Model Selection for Degree-corrected Block Models

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Abstract

A central problem in analyzing networks is partitioning them into *modules* or *communities*, clusters with a statistically homogeneous pattern of links to each other or to the rest of the network. One of the best tools for this is the *stochastic block model*, which in its basic form imposes a Poisson degree distribution on all nodes within a community or block. In contrast, *degree-corrected* block models allow for heterogeneity of degree within blocks. Since these two model classes often lead to very different partitions of nodes into communities, we need an automatic way of deciding which model is more appropriate to a given graph. We present a principled and scalable algorithm for this model selection problem, and apply it to both synthetic and real-world networks. Specifically, we use belief propagation to efficiently approximate the log-likelihood of each class of models, summed over all community partitions, in the form of the Bethe free energy. We then derive asymptotic results on the mean and variance of the log-likelihood ratio we would observe if the null hypothesis were true. We find that for sparse networks, significant corrections to the classic asymptotic likelihood-ratio theory (underlying χ^2 hypothesis testing or the AIC) must be taken into account. We test our procedure against two real-world networks and find excellent agreement with our theory.

1 Introduction

In many real-world networks, nodes divide naturally into *modules* or *functional communities*, where nodes in the same group connect to the rest of the network in similar ways. Discovering such communities is an important part of modeling networks [20], as community structure offers clues to the processes which generated the graph, on scales ranging from face-to-face social interaction [26] through social-media communications [1] to the organization of food webs [3, 14].

The *stochastic block model* (SBM) [10, 12, 2] has, deservedly, become one of the most popular generative models for community detection. It splits nodes into communities or *blocks*, within which all nodes are *stochastically equivalent* [23]. That is, the probability of an edge between any two nodes

depends only on which blocks they belong to, and all edges are independent given the nodes' block memberships. Block models are highly flexible, representing assortative, disassortative and satellite community structures, as well as combinations thereof, in a single generative framework [17, 18]. Their asymptotic properties, including phase transitions in the detectability of communities, can be determined exactly using tools from statistical physics [9].

Despite this flexibility, SBMs impose real restrictions on networks; notably, the degree distribution within each block is asymptotically Poisson. This makes the SBM implausible for many real-world networks, where the degrees within each community are highly inhomogeneous. Fitting the SBM to such networks tends to split the high- and low- degree nodes in the same community into distinct blocks; for instance, dividing both liberal and conservative political blogs into high-degree "leaders" and low-degree "followers" [1, 13]. To avoid this effect, and allow degree inhomogeneity within blocks, there is a long history of generative models where the probability of an edge depends on node attributes θ_u as well as their group memberships (e.g. [15, 21]). Here we use the variant due to [13], called the *degree-corrected* (DC) block model.

We often lack the domain knowledge to choose between the ordinary and the degree-corrected block model, and so are faced with a classic problem of statistical model selection. The classic frequentist approaches to model selection are largely based on likelihood ratios [6], and we follow that approach here. Since both SBM and DC models have many hidden variables, calculating likelihood ratios is itself non-trivial; the likelihood must be summed over all partitions of nodes into blocks, so (in statistical physics terms) the log-likelihood is a free energy. We approximate this free energy using belief propagation, giving a highly scalable algorithm that can deal with large sparse networks in nearly linear time. However, even with the likelihoods in hand, it turns out that the usual χ^2 theory for likelihood ratios relies on approximations which are invalid in our setting, because of the dependency and sparsity of network data. We derive the correct asymptotics under certain assumptions, recovering the classic asymptotics in the limit of dense graphs, but finding that significant corrections are needed in the sparse case. Numerical experiments confirm the validity of our expressions, and we apply our method to a range of real and synthetic networks.

2 Poisson Stochastic Block Models

Let us set the problem on an observed graph G with n nodes and m edges. We assume G is undirected, though the directed case is only notationally more cumbersome. We want to split the nodes into k communities, taking k to be given *a priori*. (We will address estimating k elsewhere.)

