1} \end{bmatrix}_{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}} \tag{5.61}$$

$$\bar{V}=[k_{11},k_{12},111,k_{1n},k_{22},\ldots,k_{2n},\ldots,k_{nn}]^{T}_{1x\frac{n(n+1)}{2}}$$

$$\bar{W}=[0,\ 0,\ \ldots,\ 0,\ 1]^{T}_{1x\frac{n(n+1)}{2}}$$

To demonstrate, consider n = 5 in which case

$$\bar{A}_5 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -\gamma_5 & 0 & 1 & 0 & 0 \\ 0 & -\gamma_4 & 0 & 1 & 0 \\ 0 & 0 & -\gamma_3 & 0 & 1 \\ 0 & 0 & 0 & -\gamma_2 & -\gamma_1 \end{bmatrix}$$

$$\bar{F}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_5 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\overline{Z}_4 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma_4 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_2 & -\gamma_1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2\gamma_4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma_4 & 0 & -\gamma_3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_4 & 0 & -\gamma_2 & -\gamma_1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\gamma_3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\gamma_3 & -\gamma_2 & -\gamma_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\gamma_2 & -2\gamma_1 \end{bmatrix}.$$

After some simple algebraic manipulations, we have

$$k_{ij} = 0 \quad \text{for } i \neq j, \quad i,j = 1, 2, 3, 4, 5$$

and

$$k_{ii} = \frac{1}{\gamma_{n-i+1}} k_{i+1,i+1} \quad \text{for } i = 1, 2, 3, 4, 5$$

$$k_{55} = \frac{1}{2\gamma_1}$$

thus

$$K = \text{diag}\{\frac{1}{2\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5}, \frac{1}{2\gamma_1\gamma_2\gamma_3\gamma_4}, \frac{1}{2\gamma_1\gamma_2\gamma_3}, \frac{1}{2\gamma_1\gamma_2}, \frac{1}{2\gamma_1}\}$$

In general,

$$k_{ij} = 0 \quad \text{for } i \neq j \quad \text{and} \quad i,j = 1, 2, \ldots, n$$

$$k_{ii} = \frac{1}{\gamma_{n-i+1}} k_{i+1,i+1}$$

or

$$k_{ii} = \frac{1}{2\gamma_1 \cdot \cdot \cdot \gamma_{n-i+1}} \quad \text{for } i = 1, \ldots, n-1 \qquad (5.62)$$

and

$$k_{nn} = \frac{1}{2\gamma_1}.$$

Hence,

$$K = \text{diag}\{\frac{1}{2\gamma_1 \cdot \cdot \cdot \gamma_n}, \ldots \frac{1}{2\gamma_1}\} \qquad (5.63)$$

or

$$K = \text{diag}\{\frac{1}{2r_{n-1,1}r_{n,1}}, \frac{1}{2r_{n-2,1}r_{n-1,1}}, \ldots, \frac{1}{2r_{01}r_{11}}\}.$$

Astrom [13], working in the frequency domain, has derived an equation, equivalent to the time domain equation (3.50). To introduce Astrom's result, let the impulse response energy be

$$E = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} G(s)G(-s)ds \tag{5.64}$$

where $G(s) = C(sI-A)^{-1}B$

Let g(t) be the impulse response, i.e., the inverse Laplace transform of G(s), is $g(t) = Ce^{At}B$

From Parseval's theorem, equation (5.64) can be rewritten as

$$||g||^2 = \int_0^\infty g(t)g^T(t)dt$$

or

$$E = ||g||^2 = \int_0^\infty g^2(t)dt \tag{5.66}$$

where $||g||$ is the RMS value of g(t).

Astrom [13] has shown that

$$E = \sum_{i=1}^{n} \frac{\sigma_i^2}{2\delta_i} \tag{5.67}$$

where σ_i and δ_i are obtained from the δ- and σ- Routh arrays or the D and N tables for G(s).

Let $g_k(t)$ be the impulse response of the k^{th} order Routh or Schwarz approximant, i.e., the inverse Laplace transform of $G_k(s)$. Then

$$E_k = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} G_k(s) G_k(-s) ds$$

or

$$E_k = E_{k-1} + \frac{\sigma_k^2}{2\delta_k} \tag{5.68}$$

$$= E_{k-1} + \frac{s_{k-1,1}^2}{2r_{k1}r_{k-1,1}}$$

for $k = 1, 2, \ldots, n$, and $E_0 = 0$.

Since all the δ_i's are positive, it follows from [61], that

$$0 \leq E_1 \leq E_2 \leq \ldots \leq E_n = E$$

Hence the impulse response energies of Routh or Schwarz approximants converge monotonically to the impulse response energy of the system.

To estimate the participation of each state variable, Hutton et al. [61], proposed to evaluate its contribution to the total impulse response energy to the output of the system, by computing the ratio E_k/E. The order of the model is then determined by picking the smallest k for which the RMS value $||g_k||$ is greater than about 90 percent of $||g||$. For the same purpose, Arumugan et al. [11] proposed the following comparison process:

(a) If $\frac{\gamma_{n-i}}{\gamma_{n-i-1}} > 10$ for $i = 1, \ldots, n-2$,

then the comparison is successful, where the figure 10 is found to be satisfactory for the problems considered by the authors. This figure depends upon the variables of interest to be retained in the model.

(b) If $\frac{\gamma_{n-i}}{\gamma_{n-i-1}} < 1$ the comparison process is termined.

(c) If no comparison is successful, then the eigenvalues of the system are of the same order of magnitude, which implies that no reduction is possible.

If the i^{th} comparison is the last successful one, then the system can be reduced to the k^{th} order, where $k = n-i$ and the last i state variables are assumed to have negligible effects.

This comparison process is similar to Davison's dominance criterion [34], since the diagonal elements of the matrix K dictate the contribution of the input to the states. For the purpose of this analysis, the states to be retained are those associated with the elements of the diagonal matrix K with the highest magnitude. Once the dominant state variables are selected, the γ_i's that are left are rearranged to give the following k^{th} order reduced model:

$$\dot{z}_k(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ -\gamma_k' & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & -\gamma_2' & -\gamma_1' \end{bmatrix} z_k(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \qquad (5.69)$$

$$y_k(t) = [\bar{c}_k', \ldots \bar{c}_1'] z_k(t)$$

and transfer function

$$G_k(s) = \frac{\beta_1' s^{k-1} + \ldots + \beta_k'}{s^k + \alpha_1' s^{k-1} + \ldots + \alpha_k'} . \qquad (5.70)$$

In order to determine the α_i's and β_i's, let $p_k(s)$ be the characteristic polynomial of the submatrix comprising the last k rows and columns of the Schwarz matrix A_s of equation (5.15), then

$$
\begin{aligned}
p_0(s) &= 1 \\
p_1(s) &= s + \gamma_1 \\
p_2(s) &= s^2 + \gamma_1 s + \gamma_2 \\
&\vdots \\
p_\ell(s) &= s p_{\ell-1}(s) + \gamma_\ell p_{\ell-2}(s) \\
&\vdots \\
p(s) = p_n(s) &= s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n
\end{aligned}
\tag{5.71}
$$

$p_0(s)$ is chosen so that the recursive definition of $p_\ell(s)$ is valid for $\ell \geq 1$.