Traditionally, stochastic block models are applied to simple graphs, where each entry A_{uv} of the adjacency matrix follows a Bernoulli distribution. Following e.g. [13], we use a multigraph version of the block model, where the A_{uv} are independent and Poisson-distributed. (For simplicity, we ignore self-loops.) In the sparse network regime we are most interested in, this Poisson mode differs only negligibly from the original Bernoulli model [19], but the former is easier to analyze.

2.1 The Ordinary Stochastic Block Model

In both models, each node u has a latent variable $g_u \in \{1, \ldots, k\}$ indicating which of the k blocks it belongs to. The block assignment is then $g = \{g_u\}$. The g_u are IID draws from a multinomial distribution parameterized by γ, where $\gamma_r = P(g_u = r)$ is the prior probability that a given node belongs to block r. Thus $g_u \sim \text{Multi}(\gamma)$. After it assigns nodes to blocks, each model generates the number of edges A_{uv} between each pair of nodes u and v by making an independent Poisson draw for each pair. In the ordinary stochastic block model, the means of these Poisson draws are specified by the $k \times k$ block affinity matrix ω, so $A_{uv}|g \sim \text{Poi}(\omega_{g_u g_v})$. If we could observe the block assignment g along with G, the "complete data" likelihood would be

$$P(G, g \,|\, \omega, \gamma) = \prod_u \gamma_{g_u} \prod_{u<v} \frac{\omega_{g_u g_v}^{A_{uv}} \, \mathrm{e}^{-\omega_{g_u g_v}}}{A_{uv}!} = \prod_r \gamma_r^{n_r} \prod_{r,s=1}^{k} \omega_{rs}^{m_{rs}/2} \, \mathrm{e}^{-\frac{1}{2} n_r n_s \omega_{rs}} \prod_{u<v} \frac{1}{A_{uv}!} \,. \quad (1)$$

Here n_r denotes the number of nodes in block r, and m_{rs} denotes the number of edges connecting block r to block s, or twice that number if $r = s$. The last term is constant in the parameters, and is

identically 1 for simple graphs, so we will discard it in what follows. The log-likelihood is then

$$\log P(G, g \,|\, \omega, \gamma) = \sum_r n_r \log \gamma_r + \frac{1}{2}\left(\sum_{r,s=1}^{k} m_{rs} \log \omega_{rs} - n_r n_r \omega_{rs}\right) . \tag{2}$$

Maximizing (2) over γ and ω gives

$$\widehat{\gamma}_r = \frac{n_r}{n} , \qquad \widehat{\omega}_{rs} = \frac{m_{rs}}{n_r n_s} \tag{3}$$

Of course, the block assignments g are not observed, but rather are what we most want to infer. We could try to infer g by maximizing (2) over ω, γ and g jointly; in terms borrowed from statistical physics, this amounts to finding the *ground state* $\hat{g}$ that minimizes the *energy* $-\log P(G, g \,|\, \omega, \gamma)$. When this $\hat{g}$ can be found, it recovers the correct g *exactly* if the graph is dense enough [5]. But if we wish to infer the parameters γ, ω, or to perform model selection, we are interested in the total probability that the block model generates the network at hand. This is

$$P(G \,|\, \omega, \gamma) = \sum_g P(G, g \,|\, \omega, \gamma) ,$$

where the sum is over all k^n possible block assignments. Again following the physics picture, this is the partition function of the Gibbs distribution of g, and its logarithm is (minus) the *free energy*.

As is usual with latent variable models, we can infer γ and ω using an EM algorithm [16], where the E step approximates the average over g with respect to the Gibbs distribution, and the M step estimates γ and ω in order to maximize that average. One approach to the E step would use a Monte Carlo Markov Chain (MCMC) algorithm to sample g from the Gibbs distribution. However, as we will see below, in order to determine γ and ω it suffices to estimate the marginal distributions of g_u of each u, and joint marginal distributions of (g_u, g_v) for each pair of nodes u, v [4]. As we show in §3, belief propagation efficiently approximates both the free energy $-\log P(G \,|\, \omega, \gamma)$ and these marginals, and for many networks it converges very rapidly. Other methods of approximating the E step are certainly possible, and could be used with our model-selection analysis.