Equations (5.71) can be grouped to form the following matrix equation

$$\begin{bmatrix} p_0(s) & \cdots & p_0(s) & p_0(s) \\ s & \cdots & p_1(s) & p_1(s) \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ s^{n-1} & \cdots & p_{n-1}(s) & q_{n-1}(s) \\ s^n & \cdots & sp_{n-1}(s) & p_n(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ s & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ s^{n-1} & s^{n-2} & \cdots & 1 & 0 \\ s^n & s^{n-1} & \cdots & s & 1 \end{bmatrix} T_4$$

where $q_{n-1}(s) = (p_n(s) - t_{n+1,n+1})/s.$ (5.72)

The $(n+1)x(n+1)$ matrix T_4 is given as follows:

$$T_4 = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} & t_{1n+1} \\ 0 & t_{22} & \cdots & t_{2n} & t_{2n+1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & t_{nn} & t_{nn+1} \\ 0 & 0 & \cdots & 0 & t_{n+1,n+1} \end{bmatrix} \tag{5.73}$$

where

$$t_{ij} = \begin{cases} 0 & \text{for } i > j \\ 1 & \text{for } i = 1 \text{ and } j = 1, \ldots, n+1 \\ \gamma_1 & \text{for } i = 2 \text{ and } j = 2, \ldots, n+1 \end{cases} \tag{5.74}$$

and

$$t_{ij} = t_{i,j-1} + t_{i-2,j-2}\gamma_{j-1}$$

for $i = 3, \ldots, n+1$ and $j = 3, \ldots, n+1$

Hence, $$p_\ell(s) = t_{1,\ell+1}s^\ell + t_{2,\ell+1}s^{\ell-1} + \ldots + t_{\ell+1,\ell+1} \quad (5.75)$$

for $\ell = 0, 1, \ldots, n$

Therefore, the coefficients $\alpha'_1, \ldots, \alpha'_k$ of $G_k(s)$ are determined as follows. Let

$$t'_{ij} = \begin{cases} 0 & \text{for } i > j \\ 1 & \text{for } i = 1 \text{ and } j \geq 1 \\ \gamma'_1 & \text{for } i = 2 \text{ and } j \geq 1 \end{cases}$$

and

$$t'_{ij} = t'_{i,j-1} + t'_{i-2,j-2}\gamma'_{j-1}$$

for $i = 3, \ldots, k+1$ and $j = 3, \ldots, k+1$.

Also let $\bar{p}_k(s) = s^k + \alpha'_1 s^{k-1} + \ldots + \alpha'_k$ be the characteristic polynomial of the model state matrix, then from equation (5.75),

$$\bar{p}_k(s) = t'_{1,k+1}s^k + t_{2,k+1}s^{k-1} + \ldots + t'_{k+1,k+1}$$

thus

$$\alpha'_{\ell} = t'_{\ell+1,k+1} \tag{5.76}$$

for $\ell = 0, 1, \ldots, k$

and

$$\alpha'_0 = t'_{1,k+1} = 1$$

Similarly, since the system and the model are supposed to have the same first k time moments, then from equation (5.58), the coefficients β'_i's are given by

$$\beta'_{k-\ell} = \alpha'_k m_{\ell} + \sum_{j=1}^{\ell} \alpha'_{k-j} m_{\ell-j} \tag{5.77}$$

for $\ell = 0, 1, \ldots, k-1$. To determine the output vector C_{8k}, we first need to find $\delta'_1, \ldots, \delta'_k$, using either the following sequence

$$\delta'_i = \frac{1}{\gamma'_i}, \quad \delta'_2 = \frac{\gamma'_i}{\gamma'_2}, \quad \delta'_3 = \frac{\gamma'_2}{\gamma'_1\gamma'_3}, \quad \delta'_4 = \frac{\gamma'_1\gamma'_3}{\gamma'_2\gamma'_4}, \quad \delta'_5 = \frac{\gamma'_2\gamma'_4}{\gamma'_1\gamma'_3\gamma'_5}, \quad \ldots,$$

or the δ array with $\alpha'_1, \ldots, \alpha'_k$. Similarly, $\sigma'_1, \ldots, \sigma'_k$ are found from the σ array, with $\beta'_1, \ldots, \beta'_k$. Finally, the components of the model output vector C_{8k} are given as

$$c_i^{-1} = \frac{\sigma'_i}{\delta'_1 \cdots \delta'_i} \qquad \text{for } i = 1, \ldots, k$$

This technique can be summarized by the following algorithm.

5.5.1. Modified Schwarz Approximation Algorithm

Given the state space representation (in canonical form), or the transfer function of the system (5.1),

(1) Construct the denominator and numerator Routh arrays using equations (5.27) and (5.28).

(2) From equations (5.11) - (5.15), find the similarity transformation matrix T_2 and the Schwarz system.

(3) Using equation (5.62), form the Lyapunov matrix K.

(4) Determine the time moments of the origin system, using equation (5.58).

(5) Compute the impulse response energies and the ratio $J_k = 100(E_k/E)$, for $k = 1, \ldots, n$.

(6) Find the value of k for which $(100E_k/E) > 50$, where the figure 50 seems to be very satisfactory for all the examples considered.

(7) Select the k diagonal elements of the matrix K, with the highest magnitude, and their associated states.

(8) Rearrange the remaining components of the Schwarz matrix, according to equation (5.69).

(9) Find the coefficients of the k^{th} order model transfer function (5.70), using equations (5.76) and (5.77).

We now illustrate this algorithm by the following example.

5.6. Illustrative Example

Consider the following 5^{th} order system matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2.37305 & -7.25625 & -9.49688 & -7.5625 & -3.65 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = [9.775,\ 7.125,\ 5,\ 6.5,\ 11.75]$$

and transfer function

$$H(s) = \frac{11.75s^4+6.5s^3+5s^2+7.125s+9.775}{s^5+3.65s^4+7.5625s^3+9.49688s^2+7.25625s+2.37305}.$$

The following calculations then follow.

Denominator Routh Array

1	7.5625	7.25625	0
3.65	9.49688	2.37305	0
4.96062	6.6061	0	0
4.63614	2.37305	0	0
4.06696	0	0	0
2.37305	0	0	0

Numerator Routh Array

11.75	5	9.775	0
6.5	7.125	0	0
-25.57215	2.13573	0	0
-1.53111	0	0	0
15.22507	0	0	0
0	0	0	0

The system poles are:

$\lambda_1 = -0.45 + j1.35$

$\lambda_2 = -0.45 - j1.35$

$\lambda_3 = -0.75$

$\lambda_4 = -1 + j0.75$

$\lambda_5 = -1 - j0.75$

then

i	γ_i	δ_i	σ_i
1	3.65	0.27397	3.21918
2	4.96062	0.73579	1.31032
3	1.27018	1.06999	-5.51583
4	0.81985	1.13995	-0.37648
5	0.51186	1.71381	6.41582

The similarity transformation matrix T_2 is given by

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0.51186 & 0 & 1 & 0 & 0 \\ 0 & 1.33171 & 0 & 1 & 0 \\ 0.65015 & 0 & 2.60188 & 0 & 1 \end{bmatrix}$$

The Schwarz system matrices are then:

$$A_S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -0.51186 & 0 & 1 & 0 & 0 \\ 0 & -0.81985 & 0 & 1 & 0 \\ 0 & 0 & -1.27018 & 0 & 1 \\ 0 & 0 & 0 & -4.96062 & -3.65 \end{bmatrix}$$

$$B_s = [0,\ 0,\ 0,\ 0,\ 1]^T$$

and

$$C_s = [15.22507,\ -1.53111,\ -25.57215,\ 6.5,\ 11.75].$$

This system is stable, controllable and observable. Its Lyapunov matrix is found to be

$$K = \text{diag}\{0.05181,\ 0.02652,\ 0.02174,\ 0.02761,\ 0.13699\}$$

and

i	m_i	E_i	J_i(%)
0	4.11917	0	0
1	-9.59303	18.91267	40.78835
2	14.9555	20.07939	43.30459
3	-17.72749	34.29654	73.96625
4	23.5422	34.35871	74.10032
5	-35.68313	46.36783	100

Since $J_3 = 73.96625 > 50$, the order of the model is 3 and the three diagonal components of K with the highest magnitude are k_{11}, k_{44} and k_{55}, and the corresponding states are x_1, x_4 and x_5. Therefore, the 3rd order model has the following representation

$$\dot{z}_3(t) = \begin{bmatrix} 0 & 1 & 0 \\ -1.27018 & 0 & 1 \\ 0 & -4.96062 & -3.65 \end{bmatrix} z_3(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y_3(t) = [-12.14768,\ -18.80907,\ 24.53872]\, z_3(t)$$

and transfer function

$$H_3(s) = \frac{19.09712s^2+18.80907s+24.59872}{s^3+3.65s^2+6.2308s+4.63615}$$

The 3rd order model is stable, controllable and observable.

From Figure 5.3, we can see that the model step response tracks the system step response.

For comparison purposes, we have also plotted, in Figure 5.4, the step responses of the 5th order system, 4th, 3rd and 2nd order models,

where

$$H_4(s) = \frac{10.45916s^3+2.43484s^2+3.35094s+2.61698}{s^4+3.65s^3+6.74266s^2+6.50445s+2.53914}$$

and

$$H_2(s) = \frac{20.43364s-32.55241}{s^2+3.65s+4.96062}$$

From the plots of Figure 5.4, we notice that the major difference on the responses of the models is on the transient part of the outputs, and the only model with a minor difference from the response of the system is the 3^{rd} order model.

5.7. Conclusion

In this chapter we have described the Routh and Schwarz approximations and their link. We have also introduced a new version of the Schwarz method. This new scheme combines the advantages of the Routh and Schwarz and can be used either for time domain or frequency domain design of reduced order models. An example has been worked out to illustrate the goodness of this new procedure.

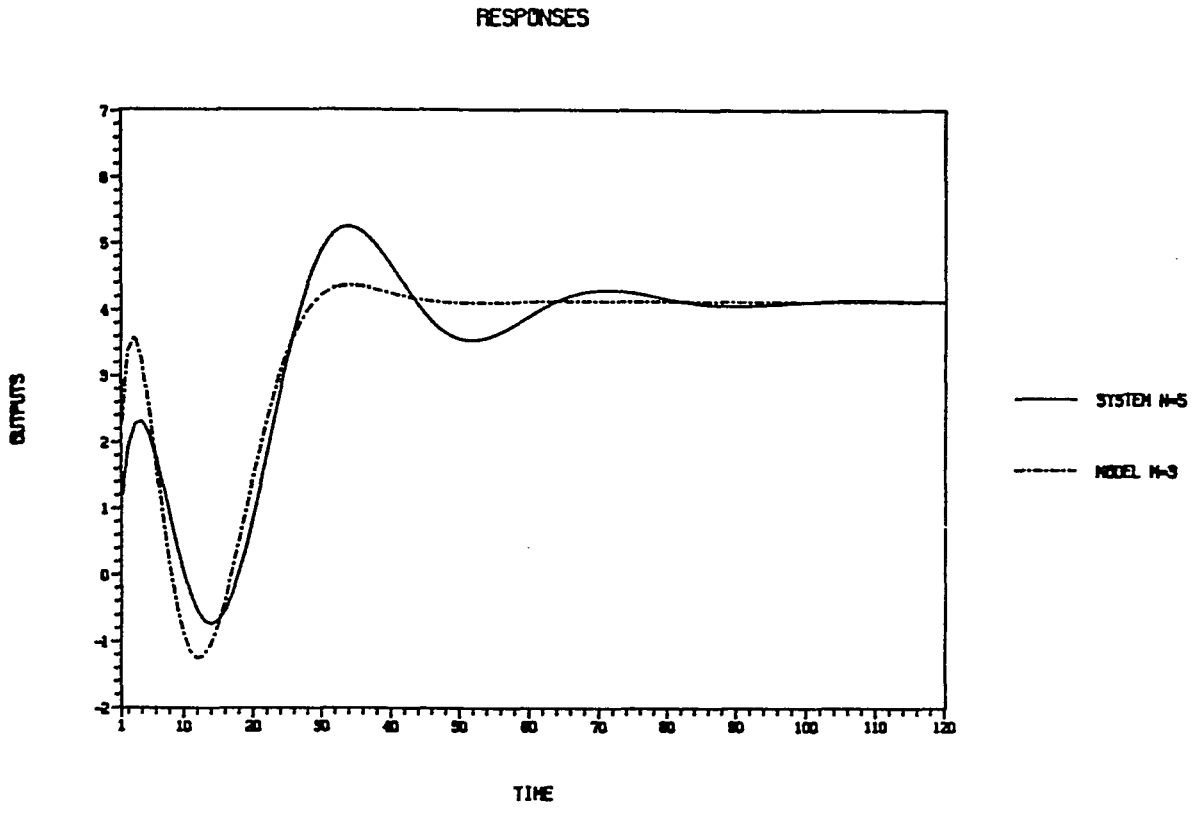


Figure 5.3. Step Responses of the 5^{th} Order System and Its 3^{rd} Order Model.

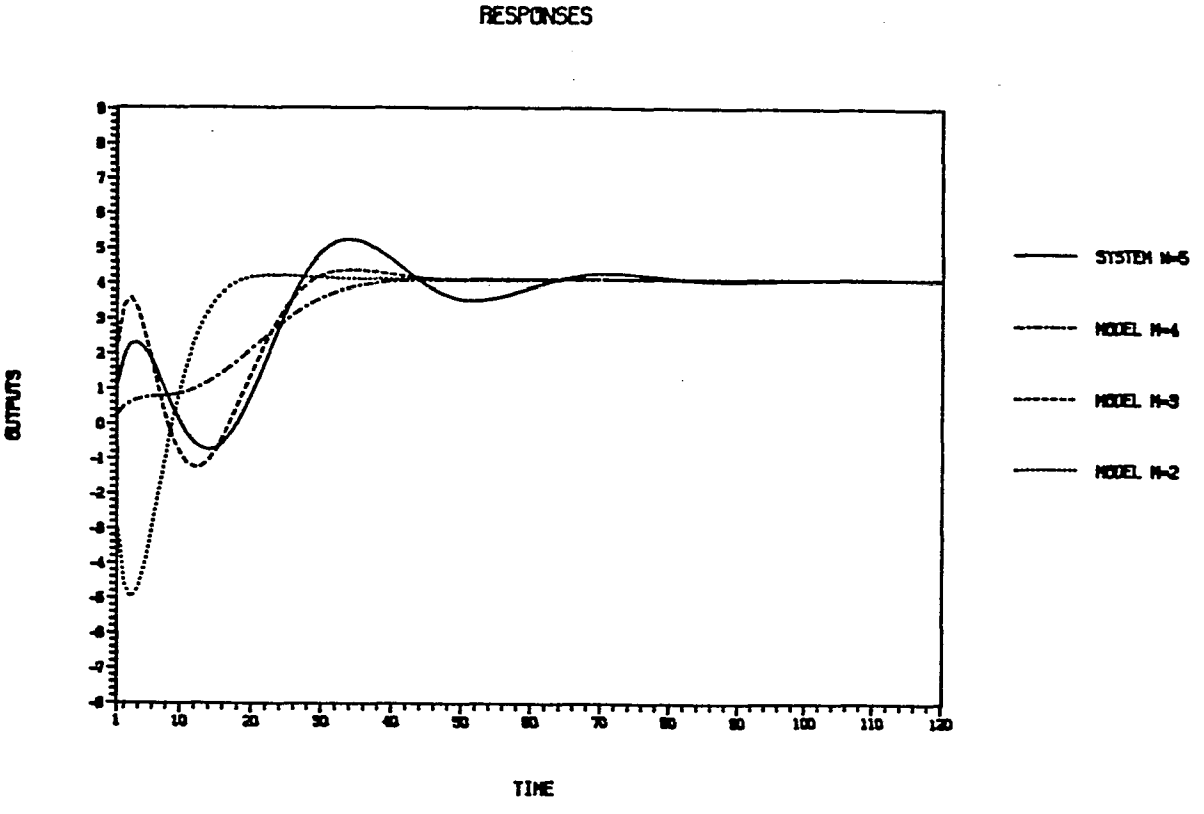


Figure 5.4. Step Responses of the 5^{th} Order System and Its 4^{th}, 3^{rd} and 2^{nd} Order Models.