2.2 The Degree-Corrected Block Model

As discussed above, in the SBM any two nodes in the same block have the same degree distribution. Moreover, their degrees are sums of independent Poisson variables, so this distribution is Poisson. As a consequence, the SBM "resists" putting nodes with very different degrees in the same block. This leads to problems with real networks where the degree distribution is highly skewed.

The degree-corrected (DC) model extends the SBM to allow for heterogeneity of degree within blocks. Nodes are assigned to blocks as before, but each node also gets an additional parameter θ_u, which scales the number of edges connecting it to other nodes. Thus

$$A_{uv} | g \sim \text{Poi}(\theta_u \theta_v \omega_{g_u g_v})$$

Varying the θ_u gives any desired degree sequence, at least in expectation. Since setting $\theta_u = 1$ for all u recovers the SBM, that model is nested inside the DC model, which is strictly more general.

The likelihood stays the same if we increase θ_u by some factor c for all nodes in block r, provided we also decrease ω_{rs} for all s by the same factor. Thus identification demands a constraint, and a convenient one forces θ_u to sum to the total degree within each block: $\sum_{u:g_u=r} \theta_u = \sum_{u:g_u=r} d_u$. We denote this total degree D_r. The complete-data likelihood of the DC model is then

$$\begin{aligned} P(G, g \,|\, \theta, \omega, \gamma) &= \prod_u \gamma_{g_u} \prod_{u<v} \frac{(\theta_u \, \theta_v \, \omega_{g_u g_v})^{A_{uv}}}{A_{uv}!} \exp(-\theta_u \, \theta_v \, \omega_{g_u g_v}) \\ &= \prod_r \gamma_r^{n_r} \prod_u \theta_u^{d_u} \prod_{rs} \omega_{rs}^{m_{rs}/2} \exp(-\frac{1}{2} D_r D_s \omega_{rs}) \prod_{u<v} \frac{1}{A_{uv}!} , \end{aligned} \tag{4}$$

where n_r and m_{rs} are as before. Again ignoring the constant term, the log-likelihood is

$$\log P(G, g \,|\, \theta, \omega, \gamma) = \sum_r n_r \log \gamma_r + \sum_u d_u \log \theta_u + \frac{1}{2}\left(\sum_{rs} m_{rs} \log \omega_{rs} - D_r D_s \omega_{rs}\right) . \tag{5}$$

Maximizing (5) yields the MLEs

$$\widehat{\theta}_u = d_u\,, \quad \widehat{\gamma}_r = \frac{n_r}{n}\,, \quad \widehat{\omega}_{rs} = \frac{m_{rs}}{D_r D_s}\,. \tag{6}$$

However, as with the ordinary SBM, we will estimate γ and ω not just for a ground state $\hat{g}$, but using belief propagation to find the marginal distributions for g_u and pairwise marginals for (g_u, g_v).

3 Belief Propagation and the Bethe Free Energy

We referred above to the use of belief propagation for computing free energies and marginal distributions of block assignments. Here we describe how belief propagation works for the degree-corrected block model, extending the treatment of the SBM in [9]. The key idea [24] is that each node u sends a message to every other node v, indicating the marginal distribution of g_u if v were absent. We write $\mu_r^{u\to v}$ for the probability that u would be of type r in the absence of v. Then $\mu^{u\to v}$ gets updated in light of the messages u gets from the *other* nodes as follows. Let

$$f(\theta_u, \theta_v, \omega_{rs}, A_{uv}) = \frac{(\theta_u\theta_v\omega_{rs})^{A_{uv}}}{A_{uv}!}\exp(-\theta_u\theta_v\omega_{rs}) \tag{7}$$

denote the probability that A_{uv} takes its observed value assuming that $g_u = r$ and $g_v = s$. Then

$$\mu_r^{u\to v} = \frac{1}{Z^{u\to v}}\,\gamma_r \prod_{w\neq u,v}\sum_{s=1}^{k}\mu_s^{w\to u} f(\theta_w, \theta_u, \omega_{rs}, A_{wu})\,, \tag{8}$$

where $Z^{u\to v}$ is a normalization factor set so that $\sum_r \mu_r^{u\to v} = 1$. As usual in belief propagation, we assume here that the block assignments g_w of the other nodes are independent conditioned on g_u.