CHAPTER VI

CONCLUSION

6.1. Work In Retrospect

In this dissertation an attempt has been made to present, in an unified way, a survey of the existing technique of system order reduction. Due to the number of more commonly used methods and their characteristics, if one is interested in investigating some aspect of performance of order reduction problem, one should select the most appropriate method or a combination of methods to achieve one's goal. Unlike in any other theory, the order reduction problems have approximate solutions only. However, most practical objectives are not concerned with exact models, but rather with the ability to approximate systems which may be in some sense exactly reducible. Furthermore, the exact models are difficult, if not impossible, to obtain. Therefore, a better understanding of the exactness of the model will assist in defining rational approaches to the approximation problems.

Some of the techniques have characteristics such as the measure of the degree of controllability and/or

observability of the system. In Chapter III, a new algorithm for the computation of the controllability and observability Grammians is presented. This chapter introduces the mathematical material to be used in the chapter following it. Chapter IV uses the combination of the aggregation and state feedback concepts to develop a technique that yields a reduced order model whose steady state output covariances and Markov parameters match those of the original system.

This aggregation method seeks to form a model whose state vector is a projection of the system state vector. The projection is made on the most controllable subspace of the state space. Therefore, the controllability Grammian is used to define a measure of the controllability in certain directions of the state space. The reduced model is then obtained by truncating the least controllable state variables of the system. The advantages of this approach over other techniques is that it uses the projection of the original system and does not change the input and output vector dimensions. Hence, the method can be used for time invariant model reduction, time varying model, as well as for controller reduction.

Chapter V describes in great detail the Routh and Schwarz approximations and introduces a new version of

the Schwarz approximation that matches the first r time moments of the system to those of the model. Unlike the Schwarz technique, this approach utilizes the impulse response energy of the system to predict the order of the model and the controllability Grammian to select the states to be retained in the model.

In Appendix A, a critical review of Eitelberg's model reduction technique is given in which it is shown that if the models obtained by this method are not controllable, then they are unstable and vice versa. In Appendix B, an algorithm for the computation of the frequency reponses of the system and the model is presented.

6.2. Directions for Future Research

The theory presented in the previous chapters relies on the use of the controllability Grammian. An interesting extension of this theory would be to include the observability Grammian to that the dominance measure would indicate the choice of the measured and manipulated variables which maximizes the controllability and observability Grammians of the system.

Similarly, the aggregation method of Chapter IV could be improved using the combination of the algorithms derived by Kwong [73] and Miminis et al. [88]. Furthermore, despite the need of model reduction of systems operating under varying conditions in areas such as

aerospace control, navigation systems, etc., the time varying model reduction has not attracted much attention. Therefore, a more general version of the time varying analogy of the method proposed in Chapter IV deserves to be investigated. Another avenue for future research involves the further development of the techniques presented in the previous chapters to include one of the following items:

(1) controller or estimator reduction

(2) distributed system reduction

(3) delay system reduction

(4) two dimensional system reduction

(5) simplification of the singular systems with uncertainty.

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APPENDIX A

Equation Error Technique

Given an n-dimensional system,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

Eitelberg [41] proposed a technique to determine the m-dimensional model:

$$\dot{\tilde{x}}(t) = A_r\tilde{x}(t) + B_r u(t)$$

$$\tilde{y}(t) = C_r\tilde{x}(t)$$

where the state vector $\tilde{x}(t)$ reproduces the desired components of $x(t)$ and/or their linear combinations, such that $x_r(t) = Rx(t)$
where R is an mxn reduction matrix and $x_r(t)$ contains all the components that influence the output directly so that $C = C_rR$

In order to determine the parameters of the model, the author took the following approximation

$$\tilde{x}(t) \simeq x_r(t) = Rx(t)$$

and used the equation error

$$d(t) = \dot{x}_r(t) - A_r x_r(t) - B_r U(t)$$

instead of

$$d(t) = x_r(t) - \tilde{x}(t)$$

To derive the following reduced order model matrices,

$$A_r = RASR^T(RSR^T)^{-1}$$

$$B_r = A_r RA^{-1}B$$

where

$$S = \int_0^\infty e^{At}(A^{-1}B)Q(A^{-1}B)^T e^{A^T t} dt$$

and Q is a weighting matrix.

Note that the author is assuming that the original system is asymptotically stable, i.e., $\mathrm{Re}\{\lambda_i(A)\} < 0$, for $1 \leq i \leq n$ and completely controllable. Hence, if λ_i is an eigenvalue of A and ν_i its associated nonzero left eigenvector such that $\nu_i\lambda_i = \nu_i A$ or $\nu_i\lambda_i A^{-1} = \nu_i I$, then $\nu_i\lambda_i A^{-1}B = \nu_i B \neq 0$ and $A^{-1}B \neq 0$.

Therefore, $RA^{-1}B \neq 0$. Otherwise, no reduction takes place. The purpose of this appendix is to show that the models obtained by the equation error approach are uncontrollable, if they are not asymptotically stable, i.e., $Re\{\lambda_i(A_r)\} \geq 0$ for $1 \leq i \leq m$, and vice versa.

(1) First, we have to show that if the model is not asymptotically stable, then it is not controllable and the transfer function of the model is reducible. In other words, more order reduction with pole shifting needs to be done. Let λ_i be an eigenvalue of A_r and suppose that $\lambda_i = 0$. Let the characteristic polynomial of A_r be:

$$p(\lambda_i) = \lambda_i^m + \xi_1\lambda^{m-1} + \ldots + \xi_{m-1}\lambda_i + \xi_m = 0.$$

From the Cayley-Hamilton theorem, we have

$$A_r^{m-1} + \xi_1 A_r + \ldots + \xi_{m-1}A_r + \xi_m I = 0$$

since

$\lambda_i = 0$ and $p(\lambda_i) = 0$, then $\xi_m = 0$ and A_r^m is a linear combination of $A_r^{m-1}, \ldots, A_r^2, A_r$. Now, consider the model controllability matrix,

$$M_c = [B_r, A_rB_r, \ldots, A_r^{m-1}B_r]$$

since

$$B_r = A_rRA^{-1}B$$

it follows that

$$M_c = [A_r, \ldots, A_r^{m-1}] \begin{bmatrix} RA^{-1}B & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & RA^{-1}B \end{bmatrix} .$$

But rank $RA^{-1}B = m$, hence

$$\text{rank} \begin{bmatrix} RA^{-1}B & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & RA^{-1}B \end{bmatrix} = m^2 \text{ and rank } [A_r, \ldots, A_r^m] < m .$$

Therefore, rank $M_c < m$, and the model is not controllable.

(2) We now show that, if the model is not controllable, then it is not asymptotically stable. The Popov-Belevitch-Hautus (PBH) controllability test stated that: If a model is not controllable then there exists a nonzero left eigenvector ν_i of A_r with λ_i, its associated eigenvalue, such that

$$\nu_i(\lambda_i I - A_r, B_r) = 0 .$$

In other words, there exists a nonzero left eigenvector of A_r orthogonal to all the columns of B_r, i.e.,

$$\lambda_i \nu_i = \nu_i A_r \quad \text{and} \quad \nu_i B_r = 0 .$$

From the Cayley-Hamilton theorem, we have

$$A_r^m + \xi_1 A_r^{m-1} + \ldots + \xi_{m-1} A_r + \xi_m I = 0 .$$

The pre- and post-multiplication of this matrix equation by ν_i and $RA^{-1}B$ yields

$$\nu_i A_r^m RA^{-1}B + \xi_1 \nu_i A_r^{m-1} RA^{-1}B + \ldots$$

$$\ldots + \xi_{m-1} \nu_i A_r RA^{-1}B + \xi_m \nu_i RA^{-1}B = 0.$$

This equation can also be written as follows:

$$\nu_i [A_r, \ldots, A_r^m] \begin{bmatrix} RA^{-1}B & \ldots & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \ldots & RA^{-1}B \end{bmatrix} \begin{bmatrix} \xi_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} + \xi_m \nu_i RA^{-1}B = 0 .$$

We can now prove the second part by contradiction. Suppose that the model is controllable, but unstable, that is $\lambda_i = 0$, then $\xi_m = 0$

and

$$\nu_i[A_r,\ldots,A_r^m]\begin{bmatrix} RA^{-1}B & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & RA^{-1}B \end{bmatrix}\begin{bmatrix} \xi_{m-1} \\ \vdots \\ 1 \end{bmatrix} = 0$$

or

$$[\nu_i A_r RA^{-1}B,\ldots,\nu_i A_r^m RA^{-1}B]\begin{bmatrix} \xi_{m-1} \\ \vdots \\ 1 \end{bmatrix} = 0 .$$

Since $B_r = A_r RA^{-1}B$ and $\xi_i \neq 0$ for $1 \leq i \leq m-1$,

we have

$$[\nu_i B_r,\ \nu_i A_r B_r,\ \ldots,\ \nu_i A_r^{m-1} B_r] = 0$$

and

$$[\nu_i B_r,\ \lambda_i \nu_i B_r,\ \ldots,\ \lambda_i^{m-1}\nu_i B_r] = 0 .$$

But $\lambda_i = 0$, hence

$$[\nu_i B_r, 0, \ldots, 0] = 0$$

or

$$[1, 0, \ldots, 0]\nu_i B_r = 0.$$

Thus, $\nu_i B_r = 0$. This contradicts the assumption that the model is controllable. The model is uncontrollable, if it is unstable.

Therefore, the equation error method cannot be used if the high order system has very sensitive eigenvalues or an ill conditioned state matrix.

APPENDIX B

Computational Method for Evaluating the Magnitude and Phase of the Frequency Response of a System

Many small computers have compilers without the capability of complex arithmetic. To overcome this drawback, we will introduce an algorithm for the computation of the magnitude and phase of the frequency response of a continuous time invariant linear system, that does not require any complex arithmetic.

Problem

Given the transfer function

$$H(s) = \frac{\beta_o s^m + \beta_1 s^{m-1} + \ldots + \beta_m}{\alpha_o s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n} = \frac{q(s)}{p(s)}$$

where $n > m$ and $\alpha_0 = 1$

or given the sequence $\{\alpha_o, \alpha_1, \ldots, \alpha_n, \beta_o, \ldots, \beta_m\}$. Determine the magnitude $|H(jw)|$ and phase $\measuredangle H(jw)$, associated with $H(s)$.

Let the complex number $s = a + jb$, where a and b are real numbers and $j = \sqrt{-1}$, and

$$E = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

be a skew symmetric matrix.

To solve the problem above, we shall derive an algorithm which is based upon the mapping $s \longleftrightarrow E$. The matrix E is also a normal matrix, it commutes with its complex conjugate transpose.

We now investigate the properties of the map

$$s = a + jb \longleftrightarrow E = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$s^* = a - jb \longleftrightarrow E^* = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$s + s^* = 2a \longleftrightarrow E + E^* = 2aI_2$$

$$ss^* = a^2 + b^2 \longleftrightarrow EE^* = E^*E = (a^2 + b^2)I_2$$

where I_2 is the 2x2 identity matrix. Then, this mapping preserves addition and multiplication, and it is actually an isomorphism. Therefore, E has all the properties of a complex number, and in particular,

the following property

$$s \longleftrightarrow E$$

$$s^2 = (a+jb)^2 = (a^2-b^2) + j2ab \longleftrightarrow E^2 = \begin{bmatrix} a^2-b^2 & 2ab \\ -2ab & a^2-b^2 \end{bmatrix}$$

$$s^3 = (a+jb)^3 = (a^3-3ab^2) + j(3a^2b-b^3) \longleftrightarrow E^3$$

where

$$E^3 = \begin{bmatrix} a^3-3ab^2 & 3a^2b-b^3 \\ -3a^2b+b^3 & a^3-3ab^2 \end{bmatrix}$$

then

$$s^i = (a+jb)^i \longleftrightarrow E^i = \begin{bmatrix} \mathrm{Re}\{(a+jb)^i\} & \mathrm{Im}\{(a+jb)^i\} \\ -\mathrm{Im}\{(a+jb)^i\} & \mathrm{Re}\{(a+jb)^i\} \end{bmatrix}$$

To determine the expressions for $\mathrm{Re}\{(a+jb)^i\}$ and $\mathrm{Im}\{(a+jb)^i\}$, we have to use either Pascale triangle or the following two tables:

Real Part

n	a^n	$a^{n-2}b^2$	$a^{n-4}b^4$	$a^{n-6}b^6$	. . .
1	1	0	0	0	. . .
2	1	-1	0	0	. . .
3	1	-3	0	0	. . .
4	1	-6	0	0	. . .
5	1	-10	5	0	. . .
6	1	-15	15	-1	. . .
7	1	-21	35	-7	. . .
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	

TABLE 1. Coefficients of $\mathrm{Re}\{(a+jb)^n\}$

Imaginary Part

n	$a^{n-1}b$	$a^{n-3}b^3$	$a^{n-5}b^5$	$a^{n-7}b^7$	. . .
1	1	0	0	0	. . .
2	2	0	0	0	. . .
3	3	-1	0	0	. . .
4	4	-4	0	0	. . .
5	5	-10	1	0	. . .
6	6	-20	6	0	. . .
7	7	-35	21	-1	. . .
.	.	.	.	.	
.	.	.	.	.	
.	.	.	.	.	

TABLE 2. Coefficients of $\mathrm{Im}\{(a+jb)^n\}$

since

$$s^i = (a+jb)^i \longleftrightarrow E^i = \begin{bmatrix} \mathrm{Re}\{s^i\} & \mathrm{Im}\{s^i\} \\ -\mathrm{Im}\{s^i\} & \mathrm{Re}\{s^i\} \end{bmatrix}$$

and $\quad p(s) = \alpha_o s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$

then $p(E) = \alpha_o E^n + \alpha_1 E^{n-1} + \ldots + \alpha_{n-1} E + \alpha_n I_2$

where

$$p(E) = \alpha_o \begin{bmatrix} \mathrm{Re}\{s^n\} & \mathrm{Im}\{s^n\} \\ -\mathrm{Im}\{s^n\} & \mathrm{Re}\{s^n\} \end{bmatrix} + \ldots + \alpha_n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p(E) = \begin{bmatrix} \alpha_o \mathrm{Re}\{s^n\} + \alpha_1 \mathrm{Re}\{s^{n-1}\} + \ldots + \alpha_n \\ -\alpha_o \mathrm{Im}\{s^n\} - \alpha_1 \mathrm{Im}\{s^{n-1}\} - \ldots - \alpha_{n-1} \mathrm{Im}\{s\} \end{bmatrix}$$

$$\begin{bmatrix} \alpha_o \mathrm{Im}\{s^n\} + \alpha_1 \mathrm{Im}\{s^{n-1}\} + \ldots + \alpha_{n-1} \mathrm{Im}\{s\} \\ \alpha_o \mathrm{Re}\{s^n\} + \alpha_1 \mathrm{Re}\{s^{n-1}\} + \ldots + \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Re}\{\alpha_o s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n\} \\ -\mathrm{Im}\{\alpha_o s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n\} \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{Im}\{\alpha_o s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n\} \\ \mathrm{Re}\{\alpha_o s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n\} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Re}\{p(s)\} & \mathrm{Im}\{p(s)\} \\ -\mathrm{Im}\{p(s)\} & \mathrm{Re}\{p(s)\} \end{bmatrix}$$

Hence $p(s) \longleftrightarrow p(E)$,

and

$$p(E) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\{p(S)\} \\ -\mathrm{Im}\{p(S)\} \end{bmatrix}$$

Let T be a similarity transformation matrix such that for $b \neq 0$,

$$T = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 0 & -1/b \\ 1 & -a/b \end{bmatrix}$$

and $E_c = T^{-1}ET$

$$E_c = \begin{bmatrix} 0 & -1/b \\ 1 & -a/b \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -(a^2+b^2) & 2a \end{bmatrix}$$

Note that the eigenvalues of E and E_c are $\lambda_1 = s$ and $\lambda_2 = s^*$.