Note that each node sends messages to every other node, not just to its neighbors, since non-edges are also informative about g_u and g_v. Thus we have a Markov random field on a weighted complete graph, as opposed to just on the network itself. However, keeping track of n^2 messages is cumbersome. For sparse networks, we can restore scalability by noticing that, up to $O(1/n)$ terms, each node u sends the same message to all of its non-neighbors. That is, for any v such that $A_{uv} = 0$, we have $\mu_r^{u\to v} = \mu_r^u$ where

$$\mu_r^u = \frac{1}{Z^u}\,\gamma_r \prod_{w\neq u}\sum_{s=1}^{k}\mu_s^{w\to u} f(\theta_w, \theta_u, \omega_{rs}, A_{wu})\,. \tag{9}$$

This simplification reduces the number of messages to $O(n+m)$. We can then write

$$\mu_r^{u\to v} = \frac{1}{Z^{u\to v}}\,\gamma_r \prod_{w\neq v, A_{uw}\neq 0}\frac{\sum_{s=1}^{k}\mu_s^{w\to u} f(\theta_w, \theta_u, \omega_{rs}, A_{wu})}{\sum_{s=1}^{k}\mu_s^{w} f(\theta_w, \theta_u, \omega_{rs}, 0)} \times \prod_w \sum_{s=1}^{k}\mu_s^w f(\theta_w, \theta_u, \omega_{rs}, 0)\,.$$

Since the second product depends only on θ_u, we can compute it once for each degree in the network, and then update the messages for each u in $O(k^2 d_u)$ time. Thus, for fixed k, the total time it takes to update all the messages is $O(m + \ell n)$, where ℓ is the number of distinct degrees. As discussed in [9], for many networks only a constant number of updates are necessary in order to reach a fixed point, making the entire algorithm quite scalable. Please refer to [9] for details.

The BP estimate of the joint marginals $\Pr[g_u = r, g_v = s]$ is $b_{rs}^{uv} \propto f(\theta_u, \theta_v, \omega_{rs}, A_{uv})\mu_r^{u\to v}\mu_s^{v\to u}$, normalized so that $\sum_{rs} b_{rs}^{uv} = 1$. The M step of the EM algorithm sets γ and ω analogously to (6),

$$\gamma_r = \frac{\bar{n}_r}{n} = \frac{\sum_u \mu_r^u}{n}\,, \quad \omega_{rs} = \frac{\overline{m}_{rs}}{\overline{D}_r\overline{D}_s} = \left(\sum_{u\neq v: A_{uv}\neq 0} A_{uv} b_{rs}^{uv}\right) \Big/ \left(\sum_u d_u\mu_r^u \sum_u d_u \mu_s^u\right)\,. \tag{10}$$

Belief propagation also lets us approximate the total probability summed over g that the model generates G. The *Bethe free energy* is the following approximation to the log partition function [25]:

$$\log P(G\,|\,\theta,\omega,\gamma) \approx \sum_u \log Z^u - \sum_{u\neq v, A_{uv}\neq 0}\log\left[\sum_{rs} f(\theta_u, \theta_v, \omega_{rs}, A_{uv})\mu_r^{u\to v}\mu_s^{v\to u}\right] + \frac{1}{2}\sum_{rs}\omega_{rs}\overline{D}_r\overline{D}_s\,.$$

We reiterate that while we use belief propagation in our numerical work, our results on model selection in the next section are quite indifferent as to *how* the likelihood or free energy is computed.

4 Model Selection

When the degree distribution is relatively homogeneous within each block (e.g. [10, 12]), the ordinary stochastic block model is better than the degree-corrected model, since the extra parameters θ_u simply lead to over-fitting. On the other hand, when degree distributions within blocks are highly heterogeneous, DC is better. However, without prior knowledge about the communities, and thus the block degree distributions, we need to use the data to pick a model, i.e., to do model selection [6].

From the machine-learning perspective, the natural impulse is to reach for multi-fold cross-validation. Unfortunately, because network data is globally dependent, there is as yet no good way to split a given into training and testing sets for cross-validation. Predicting missing links or tagging false positives are popular forms of leave-k-out cross-validation in the network literature [7], but the latter does not converge on the true model even for IID data [6].