Furthermore, $E_c^{\ell} = (T^{-1}ET)^{\ell} = T^{-1}E^{\ell}T$

thus $T^{-1}p(E)T = T^{-1}(\alpha_o E^n+\alpha_1 E^{n-1} + \ldots + \alpha_{n-1}E+\alpha_n I_2)T$

$$= \alpha_o(T^{-1}E^nT)+\alpha_1(T^{-1}E^{n-1}T) + \ldots +$$

$$\alpha_{n-1}(T^{-1}ET)+\alpha_n I_2$$

$$= \alpha_o(T^{-1}ET)^n+\alpha_1(T^{-1}ET)^{n-1} + \ldots +$$

$$\alpha_{n-1}(T^{-1}ET)+\alpha_n I_2$$

$$= p(T^{-1}ET) = p(E_c)$$

and

$$p(E)\begin{bmatrix}1\\0\end{bmatrix} = TT^{-1}p(E)TT^{-1}\begin{bmatrix}1\\0\end{bmatrix} = T[T^{-1}p(E)T]T^{-1}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= Tp(E_c)\begin{bmatrix}0\\1\end{bmatrix}$$

then

$$\begin{bmatrix}\text{Re} & p(s)\\ -\text{Im} & p(s)\end{bmatrix} = Tp(E_c)\begin{bmatrix}0\\1\end{bmatrix}, \qquad T = \begin{bmatrix}-a & 1\\ -b & 0\end{bmatrix}$$

To compute $p(E_c)\begin{bmatrix}0\\1\end{bmatrix}$, let $z_0 = \begin{bmatrix}0\\0\end{bmatrix}$ such that $E_c^{n+1}z_0 = \begin{bmatrix}0\\0\end{bmatrix}$ and let, for example, $n = 4$, then

$$p(E_c) = \alpha_0 E_c^4 + \alpha_1 E_c^3 + \alpha_2 E_c^2 + \alpha_3 E_c + \alpha_4 I_2$$

and

$$p(E_c)\begin{bmatrix}0\\1\end{bmatrix} = \alpha_0 E_c^4\begin{bmatrix}0\\1\end{bmatrix} + \alpha_1 E_c^3\begin{bmatrix}0\\1\end{bmatrix} + \alpha_2 E_c^2\begin{bmatrix}0\\1\end{bmatrix} + \alpha_3 E_c\begin{bmatrix}0\\1\end{bmatrix} + \alpha_4\begin{bmatrix}0\\1\end{bmatrix}$$

or

$$p(E_c)\begin{bmatrix}0\\1\end{bmatrix} = E_c^5 z_0 + \alpha_0 E_c^4\begin{bmatrix}0\\1\end{bmatrix} + \alpha_1 E_c^3\begin{bmatrix}0\\1\end{bmatrix} + \alpha_2 E_c^2\begin{bmatrix}0\\1\end{bmatrix} + \alpha_3 E_c\begin{bmatrix}0\\1\end{bmatrix} + \alpha_4\begin{bmatrix}0\\1\end{bmatrix}.$$

We now apply Horner's rule to this expression:

$$p(E_c)\begin{bmatrix}0\\1\end{bmatrix} = E_c\{E_c^4 z_0+\alpha_0 E_c^3\begin{bmatrix}0\\1\end{bmatrix}+\alpha_1 E_c^2\begin{bmatrix}0\\1\end{bmatrix}+\alpha_2 E_c\begin{bmatrix}0\\1\end{bmatrix}+\alpha_3\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_4\begin{bmatrix}0\\1\end{bmatrix}$$

$$= E_c\{E_c\{E_c^3 z_0+\alpha_0 E_c^2\begin{bmatrix}0\\1\end{bmatrix}+\alpha_1 E_c\begin{bmatrix}0\\1\end{bmatrix}+\alpha_2\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_3\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_4\begin{bmatrix}0\\1\end{bmatrix}$$

$$p(E_c)\begin{bmatrix}0\\1\end{bmatrix} = \{E_c\{E_c\{E_c\{E_c^2 z_0+\ {}_0E_c\begin{bmatrix}0\\1\end{bmatrix}+\alpha_1\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_2\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_3\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_4\begin{bmatrix}0\\1\end{bmatrix}\}$$

$$= \{E_c\{E_c\{E_c\{E_c\{E_c z_0+\alpha_0\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_1\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_2\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_3\begin{bmatrix}0\\1\end{bmatrix}\}+\alpha_4\begin{bmatrix}0\\1\end{bmatrix}\}$$

From this rearrangement, we have

$$z_1 = E_c z_0 + \alpha_0\begin{bmatrix}0\\1\end{bmatrix} \qquad \text{with } z_0 = \begin{bmatrix}0\\0\end{bmatrix}$$

$$z_2 = E_c z_1 + \alpha_1\begin{bmatrix}0\\1\end{bmatrix}$$

$$z_3 = E_c z_2 + \alpha_2\begin{bmatrix}0\\1\end{bmatrix}$$

$$z_4 = E_c z_3 + \alpha_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z_5 = E_c z_4 + \alpha_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In general,

$$z_{\ell+1} = E_c z_\ell + \alpha_\ell \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{for } \ell = 0, 1, \ldots, n$$

and

$$p(E_c) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = z_{n+1} \qquad \text{with } z_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\alpha + j\beta = p(s) = \text{Re}\{p(s)\} + j\text{Im}\{p(s)\}$,

thus

$$\begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = \begin{bmatrix} \text{Re}\{p(s)\} \\ -\text{Im}\{p(s)\} \end{bmatrix} = p(E) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Tp(E_c) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = Tz_{n+1} = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix} z_{n+1}$$

The same procedure can be repeated for the numerator of H(s).

Let AN and AD be the numerator and denominator arrays of H(s) defined as

$$AN = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{m-1} \\ \beta_m \end{bmatrix} \quad \text{and} \quad AD = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix}$$

In order to use this algorithm, we have to introduce the following notation

$$AN(j) = \beta_{j-1} \qquad \text{for } j = 1, \ldots, m+1$$

$$AD(i) = \alpha_{i-1} \qquad \text{for } i = 1, \ldots, n+1$$

and

$$z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Algorithm

Given the array AR which is either AN or AD and NM which is either M or N, compute the magnitude and phase of q(s) or p(s)

(1) Let NM1 = NM + 1

(2) x1 = 0

(3) x2 = 0

(4) TA = 2a

(5) TB = $a^2 + b^2$

(6) Begin I = 1, . . ., NM1

(7) Let TC = x1

(8) x1 = x2

(9) x2 = TA·x2 - TB·TC + AR(I)