Instead, we approach this problem statistically, as a hypothesis testing problem. Since the ordinary SBM is nested within the DC model, any given graph G is at least as likely under the latter as under the former. Moreover, if the SBM really is the better model, the DC should converge to it, at least in the limit of large networks. Our null model $H_0 = \{\gamma, \omega\}$ is then the SBM, and the larger, nesting alternative $H_1 = \{\theta, \gamma, \omega\}$ is the DC model. The appropriate test statistic is the log-likelihood ratio,

$$\Lambda(G) \quad = \quad \log \frac{\sup_{H_1} \sum_g \prod_r \gamma_r^{n_r} \prod_u \theta_u^{d_u} \prod_{rs} \omega_{rs}^{m_{rs}/2} \exp(-\frac{1}{2} D_r D_s \omega_{rs})}{\sup_{H_0} \sum_g \prod_r \gamma_r^{n_r} \prod_{rs} \omega_{rs}^{m_{rs}/2} \exp(-\frac{1}{2} n_r n_s \omega_{rs})} \tag{11}$$

We reject the null model in favor of the more elaborate alternative when Λ exceeds some threshold. This threshold, in turn, is fixed by our desired error rate, and by the distribution of Λ when G is generated from the null model. When G is small, the null-model distribution of Λ can be found through parametric bootstrapping [8]: fitting H_0, generating new graphs $\tilde{G}$ from it, and evaluating $\Lambda(\tilde{G})$. When n is large, however, it would be helpful to replace bootstrapping with analytic calculations.

A classic result in asymptotic statistics [22] asserts that in hypothesis-testing problems like this, the large-sample null distribution of Λ is $2\Lambda(G) \sim \chi^2_\ell$, where ℓ is the number of constraints that must be imposed on H_1 to recover H_0. In this case we have $\ell = n - k$, as we must set all n of the θ_u to 1, while our identifiability convention $\sum_{u:g_u=r} \theta_u = D_r$ already imposed k constraints.

However, deriving the χ^2 distribution relies on a key assumption [22, 11]: namely, that the log-likelihood of both models is well-approximated by a quadratic function in the vicinity of its maximum, so that the parameter estimates have Gaussian distributions around the true model. The most common grounds for this assumption are central limit theorems for IID data, or more generally, being in a "large data limit." We will see that, for sparse networks, this assumption does not hold for the parameters θ_u. Nevertheless, with some work we are able to compute the mean and variance of Λ's null distribution. While we recover the classical χ^2 distribution in the the limit of large, dense graphs, there are significant corrections when the average degree of the graph is small. In particular, χ^2 testing commits type I errors in the sparse case whenever the graph is sufficiently large, rejecting the SBM in favor of DC even for graphs generated by the SBM (see Fig. 2). In essence, it underestimates the amount of degree inhomogeneity we would get simply from random noise, incorrectly concluding that the inhomogeneity must come from underlying properties of the nodes.

To obtain theoretical estimates of the null distribution of Λ, we assume that the Gibbs distribution of both models is concentrated on the same block assignment g. This is a major assumption, but it is borne out by our experiments (Fig. 1 and Fig. 3), and the fact that under some conditions [5] the SBM recovers the underlying block assignment exactly. Under this assumption, while the free energy differs from the ground state energy by an entropy term, the free energy *difference* between the two models has the same distribution as the ground state energy difference. The MLE estimates for H_0 and H_1 are then given by (3) and (6) respectively. Substituting these into (11) gives Λ the form of a Kullback-Leibler divergence,

$$\Lambda(G) = \log \prod_u d_u^{d_u} \prod_{rs} \left(\frac{n_r n_s}{D_r D_s} \right)^{m_{rs}/2} = \log \prod_u \left(\frac{d_u}{\overline{d}_{g_u}} \right)^{d_u} = \sum_u d_u \log \frac{d_u}{\overline{d}_{g_u}} \,, \tag{12}$$

where $\overline{d}_{g_u} = D_{g_u}/n_u$ is the average degree of u's block. Note that $\overline{d}_r$ is the empirical mean, not the expected degree $\mu_r = \sum_s \gamma_s \omega_{rs}$ of the true underlying SBM.

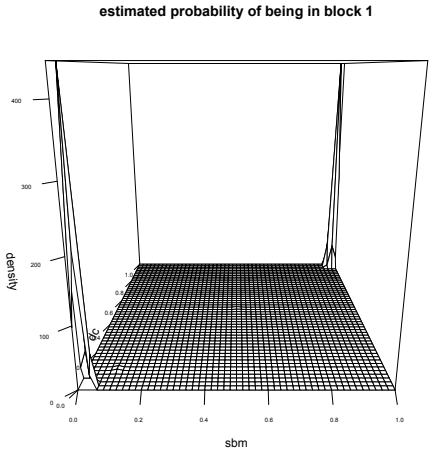


Figure 1: Joint density of posterior probabilities over block assignments, showing that the SBM and the DC are concentrated around the *same* ground state. The synthetic network has $n = 10^3$, $k = 2$ groups of equal size $\gamma_1 = \gamma_2 = 1/2$, average degree $\mu_r = 11$, and associative structure with $\omega_{12}/\omega_{11} = \omega_{21}/\omega_{22} = 1/11$. The x and y axes are the marginal probabilities of being in block 1 according to SBM and DC.

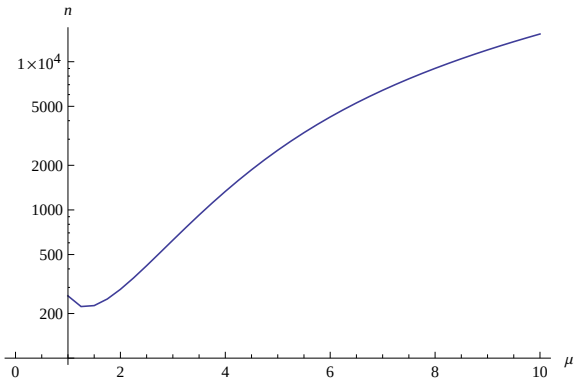


Figure 2: The size n, as a function of the average degree μ, above which naive χ^2 testing commits a type I error with 95% confidence, rejecting the SBM for almost all graphs generated by the SBM. For instance, for $\mu = 5$, χ^2 commits a type I error at roughly $n > 3000$, while for $\mu = 3$, it does so for $n > 700$. Here we assume the asymptotic analysis of (13)–(16) for the mean and variance of the likelihood ratio; see Fig. 3 for comparison with experiment.

We can understand the asymptotic null distribution of Λ by assuming that the d_u in each block r are IID and Poisson with expectation μ_r. This assumption is sound in the limit $n \to \infty$, since the correlations between node degrees are $O(1/n)$. In that case, we can compute the expectation and variance of Λ analytically (see full paper in arXiv). These results show how the behavior of Λ differs from naive χ^2 asymptotics, as well as revealing the limits where the naive results apply. Specifically,

$$\mathbb{E}[\Lambda] = \sum_r n_r f(\mu_r) - f(n_r \mu_r) \tag{13}$$

where if d is Poisson with mean μ,

$$f(\mu) = \mathbb{E}[d \log d] - \mu \log \mu = \sum_{d=1}^{\infty} \frac{\mathrm{e}^{-\mu} \mu^d}{d!} \, d \log d - \mu \log \mu \, . \tag{14}$$

In the limit $\mu \to \infty$, i.e., for dense graphs, both $f(\mu)$ and $f(n\mu)$ approach $1/2$, and (13) gives $\mathbb{E}[\Lambda] = (n-k)/2$ just as in the standard χ^2 analysis. However, when μ is finite, $f(\mu)$ differs significantly from $1/2$.