(10) end loop in I

(11) α = x2 - x1·a

(12) β = x1·b

(13) magnitude = $(\alpha^2 + \beta^2)^{1/2}$

(14) phase = $\tan^{-1}(\beta/\alpha)$

Comments

since $z_{\ell+1} = E_c z_\ell + \alpha_\ell \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $E_c = \begin{bmatrix} 0 & 1 \\ -(a^2+b^2) & 2a \end{bmatrix}$

and $z_\ell = \begin{bmatrix} x1_\ell \\ x2_\ell \end{bmatrix}$

then

$$\begin{bmatrix} x1_{\ell+1} \\ x2_{\ell+1} \end{bmatrix} = \begin{bmatrix} x2_{\ell} \\ -(a^2+b^2)\cdot x1_{\ell}+2\cdot a\cdot x2 +\alpha_{\ell} \end{bmatrix}$$

or

$$\begin{bmatrix} x1_{\ell+1} \\ x2_{\ell+1} \end{bmatrix} = \begin{bmatrix} x2_{\ell} \\ -TB\cdot x1_{\ell}+TA\cdot x2_{\ell}+AR(\ell) \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha \\ -\beta \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -b & 0 \end{bmatrix} z_{n+1}$$

$$= \begin{bmatrix} -a\cdot x1_{n+1} + x2_{n+1} \\ -b\cdot x1_{n+1} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x2_{n+1} - a\cdot x1_{n+1} \\ b\cdot x1_{n+1} \end{bmatrix}$$

In control systems theory, the frequency response of a continuous time invariant system is obtained by replacing s with jw (i.e., a = 0 and b = w), in the

transfer function H(s), as w varies between two fixed values.

$$\text{Since } H(s) = \frac{q(s)}{p(s)}$$

and

$$H(jw) = H(s)\big|_{s=jw} = |H(jw)|e^{j\angle H(jw)}$$

$$H(jw) = \frac{|q(jw)|e^{j\angle q(jw)}}{|p(jw)|e^{j\angle p(jw)}}$$

$$= \left|\frac{q(jw)}{p(jw)}\right| e^{j(\angle q(jw) - \angle p(jw))}$$

then

$$|H(jw)| = \left|\frac{q(jw)}{p(jw)}\right| \quad \text{and} \quad \angle H(jw) = \angle q(jw) - \angle p(jw)$$

$$\text{Let } q(jw) = \alpha_q + j\beta_q \qquad \text{and } p(jw) = \alpha_p + j\beta_p$$

Therefore the algorithm should be used twice to compute α_q, β_q, α_p and β_p so that

$$|H(jw)| = \frac{\sqrt{\alpha_q^2+\beta_q^2}}{\sqrt{\alpha_p^2+\beta_p^2}} = \sqrt{\frac{\alpha_q^2+\beta_q^2}{\alpha_p^2+\beta_p^2}}$$

$$\angle H(jw) = \tan^{-1}\left(\frac{\beta_q}{\alpha_q}\right) - \tan^{-1}\left(\frac{\beta_p}{\alpha_p}\right)$$

$$\mathrm{Re}\{H(jw)\} = |H(jw)| \cos(\angle H(jw))$$

and

$$\mathrm{Im}\{H(jw)\} = |H(jw)| \sin(\angle H(jw))$$

For absolute stability analysis, it is usually enough to have a rough sketch of H(jw), since only the intersects on the real axis of the complex plane are to be determined. However, for complex systems and system designs, these computations may be too tedious to handle and sometimes it may be necessary to know more about the plot itself before the final sketch is made correctly. Therefore, based on the previous algorithm, a FORTRAN computer program is given in Table B.1, for the computation of the magnitude and phase to be used for the Bode plots, as well as the real and imaginary parts of H(jw) for the Nyquist plot. The time delay elements can be included only if a finite number of terms of the exponential function expansion are retained in the transfer function.

```
*       THIS PROGRAM COMPUTES THE MAGNITUDE AND PHASE
*       OF THE FREQUENCY RESPONSE FOR VARIOUS FREQUENCIES
*       N IS THE DEGREE OF THE DENOMINATOR
*       M IS THE DEGREE OF THE NUMERATOR
*       NP IS THE NUMBER OF POINTS IN EACH DECADE
*       ND IS THE NUMBER OF DECADES IN THE FREQUENCY VARIATION
*       ST IS THE STARTING FREQUENCY
        DIMENSION P(10),AD(3),AN(2)
        REAL MREAL,MIMAG
        REWIND 6
        NP=10
        ND=5
        N=2
        M=1
        N1=N+1
        M1=M+1
        ST=0.1
        PI=4.0*ATAN(1.0)
        GAIN=1.0
        WRITE(6,10)
10      FORMAT(25X,'FREQUENCY RESPONSE',/)
        WRITE(6,20)
20      FORMAT(5X,'OMEGA',7X,'AMGS',8X,'AMDB',8X,'MREAL',
     *    7X,'MIMAG',6X,'PHASE',/)
        DO 30 I=1,N1
        READ(5,40) AD(I)
30      CONTINUE
40      FORMAT(F10.5)
        DO 50 I=1,M1
        READ(5,60) AN(I)
50      CONTINUE
60      FORMAT(F10.5)
        DO 70 L=1,NP
70      P(L)=FLOAT(L)
        SIGMA=0.0
        DO 80 I=1,ND
        DO 80 J=1,NP
        OMEGA=ST*P(J)*(10.0**(I-1))
        CALL SPECT(AD,SIGMA,OMEGA,ALPHD,BETD,N1)
        AMGD=SQRT(ALPHD**2+BETD**2)
        CALL CMPHS(ALPHD,BETD,PHASD)
        CALL SPECT(AN,SIGMA,OMEGA,ALPHN,BETN,M1)
        AMGN=SQRT(ALPHN**2+BETN**2)
        CALL CMPHS(ALPHN,BETN,PHASN)
        AMGS=GAIN*AMGN/AMGD
        AMDB=20.0*ALOG10(AMGS)
        PHASE=(PHASN-PHASD)*180.0/PI
        MREAL=AMGS*COS(PHASN-PHASD)
        MIMAG=AMGS*SIN(PHASN-PHASD)
        WRITE(6,90) OMEGA,AMGS,AMDB,MREAL,MIMAG,PHASE
80      CONTINUE
90      FORMAT(5E12.3,1E11.3)
        STOP
        END
```

TABLE B1. Main Program

```
      SUBROUTINE SPECT(AR,SIGMA,OMEGA,ALPHA,BETA,N)
      DIMENSION AR(N)
      X1=0.0
      X2=0.0
      TA=2.0*SIGMA
      TB=SIGMA**2+OMEGA**2
      DO 10 L=1,N
      TC=X1
      X1=X2
      X2=TA*X2-TB*TC+AR(L)
10    CONTINUE
      ALPHA=X2-X1*SIGMA
      BETA=X1*OMEGA
      RETURN
      END
      SUBROUTINE CMPHS(ALPHA,BETA,PHASE)
      PI=4.0*ATAN(1.0)
      ALP=ABS(ALPHA)
      BET=ABS(BETA)
      IF (ALPHA.EQ.0.0.AND.BETA.EQ.0.0) THEN
      WRITE(6,10)
10    FORMAT(10X,'ERROR IN THE COMPUTATION',/)
      ELSE IF (ALPHA.LT.0.0.AND.BETA.EQ.0.0) THEN
      PHASE=PI
      ELSE IF (ALPHA.GT.0.0.AND.BETA.EQ.0.0) THEN
      PHASE=0.0
      ELSE IF (ALPHA.EQ.0.0.AND.BETA.GT.0.0) THEN
      PHASE=PI/2.0
      ELSE IF (ALPHA.EQ.0.0.AND.BETA.LT.0.0) THEN
      PHASE=-PI/2.0
      ELSE IF (ALPHA.LT.0.0.AND.BETA.GT.0.0) THEN
      PHASE=PI-ATAN(BET/ALP)
      ELSE IF (ALPHA.LT.0.0.AND.BETA.LT.0.0) THEN
      PHASE=PI+ATAN(BET/ALP)
      ELSE IF (ALPHA.GT.0.0.AND.BETA.LT.0.0) THEN
      PHASE=-ATAN(BET/ALP)
      ELSE
      PHASE=ATAN(BET/ALP)
      END IF
      RETURN
      END
```

TABLE B1 (Continued). Subroutines.