The variance of Λ is more complicated, but still calculable. The limiting variance per node is

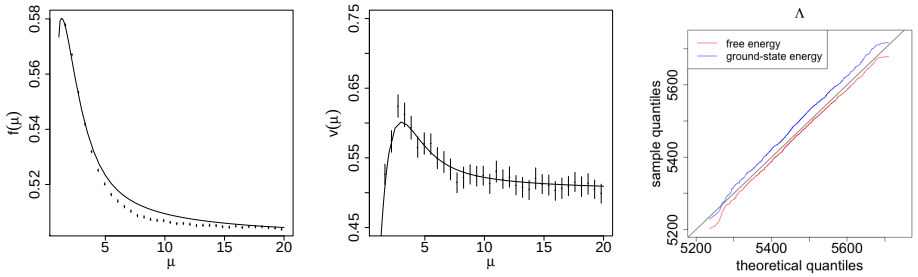
$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Var}[\Lambda] = \sum_r \gamma_r v(\mu_r) \, , \tag{15}$$

where, again taking d to be Poisson with mean μ,

$$v(\mu) = \mu(1 + \log \mu)^2 + \mathrm{Var}[d \log d] - 2(1 + \log \mu) \, \mathrm{Cov}[d, d \log d] \, . \tag{16}$$

Since the variance of χ^2_ℓ is 2ℓ, the χ^2 analysis would predict $(1/n)\mathrm{Var}[\Lambda] = 1/2$. Indeed $v(\mu)$ approaches $1/2$ in the limit $\mu \to \infty$, but like $f(\mu)$ it differs significantly from $1/2$ for finite μ. Plots of $f(\mu)$ and $v(\mu)$ can be found in Fig. 3(a,b). More details are available in the full paper.

Why exactly does the null distribution of Λ differ from the usual χ^2 distribution? The reason is that the parameters θ_u are not in the large data limit. We have one observation for each node, i.e., its



(a) (b) (c)

Figure 3: (a) $f(\mu)$ from (14), the expected log-likelihood difference per node, compared to simulation results; (b) the asymptotic variance of the log-likelihood difference per node, from (16), with simulation results; (c) QQ plots comparing the distribution of log-likelihood differences from 10^4 synthetic networks with $\mu = 3$ to a Gaussian with the theoretical mean and variance, showing that the free energy difference and the ground state energy difference have similar distributions. All simulations have $n = 10^4$, $k = 2$, $\gamma_1 = \gamma_2 = 1/2$, and $\omega_{12}/\omega_{11} = 0.15$, $\omega_{11}/\omega_{22} = 1$. In (a) and (b), each point is the average over 10^3 networks, including 95% bootstrap confidence intervals.

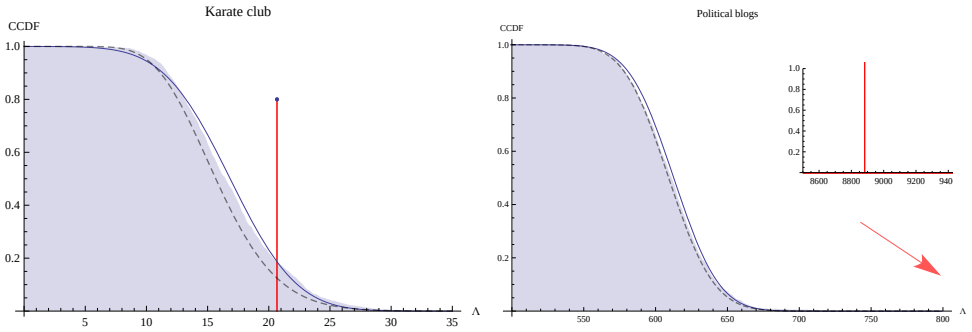
degree d_u. If a Poisson distribution has small mean, its shape differs significantly from a Gaussian, and so does the posterior distribution of the mean based on a single sample. In particular, $P(\theta \mid d)$ follows a Gamma distribution, if the prior on θ is uninformative [27]. When the degrees are large, both the sample distribution and the posterior become Gaussian, and the χ^2 analysis takes over; but when they are small, the geometry is simply different, causing $f(\mu)$ and $v(\mu)$ to differ from $1/2$. This would eventually lead to a type I error for the χ^2 testing, rejecting the SBM for almost all graphs generated by the SBM. Since χ^2 distribution is tightly peaked around $0.5n$, the situation also becomes worse with bigger n (see Fig. 2).