OMEGA	AMGB	AMDB	MREAL	MIMAG	PHASE
.100E+00	.100E+01	.428E-01	.100E+01	-.424E-01	-.242E+01
.200E+00	.102E+01	.163E+00	.101E+01	-.907E-01	-.510E+01
.300E+00	.104E+01	.339E+00	.103E+01	-.150E+00	-.829E+01
.400E+00	.106E+01	.535E+00	.104E+01	-.224E+00	-.122E+02
.500E+00	.108E+01	.706E+00	.104E+01	-.313E+00	-.167E+02
.600E+00	.110E+01	.807E+00	.102E+01	-.411E+00	-.220E+02
.700E+00	.110E+01	.798E+00	.970E+00	-.510E+00	-.277E+02
.800E+00	.108E+01	.659E+00	.898E+00	-.598E+00	-.337E+02
.900E+00	.105E+01	.387E+00	.807E+00	-.665E+00	-.395E+02
.100E+01	.100E+01	.131E-02	.707E+00	-.707E+00	-.450E+02
.100E+01	.100E+01	.131E-02	.707E+00	-.707E+00	-.450E+02
.200E+01	.542E+00	-.531E+01	.156E+00	-.519E+00	-.733E+02
.300E+01	.349E+00	-.914E+01	.576E-01	-.344E+00	-.805E+02
.400E+01	.257E+00	-.118E+02	.297E-01	-.255E+00	-.834E+02
.500E+01	.204E+00	-.138E+02	.181E-01	-.203E+00	-.849E+02
.600E+01	.169E+00	-.154E+02	.123E-01	-.168E+00	-.858E+02
.700E+01	.144E+00	-.168E+02	.886E-02	-.144E+00	-.865E+02
.800E+01	.126E+00	-.180E+02	.671E-02	-.126E+00	-.869E+02
.900E+01	.112E+00	-.190E+02	.526E-02	-.112E+00	-.873E+02
.100E+02	.100E+00	-.200E+02	.424E-02	-.100E+00	-.876E+02
.100E+02	.100E+00	-.200E+02	.424E-02	-.100E+00	-.876E+02
.200E+02	.501E-01	-.260E+02	.104E-02	-.501E-01	-.888E+02
.300E+02	.334E-01	-.295E+02	.461E-03	-.333E-01	-.892E+02
.400E+02	.250E-01	-.320E+02	.259E-03	-.250E-01	-.894E+02
.500E+02	.200E-01	-.340E+02	.166E-03	-.200E-01	-.895E+02
.600E+02	.167E-01	-.356E+02	.115E-03	-.167E-01	-.896E+02
.700E+02	.143E-01	-.369E+02	.845E-04	-.143E-01	-.897E+02
.800E+02	.125E-01	-.381E+02	.647E-04	-.125E-01	-.897E+02
.900E+02	.111E-01	-.391E+02	.511E-04	-.111E-01	-.897E+02
.100E+03	.100E-01	-.400E+02	.414E-04	-.100E-01	-.898E+02
.100E+03	.100E-01	-.400E+02	.414E-04	-.100E-01	-.898E+02
.200E+03	.500E-02	-.460E+02	.104E-04	-.500E-02	-.899E+02
.300E+03	.333E-02	-.495E+02	.460E-05	-.333E-02	-.899E+02
.400E+03	.250E-02	-.520E+02	.259E-05	-.250E-02	-.899E+02
.500E+03	.200E-02	-.540E+02	.166E-05	-.200E-02	-.900E+02
.600E+03	.167E-02	-.556E+02	.115E-05	-.167E-02	-.900E+02
.700E+03	.143E-02	-.569E+02	.845E-06	-.143E-02	-.900E+02
.800E+03	.125E-02	-.581E+02	.647E-06	-.125E-02	-.900E+02
.900E+03	.111E-02	-.591E+02	.511E-06	-.111E-02	-.900E+02
.100E+04	.100E-02	-.600E+02	.414E-06	-.100E-02	-.900E+02

TABLE B2. Frequency Response

Table B2 gives the results for the transfer function

$$H(s) = \text{Gain}\ \frac{s+1}{s^2+1.414s+1}\ , \quad \text{Gain} = 1.0$$

with s = jw and w varies from 0.1 to 10^4 rad/sec.

In Table B2,

OMEGA represents the frequency w rad/sec.

AMGS represents the magnitude $|H(jw)|$.

AMBD represents $20 \log_{10} |H(jw)|$.

MREAL represents the real part of H(jw), i.e.,{Re H(jw)}.

MIMAG represents the imaginary part of H(jw), i.e., Im{H(jw)}.

PHASE represents the phase of H(jw), i.e., $\measuredangle H(jw)$ in degrees.

Observations

(1) If the numerator and/or the denominator of the transfer function H(s) are given as products of first order factors, i.e.,

$$H(s) = \frac{(1+T_{n1}s)(1+T_{n2}s) \ . \ . \ . \ (1+T_{nm}s)}{(1+T_{d1}s)(1+T_{d2}s) \ . \ . \ . \ (1+T_{dn}s)}$$

or

$$H(s) = \frac{T_{n1}T_{n2} \cdot \cdot \cdot T_{nm}}{T_{d1}T_{d2} \cdot \cdot \cdot T_{dn}} \cdot \frac{(\frac{1}{T_{n1}} + s)(\frac{1}{T_{n2}} + s) \ . \ . \ . \ (\frac{1}{T_{nm}} + s)}{(\frac{1}{T_{d1}} + s)(\frac{1}{T_{d2}} + s) \ . \ . \ . \ (\frac{1}{T_{dn}} + s)}$$

Then H(s) has to be changed to the following form:

$$H(s) = \text{Gain} \ \frac{\beta_0 s^m + \beta_1 s^{m-1} + \ldots + \beta_m}{\alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n}$$

where

$$\text{Gain} = \frac{T_{n1}T_{n2} \cdot \cdot \cdot T_{nm}}{T_{d1}T_{d2} \cdot \cdot \cdot T_{dn}} \ , \ \alpha_0 = 1 \quad \text{and} \quad n > m$$

The coefficients of the numerator and/or the denominator are then computed as follows:

$$\text{coef}(\ell) = (-1)^{\ell} \sum \begin{pmatrix}\text{products of the roots taken} \\ \ell \text{ at the time}\end{pmatrix}$$

for $\ell = 0, 1, \ldots, r$

(a) Numerator

roots are $-\frac{1}{T_{ni}}$ for $i = 1, \ldots, m$

$\text{coef}(i) = \beta_i$ for $i = 0, 1, \ldots, m$

(b) Denominator

roots are $-\frac{1}{T_{di}}$ for $i = 1, \ldots, n$

$\text{coef}(i) = \alpha_i$ for $i = 0, 1, \ldots, n$

Using the previous expression, a computer program can be written to calculate the coefficients of the transfer function to be used in the algorithm.

(2) The algorithm can be used to compute the frequency response of a discrete time system, by replacing $s = a + jb = \sigma + jw$ with $z = e^{jwT} = \cos wT + j\sin wT$, then

$$E = \begin{bmatrix} \sigma & w \\ -w & \sigma \end{bmatrix}$$

becomes

$$E = \begin{bmatrix} \cos wT & \sin wT \\ -\sin wT & \cos wT \end{bmatrix}$$

E is a rotation matrix, and it satisfies all the properties of e^{jwT}. The map $e^{jwT} \leftrightarrow E$ is an isomorphic map.

VITA

Khier Benmahammed was born in Algeria on June 5, 1952. He received a diploma of engineer in electrical engineering from the University of Annaba, Algeria in 1976. In 1980, he earned the M.S. in electrical engineering from the University of Colorado at Boulder, and in 1984, the M.A. in mathematics from Louisiana State University. Since then, he has been pursuing his Ph.D. at L.S.U. and teaching in the Computer Engineering Department at the University of Minnesota, Duluth.

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