As shown in Fig. 3, experiments on synthetic networks generated from the SBM show that the mean and variance of Λ are very well fit by our theoretical results. We have not attempted to compute higher moments of Λ. However, if we assume that d_u are independent, then the central limit theorem applies, and Λ follows a Gaussian distribution in the limit of large n. Quantile plots from the same experiments (Fig. 3(c)) show that a Gaussian with mean and variance given by (13) and (15) is indeed a good fit. Moreover, the free energy difference and the ground state energy difference have similar distributions, as implied by our assumption that both Gibbs distributions are concentrated around the ground state. Interestingly, in Fig. 3(c), the degree is low enough that this concentration must be imperfect, but our theory still holds remarkably well. Notice all the synthetic experiments done in this paper took the simplifying assumptions that γ and μ for each block is the same.

5 Results on real networks

We have derived the theoretical null distribution of Λ, and backed up our calculations with simulations. We now apply our theory to the real world, considering two examples studied in [13].

The first is a social network consisting of 34 members of a karate club, where undirected edges represent friendships [26]. The network is made up of two assortative blocks centered around the instructor and the club president, each with a high degree hub and lower-degree peripheral nodes. The authors of [13] compared the perfomance of SBM and DC on this network, and heavily favored DC over SBM because the former leads to a community structure agreeing with the ground truth. Our test, however, shows that the evidence is not strong enough to reject the null SBM model with any great confidence. As shown in Fig. 4(a), the distribution of Λ from bootstrap experiments is fit reasonably will by a Gaussian with our predicted mean and variance. The observed $\Lambda = 20.7$ has a p-value of 0.19 according to the theoretical Gaussian, and 0.15 according to the bootstrap distribution. Thus a prudent statistician would think twice before embracing the additional n parameters of DC. Indeed, in a study of active learning, the authors of [14] found that SBM labels most of the nodes correctly if we fix the block assignment of the instructor and the president to 1 and 2 respec-



(a) (b)

Figure 4: Hypothesis testing of real world networks. (a): Zachary's karate club [26], where $n = 34$. The CCDF (complementary cumulative distribution) of the log-likelihood ratio Λ under the null model is estimated using bootstrapping (shaded), and is fit reasonably well by the CCDF of a Gaussian (curve) with our theoretically predicted mean and variance. The observed $\Lambda = 20.7$ (marked with the red line) has p-values of 0.15 and 0.19 according to the bootstrap and theoretical distributions respectively. (b): A network of political blogs [1] where $n = 1222$. The bootstrap distribution (shaded) is very well fit by a Gaussian (curve) with our predicted mean and variance. The actual log-likelihood ratio is so far in the tail (see inset) that its p-value is effectively zero. Thus for the blog network, we can decisively reject the ordinary block model in favor of the degree-corrected model, while for the karate club, the evidence is less clear.

tively. This implies that the degree inhomogeneity is not too extreme, and that only a handful of nodes are responsible for the better performance of DC.

The second example is a network of political blogs in the US assembled by Adamic and Glance [1]. As in [13], we focus on the giant component, which consists of 1222 blogs and 19087 links between them. The blogs have known political leanings, and were labeled as either liberal or conservative. The network is assortative and has a highly right-skewed degree distribution within each block. In its agreement with ground truth, DC substantially outperforms SBM, as observed in [13]. This time around, our hypothesis testing procedure completely agrees with their choice of model. As shown in Fig. 4(b), the bootstrap distribution of Λ is very well fit by a Gaussian with our theoretical prediction of the mean and variance. The observed log-likelihood ratio $\Lambda = 8883$ is 330 standard deviations above the mean. It is essentially impossible to produce such extreme results through mere fluctuations under the null model. Thus, for this network, introducing n extra parameters to capture the degree heterogeneity, and rejecting SBM in favor of DC, is fully justified.

6 Conclusion

We have presented a mathematically principled procedure for determining whether the degree-corrected block model is justified over the ordinary stochastic block model. We found that for sparse networks, the distribution of log-likelihood ratios differs significantly from the naive χ^2 analysis, and showed how to compute its mean and variance exactly in the large-n limit where node degrees are essentially independent and Poisson. We confirmed our calculations with experiments on synthetic networks, and applied our procedure to two real-world networks; one where the ordinary block model can be decisively rejected, and another where the evidence is less clear. We hope that similar approaches will let us choose between competing generative models for network data, and in particular between other variants of the block model such as those in [27].

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